“Experience is that marvelous thing that enables you to recognize a mistake when you make it again.”
-Franklin P. Jones

Objectives

- Understand how the Cartesian coordinate system can lead to the idea of a Cartesian representation
- Understand the idea of a unit vector and how it is applied to develop Cartesian representation
- Understand the use of a position vector to develop the unit vector along a line of action
Tools

- Basic Trigonometry
- Pythagorean Theorem

Review

Remember, it how you draw the picture that can determine the angle if you don’t stick to CCW from the +x-axis

\[ \overrightarrow{F} = \overrightarrow{F}_x + \overrightarrow{F}_y \]
\[ F_y = F \sin(\alpha) \]
\[ F_x = F \cos(\alpha) \]
\[ F = \sqrt{(F_x)^2 + (F_y)^2} \]
Three Dimensional Representation

- Now we can extend our Cartesian representation to a third mutually perpendicular axis
- We have to be careful in this case as to how we construct our axes because the vector rules require a certain configuration

Three Dimensional Representation

- Previously we had an x and a y axis
- Now we are going to add a z axis
- One of the biggest problems with this is the difficulty of representing this third dimension on a two-dimensional field such as screens, white-boards, paper, or chalk-boards
Three Dimensional Representation

- When we do represent a three-dimensional system, we try to show the third axis as either entering or leaving the page.

You may see the axes represented either of the two ways shown below or in some other form to try and show a three dimensional view.
Labeling the axes, you need to follow the “right hand rule.”

Informally, the right hand rule states that if you lay the heel of your right hand along the x-axis with the fingers pointing in the +x direction.

Then flex your fingers to point in the +y direction (no more than ninety degrees).

Your thumb must point in the +z direction.
Three Dimensional Representation

- For example in the two figures shown below, we have correct right hand systems

![Diagram](attachment:figure.png)

- We can extend the same arguments that we made in developing two dimensional resultant vectors from components along the x and y axes into three dimensions
- We will start with the x and y components of the force
Three Dimensional Representation

- We can start by extending the idea of a resultant vector as the sum of its component vectors.
- In this case, we have three component vectors.
  - One parallel to the x-axis, one parallel to the y-axis, and one parallel to the z-axis.

\[
\vec{F} = \vec{F_x} + \vec{F_y} + \vec{F_z}
\]

If we start with the components parallel to the x and y axes.
We can use our usual method and add these two components together.

This generates a new vector $F_{xy}$. This isn’t the final resultant, it is an intermediate that we will use in the next step.
Three Dimensional Representation

We use the same method to add the z-component.

\[ \vec{F}_{xy} = \vec{F}_x + \vec{F}_y \]

\[ \vec{F} = \vec{F}_{xy} + \vec{F}_z \]

Useful Formula

- The magnitude of the resultant of the three components is

\[ F = \sqrt{(F_x)^2 + (F_y)^2 + (F_z)^2} \]

\[ \vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z \]
Unit Vectors

- It is often convenient to extract the direction from a vector, especially if it is in a three-dimensional space.
- We can do this by recognizing a common multiple of the vector and all vectors which lay along the same line of action.

Unit Vectors

- If we were to travel on a simple map from point A to point B.
Unit Vectors

- We could describe the trip as moving 400 miles east and 300 miles north

![Diagram](image1)

- The total journey would be 500 miles (good old 3-4-5 triangle)

![Diagram](image2)
Unit Vectors

Now what if we know that a journey from A to C has to travel in the exact same direction and that the distance is 1200 miles.

- How far east do we have to travel?
- How far north?

The east distance will have the same ratio in both cases.

\[
\frac{400 \text{ miles}}{500 \text{ miles}} = \frac{? \text{ miles}}{1200 \text{ miles}}
\]
Unit Vectors

Since the units cancel on the left side of the expression we have:

\[
\frac{400\text{ miles}}{500\text{ miles}} = \frac{?\text{ miles}}{1200\text{ miles}}
\]

\[
0.8 = \frac{?\text{ miles}}{1200\text{ miles}}
\]

So the amount east we have to travel will always be the total distance we travel times 0.8.

\[
0.8 \times 1200\text{ miles} = 960\text{ miles}
\]
Unit Vectors

- In the same manner, the amount we have travel north can be calculated as:

$$\frac{300\text{miles}}{500\text{miles}} \times 1200\text{miles} = 0.6 \times 1200\text{miles} = 720\text{miles}$$

We can formalize this idea of the ratios giving us the direction, and therefore the components of a vector using the idea of a unit vector.

