Learning Objectives - Confidence Intervals

- Define confidence intervals, and explain their significance to point estimates.
- Identify and apply the appropriate confidence interval for engineering-oriented problems.
Introduction

- We have discussed point estimates:
  - as an estimate of a success probability, $p$
  - as an estimate of population mean, $\mu$

- These point estimates are almost never exactly equal to the true values they are estimating.

- In order for the point estimate to be useful, it is necessary to describe just how far off from the true value it is likely to be.
Confidence Intervals

- Since the population mean will not be exactly equal to the sample mean, $\bar{x}$, it is best to construct a confidence interval around $\bar{x}$ that is likely to cover the population mean.

- We can then quantify our level of confidence that the population mean is actually covered by the interval.
The Central Limit Theorem

Suppose we have a population described by a random variable $X$ with a mean $\mu$ and a standard deviation $\sigma$. We place no restrictions on the probability distribution of $X$. It may be normally distributed, uniformly distributed, exponentially distributed, it doesn’t matter.

Suppose we now take random samples from this population, each with a fixed and large sample size $n$. Each sample will have a sample mean $\bar{X}$, and this $\bar{X}$ will not, in general, be equal to the population mean $\mu$.

After repeated samplings, we will have built a population of $\bar{X}$s. The $\bar{X}$s are themselves random variables and they have their own probability distribution!

The **Central Limit Theorem** says that, as long as $n$ is reasonably large,

$$\bar{X} \sim N\left[\mu, \frac{\sigma^2}{n}\right]$$

If $\sigma^2/n$ is the variance of the sampling distribution, then the standard deviation is $\sigma/\sqrt{n}$. This is commonly referred to as the **standard error of the mean**.
Confidence Interval on a Mean
(n large)

An equation for the \((1 - \alpha) \times 100\%\) confidence interval on a mean:

\[
\bar{x} \pm z_{\alpha/2} \left( \frac{\sigma}{\sqrt{n}} \right)
\]

where \(z_{\alpha/2}\) is the critical point corresponding to a tail area of \(\alpha/2\)

This equation can be used as long as \(n \geq 30\), even if \(\sigma\) is unknown.
Example

a. Compute a 90% confidence interval for $\mu$ when $\sigma = 3.0$, $\bar{x} = 58.3$, and $n = 25$.

b. Compute a 99% confidence interval for $\mu$ when $\sigma = 3.0$, $\bar{x} = 58.3$, and $n = 100$.

c. How large must $n$ be for the width of the 99% confidence interval to be less than 1.0?
What if $n$ is small?

Student’s t Distribution

- As the sample size becomes smaller, the sample standard deviation becomes an increasingly poor approximation of the population standard deviation. The end result is that a 95% confidence interval computed using $s$ instead of $\sigma$ may actually only contain the population mean 90% of the time, or 85% of the time, or even less.

- William Gosset developed a new probability distribution, which he called the t distribution, to describe the probabilities associated with the statistic

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$
What if $n$ is small?

Student’s t Distribution

Figure 5.9
Confidence Interval on a Mean

(σ UNKNOWN, n small)

An equation for the (1−α)×100% confidence interval on a mean:

$$\bar{x} \pm t_{\frac{\alpha}{2}, n-1} \left( \frac{s}{\sqrt{n}} \right)$$

Where $t_{\frac{\alpha}{2}, n-1}$ is the critical point corresponding to a tail area of $\frac{\alpha}{2}$.
## Student’s t Distribution

Upper critical values of Student's t distribution with \( v \) degrees of freedom

<table>
<thead>
<tr>
<th>Probability of exceeding the critical value</th>
<th>( v )</th>
<th>0.10</th>
<th>0.05</th>
<th>0.025</th>
<th>0.01</th>
<th>0.005</th>
<th>0.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>3.078</td>
<td>6.314</td>
<td>12.706</td>
<td>31.821</td>
<td>63.657</td>
<td>318.313</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>1.886</td>
<td>2.920</td>
<td>4.303</td>
<td>6.965</td>
<td>9.925</td>
<td>22.327</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>1.638</td>
<td>2.353</td>
<td>3.182</td>
<td>4.541</td>
<td>5.841</td>
<td>10.215</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>1.533</td>
<td>2.132</td>
<td>2.776</td>
<td>3.747</td>
<td>4.604</td>
<td>7.173</td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td>1.476</td>
<td>2.015</td>
<td>2.571</td>
<td>3.365</td>
<td>4.032</td>
<td>5.893</td>
<td></td>
</tr>
<tr>
<td>6.</td>
<td>1.440</td>
<td>1.943</td>
<td>2.447</td>
<td>3.143</td>
<td>3.707</td>
<td>5.208</td>
<td></td>
</tr>
<tr>
<td>7.</td>
<td>1.415</td>
<td>1.895</td>
<td>2.365</td>
<td>2.998</td>
<td>3.499</td>
<td>4.782</td>
<td></td>
</tr>
<tr>
<td>8.</td>
<td>1.397</td>
<td>1.860</td>
<td>2.306</td>
<td>2.896</td>
<td>3.355</td>
<td>4.499</td>
<td></td>
</tr>
<tr>
<td>9.</td>
<td>1.383</td>
<td>1.833</td>
<td>2.262</td>
<td>2.821</td>
<td>3.250</td>
<td>4.296</td>
<td></td>
</tr>
<tr>
<td>10.</td>
<td>1.372</td>
<td>1.812</td>
<td>2.228</td>
<td>2.764</td>
<td>3.169</td>
<td>4.143</td>
<td></td>
</tr>
</tbody>
</table>
Example

