OPTIMAL DESIGN OF FRAMES TO RESIST BUCKLING UNDER MULTIPLE LOAD CASES

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ABSTRACT: This paper suggests an optimization-based design methodology for improving the strength and the overall stability characteristics of framed structures whose capacities are governed by limit-load behavior. Attention is focused on space-framed structures under static loads, and represents a natural extension of the basic ideas presented in the companion paper. The optimization objective function is taken here to be a linear combination of the critical buckling eigenvalues of the structure. A constant volume constraint with bounds on the design variables is used in conjunction with an optimality-criterion approach for search. The present approach avoids the need to know the eigenvector that dominates the response as the structure passes a limit load. A novel approach to solving problems with multiple loading conditions is introduced wherein each eigenvalue in the objective is weighted in accordance with the participation of that mode in the loading. Several examples are given to demonstrate aspects of the behavior of the proposed design procedure.

INTRODUCTION

The strength of a framed structure carrying gravity loads is often limited by inelastic buckling of the structure as a whole. While computational methods are currently available to estimate the limit load of a structure under a given set of loads, such an estimate is rarely done to support or check a structural design. The actual strength of a structure is addressed indirectly, at best, in most current design specifications, even those called limit-design methods. In general, the strength and stability of the structure as a whole is presumed as a consequence of insuring adequate strength of the individual members comprising the structure or through the artifice of structural drift limitations. While these traditional design procedures have a good record of providing adequate safety, they yield no information on how the strength of the structure might be improved by altering the design.

Virtually all of the methods that have been developed to improve the limit strength of structures are based upon a geometrically linear model of structural response. Consequently, they all overestimate the strength of the design they produce. When structural stability effects are important this conservative overestimate can be substantial. Those methods that do address the stability issue (e.g., those proposed to minimize the weight of the structure with a fixed buckling load constraint) have all been restricted to elastic structures. Even though the limit capacity of a structure is computable, it would be prohibitively expensive to maximize directly the limit load of the structure because a nonlinear analysis of the structure would be required at each de-

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sign iteration. There remains a clear need for an efficient method for designing elastoplastic structures to enhance stability.

In this paper we present an optimization-based design procedure capable of improving the limit behavior of space frames without resorting to nonlinear analyses during the optimization phase of the design cycle. The approach is a rational extension of the basic idea proffered in the companion paper (Pezeshk and Hjelmstad 1991) where the inelastic stability of a structure is enhanced by maximizing its linearized buckling eigenvalues. The principal contribution of the present extension is that it provides a formal approach to the problem of designing structures with robust strength characteristics under multiple loading conditions. The essential idea behind the algorithm is that the likelihood of a certain buckling mode dominating the limit response of the structure under a given loading is reflected by the product of the buckling eigenvector and the load vector. The eigenvalue of the most likely mode is then weighted more heavily in a design-objective function including contributions of all possible modes. This approach solves the problem of selecting the most dominant buckling mode a priori and at the same time establishes a rational framework for dealing with multiple load cases. The method provides a general tool for solving complex design problems and generally leads to structures with better limit strength and stability characteristics than other structures of similar weight.

In the following, we formally state the problem as a constrained optimization problem with a single equality constraint. The first-order necessary conditions are derived and used as the basis of a fixed-point iterative method to search for the optimal design. The method is illustrated by two examples of three-dimensional framed structures subjected to multiple load cases.

**Formulation and Development**

Based on analyses of the limit and postlimit behavior of elastoplastic frames, Hjelmstad and Pezeshk (1988) observed that the overall stability and strength of a structure can be improved by maximizing its linearized buckling eigenvalues. This concept was successfully applied by Pezeshk and Hjelmstad (1991) to improve the performance of planar-framed structures. Here again we posit that, in order to improve the limit-load and postlimit behavior of framed structures, we should endeavor to maximize the linearized buckling eigenvalues. However, now we wish to increase the strength of the structure for multiple load cases.