- Remember, all vectors along a line of action will have the same direction even though they may vary in their magnitude.
Unit Vectors

- We can start by looking at the resultant and the three component vectors

If we create a triangle with $\mathbf{F}$ as the hypotenuse and $\mathbf{F}_x$ as one of the sides
Then we can use the angle $\alpha$ to relate the two sides together:

$$\cos(\alpha) = \frac{F_x}{F}$$

90 degree angle

We can develop the same relationship between $F_y$ and $F$:

$$\cos(\beta) = \frac{F_y}{F}$$

90 degree angle
Unit Vectors

- And between $F_z$ and $F$

$$\cos(\gamma) = \frac{F_z}{F}$$

90 degree angle

- Collecting all these relationships we have

$$\cos(\alpha) = \frac{F_x}{F}$$

$$\cos(\beta) = \frac{F_y}{F}$$

$$\cos(\gamma) = \frac{F_z}{F}$$
Unit Vectors

- The cosine of \(a\) is known as the direction cosine of \(F\) with the x-axis
- The cosine of \(b\) is known as the direction cosine of \(F\) with the y-axis
- The cosine of \(g\) is known as the direction cosine of \(F\) with the z-axis

\[
\cos(\alpha) = \frac{F_x}{F}
\]
\[
\cos(\beta) = \frac{F_y}{F}
\]
\[
\cos(\gamma) = \frac{F_z}{F}
\]

Unit Vectors

- If we go back to our original formulation of a force as the vector sum of its components we have

\[
F = \sqrt{(F_x)^2 + (F_y)^2 + (F_z)^2}
\]

\[
\cos(\alpha) = \frac{F_x}{F}
\]
\[
\cos(\beta) = \frac{F_y}{F}
\]
\[
\cos(\gamma) = \frac{F_z}{F}
\]
Unit Vectors

Substituting

\[ F = \sqrt{(F \cos(\alpha))^2 + (F \cos(\beta))^2 + (F \cos(\gamma))^2} \]

\[ F = F \sqrt{(\cos(\alpha))^2 + (\cos(\beta))^2 + (\cos(\gamma))^2} \]

\[ 1 = \sqrt{(\cos(\alpha))^2 + (\cos(\beta))^2 + (\cos(\gamma))^2} \]

\[ 1 = (\cos(\alpha))^2 + (\cos(\beta))^2 + (\cos(\gamma))^2 \]

\[ \cos(\alpha) = \frac{F_x}{F} \]
\[ \cos(\beta) = \frac{F_y}{F} \]
\[ \cos(\gamma) = \frac{F_z}{F} \]

Unit Vectors

We can now introduce the concept of a unit vector

A unit vector, \( \mathbf{u} \), is a vector which has a magnitude of 1 and has no units

It is used to convey direction and actually contains the ratios that we developed in the map example.
Unit Vectors

- A scalar times a vector is a vector
- This allows us to rewrite a force vector $\vec{F}$ as its magnitude $F$ times a unit vector $\vec{u}_F$ that is along the line of action of and in the same direction as $\vec{F}$

$$\vec{F} = F \vec{u}_F$$

Unit Vectors

- This is a fundamental idea
- $F$ is the magnitude of the vector $\vec{F}$
- $\vec{u}_F$ is a unit vector which has the same direction as $\vec{F}$

$$\vec{F} = F \vec{u}_F$$
Unit Vectors

- We have assigned special labels to three unit vectors
  - $\mathbf{i}$ is the unit vector in the $+x$-direction
  - $\mathbf{j}$ is the unit vector in the $+y$-direction
  - $\mathbf{k}$ is the unit vector in the $+z$-direction

$$\mathbf{F} = \mathbf{F}_\mathbf{u}_\mathbf{F}$$

Unit Vectors

- We can use these definitions to generate some very useful formulas

$$\mathbf{F} = \mathbf{F}_\mathbf{u}_\mathbf{F}$$
$$\mathbf{F}_x = \mathbf{F}_x \mathbf{u}_x = \mathbf{F}_x \mathbf{i}$$
$$\mathbf{F}_y = \mathbf{F}_y \mathbf{u}_y = \mathbf{F}_y \mathbf{j}$$
$$\mathbf{F}_z = \mathbf{F}_z \mathbf{u}_z = \mathbf{F}_z \mathbf{k}$$
Unit Vectors

- We can use these definitions to generate some very useful formulas

\[ \vec{F} = F \vec{u}_F = F_x \vec{i} + F_y \vec{j} + F_z \vec{k} \]

\[ \vec{u}_F = \frac{F_x}{F} \vec{i} + \frac{F_y}{F} \vec{j} + \frac{F_z}{F} \vec{k} \]

\[ \vec{u}_F = \cos(\alpha) \vec{i} + \cos(\beta) \vec{j} + \cos(\gamma) \vec{k} \]

- So depending on what we know
  - Direction cosines
  - Component magnitudes
  - Unit vector and magnitude