An unconfined compression test performed on 15 concrete cylinders produced the following strength results (in psi):

2670  2580  2400
2490  2640  2590
2440  2170  2410
2590  2730  2690
2730  2480  2360

Find a 95% confidence interval for the true average strength of the concrete.
Confidence Interval on Differences

(*σ₁* and *σ₂* KNOWN)

An equation for the (1 − α)×100% confidence interval on a difference in means:

\[
(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}
\]

where *z*_{α/2} is the critical point corresponding to a tail area of *α*/2

This relationship is exact if the two populations are normally distributed. Otherwise, the confidence interval is approximately valid for large sample sizes (*n₁ ≥ 30* and *n₂ ≥ 30*).
Example

Aluminum spars from two different suppliers are used in manufacturing the wing of a commercial aircraft. You have been asked to determine if the latest shipments from each supplier are equally strong. From past experience, the standard deviations of the tensile strengths are known to be 1.5 kg/mm² for Supplier 1 and 1.0 kg/mm² for Supplier 2 (who has tighter quality control). A sample of 12 spars from Supplier 1 has a mean tensile strength of 87.6 kg/mm² and a sample of 10 spars from Supplier 2 has a mean tensile strength of 72.5 kg/mm². If \( \mu_1 \) and \( \mu_2 \) denote the true mean tensile strengths for the two shipments of spars, find a 90% confidence interval on the difference in mean strength, \( \mu_1 - \mu_2 \).
Confidence Interval on Differences
\((\sigma_1 \text{ and } \sigma_2 \text{ UNKNOWN but equal})\)

If random samples of size \(n_1\) and \(n_2\) are drawn from two normal populations with equal but unknown variances, a \(100(1-\alpha)\%\) confidence interval on the difference between the sample means, \(\mu_1 - \mu_2\) is:

\[
(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2, n_1 + n_2 - 2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}
\]

where \(S_p\) is a “pooled” estimator of the unknown standard deviation and is calculated as:

\[
S_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}
\]

But this can only be used if both populations are normally distributed.
Example

The drying time of pavement marking paint is of concern to transportation engineers. Of two such paints from a particular manufacturer, it is suspected that yellow paint dries faster than white paint. Sample measurements of the drying times of both paints (in minutes) are given below.

White: 120, 132, 123, 122, 140, 110, 120, 107
Yellow: 126, 124, 116, 125, 109, 130, 125, 117, 129, 120

Find a 95% confidence interval on the difference in mean drying times, assuming that the drying times are normally distributed and the standard deviations of the drying times are equal.
Confidence Intervals on Paired Samples

An equation for the \((1-\alpha)\times100\%\) confidence interval on \(\bar{d}\) for a paired sample:

\[
\bar{d} \pm t_{\alpha/2,n-1} \left( \frac{s_d}{\sqrt{n}} \right)
\]

But this can only be used if both populations are \textit{normally distributed}. 
Example

The manager of a fleet of automobiles is testing two brands of radial tires. He assigns one tire of each brand at random to the two front wheels of eight different cars and runs the cars until the tires wear out. The tire lives (in miles) are shown below. Assuming that the tire lives for both brands are normally distributed, find a 99% confidence interval on the difference in mean life.

<table>
<thead>
<tr>
<th>Car</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brand 1</td>
<td>36,925</td>
<td>45,300</td>
<td>36,240</td>
<td>32,100</td>
<td>37,210</td>
<td>48,360</td>
<td>38,200</td>
<td>33,500</td>
</tr>
<tr>
<td>Brand 2</td>
<td>34,318</td>
<td>42,280</td>
<td>35,500</td>
<td>31,950</td>
<td>38,015</td>
<td>47,800</td>
<td>37,810</td>
<td>33,215</td>
</tr>
</tbody>
</table>
Confidence Interval on the Variance

If a random sample of size n is taken from a normally distributed population, a 100(1–α)% confidence interval on the variance of the population is:

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}$$

But this can only be used if the population is normally distributed.

Here, $\chi^2_{\alpha/2,n-1}$ and $\chi^2_{1-\alpha/2,n-1}$ are the upper and lower critical points of the chi-square distribution with $n-1$ degrees of freedom. Because the $\chi^2$ distribution is asymmetrical, the upper and lower tails are not the same.
Example

The compressive strength of concrete is being tested by a civil engineer. He tests 12 specimens and obtains the following data:

2216  2225  2318
2237  2301  2255
2249  2281  2275
2204  2263  2295

Find the 95% confidence interval on the population variance.
Confidence Interval on Ratio of Variances (\(\sigma_1\) and \(\sigma_2\) UNKNOWN):

A 100(1 – \(\alpha\))% confidence interval on the ratio of variances (assuming both populations are normally distributed) is:

\[
\frac{s_1^2}{s_2^2} \cdot \frac{1}{F_{\alpha/2, \nu_1, \nu_2}} \leq \frac{\sigma_1^2}{\sigma_2^2} \leq \frac{s_1^2}{s_2^2} \cdot F_{\alpha/2, \nu_2, \nu_1}
\]

where:

\(\nu_1 = n_1 - 1; \nu_2 = n_2 - 1\)

\[F_{1-\alpha, \nu} = \frac{1}{F_{\alpha, \nu, \nu}}\]
Example

The diameter of steel rods manufactured on two different extrusion machines is being investigated. Two random samples of sizes $n_1 = 15$ and $n_2 = 18$ were selected from the two machines. The sample means and variances are $m_1 = 8.73$, $s^2_1 = 0.35$, $m_2 = 8.68$, $s^2_2 = 0.40$. Construct a 95% confidence interval on the ratio of the population variances.