To set the stage for the following developments we consider the linearized buckling eigenvalue problem

\[ \mathbf{K}\phi = \mu \mathbf{G}\phi \]  \hspace{1cm} (1)

where \( \mathbf{K} \) = the (positive definite) elastic stiffness matrix; and \( \mathbf{G} \) = the (possibly indefinite) geometric stiffness matrix. Both matrices are symmetric and depend on the design variables \( x \). If the dimension of the matrices \( \mathbf{K} \) and \( \mathbf{G} \) is \( N \), then Eq. 1 gives rise to solution pairs \( (\mu_i, \phi_i) \), \( i = 1, \ldots, N \). The eigenvectors are orthogonal and, hence, span \( N \)-dimensional space. Further assume that the eigenvectors are normalized such that \( \phi_i^T \mathbf{G}\phi_i = \text{constant} \). The value of the constant, which may be negative since matrix \( \mathbf{G} \) is possibly indefinite, is not important to the present derivation. The possibility that \( \phi_i^T \mathbf{G}\phi_i = 0 \) exists, but since the infinite eigenvalues are of no interest in the
present setting, we will simply constrain our vectors to be perpendicular to the null-space of $G$ to avoid the problem.

The buckling mode shapes can be used as a basis (in the mathematical sense) for describing the deformation of the system in the nonlinear range. Hjelmstad and Pezeshk (1988) observed that the buckling mode that contributes most strongly to the nonlinear response is not necessarily the fundamental mode. Furthermore, the strength of the structure depends upon which mode is dominant at the limit load. The load form is the primary indicator of the dominant mode (e.g., a laterally loaded frame will tend to buckle in a sidesway mode). On the basis of the preceding observations we hypothesize that the load vectors should play a role in the optimization as a measure of the likelihood of the actual nonlinear response being associated with a particular buckling mode. In particular, if the work done by load case $i$ going through the displacements of buckling mode $j$ is large then we can expect mode $j$ to contribute to the nonlinear response of the structure subjected to that load case. Hence, we will seek to maximize the buckling eigenvalues of the structure using the work of the various load cases going through modal displacements as weighting factors.

Let the matrix $\Phi = (\phi_1, \phi_2, \ldots, \phi_n)$ have columns composed of the buckling eigenvectors corresponding to the buckling eigenvalues $\mu' = (\mu_1, \mu_2, \ldots, \mu_n)$ where $n$ is the number of eigenpairs considered. Also let the matrix $F = (f_1, f_2, \ldots, f_n)$ have columns composed of the nodal force vectors from $v$ different load cases. In general, the eigenvalues and eigenvectors depend upon the design variables $x$, but the load vectors do not. Define a modal participation matrix $B(x)$ to be the inner products among the eigenvectors and load vectors as follows:

$$B(x) = \Phi(x)F$$

The matrix $B$ has components given by $B_{ij} = |\phi_i' f_j|$. We also define a vector $c(x) \in \mathbb{R}^v$ of load case weighting factors, or load factors. The choice of these load factors, which may depend upon the design variables $x$, will be discussed later.

We will consider the members of the structures to be grouped into $M$ distinct groups. Each group is associated with a set of design variables that describe the geometry of the cross section of that group. For example, an I-beam can be described by its depth $h$, flange width $b$, web thickness $t$, and flange thickness $t_f$. Consequently, the I-beam has four design variables. A rectangular cross section has two design variables: the width $b$ and the height $h$ of the cross section. The vector of design variables will be designated as $x = (x_1, x_2, \ldots, x_{dv})$, where $dv$ is the total number of design variables, computed as the sum over all groups of the number of design variables per group. While it is not necessary for our discussion, we will assume a particular ordering of the design variables in which the variables in group $m$ are listed sequentially in the array. To simplify the notation, we designate the specific weight of the $m$th group as the weight per unit of cross-sectional area of the entire group

$$w_m = \sum_{i \in m} \rho_i L_i$$

where $L_i$ = the length of member $i$; and its density is $\rho_i$. The sum is taken over all members associated with group $m$. 

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We will consider that a target value of the total structural weight is known, for example from some initial design procedure, and we endeavor to re-portion that weight among the members to improve the limit strength of the structure. With the above definitions, the optimization problem can be stated in the following way:

$$\text{max } \mu'(x) B(x)c(x)$$

such that

$$h(x) = w'a(x) - \omega_o = 0$$

and

$$\underline{x} \leq x \leq \bar{x}$$

where \(\omega_o\) = the prescribed weight of the structure; \(w = (w_1, w_2, \ldots, w_M)\) is the vector of specific weights; and \(a(x) = [a_1(x), a_2(x), \ldots, a_M(x)]\) is the vector of cross-sectional areas of the \(M\) groups. Each function \(a_m(x)\) depends only on the design variables from group \(m\). The inequality constraints indicate that each design variable has a minimum permissible size, \(x_i\), and a maximum permissible size, \(\bar{x}_i\).

