Deflection – Part 2

Blessed are they who can laugh at themselves for they shall never cease to be amused.

Statically Indeterminate Beams

- We can use the same method that we used for deflection to analyze statically indeterminate beams

Statically Indeterminate Beams

- If we start with a beam loaded as shown
  - The left end is supported as a fixed end support
  - The right end is supported on a roller

Statically Indeterminate Beams

- If we remove the supports and look at the reactions we have
Statically Indeterminate Beams

We have four unknowns and three equilibrium conditions.

We use up one of those conditions solving for \( A_x = 0 \).

We still have three unknown reactions and only two equilibrium conditions to solve for them.

We can reduce the distributed load to a point load but that probably won't help much.
Statically Indeterminate Beams

- We do know the loading rate on the beam so we can start from that

\[ w(x) = w_0 \]

- From that we can look at the change in the shear across the beam

\[ w(x) = w_0 \]

\[ \frac{dV}{dx} = -w(x) = -w_0 \]

- Integrating to find the shear at any distance into the beam we have

\[ V(x) = -w_0x + C_1 \]

- We have two possible boundary conditions that we can use to solve for \( C_1 \)

\[ V(x) = -w_0x + C_1 \]

- at \( x = 0, V = A_y \) and \( x = L, V = B_y \)
Statically Indeterminate Beams

Using the left hand boundary condition we have

\[ V(x) = -w_0 x + C_1 \]
\[ V(0) = A_y = -w_0 (0) + C_1 \]
\[ M_A C_1 = A_y \]
\[ A \quad B \]
\[ y \]
\[ M_A \]
\[ A \quad L \quad B \]
\[ y \]
\[ V(x) = -w_0 x + A_y \]

Statically Indeterminate Beams

So our expression for the shear is

\[ V(x) = -w_0 x + A_y \]

Statically Indeterminate Beams

If we now utilize the expression for the shear to develop the expression for the moment at any \( x \) we have

\[ V(x) = -w_0 x + A_y \]
\[ \frac{dM}{dx} = V(x) = -w_0 x + A_y \]
\[ M(x) = -\frac{w_0 x^2}{2} + A_y x + C_2 \]

Statically Indeterminate Beams

We can use our boundary condition again to solve for \( C_2 \)

\[ M(x) = -\frac{w_0 x^2}{2} + A_y x + C_2 \]
\[ M(0) = -M_A = -\frac{w_0 (0)^2}{2} + A_y (0) + C_2 \]
\[ -M_A = C_2 \]

Notice that the sign of \( M_A \) is negative. This is due to the fact that we are directing our \( y \)-axis downward so a positive moment will be into the page or \( CW \). (i cross j) = positive \( k \)
Statically Indeterminate Beams

- So our expression for the moment at any point in the beam is

\[ M(x) = -\frac{wx^2}{2} + A_x x - M_A \]

- We can continue with the process utilizing our slope equation

\[ M(x) = -\frac{wx^2}{2} + A_x x - M_A \]

\[ \frac{d\theta}{dx} = \frac{M(x)}{EI} \]

- An easier to use form can be generated by multiplying both sides by EI and taking the negative sign inside of the expression for the moment

\[ EI \frac{d\theta}{dx} = -\frac{wx^2}{2} - A_x x + M_A \]

- Integrating for the slope we have

\[ EI \theta(x) = -\frac{wx^3}{6} - \frac{A_x x^2}{2} + M_A x + C_1 \]
Statically Indeterminate Beams

- We have a boundary condition that we can utilize here to solve for $C_3$
- At the support at A, the slope is equal to 0

$$EI\theta(x) = \frac{w_0 x^3}{6} - \frac{A_3 x^2}{2} + M_3 x + C_3$$

$$EI\theta(0) = 0 = \left( \frac{w_0 (0)^3}{6} - \frac{A_3 (0)^2}{2} + M_3 (0) + C_3 \right)$$

