Chapter 3b – Development of Truss Equations

Learning Objectives

- To derive the stiffness matrix for a bar element.
- To illustrate how to solve a bar assemblage by the direct stiffness method.
- To introduce guidelines for selecting displacement functions.
- To describe the concept of transformation of vectors in two different coordinate systems in the plane.
- To derive the stiffness matrix for a bar arbitrarily oriented in the plane.
- To demonstrate how to compute stress for a bar in the plane.
- To show how to solve a plane truss problem.
- To develop the transformation matrix in three-dimensional space and show how to use it to derive the stiffness matrix for a bar arbitrarily oriented in space.
- To demonstrate the solution of space trusses.

Symmetry and Bandwidth

In this section, we will introduce the concepts of symmetry to reduce the size of a problem and of banded-symmetric matrices and bandwidth.

In many instances, we can use symmetry to facilitate the solution of a problem.

Symmetry means correspondence in size, shape, and position of loads; material properties; and boundary conditions that are mirrored about a dividing line or plane.

Use of symmetry allows us to consider a reduced problem instead of the actual problem. Thus, the order of the total stiffness matrix and total set of stiffness equations can be reduced.
Symmetry and Bandwidth - Example 1

Solve the plane truss problem shown below. The truss is composed of eight elements and five nodes.

A vertical load of 2P is applied at node 4. Nodes 1 and 5 are pin supports. Bar elements 1, 2, 7, and 8 have an axial stiffness of $AE$ and bars 3, 4, 5, and 6 have an axial stiffness of $AE$.

Symmetry and Bandwidth - Example 1

In this problem, we will use a plane of symmetry.

The vertical plane perpendicular to the plane truss passing through nodes 2, 4, and 3 is the plane of symmetry because identical geometry, material, loading, and boundary conditions occur at the corresponding locations on opposite sides of this plane.
Symmetry and Bandwidth - Example 1

For loads such as $2P$, occurring in the plane of symmetry, one-half of the total load must be applied to the reduced structure.

For elements occurring in the plane of symmetry, one-half of the cross-sectional area must be used in the reduced structure.

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**Symmetry and Bandwidth - Example 1**

<table>
<thead>
<tr>
<th>Element</th>
<th>$\theta$</th>
<th>C</th>
<th>S</th>
<th>$C^2$</th>
<th>$S^2$</th>
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<td>1</td>
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</table>
Symmetry and Bandwidth - Example 1

\[ k^{(1)} = \frac{AE}{2L} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \]

\[ k^{(2)} = \frac{AE}{2L} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \]

<table>
<thead>
<tr>
<th>Element</th>
<th>( \theta )</th>
<th>C</th>
<th>S</th>
<th>C²</th>
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</tbody>
</table>

Symmetry and Bandwidth - Example 1

\[ k^{(3)} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ k^{(4)} = \frac{AE}{2L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \]

<table>
<thead>
<tr>
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</table>
Symmetry and Bandwidth - Example 1

\[ k^{(5)} = \frac{AE}{2L} \begin{bmatrix} u_3 & v_3 & u_4 & v_4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \]

<table>
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<tr>
<th>Element</th>
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<th>C</th>
<th>S</th>
<th>( C^2 )</th>
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<th>CS</th>
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<td>1</td>
<td>0</td>
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</tr>
</tbody>
</table>

Symmetry and Bandwidth - Example 1

Since elements 4 and 5 lie in the plane of symmetry, one half of their original areas have been used in developing the stiffness matrices.

The displacement boundary conditions are:

\[ u_1 = v_1 = u_2 = u_3 = u_4 = 0 \]

By applying the boundary conditions the force-displacement equations reduce to:

\[ \frac{AE}{2L} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ P \end{bmatrix} \]
**Symmetry and Bandwidth - Example 1**

We can solve the above equations by separating the matrices in submatrices (indicated by the dashed lines). Consider a general set of equations shown below:

\[
\begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix}
\begin{bmatrix}
d_1 \\
d_2
\end{bmatrix}
=
\begin{bmatrix}
0 \\
F
\end{bmatrix}
\]

\[
K_{11}d_1 + K_{12}d_2 = 0 \\
K_{21}d_1 + K_{22}d_2 = F
\]

Solving the first equation for \(d_1\) gives: \(d_1 = -K_{11}^{-1}K_{12}d_2\)

