Moment of Inertia

- When we calculated the centroid of a shape, we took the moment generated by the shape and divided it by the total area of the shape.
- This gave us a distance, which was the distance to the centroid of the shape.
The moment of inertia is actually the second moment of an area or mass about an axis

Notice that it is not a distance, it is a moment of a moment

That may sound strange

It should

There is really nothing that can easily be used to describe the moment of inertia

For an area, it will have units of length^4 which is very difficult to map to a physical quantity
The symbol for the moment of inertia is $I$ with a subscript describing about which axis the moment is being calculated.

The moment of inertia about the x-axis would be $I_x$, about the y-axis, $I_y$.

There is also a moment of inertia about the origin, known as the polar moment of inertia designated as $J_O$.

The moment of inertia is a physical property and determines the behavior of a material under certain loading and dynamic conditions.

Remember, we are taking the moment of the moment (the second moment) of an area about an axis.

Keep this in mind and you won't have any trouble here.
The first moment of a shape about an axis was calculated by taking the moment arm to the local centroid of the shape and multiplying that by the area of the shape.

The second moment will be generated in a similar manner. We will take a moment arm from the axis to the centroid of the shape, square that moment arm, and multiply that product by the area.
For a moment of inertia about (around) a y-axis, the moment arm will be measured perpendicular to the y-axis, so it will be an x-distance

So for $I_y$ we would have

$$I_y = x^2 A$$

An Example

Consider the following figure
An Example

- We will start with the $I_y$, or the moment of inertia about the y-axis

To take a moment about the y-axis, we will need to have a moment arm that has an $x$-distance
An Example

Again, we will begin by generating a differential area, $dA$

\[ y^2 = 4x \]
\[ y_{\text{top}} - y_{\text{bottom}} = 4m \]

Point to Note

- You must be careful that the side of the rectangle describing the differential area that does not have the differential component is parallel to the axis about which you are taking the moment of inertia
Point to Note

- If you do not set up the problem this way, the calculations are a bit different as you have seen from the example we did in class.

An Example

- In this case, the height is parallel to the y-axis
An Example

- If this isn’t so, the method breaks down

\[ y^2 = 4x \]

If \( y = \frac{1}{4} x^2 \) then
\[ 4m \]
\[ y_{\text{top}} - y_{\text{bottom}} \]
\[ dx \]

An Example

- Once we have the differential area, we locate the moment arm from the axis

\[ y^2 = 4x \]

If \( y = \frac{1}{4} x^2 \) then
\[ 4m \]
\[ y_{\text{top}} - y_{\text{bottom}} \]
\[ dx \]
An Example

- Now the second moment of this differential area will be the moment arm squared times the differential area

\[ x^2 \, dA \]

\[
\frac{dA}{dx} = \left( y_{\text{TOP}} - y_{\text{BOTTOM}} \right) \, dx
\]

\[
\frac{dA}{dx} = \left( 2\sqrt{x} - \frac{x^2}{4} \right) \, dx
\]
An Example

- The moment of inertia of the differential area is the square of the moment arm times the differential area

\[
I_{y, \Delta A} = x^2 \Delta A
\]

\[
I_{x, \Delta A} = x^2 \left( 2\sqrt{x} - \frac{x^2}{4} \right) \Delta x
\]

\[
I_{y, \Delta A} = x^2 \left( 2\sqrt{x} - \frac{x^2}{4} \right) \Delta x
\]

An Example

- The moment of inertia for the complete shape, \( I_y \), is the sum of all the moments of inertia of the differential areas

\[
I_y = \int_A x^2 \, dA
\]

\[
I_y = \int_A x^2 \left( y_{\text{top}} - y_{\text{bottom}} \right) \, dx
\]

\[
I_y = \int_{-4m}^{4m} x^2 \left( 2\sqrt{x} - \frac{x^2}{4} \right) \, dx
\]
An Example

- Notice that we are calculating $I_y$ but the distances are in the x-direction, be careful to remember this

\[ I_y = \int_A x^2 \, dA \]
\[ I_y = \int_A x^2 \left( y_{\text{TOP}} - y_{\text{BOTTOM}} \right) \, dx \]
\[ I_y = \int_{0m}^{4m} x^2 \left( 2\sqrt{x} - \frac{x^2}{4} \right) \, dx \]

Evaluating the integral, we have

\[ I_y = \int_{0m}^{4m} \left( \frac{5}{2} - \frac{x^4}{4} \right) \, dx \]
\[ I_y = \frac{2}{7} \left[ 2x^2 - \frac{1}{5} x^5 \right]_{0m}^{4m} \]
\[ I_y = 73.14 - 51.20 - 0 + 0 \]
\[ I_y = 21.94 m^4 \]
An Example

- Using the same method, we can calculate the $I_x$

- Start by drawing the differential area

\[ y^2 = 4x \]

\[ y = 1/4 \times 4^2 \]
An Example

- Draw the moment arm from the x-axis

The second moment for this differential area is

\[ y^2 \, dA \]
An Example

- The second moment for this differential area is

\[ y^2 \, dA \]
\[ y^2 \left( x_{\text{RIGHT}} - x_{\text{LEFT}} \right) \, dy \]
\[ y^2 \left( 2\sqrt{y} - \frac{y^2}{4} \right) \, dx \]

An Example

- The \( I_x \) for the composite area is the sum of the \( I_x \)'s for the individual differential areas

\[ I_x = \int_{0}^{4m} \left( 2y^2 - \frac{y^4}{4} \right) \, dy \]
\[ I_x = \frac{2}{7} \cdot \frac{7}{4} - \frac{5}{4} \cdot \frac{4}{4} \]
\[ I_x = 73.14 - 51.20 - 0 + 0 \]
\[ I_x = 21.94 \, m^4 \]
An Example

- The polar moment of inertia, $J_O$, is the sum of the moments of inertia about the x and y axis

$$J_O = I_x + I_y$$

$$J_O = 21.94m^4 + 21.94m^4$$

$$J_O = 43.88m^4$$

An Aside

- Just for your information, you are not required to know this method, you can use a double integral to find the moment of inertia
An Aside

- The difference is how you describe the differential area, in this case the differential area would be

\[ dA = (dx)(dy) \]

- The second moment of this differential area about the y-axis would be

\[ x^2 \, dA = x^2 \, (dx)(dy) \]
An Aside

- As we sum the differential areas through the composite, we are integrating in two directions, x and y

\[ \int_A x^2 \, dA = \iint_A x^2 \, (dx) \, (dy) \]

Since we have an \( x^2 \), we can choose to the y-direction as the inner integral and move y from bottom to top

\[
\int_{0}^{4m} \int_{y_{\text{BOTTOM}}}^{y_{\text{TOP}}} \left[ x^2 \, (dx) \right] (dy)
\]

\[
\int_{0}^{4m} \int_{\frac{\sqrt{4x}}{x^2}}^{\frac{\sqrt{4x}}{x^2/4}} \left[ x^2 \, (dx) \right] (dy)
\]

\[
\int_{0}^{4m} x^2 y \, (dx) \left[ \frac{\sqrt{4x}}{x^2} \right] \left( \frac{\sqrt{4x}}{x^2/4} \right)
\]
An Aside

- Making the inner integration, we have

\[
\int_{0}^{4m} x^2 \left( \sqrt{4x - \frac{x^2}{4}} \right) dx
\]

Which is the same form as we had before for \( I_y \)

\[
I_y = \int_{0}^{4m} x^2 \left( \sqrt{4x - \frac{x^2}{4}} \right) dx
\]
Homework

- Problem 10-1
- Problem 10-2
- Problem 10-7