Reinforced Concrete Beams
Mathematical modeling of reinforced concrete is essential to civil engineering

Concrete as a material
Concrete in a structure

Stress distribution in a reinforced concrete beam

Reinforced Concrete Beams
Mathematical modeling of reinforced concrete is essential to civil engineering

Geometric model a reinforced concrete bridge

Blast failure of a reinforced concrete wall

Reinforced Concrete Beams
Mathematical model for failure in an unreinforced concrete beam

Blast failure of a reinforced concrete wall
In the reinforced concrete beam project, there are three different failure modes we need to investigate.

The purpose of RC is the reinforcement of areas in concrete that are weak in tension.

Let's look at the internal forces acting on the beam and locate the tension zones.

The shear between the applied load and the support is constant $V = P/2$.

$\sum F = \frac{P}{2} - V \Rightarrow V = \frac{P}{2}$
Reinforced Concrete Beams

The shear between the applied load and the support is constant $V = P/2$

Let's look at the internal moment at section between the supports and applied load

$x_{\text{max}} = 8 \text{ in.}$

$M = 4P \text{ (lb.-in.)}$

The top of the beam is in compression and the bottom of the beam is in tension

To model the behavior of a reinforced concrete beam we will need to understand three distinct regions in the beam.

Two are illustrated below; the third is called shear.
We need models to help us with compression, tension, and shear failures in concrete.
Shear failure in a reinforced concrete beam

Let's focus on how to model the ultimate tensile load in a reinforced concrete beam.

Typical rebar configuration to handle tension and shear loads.
In the 1930s, Whitney proposed the use of a rectangular compressive stress distribution.

Assume that the concrete contributes nothing to the tensile strength of the beam.

The height of the stress box, $a$, is defined as a percentage of the depth to the neutral axis.

The height of the stress box, $a$, is defined as a percentage of the depth to the neutral axis.

$\beta_1 = \begin{cases} 0.85 & \text{if } f'c \leq 4000 \text{ psi} \\ 0.85 - 0.05 \left( \frac{f'c - 4000}{1000} \right) & \text{if } f'c \geq 4000 \text{ psi} \end{cases}$
Whitney Rectangular Stress Distribution

The values of the tension and compression forces are:

\[ C = 0.85f'c \cdot ba \]

\[ T = A_f y \]

\[ \sum F = 0 = T - C \]

\[ a = \frac{A_f y}{0.85f'c \cdot b} \]

Whitney Rectangular Stress Distribution

If the tension force capacity of the steel is too high, then the value of \( a \) is large

\[ a = \frac{A_f y}{0.85f'c \cdot b} \]

If \( a > d \), then you have too much steel

\[ \sum M = T \left( d - \frac{a}{2} \right) \]

\[ M = A_f y \left( d - \frac{a}{2} \right) \]

Whitney Rectangular Stress Distribution

The internal moment is the value of either the tension or compression force multiplied the distance between them.

\[ M = A_f y \left( d - \frac{a}{2} \right) \]

Substitute the value for \( a \)

\[ M = A_f y \left( d - 0.59 \cdot \frac{A_f y}{f'c \cdot b} \right) \]

\[ M = 4P \]

Whitney Rectangular Stress Distribution

The internal moment is the value of either the tension or compression force multiplied the distance between them

\[ M = A_f y \left( d - 0.59 \cdot \frac{A_f y}{f'c \cdot b} \right) \]

We know that the moment in our reinforced concrete beams is

\[ M = 4P \]

\[ P_{tension} = \frac{A_f y}{4} \left( d - 0.59 \cdot \frac{A_f y}{f'c \cdot b} \right) \]

Reinforced Concrete Beams

Let’s focus on how to model the ultimate shear load in a reinforced concrete beam
Reinforced Concrete Beams

We can approximate the shear failure in unreinforced concrete as:

\[ V_c = 2\sqrt{f'_c bd} \]

If we include some reinforcing for shear the total shear capacity of a reinforce concrete beam would be approximated as:

\[ V_n = V_c + V_s \]