Substituting the above equation in the second matrix equation gives:

\[
K_{21}(-K_{11}^{-1}K_{12}d_2) + K_{22}d_2 = F
\]

Simplifying this expression gives:

\[
(K_{22} - K_{21}K_{11}^{-1}K_{12})d_2 = F
\]

**Symmetry and Bandwidth - Example 1**

The previous equations can be written as: \(k_c d_2 = F\)

where: \(k_c = K_{22} - K_{21}K_{11}^{-1}K_{12}\)

Therefore, the displacements \(d_2\) are: \(d_2 = (k_c)^{-1}F\)

If we apply this solution technique to our example global stiffness equations we get:

\[
k_c = \frac{AE}{L} \left\{ \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} - \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \right\}
\]

Simplifying:

\[
k_c = \frac{AE}{L} \left\{ \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \right\} = \frac{AE}{L} \left\{ \frac{1}{2} \right\} \quad (k_c)^{-1} = \frac{2L}{AE}
\]
**Symmetry and Bandwidth - Example 1**

Therefore, the displacements $d_2$ are:  
\[ d_2 = v_4 = -\frac{2PL}{AE} \]

The remaining displacements can be found by substituting the result for $v_4$ in the global force-displacement equations.

\[
\begin{bmatrix}
  v_2 \\
  v_3 
\end{bmatrix} = - \begin{bmatrix}
  1 & 0 \\
  0 & 1 
\end{bmatrix} \begin{bmatrix}
  1 \\
  2 
\end{bmatrix} \begin{bmatrix}
  -\frac{2PL}{AE} \\
  1 \\
  -\frac{2}{2} 
\end{bmatrix}
\]

Expanding the above equations gives the values for the displacements.

\[
\begin{bmatrix}
  v_2 \\
  v_3 
\end{bmatrix} = \begin{bmatrix}
  -\frac{PL}{AE} \\
  \frac{PL}{AE} \\
  -\frac{PL}{AE} \\
\end{bmatrix}
\]

**Symmetry and Bandwidth**

The coefficient matrix (stiffness matrix) for the linear equations that occur in structural analysis is always symmetric and banded.

Because a meaningful analysis generally requires the use of a large number of variables, the implementation of compressed storage of the stiffness matrix is desirable both from the viewpoint of fitting into memory (immediate access portion of the computer) and computational efficiency.
Symmetry and Bandwidth

Another method, based on the concept of the skyline of the stiffness matrix, is often used to improve the efficiency in solving the equations.

The **skyline** is an envelope that begins with the first nonzero coefficient in each column of the stiffness matrix (see the following figure).

In skyline, only the coefficients between the main diagonal and the skyline are stored.

In general, this procedure takes even less storage space in the computer and is more efficient in terms of equation solving than the conventional banded format.

**Symmetry and Bandwidth**

A matrix is **banded** if the nonzero terms of the matrix are gathered about the main diagonal.

To illustrate this concept, consider the plane truss shown below.

We can see that element 2 connects nodes 1 and 4.

Therefore, the 2 x 2 submatrices at positions 1-1, 1-4, 4-1, and 4-4 will have nonzero coefficients.
Symmetry and Bandwidth

The total stiffness matrix of the plane truss, shown in the figure below, denotes nonzero coefficients with X’s.

The nonzero terms are within the same band. Using a banded storage format, only the main diagonal and the nonzero upper codiagonals need be stored.

We now define the semibandwidth: \( n_b \) as

\[
\begin{align*}
    n_b &= n_d (m + 1) \\
    n_d &= \text{number of degrees of freedom per node} \\
    m &= \text{maximum difference in node numbers}
\end{align*}
\]

where \( n_d \) is the number of degrees of freedom per node and \( m \) is the maximum difference in node numbers determined by calculating the difference in node numbers for each element of a finite element model.

In the example for the plane truss shown above,

\[
\begin{align*}
    m &= 4 - 1 = 3 \quad \text{and} \quad n_d = 2; \\
    n_b &= 2(3 + 1) = 8
\end{align*}
\]
Symmetry and Bandwidth

Execution time (primarily, equation-solving time) is a function of the number of equations to be solved.

Without using banded storage of global stiffness matrix K, the execution time is proportional to \((1/3)n^2\), where \(n\) is the number of equations to be solved.

Using banded storage of K, the execution time is proportional to \(n(n_b)^2\).