$0 = C_3$

Statically Indeterminate Beams

So the slope equation has the form

$$EI\theta(x) = \frac{w_0 x^3}{6} - \frac{A_3 x^2}{2} + M_3 x$$

Statically Indeterminate Beams

- Finally utilizing the deflection relationship we have

$$\theta(x) = \frac{dv}{dx}$$

$$EIv(x) = \frac{w_0 x^4}{24} - \frac{A_3 x^3}{6} + \frac{M_3 x^2}{2} + C_4$$

Statically Indeterminate Beams

- Utilizing our boundary condition at the support at A to solve for $C_4$
- At the support we have a deflection of 0

$$EIv(x) = \frac{w_0 x^4}{24} - \frac{A_3 x^3}{6} + \frac{M_3 x^2}{2} + C_4$$

$$EIv(0) = 0 = \left( \frac{w_0 (0)^3}{6} - \frac{A_3 (0)^2}{2} + M_3 (0) \right) + C_4$$

$0 = C_4$
Statically Indeterminate Beams

- So our final form for the deflection is

$$EIV(x) = + \frac{w_0x^4}{24} - \frac{A_x x^3}{6} + \frac{M_A x^2}{2}$$

- There are actually still two boundary conditions that we know but we haven’t used:
  - At \( x = L, M = 0 \)
  - At \( x = L, v = 0 \)

$$EIV(x) = + \frac{w_0x^4}{24} - \frac{A_x x^3}{6} + \frac{M_A x^2}{2}$$

$$M(x) = - \frac{w_0x^2}{2} + A_x x - M_A$$

- If we use these two conditions in the expression below, we will have two equations with two unknowns:
  - At \( x = L, M = 0 \)
  - At \( x = L, v = 0 \)

$$EIV(L) = 0 = + \frac{w_0L^4}{24} - \frac{A_x(L)^3}{6} + \frac{M_A(L)^2}{2}$$

$$M(L) = 0 = - \frac{w_0(L)^2}{2} + A_x(L) - M_A$$

- Working on the math

\[
0 = + \frac{w_0L^2}{12} - \frac{A_x L}{3} + M_A \\
0 = - \frac{w_0(L)^2}{2} + A_x(L) - M_A
\]
Statically Indeterminate Beams

- Adding the equations together

\[ 0 = \frac{5w_0L^2}{12} + \frac{2A_yL}{3} \]
\[ A_y = \frac{5w_0L^2}{12} - \frac{3}{2L} = \frac{5w_0L}{8} \]

- Substituting the \( A_y \) into the second equation we have

\[ 0 = -\frac{w_0(L)^2}{2} + A_y(L) - M_A \]
\[ 0 = -\frac{w_0(L)^2}{2} + \frac{5w_0L^2}{8}(L) - M_A \]
\[ M_A = -\frac{w_0L^2}{2} + \frac{5w_0L^2}{8} = \frac{w_0L^2}{8} \]

Statically Indeterminate Beams

- You can substitute the values for \( M_A \) and \( A_y \) in to the expressions for slope and deflection
- You can also solve for \( B_y \) using the equilibrium expression

\[ M_A = -\frac{w_0L^2}{2} + \frac{5w_0L^2}{8} = \frac{w_0L^2}{8} \]

Boundary Conditions

- If you can write the expressions for the loading, shear, moment, slope, and deflection you have very powerful tools for solving indeterminate structures
- To be able to use these tools you must be able to identify the boundary conditions that can be used
Boundary Conditions
- If you are at a fixed end support
  - The slope ($\theta$) is equal to 0
  - The deflection ($v$) is equal to 0
- If you are at a pin or roller
  - The deflection ($v$) is equal to 0
  - The moment ($M$) is equal to 0

Boundary Conditions
- If you are at a point in a beam where two solutions meet
  - The slope ($\theta$) from one direction is equal to the slope from the other direction
  - The deflection ($v$) from one direction is equal to the deflection from the other direction
  - The moment ($M$) from one direction is equal to the moment from the other direction
  - The shear ($V$) from one direction is equal to the shear from the other direction

Homework
- 9-2.1
- 9-2.4
- 9-2.7
- In Class test next Wed