Reinforced Concrete Beams

Let's consider shear failure in reinforced concrete

\[ V_s = \frac{A_v f_y d}{s} \]

\[ V_n = \frac{P}{2} \]

\[ P_{shear} = 2\left( \frac{A_v f_y d}{s} + 2\sqrt{f'_c bd} \right) \]

Reinforced Concrete Beams

Let's focus on how to model the ultimate compression load in a reinforced concrete beam

\[ P \]

Compression

Reinforced Concrete Beams

There is a "balanced" condition where the stress in the steel reinforcement and the stress in the concrete are both at their yield points

The amount of steel required to reach the balanced strain condition is defined in terms of the reinforcement ratio:

\[ \rho = \frac{A_s}{bd} \]

Reinforced Concrete Beams

The limits of the reinforcement ratio are established as:

\[ \rho = \frac{A_s}{bd} \]

\[ \rho = 0.85 \beta_1 \frac{c f'_c}{d f_y} \]

\[ \rho \text{ as function of } c/d \]

\[ \rho \text{ as function of } c/d \]

Reinforced Concrete Beams

The limits of the reinforcement ratio are established as:

\[ \frac{c}{d} > 0.600 \quad \text{Beam failure is controlled by compression} \]

\[ 0.375 < \frac{c}{d} < 0.600 \quad \text{Transition between tension and compression control} \]

\[ \frac{c}{d} < 0.375 \quad \text{Beam failure is controlled by tension} \]
Reinforced Concrete Beams

Let's consider compression failure in over reinforced concrete. First, let define an equation that given the stress in the tensile steel when concrete reaches its ultimate strain.

\[ f_{\text{steel}} = 87,000 \, \text{psi} \left( \frac{d - c}{c} \right) \]

If \( f_{\text{steel}} < f_y \) then or \( \frac{c}{d} > 0.600 \)

\[ M_{\text{compression}} = A_s \left( \frac{d - c}{c} \right) \left( \frac{d - a}{2} \right) 87,000 \, \text{psi} \]

Reinforced Concrete Beams

Consider the different types of failures in reinforced concrete:

1. Archcrack failure
2. Bearing failure
3. Flexural failure
4, 5. Failure of compression strut

Reinforced Concrete Beam Analysis

Let's use the failure models to predict the ultimate strength-to-weight (SWR) of one of our reinforced concrete beams from lab.

Consider a beam with the following characteristics:

- Concrete strength \( f'c = 5,000 \, \text{psi} \)
- Steel strength \( f_y = 60,000 \, \text{psi} \)
- The tension reinforcement will be 2 #3 rebars
- The shear reinforcement will be #3 rebars bent in a U-shape spaced at 4 inches.

Use the minimum width to accommodate the reinforcement.

Reinforcing bars are denoted by the bar number. The diameter and area of standard rebars are shown below.

<table>
<thead>
<tr>
<th>Bar #</th>
<th>Diameter (in.)</th>
<th>A_s (in.²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.375</td>
<td>0.17</td>
</tr>
<tr>
<td>4</td>
<td>0.500</td>
<td>0.20</td>
</tr>
<tr>
<td>5</td>
<td>0.625</td>
<td>0.31</td>
</tr>
<tr>
<td>6</td>
<td>0.750</td>
<td>0.44</td>
</tr>
<tr>
<td>7</td>
<td>0.875</td>
<td>0.60</td>
</tr>
<tr>
<td>8</td>
<td>1.000</td>
<td>0.79</td>
</tr>
<tr>
<td>9</td>
<td>1.125</td>
<td>1.00</td>
</tr>
<tr>
<td>10</td>
<td>1.270</td>
<td>1.27</td>
</tr>
<tr>
<td>11</td>
<td>1.410</td>
<td>1.56</td>
</tr>
</tbody>
</table>