- We can develop complete vector descriptions

\[ \vec{F} = F \vec{u}_F = F_x \vec{i} + F_y \vec{j} + F_z \vec{k} \]

\[ \vec{u}_F = \frac{F_x}{F} \vec{i} + \frac{F_y}{F} \vec{j} + \frac{F_z}{F} \vec{k} \]

\[ \vec{u}_F = \cos(\alpha) \vec{i} + \cos(\beta) \vec{j} + \cos(\gamma) \vec{k} \]
Position Vectors

- Position vectors are vectors which connect two points in space.
- If we know two points on the line of action of the force vector we are interested in we can use a position vector between these two points to develop the unit vector for the force.
- Here is where you have to be careful about units.

Position Vectors

- If we have two points in space along the line of action of a force we are interested in.
  - Point A
  - Point B
We can develop a vector along the line segment from A to B by calculating the changes in the x, y, and z direction made traveling from A to B.

This is especially easy if A and B are given as coordinate triplets.

\[ A = \{-2, 4, 3\} \] in
\[ B = \{5, -1, 5\} \] in
Position Vectors

- Then the directed line segment which goes from A to B can be found by

\[
A = \{-2, 4, 3\} in \\
B = \{5, -1, 5\} in \\
\vec{r}_{AB} = \{B\} - \{A\}
\]

- Take care about the order of the coordinates of A and B
- The position vector \( \vec{r}_{AB} \) goes from A to B so we subtract the coordinates of A from the coordinates of B

\[
A = \{-2, 4, 3\} in \\
B = \{5, -1, 5\} in \\
\vec{r}_{AB} = \{B\} - \{A\}
\]
Position Vectors

- The difference in the x coordinates of A and B gives the magnitude of the position vector in the x or i direction.
- Likewise, the difference in the y coordinates corresponds to the magnitude in the y or j direction and the difference in the z coordinates corresponds to the magnitude in the z of k direction.

For our problem, we can calculate the position vector, \( \mathbf{r}_{AB} \)

\[
\begin{align*}
A &= \{-2, 4, 3\} in \\
B &= \{5, -1, 5\} in \\
\mathbf{r}_{AB} &= \{B\} - \{A\} \\
\mathbf{r}_{AB} &= \left[\{5 - (-2)\} \mathbf{i} + \{-1\} - 4 \mathbf{j} + \{5 - 3\} \mathbf{k}\right] in \\
\mathbf{r}_{AB} &= \left[7 \mathbf{i} - 5 \mathbf{j} + 2 \mathbf{k}\right] in
\end{align*}
\]
Position Vectors

- Another way to read this is to say that to get from A to B we had to move 7 inches in the x direction, -5 inches in the y direction, and 2 inches in the z direction.

\[ A = \{-2,4,3\} \text{ in} \]
\[ B = \{5,-1,5\} \text{ in} \]
\[ \vec{r}_{AB} = \{B\} - \{A\} \]
\[ \vec{r}_{AB} = \left[\{5-(-2)\}i + \{(1)-4\}j + \{5-3\}k\right] \text{ in} \]
\[ \vec{r}_{AB} = \left[7i - 5j + 2k\right] \text{ in} \]

- We can now use this position vector to generate a unit vector who will give the direction from A to B.
- This unit vector is the same as the unit vector of \( \mathbf{F} \) of \( \mathbf{u}_{AB} = \mathbf{u}_F \)

\[ A = \{-2,4,3\} \text{ in} \]
\[ B = \{5,-1,5\} \text{ in} \]
\[ \vec{r}_{AB} = \{B\} - \{A\} \]
\[ \vec{r}_{AB} = \left[\{5-(-2)\}i + \{(1)-4\}j + \{5-3\}k\right] \text{ in} \]
\[ \vec{r}_{AB} = \left[7i - 5j + 2k\right] \text{ in} \]
Position Vectors

- Back to our utilization of a scalar times a vector, we can rewrite the vector description of the position vector from A to B as

\[ \overrightarrow{r}_{AB} = r_{AB} \overrightarrow{u}_{AB} \]

\[ r_{AB} = 7\text{i} - 5\text{j} + 2\text{k} \]

\[ r_{AB} = \sqrt{(7\text{i})^2 + (-5\text{j})^2 + (2\text{k})^2} = \sqrt{78} \text{i} \]

\[ \overrightarrow{u}_{AB} = \frac{r_{AB}}{r_{AB}} = \frac{7\text{i}}{\sqrt{78}} - \frac{5\text{j}}{\sqrt{78}} + \frac{2\text{k}}{\sqrt{78}} \]

Homework

- Problem 2-67
- Problem 2-69
- Problem 2-77