The advantage of stating the problem as we have should be evident. With a single equality constraint, the onerous task of estimating Lagrange multipliers becomes almost trivial, in stark contrast to stress and displacement constrained procedures. The procedure can easily be iterated by relaxing the weight constraint, taking the previous optimal design as a starting design, and computing a new, heavier design. The final choice could then be made from among the designs of differing weight.

**Remark**

The optimization problem can be interpreted as a multiobjective optimization problem in which we wish to maximize the vector \(\mu' = (\mu_1, \mu_2, \ldots, \mu_n)\) subject to the above constraints. The individual objectives \(\mu_m(x)\) are homogeneous, yet they can be conflicting since maximizing one buckling eigenvalue will generally result in a decrease in another one if the volume of the structure is held constant. In such a case one might consider finding the noninferior frontier or Pareto optimum set. A vector \(\hat{x}\) is called noninferior or Pareto optimal if there exists no feasible vector \(x\) that would increase some objective functions without causing a simultaneous decrease in at least one other objective.

There are many different generating techniques to determine the noninferior frontier. A survey of different generating techniques can be found in Atrek et al. (1984) and Cohon (1979). The formulation considered here can be viewed as a weighting method where the weight vector \(B(x)c(x)\) depends upon the design variables. Because we have additional insight into the problem regarding the importance of the various objectives, it is not necessary to determine the entire noninferior frontier. For a specific selection of the vector \(c(x)\) a single point on the noninferior frontier is determined. In the sequel different choices will be made for the weight vector \(c(x)\). These variations can be interpreted as attempts to find additional points on the noninferior frontier.

**Choice of Load Factors**

The objective given by Eq. 4 can be given a geometric interpretation as the dot product of a linear combination of the eigenvectors with a linear combination of the force vectors as
\[ u'(x)B(x)c(x) = \mu'(x)\Phi'(x)Fe(x) = v(x) \cdot r(x) \] 

where the vector \( v(x) = \Phi(x)\mu(x) \) can be thought of as an effective displacement vector; and \( r(x) = Fe(x) \) can be thought of as an effective force vector. When expressed in this manner, the optimization problem can be interpreted as finding the vector \( x \) that maximizes the work done by the effective loads going through the effective displacements. Furthermore, the above expression gives us some geometric insight into how the optimum might be achieved. First, assuming that the weights \( c(x) \) and force vectors are fixed, and that the eigenvectors are normalized, only the length of vector \( v(x) \) can change, and hence the objective will be increased by increasing the magnitude of the eigenvalues. Second, since the objective is the dot product between the two vectors, the maximum will occur when they tend to line up.

Many possible choices exist for the selection of \( c(x) \). The simplest choice for \( c(x) \) would be constants that reflect the relative importance of the various load cases. Such constants may be difficult to establish, since it is generally not clear which load cases will be important from the point of view of stability. To remedy this problem, we suggest that the weights should depend upon the buckling eigenvalues themselves. For example, a complete quadratic combination of the eigenvalues can be obtained by choosing \( c(x) = B'(x)\mu(x) \) giving the quadratic form for the objective of \( \mu'(x)C(x)\mu(x) \), where \( C(x) = B(x)B'(x) \) is a symmetric matrix. Such a choice has considerable theoretical appeal and it completely fixes the objective, reducing the non-inferior set to a point in design space. In the sequel we adopt a slightly more flexible choice for \( c(x) \). Specifically, we choose \( c_i(x) = \mu_{\pi_i}(x) \), where \( \pi = \{\pi_1, \pi_2, \ldots, \pi_v\} \) is a set of index numbers. Thus the entries in the \( c(x) \) array are simply eigenvalues selected from the subspace. Note that a uniform weighting can be accomplished by selecting all indexes to be the same, i.e., \( \pi = \{1,1,\ldots,1\} \). Some different choices will be examined in the examples at the end of the paper.

