A math professor in an unheated room is \textit{cold} and calculating.

\section*{Radius of Gyration}

- This actually sounds like some sort of rule for separation on a dance floor.
- It actually is just a property of a shape and is used in the analysis of how some shapes act in different conditions.
Radius of Gyration

- The radius of gyration, \( k \), is the square root of the ratio of the moment of inertia to the area

\[
k_x = \sqrt{\frac{I_x}{A}}
\]
\[
k_y = \sqrt{\frac{I_y}{A}}
\]
\[
k_O = \sqrt{\frac{J_O}{A}} = \sqrt{\frac{I_x + I_y}{A}}
\]

Parallel Axis Theorem

- If you know the moment of inertia about a centroidal axis of a figure, you can calculate the moment of inertia about any parallel axis to the centroidal axis using a simple formula

\[
I_y = I_{\bar{y}} + Ax^2
\]
\[
I_x = I_{\bar{x}} + Ay^2
\]
Parallel Axis Theorem

- Since we usually use the bar over the centroidal axis, the moment of inertia about a centroidal axis also uses the bar over the axis designation.

\[
I_y = I_{\overline{y}} + Ax^2 \\
I_x = I_{\overline{x}} + Ay^2
\]

Parallel Axis Theorem

- If you look carefully at the expression, you should notice that the moment of inertia about a centroidal axis will always be the minimum moment of inertia about any axis that is parallel to the centroidal axis.

\[
I_y = I_{\overline{y}} + Ax^2 \\
I_x = I_{\overline{x}} + Ay^2
\]
Parallel Axis Theorem

- In a manner similar to that which we used to calculate the centroid of a figure by breaking it up into component areas, we can calculate the moment of inertia of a composite area

\[ I_y = I_{\bar{y}} + Ax^2 \]

\[ I_x = I_{\bar{x}} + Ay^2 \]

Parallel Axis Theorem

- Inside the back cover of the book, in the same figure that we used for the centroid calculations we can find calculations for moments of inertia

\[ I_y = I_{\bar{y}} + Ax^2 \]

\[ I_x = I_{\bar{x}} + Ay^2 \]
Parallel Axis Theorem

\[ I_y = I_y + A \alpha^2 \]
\[ I_x = I_x + A \beta^2 \]

- HERE IS A CRITICAL MOMENT OF CAUTION
- REMEMBER HOW THE PARALLEL AXIS IS WRITTEN
- IF THE AXIS SHOWN IN THE TABLE IS NOT THROUGH THE CENTROID, THEN THE FORMULA DOES NOT GIVE YOU THE MOMENT OF INERTIA THROUGH THE CENTROIDAL AXIS

By example

- The \( I_y \) given for the Semicircular area in the table is about the centroidal axis
- The \( I_x \) given for the same Semicircular area in the table is not about the centroidal axis
Using The Table

- We want to locate the moment of inertia in the position shown of a semicircular area as shown about the x and y axis, $I_x$ and $I_y$

Using the Table

- First, we can look at the table and find the $I_x$ and $I_y$ about the axis as shown
Using the Table

- In this problem, the y axis is 8" from the y centroidal axis and x axis is 6" below the base of the semicircle, this would be usually evident from the problem description.

\[ I_y = \frac{1}{8} \pi r^4 \]

\[ I_y = \frac{1}{8} \pi (5\text{in})^4 \]

\[ I_y = 245.44\text{in}^4 \]
Using the Table

Next we can calculate the area

\[ A = \frac{\pi (5\text{in})^2}{2} \]

\[ A = 39.27\text{in}^2 \]

Using the Table

If we know that distance between the y axis and the ybar axis, we can calculate the moment of inertia using the parallel axis theorem

\[ I_y = I_y + Ad_x^2 \]

\[ I_x = I_x + Ad_y^2 \]
Using the Table

- I changed the notation for the distances moved to avoid confusion with the distance from the origin.

\[ I_y = I_{y} + A d_x^2 \]
\[ I_x = I_{x} + A d_y^2 \]

Using the Table

- The axis we are considering may not always be at the origin.
Using the Table

- If the y axis is 8 inches to the left of the centroidal axis, then the moment of inertia about the y axis would be

\[ I_y = I_y + Ad_x^2 \]
\[ I_y = 245.44\text{in}^4 + (39.27\text{in}^2)(8\text{in})^2 \]
\[ I_y = 2758.72\text{in}^4 \]

Using the Table

- The moment of inertia about the x axis is a slightly different case since the formula presented in the table is the moment of inertia about the base of the semicircle, not the centroid.
Using the Table