The ratio of time of execution without banded storage to that using banded storage is then \((1/3)(n/n_b)^2\).
Symmetry and Bandwidth

For the plane truss example, this ratio is \((1/3)(24/8)^2 = 3\)

Therefore, it takes about three times as long to execute the solution of the example truss if banded storage is not used.

Symmetry and Bandwidth

Several automatic node renumbering schemes have been computerized.

This option is available in most general-purpose computer programs. Alternatively, the wavefront or frontal method are popular for optimizing equation solution time.

In the wavefront method, elements, instead of nodes, are automatically renumbered.

In the wavefront method the assembly of the equations alternates with their solution by Gauss elimination.
**Symmetry and Bandwidth**

The sequence in which the equations are processed is determined by element numbering rather than by node numbering.

The first equations eliminated are those associated with element 1 only.

Next the contributions to stiffness coefficients from the adjacent element, element 2, are eliminated.

If any additional degrees of freedom are contributed by elements 1 and 2 only these equations are eliminated (condensed) from the system of equations.

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**Symmetry and Bandwidth**

As one or more additional elements make their contributions to the system of equations and additional degrees of freedom are contributed only by these elements, those degrees of freedom are eliminated from the solution.

This repetitive alternation between assembly and solution was initially seen as a **wavefront** that sweeps over the structure in a pattern determined by the element numbering.

The **wavefront method**, although somewhat more difficult to understand and to program than the banded-symmetric method, is computationally more efficient.
**Symmetry and Bandwidth**

A banded solver stores and processes any blocks of zeros created in assembling the stiffness matrix.

These blocks of zero coefficients are not stored or processed using the wavefront method.

Many large-scale computer programs are now using the wavefront method to solve the system of equations.

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**Homework Problems**

5. Do problem B.9 on pages 864 in your textbook “A First Course in the Finite Element Method” by D. Logan. Determine the bandwidths of the plane trusses shown in the figure below. What conclusions can you draw regarding labeling of nodes?
Homework Problems

6. Solve the following truss problems. You may use SAP2000 to do truss analysis.

a) For the plane truss shown below, determine the nodal displacements and element stresses.

Nodes 1 and 2 are pin joints.
Let $E = 10^7$ psi and the $A = 2.0$ in$^2$ for all elements.

![Diagram of the plane truss](image)

Homework Problems

b) For the 25-bar truss shown below, determine the displacements and elemental stresses. Nodes 7, 8, 9, and 10 are pin connections.
Let $E = 10^7$ psi and the $A = 2.0$ in$^2$ for the first story and $A = 1.0$ in$^2$ for the top story. Table 1 lists the coordinates for each node. Table 2 lists the values and directions of the two loads cases applied to the 25-bar space truss.

![Diagram of the 25-bar truss](image)

<table>
<thead>
<tr>
<th>Node</th>
<th>$x$ (in)</th>
<th>$y$ (in)</th>
<th>$z$ (in)</th>
</tr>
</thead>
<tbody>
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<td>200.0</td>
</tr>
<tr>
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<td>3</td>
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<tr>
<td>10</td>
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<td>-100.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Note: 1 in = 2.54 cm
Homework Problems

b) For the 25-bar truss shown below, determine the displacements and elemental stresses. Nodes 7, 8, 9, and 10 are pin connections. Let $E = 10^7$ psi and the $A = 2.0 \text{ in}^2$ for the first story and $A = 1.0 \text{ in}^2$ for the top story. Table 1 lists the coordinates for each node. Table 2 lists the values and directions of the two loads cases applied to the 25-bar space truss.

<table>
<thead>
<tr>
<th>Case</th>
<th>Node</th>
<th>$F_x (kip)$</th>
<th>$F_y (kip)$</th>
<th>$F_z (kip)$</th>
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<td>1</td>
<td>1.0</td>
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</tr>
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</table>

Note: 1 kip = 4.45 kN

Homework Problems

c) For the 72-bar truss shown below, determine the displacements and elemental stresses. Nodes 1, 2, 3, and 4 are pin connections. Let $E = 10^7$ psi and the $A = 1.0 \text{ in}^2$ for the first two stories and $A = 0.5 \text{ in}^2$ for the top two stories. Table 3 lists the values and directions of the two loads cases applied to the 72-bar space truss.

<table>
<thead>
<tr>
<th>Case</th>
<th>Node</th>
<th>$F_x (kip)$</th>
<th>$F_y (kip)$</th>
<th>$F_z (kip)$</th>
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Note: 1 kip = 4.45 kN
End of Chapter 3b