**Optimality Criteria**

The optimality criteria can be obtained from the first-order necessary conditions for a constrained optimum. The Lagrangian functional corresponding to the optimization problem given in Eq. 4 can be written as:

\[ l(x, \lambda) = \mu'(x)B(x)c(x) - \lambda[w'a(x) - \omega_0] \] 

where \( \lambda \) is the Lagrange multiplier for the equality constraint. The element size limit constraints are not included in the Lagrangian and therefore do not have corresponding Lagrange multipliers. The explicit size constraints can be handled efficiently with an active-set strategy. Whenever a design variable violates a size constraint, it is assigned its limiting value and is removed from the active set (i.e., it is no longer viewed as a design variable).

The first-order necessary conditions for an optimum are obtained by differentiating Eq. 6 with respect to design variables \( x \) and setting the corresponding equation to zero

\[ \nabla l(x, \lambda) = \nabla \mu' Bc + \mu' \nabla Bc + \mu' B \nabla c - \lambda w' \nabla a = 0 \] 

where \( [\nabla(\ )]_i = \partial(\ )/\partial x_i \) is the ordinary gradient operator.

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SOLUTION PROCEDURE

The optimum structure must satisfy the optimality criteria and the weight constraint. Since these equations are nonlinear, they must be solved by an iterative scheme. The algorithm suggested here is a fixed-point iteration based on the first-order necessary conditions (optimality criteria). The fixed-point iteration, used in conjunction with a scaling procedure, will move the initial design toward a configuration that satisfies the optimality criteria and the constraints. An iteration step comprises three stages: (1) Estimation of the Lagrange multiplier; (2) selecting a design vector direction; and (3) performing a line search (scaling) to satisfy the weight constraint. A brief overview of these steps is presented in the following sections.

Fixed-Point Iteration

Various forms of recurrence relations have been developed and used for the configuration update in structural optimization problems. Berke (1970) used a recursion relation in a virtual strain energy formulation for minimizing weight with prescribed displacement constraints. The same recurrence relation was effectively used by Gellaty and Berke (1971) for design problems with stress and displacement constraints. Later, Khot et al. (1973, 1976) derived different forms of the recurrence relations for displacement constraints, stress constraints, and dynamic stiffness requirements. The recursive approach has the advantage that it eliminates the need for the Hessian of the Lagrangian functional required in a nonlinear programming algorithm. The Hessian in the present problem is difficult to compute because it involves second derivatives of the eigenvectors of the system.

To make the formulation easier to follow we will make some definitions. Define the vector $Q$ to be the gradient of the objective

$$Q(x) = \nabla \mu' B c + \mu' \nabla B c + \mu' B \nabla c$$

Further, let $q$ be the gradient of the constraint

$$q(x) = w' \nabla a(x) = w' A(x)$$

where the components of the newly defined matrix $A$ are given by $A_{ij} = \partial a_i / \partial x_j$. With this notation, the optimality conditions take the form

$$Q(x) - \lambda q(x) = 0$$

The geometric significance of the optimality criterion is clear: the optimum is achieved for a design $x$ where the vectors $Q$ and $q$ are colinear. To set up the recurrence algorithm let us define a diagonal quotient matrix $D(x)$ in the following manner

$$D_{ij} = \frac{Q_i}{q_i}, \quad i = j$$

$$D_{ij} = 0, \quad i \neq j$$

From its definition, $D$ satisfies the relationship $Q = Dq$. We generate a new design vector from the previous one with the exponential recursion relation

$$x^{k+1} = \left[ \frac{D(x^k)}{\lambda^k} \right]^{1/r} x^k$$
where \( k \) = the iteration counter; \( r \) = a parameter that determines the magnitude of the adjustment in the design variable; and exponentiation of the diagonal matrix \( D \) is accomplished by exponentiating each term independently. The value of \( \lambda^k \) is the current estimate of the Lagrange multiplier.

The central idea of the recurrence relationship is that the matrix \( D/\lambda \) approaches the identity at the optimum, and any deviation from the identity indicates a need for adjustment in the design variables. If an element of \( D/\lambda \) is less than one, the associated design variable is dominating and needs to be reduced. If the value is greater than one, the associated design variable needs to be increased. The update formula treats the design variables as uncoupled when, in fact, a change in design variable \( i \) will change the value of \( Q_j \) for \( j \neq i \). The parameter \( r > 1 \) keeps the adjustment factors closer to unity to avoid divergence. At the optimum, the design variables will be unchanged by Eq. 12. A parameter study on the magnitude of the step length and its effect on the convergence of the algorithm can be found in the paper by No and Aguinagalde (1987).