Based on the choice of reinforcement we can compute an estimate of \( b \) and \( d \):

\[ b \geq 2(0.375 \, \text{in.}) + 2(0.75 \, \text{in.}) + 2(0.375 \, \text{in.}) + 0.75 \, \text{in.} = 3.75 \, \text{in.} \]
If we allow a minimum cover under the rebars we can estimate 
d

Half of #3 bar diameter

Minimum cover

\[ d = 6 - \frac{0.375}{2} - 0.375 - 0.75 \]

\[ d = 4.69 \text{ in.} \]

We now have values for \( b, d, \) and \( A_s \)

\[ M = A_s f_y \left( d - 0.59 \frac{A_s f_y}{f' c b} \right) \]

The \( A_s \) for two #3 rebars is:

\[ A_s = 2(0.11 \text{ in.}^2) = 0.22 \text{ in.}^2 \]

Compute the moment capacity

\[ M = A_s f_y \left( d - 0.59 \frac{A_s f_y}{f' c b} \right) \]

\[ = 0.22 \text{ in.}^2 (60 \text{ ksi}) \left( 4.69 \text{ in.} - 0.59 \frac{0.22 \text{ in.}^2 (60 \text{ ksi})}{5 \text{ ksi} (3.75 \text{ in.})} \right) \]

\[ = 56.4 \text{ k} \cdot \text{in.} \quad \Rightarrow \quad P = \frac{M}{4} = 14.1 \text{ kips} \]

Let's check the shear model

\[ P_{\text{shear}} = 2 \left( \frac{A_f d}{s} + 2 \sqrt{f' c b d} \right) \]

Shear reinforcement spacing

Area of two #3 rebars

\[ P_{\text{shear}} = 2 \left( 2(0.11 \text{ in.}^2) (60,000 \text{ psi}) (4.69 \text{ in.}) \right) \]

\[ = 35,928 \text{ lb.} = 35.9 \text{ kips} \]

Since \( P_{\text{tension}} < P_{\text{shear}} \) therefore \( P_{\text{tension}} \) controls

Let's check the reinforcement ratio

\[ \rho = \frac{A_s}{bd} \]

\[ \rho = 0.85 \beta_i \frac{f'_c}{d} \frac{f_y}{f'_c} \]

To compute \( \rho \), first we need to estimate \( \beta_i \)

An \( \beta_i \) estimate is given as:

\[ f'_c \leq 4000 \text{ psi} \quad \Rightarrow \quad \beta_i = 0.85 \]

\[ f'_c \geq 4000 \text{ psi} \]

\[ \beta_i = 0.85 - 0.05 \left( \frac{f'_c - 4000}{1000} \right) \geq 0.65 \]

\[ \beta_i = 0.85 - 0.05 \left( \frac{5000 - 4000}{1000} \right) = 0.80 \]
Check the reinforcement ratio for the maximum steel allowed for tension controlled behavior or $c/d = 0.375$

$$\rho = 0.85\beta_s \frac{c}{d} \frac{f'_c}{f_y} = 0.85(0.80)0.375 \frac{5ksi}{60ksi} = 0.021$$

$$\rho = \frac{A_s}{bd} = \frac{0.22 \text{ in.}^2}{3.75 \text{ in.}(4.69 \text{ in.})} = 0.0125$$

The amount of steel in this beam is tension-controlled behavior.

An estimate of the weight of the beam can be made as:

$$W = \frac{bhL}{1728 \text{ in.}^3/\text{ft.}^3} (\frac{145 \text{ lb.}}{\text{ft.}^3})$$

$$+ \frac{A_L}{1728 \text{ in.}^3/\text{ft.}^3} (\frac{490 \text{ lb.} - 145 \text{ lb.}}{\text{ft.}^3})$$

$$= 56.64 \text{ lb.} + 1.32 \text{ lb.} = 57.96 \text{ lb.}$$

In summary, this reinforced concrete beam will fail in tension:

$$S \Rightarrow P = 14.1 \text{ kips}$$

$$W = 57.96 \text{ lb.}$$

$$SWR = \frac{14,100 \text{ lb.}}{57.96 \text{ lb.}} = 243$$

Questions?