- To move it to the moment of inertia about the x-axis, we have to make two steps

\[ I_x = I_{base} - A\left(d_{base \ to \ centroid}\right)^2 \]

\[ I_x = I_{\bar{x}} + A\left(d_{centroid \ to \ x-axis}\right)^2 \]

Using the Table

- We can combine the two steps

\[ I_x = I_{base} - A\left(d_{base \ to \ centroid}\right)^2 \]

\[ I_x = I_{\bar{x}} + A\left(d_{centroid \ to \ x-axis}\right)^2 \]

\[ I_x = I_{base} - A\left(d_{base \ to \ centroid}\right)^2 + A\left(d_{centroid \ to \ x-axis}\right)^2 \]
Using the Table

- Don’t try and cut corners here
- You have to move to the centroid first

\[ I_x = I_{\text{base}} - A(d_{\text{base to centroid}})^2 \]
\[ I_x = I_x + A(d_{\text{centroid to x-axis}})^2 \]
\[ I_x = I_{\text{base}} - A(d_{\text{base to centroid}})^2 + A(d_{\text{centroid to x-axis}})^2 \]

In this problem, we have to locate the y centroid of the figure with respect to the base.

- We can use the table to determine this

\[ \bar{y} = \frac{4r}{3\pi} = \frac{4(5in)}{3\pi} \]
\[ \bar{y} = 2.12in \]

This ybar is with respect the base of the object, not the x-axis.
Using the Table

- Now the $I_x$ in the table is given about the bottom of the semicircle, not the centroidal axis.
- That is where the $x$ axis is shown in the table.

\[ I_{base} = \frac{1}{8} \pi r^4 \]

\[ I_{base} = \frac{1}{8} \pi (5\text{in})^4 \]

\[ I_{base} = 245.44\text{in}^4 \]
Using the Table

- Now we can calculate the moment of inertia about the x centroidal axis

\[ I_{\text{base}} = I_{\bar{x}} + A d_{\text{base to centroid}}^2 \]
\[ I_{\bar{x}} = I_{\text{base}} - A d_{\text{base to centroid}}^2 \]
\[ I_{\bar{x}} = 245.44 \text{in}^4 - \left(39.27 \text{in}^2\right)(2.12 \text{in})^2 \]
\[ I_{\bar{x}} = 68.60 \text{in}^4 \]

Using the Table

- And we can move that moment of inertia the the x-axis

\[ I_x = I_{\bar{x}} + A d_{\text{centroid to x-axis}}^2 \]
\[ I_x = 68.60 \text{in}^4 + \left(39.27 \text{in}^2\right)(6\text{in} + 2.12 \text{in})^2 \]
\[ I_x = 2657.84 \text{in}^4 \]
Using the Table

- The polar moment of inertia about the origin would be

\[ J_O = I_x + I_y \]

\[ J_O = 2657.84in^4 + 2758.72in^4 \]

\[ J_O = 5416.56in^4 \]

Another Example

- We can use the parallel axis theorem to find the moment of inertia of a composite figure
Another Example

- We can divide up the area into smaller areas with shapes from the table.
Another Example

Since the parallel axis theorem will require the area for each section, that is a reasonable place to start.

<table>
<thead>
<tr>
<th>ID</th>
<th>Area (in²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>36</td>
</tr>
<tr>
<td>II</td>
<td>9</td>
</tr>
<tr>
<td>III</td>
<td>27</td>
</tr>
</tbody>
</table>

We can locate the centroid of each area with respect to the y axis.

<table>
<thead>
<tr>
<th>ID</th>
<th>Area (in²)</th>
<th>xbar_i (in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>36</td>
<td>3</td>
</tr>
<tr>
<td>II</td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>III</td>
<td>27</td>
<td>6</td>
</tr>
</tbody>
</table>
Another Example

From the table in the back of the book we find that the moment of inertia of a rectangle about its y-centroid axis is

\[ I_y = \frac{1}{12} b^3 h \]

<table>
<thead>
<tr>
<th>ID</th>
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</thead>
<tbody>
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<td>6</td>
</tr>
</tbody>
</table>

In this example, for Area I, \( b = 6" \) and \( h = 6" \)

\[ I_y = \frac{1}{12} (6in)(6in)^3 \]

\[ I_y = 108in^4 \]
Another Example

For the first triangle, the moment of inertia calculation isn’t as obvious

The way it is presented in the text, we can only find the $I_x$ about the centroid
Another Example

The change may not seem obvious but it is just in how we orient our axis. Remember an axis is our decision.