Near the optimum, the term in brackets in Eq. 12 will approximate the identity. Linearizing the exponential about \( D/\lambda = I \) gives the alternative update formula

\[
x^{k+1} = \left\{ I + \frac{1}{r} \left[ \frac{D(x^k)}{\lambda^k} - I \right] \right\} x^k
\]

This equation is referred to as the linear recurrence relation for the design variables and can be used in place of Eq. 12 to update the design or to help estimate the Lagrange multiplier as shown below.

The optimization converges when the optimality criteria of all active variables are satisfied. After each iteration the deviation of the optimality criteria from unity is calculated, and if the Euclidean norm of the deviation is less than a specified tolerance the iteration is terminated.

**Estimation of Lagrange Multiplier**

To update the design variables according to the recurrence relation given by Eq. 12, the Lagrange multiplier \( \lambda \) must be determined. The simplest estimate is the least squares projection of Eq. 10 which gives

\[
\lambda = \frac{q'Q}{q'q} = \frac{w'ADA'w}{w'AA'w}
\]

The Lagrange multiplier can alternatively be determined from the condition that the linearized constraint should be satisfied at the end of a design iteration. An explicit equation for the Lagrange multiplier can be obtained by linearizing the constraint about the current iterate and substituting an estimate for the new design from Eq. 12. Linearizing the volume constraint \( h(x) = w'a(x) - \omega_o = 0 \) about the configuration \( x^K \) one obtains:

\[
L(h)_{x=x^K} = w'a(x^K) - \omega_o + w'A(x^K)(x - x^K) = 0
\]

The Lagrange multiplier can be obtained by satisfying the linearized constraint equation at the new iterate \( x^{k+1} \). Substituting Eq. 12 into Eq. 15 and solving for the Lagrange multiplier one gets:

\[
\lambda = \left[ \frac{w'AD^{1/r}}{\omega_o + w'(Ax - a)} \right]'
\]
in which all terms are evaluated at the current estimate of the design variable, \( x^K \). Because the constant volume constraint is an equality, the Lagrange multiplier can be either positive or negative. If the linearized recurrence update is used in favor of the exponential update the following estimate of the Lagrange multiplier results:

\[
\lambda = \frac{w'ADx}{r_0 + w'(Ax - ra)} \tag{17}
\]

Scaling Procedure

After each iteration, the design variables must be scaled to ensure satisfaction of the volume constraint, \( h(x) = 0 \). Scaling is necessary because the design variable update formula does not ensure feasibility. The constraint can be used to estimate the Lagrange multiplier, but even in this instance feasibility is not guaranteed because the constraint is linearized. In general, the constraint is nonlinear in the design variables so that an iterative method must be used for scaling. Since there is a single scalar constraint for the present application, scaling is relatively straightforward.

The problem of scaling is essentially a line search in the direction of the current design vector \( x \). There are many methods available for carrying out this line search. Among these is Newton’s method. We wish to satisfy the constraint exactly for the scaled design variables \( \xi x \), where the following notation has been introduced to accommodate the active-set strategy:

\[
(\xi x)_i = \xi x_i \quad \text{for } i \text{ in the active set} \tag{18a}
\]
\[
(\xi x)_i = x_i \quad \text{for } i \text{ not in the active set} \tag{18b}
\]

If we linearize the scaled constraint with respect to the scaling parameter \( \xi \) we obtain the iteration equation

\[
\xi^{v+1} = \xi^v - \frac{h(\xi^v x)}{\nabla h(\xi^v x) \cdot x} \tag{19}
\]

where \( v = \) the iteration counter; and \( \nabla h = \) the ordinary gradient with respect to \( x \).

For rectangular members the scale factor can be computed in closed form because the constraint is quadratic in the scale parameter. The weight of each member is proportional to the product of the two design variables (height and width) describing the cross section. Neither, both, or either one of the two variables will be active at any given state of the optimization process. The members of the structure can be grouped according to the number of active variables it has. The total weight is given by:

\[
\omega = \omega^{aa} + \omega^{ap} + \omega^{pp} \tag{20}
\]

where \( \omega^{aa} = \) the total weight of members with both design variables active; \( \omega^{ap} = \) the total weight of members with one design variable active; and \( \omega^{pp} = \) the total weight of members with neither design variables active. The scaling factor \( \xi \) is determined by the equation:

\[
\xi = \sqrt{\frac{\omega_0 - \omega^{pp}}{\omega^{aa}}} + \left( \frac{\omega^{ap}}{2\omega^{aa}} \right)^2 - \frac{\omega^{ap}}{2\omega^{aa}} \tag{21}
\]
The scaling equation for a general cross section depends upon the representation of the cross-sectional area in terms of the design variables. If the design variables are chosen to be linear dimensions of the cross section, then the resulting equation will always be quadratic. However, for complex cross sections like the I-beam, the grouping strategy becomes cumbersome and Newton's method is preferred. Care must be exercised to adjust the active set to reflect variables that become passive as a result of scaling.

**Active-Set Constraint Strategy**

After each iteration a new set of design variables is obtained. If a design variable lies within its permissible range, it is placed in the active set, otherwise it is placed in the passive set so that a proper scaling can be performed before the next iteration. At the start of each iteration, formerly passive variables can either remain in the passive set or be reactivated. In general, it is not known a priori if a variable will be active at the optimum. Allwood and Chung (1984) have suggested that if a design variable is moved to the passive set in two consecutive iterations, it will probably be passive at the optimum. In principle, the method suggested by Allwood and Chung was followed in the computations reported here. However, it was found that in the early stages of optimization the iterations can be erratic and it is prudent to return all variables to the active set at each iteration until the algorithm settles down.

**Eigenvalue and Eigenvector Sensitivity Analysis**

Evaluation of the optimality conditions requires knowledge of the sensitivity, or rate of change, of the buckling eigenvalues and eigenvectors with respect to the design variables. Procedures for computing these sensitivities have been known for some time, but efficient methods of computation continue to be of interest to researchers. A complete and detailed discussion of the problem has been given recently by Dailey (1989). Some of the basic ideas are outlined below.

Consider that the matrices \( K(s) \) and \( G(s) \) depend upon a parameter, \( s \) (possibly, but not necessarily, a design variable or a linear combination of design variables), and that these matrices are differentiable with respect to the parameter. The eigenvalues and eigenvectors must also depend implicitly upon the same parameter. The expressions for the sensitivities for the \( i \)th eigenvector and eigenvalue are given as

\[
\mu'_i = \frac{\phi'_i (K' - \mu'_i G') \phi_i}{\phi'_i G \phi_i} \quad (22)
\]

\[
\phi'_i = \sum_{j=1}^{N} \frac{\phi'_j (K' - \mu'_i G') \phi_j}{(\mu_i - \mu_j) \phi'_j G \phi_j} - \frac{1}{2} \frac{\phi'_i G \phi_i}{\phi'_i G \phi_i} \phi_i \quad (23)
\]

Clearly, the parameter with respect to which differentiation is done can be any of the design variables. Hence, Eqs. 22 and 23 can be used to compute the rate of change of the eigenproperties with respect to the design variables. The matrix \( G' \) is identically zero for statically determinate structures since the distribution of force through the structure does not depend upon the element rigidities. For indeterminate structures the derivative of the
geometric stiffness matrix is usually small in comparison to the derivative of the elastic stiffness matrix and is, therefore, generally neglected in practical computations.

The above expression for the sensitivity of an eigenvector suffers from the practical drawback that it requires knowledge of all of the eigenvectors and eigenvalues of the system. The method becomes prohibitively expensive for large systems since the determination of all $N$ eigenpairs becomes practically impossible. Nelson (1976) presented a powerful algorithm for computing eigenvector and eigenvalue derivatives of general matrices with non-repeated eigenvalues in which the derivative of the eigenvector of any mode requires only the eigenvalue and eigenvector of that mode. The eigenvector derivative satisfies the following system of equations, obtained by differentiating Eq. 1

$$(K - \mu G)\phi' = -(K' - \mu' G - \mu G)\phi \tag{24}$$

The quantities on the right-hand side can all be considered known since the derivative of the eigenvalue is easily obtained from Eq. 22. The coefficient matrix $K - \mu G$ is one-degree rank deficient (for a distinct eigenvalue), and the null-space is spanned by the eigenvector $\phi$. Thus, the solution to Eq. 24 is given by the sum of a particular solution, $v$, plus a component in the null-space as