So the moment of inertia of the II triangle can be calculated using the formula with the correct orientation.

\[
I_y = \frac{1}{36} bh^3 \\
I_y = \frac{1}{36} (6in)(3in)^3 \\
I_y = 4.5in^4
\]
Another Example

The same is true for the III triangle

\[ I_y = \frac{1}{36} bh^3 \]
\[ I_y = \frac{1}{36} (6in)(9in)^3 \]
\[ I_y = 121.5in^4 \]

Now we can enter the \( I_{ybar} \) for each sub-area into the table

<table>
<thead>
<tr>
<th>Sub-Area</th>
<th>Area ((\text{in}^2))</th>
<th>(x_{bar_i}) ((\text{in}))</th>
<th>(I_{ybar}) ((\text{in}^4))</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>36</td>
<td>3</td>
<td>108</td>
</tr>
<tr>
<td>II</td>
<td>9</td>
<td>7</td>
<td>4.5</td>
</tr>
<tr>
<td>III</td>
<td>27</td>
<td>6</td>
<td>121.5</td>
</tr>
</tbody>
</table>
Another Example

We can then sum the $I_y$ and the $A(d_x)^2$ to get the moment of inertia for each sub-area.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{Sub-Area} & \text{Area} & x_{\text{bar}_i} & I_{y\text{bar}} & A(d_x)^2 & I_{y\text{bar}} + A(d_x)^2 \\
\hline
\text{I} & 36 & 3 & 108 & 324 & 432 \\
\text{II} & 9 & 7 & 4.5 & 441 & 445.5 \\
\text{III} & 27 & 6 & 121.5 & 972 & 1093.5 \\
\hline
\end{array}
\]

And if we sum that last column, we have the $I_y$ for the composite figure.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{Sub-Area} & \text{Area} & x_{\text{bar}_i} & I_{y\text{bar}} & A(d_x)^2 & I_{y\text{bar}} + A(d_x)^2 \\
\hline
\text{I} & 36 & 3 & 108 & 324 & 432 \\
\text{II} & 9 & 7 & 4.5 & 441 & 445.5 \\
\text{III} & 27 & 6 & 121.5 & 972 & 1093.5 \\
\hline
\end{array}
\]

\[1971\]
Another Example

We perform the same type analysis for the $I_x$

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Another Example

Locating the y-centroids from the x-axis

<table>
<thead>
<tr>
<th>Sub-Area</th>
<th>Area (in²)</th>
<th>$y_{bar_i}$ (in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
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<td>3</td>
</tr>
<tr>
<td>II</td>
<td>9</td>
<td>2</td>
</tr>
<tr>
<td>III</td>
<td>27</td>
<td>-2</td>
</tr>
</tbody>
</table>
### Another Example

#### Determining the $I_x$ for each sub-area

<table>
<thead>
<tr>
<th>Sub-Area</th>
<th>Area $(\text{in}^2)$</th>
<th>$y_{\text{bar},i}$ (in)</th>
<th>$I_{\text{xbar}}$ $(\text{in}^4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
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<td>108</td>
</tr>
<tr>
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<td>9</td>
<td>2</td>
<td>18</td>
</tr>
<tr>
<td>III</td>
<td>27</td>
<td>-2</td>
<td>54</td>
</tr>
</tbody>
</table>

### Another Example

#### Making the $A(d_y)^2$ multiplications

<table>
<thead>
<tr>
<th>Sub-Area</th>
<th>Area $(\text{in}^2)$</th>
<th>$y_{\text{bar},i}$ (in)</th>
<th>$I_{\text{xbar}}$ $(\text{in}^4)$</th>
<th>$A(d_y)^2$ $(\text{in}^4)$</th>
</tr>
</thead>
<tbody>
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<td>-2</td>
<td>54</td>
<td>108</td>
</tr>
</tbody>
</table>
Another Example

Summing and calculating $I_x$

<table>
<thead>
<tr>
<th>Sub-Area</th>
<th>Area (in$^2$)</th>
<th>$y_{bar_i}$ (in)</th>
<th>$I_{x_{bar}}$ (in$^4$)</th>
<th>$A(d_y)^2$ (in$^4$)</th>
<th>$I_{x_{bar}} + A(d_y)^2$ (in$^4$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
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<td>3</td>
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<td>III</td>
<td>27</td>
<td>-2</td>
<td>54</td>
<td>108</td>
<td>162</td>
</tr>
</tbody>
</table>

Homework

- Problem 10-27
- Problem 10-29
- Problem 10-47