$$\phi' = v + c\phi \tag{25}$$

The particular solution can be found with the aid of the singular value decomposition or, as Nelson (1976) suggested, by setting one component of $v$ to zero (usually the one corresponding to the largest component of $\phi$) and solving the remaining equations. The constant $c$ is determined by substituting Eq. 25 into the derivative of the normality condition, $2\phi'G\phi' + \phi'G'\phi = 0$. The resulting expression for the eigenvector derivative is then

$$\phi' = v - \frac{\phi'G'\phi + 2\phi'Gv}{2\phi'G\phi} \phi \tag{26}$$

The analysis of the sensitivity for systems with repeated eigenvalues is complicated by the fact that, while the eigenvectors associated with the repeated eigenvalue are not unique (any linear combination of eigenvectors is also an eigenvector) the eigenvector derivatives are. A subsidiary calculation of the directions in which those derivatives exist must be done. Once established, results analogous to those given above can be obtained. Sensitivity analysis with repeated eigenvalues is discussed by Haug et al. (1986), Dailey (1989) and Mills-Curran (1989) independently extended Nelson’s (1976) method to systems having repeated eigenvalues.

Repeated eigenvalues in an unfortunately chosen symmetric initial design can be easily diagnosed and easily cured. For the optimization problems solved here the solutions will, at worst, converge to a configuration with a repeated eigenvalue. In those cases convergence of the optimization algorithm will generally be achieved before numerical difficulties set in. As a consequence of the preceding observations, the implementation used for the computations presented subsequently treats the spectrum as completely distinct.

The approach followed in computing eigenvector derivatives for practical
problems in the present study was to truncate the sum in Eq. 23 after a finite (small) number of terms. Specifically, the eigenvector derivatives are determined from Eq. 23 by including only those eigenvectors in the subspace used to define the objective function. Although using only a few eigenvectors does not give the exact eigenvector derivative, computational experience has shown that the results are adequate for the optimization algorithm.

The linear elastic structural stiffness matrix $K$ exhibits the following explicit form

$$K(x) = \sum_{m=1}^{M} \left[ a_m(x) \sum_{i \in m} K_i^a + \bar{l}_m(x) \sum_{i \in m} K_i^l + \sum_{i \in m} L_i(x) \sum_{i \in m} K_i^l + J_m(x) \sum_{i \in m} K_i^l \right]$$

where $a_m(x)$, $\bar{l}_m(x)$, $L_m(x)$, and $J_m(x)$ = the area, major moment of inertia, minor moment of inertia, and torsion constant of the cross section for group $m$, respectively, all of which depend on the design variables; and $K_i^a$, $K_i^l$, $K_i^f$, and $K_i^f$ = the stiffness kernels corresponding to the four cross-sectional parameters for element $i$. These kernels do not depend on the design variables, and hence can be summed over all elements in each group. The sensitivity of the structural stiffness matrix can be computed by differentiating the structural stiffness matrix given in Eq. 27 with respect to design variable $x$, and involves only differentiation of the scalar cross-sectional parameters.

In the following sections we apply the proposed optimization method to two framed structures. The purpose of the examples is to demonstrate how the algorithm performs. In these studies we examine the effects of varying the size of the eigenspace and different weighting schemes used to form the objective, in addition to the main goal of examining the performance with multiple load cases. We also examine the effect of choosing different sets of design variables, and we compare the designs obtained with those obtained by specifying a displacement constraint.

**TWO-STORY-FRAME EXAMPLE**

Consider the two-story, single-bay, three-dimensional, moment-resisting frame shown in Fig. 1. The story heights were chosen tall to accentuate the importance of stability in the design. The frame is exposed to two possible loading conditions. Load case I consists of proportionally applied, triangular-shaped “earthquake” load in the $x$-direction, a distributed (nonproportional) gravity loading applied along the girders, and a proportional vertical loading applied at the columns. Lateral loads were obtained following the specifications set out in the Uniform Building Code (1979). Load case II consists of torsional loads applied at the top-story level in the $z$-direction. The two loading conditions are also shown in Fig. 1. The member properties are grouped into three material sets: (1) First-story columns; (2) second-story columns; and (3) all girders. The members are assumed to have rectangular cross sections. Thus, each material set has two associated design variables (width and height), and the structure has a total of six design variables.
Optimization under Single Load Case

The two-story frame was first optimized under a single loading condition (load case I) with the size of the subspace used in the objective function varied among numbers \( n = 6, 8, \) and 10. The properties of the optimized designs are presented in Table 1. The nonlinear responses of the four structures under load case I are shown in Fig. 2. [Note: The analyses of the frames studied here are based upon the geometrically nonlinear rod model proposed by Simo (1984) and Simo and Vu-Quoc (1986). The model was extended to account for rate-independent elastoplasticity (in stress resultant space) and implemented in the general purpose finite element program FEAP (Taylor 1977). The yield surface was taken to be an ellipsoid in stress-resultant space with principle axes proportional to the fully plastic yield values of each of the stress resultants in absence of the others.]

The member dimensions of the optimized designs show a clear preference for the loading direction. An interesting feature of the optimal designs is that the girder depths are at their lower limit, indicating that the girders contribute little to the stability of this structure. One should note, however, that girder properties are not entirely passive in these designs. All of the optimized designs show a great improvement in the load-deflection behavior over the initial design. While the member sizes of the cases \( n = 6 \) and \( n = 8 \) are not the same, the nonlinear performance of the two is nearly identical. The case \( n = 10 \) is considerably better than the other two, illustrating that the size of the subspace can have an important influence on the design.

The evolution of the buckling eigenvalues during the course of optimization for the case \( n = 6 \) is shown in Fig. 3. This spectral evolution allows one to observe easily how the eigenvalues change, and it gives insight into the behavior of the structure as the optimization progresses. The eigenvalues corresponding to a given mode shape are connected together with lines throughout the iteration history.

The optimization was started with a step length of \( r = 8, \) and all the design variables were active. After the 10th iteration the height of the girders became passive and adopted the minimum allowable value. To settle the oscillations of the algorithm that manifested in iterations 10 through 15, the
TABLE 1. Properties of Optimized Two-Story Frame for Different Weighting Functions

<table>
<thead>
<tr>
<th>Property</th>
<th>$d \nu = 6$</th>
<th>$d \nu = 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\pi = (1, 1)$</td>
<td>$\pi = (1, 2)$</td>
</tr>
<tr>
<td>$\pi = (1, 2)$</td>
<td>$\pi = (2, 3)$</td>
<td>$\pi = (1, 1)$</td>
</tr>
<tr>
<td>$\pi = (2, 3)$</td>
<td>$\pi = (1, 1)$</td>
<td></td>
</tr>
</tbody>
</table>

(a) First-Story Columns
- Width (in.): 5.04, 5.91, 6.94, 7.54
- Height (in.): 6.92, 5.43, 4.90, 5.46

(b) Second-Story Columns
- Width (in.): 5.04, 5.91, 6.94, 7.63
- Height (in.): 6.92, 5.43, 4.90, 3.69

(c) First-Story Girders
- Width (in.): 7.60, 8.86, 7.99, 10.33
- Height (in.): 3.00, 3.00, 3.00, 3.00

(d) Second-Story Girders
- Width (in.): 7.60, 8.86, 7.99, 5.00
- Height (in.): 3.00, 3.00, 3.00, 3.00

(e) Eigenvalues at Optimum
- Mode 1: 410, 257, 220, 282
- Mode 2: 417*, 420*, 356, 345
- Mode 3: 572, 423, 491*, 444
- Mode 4: 705, 550, 493, 700
- Mode 5: 710, 670, 567, 771
- Mode 6: 718, 757, 696, 778*

*Dominant mode.
Note: 1 in. = 25.4 mm.

FIG. 2. Response of Two-Story Frame for Different Subspace Sizes

step length was changed from 8 to 20. The optimization converged smoothly from that point to a structure with a relatively evenly spaced spectrum (quite different from the initial spectrum). The mode with the highest participation with respect to the single load vector was marked with an x at each iteration. Note that the second mode participated most in the initial design whereas the fourth mode was dominant in the optimal design.

The spectral evolutions for subspace sizes of $n = 8$ and $n = 10$ are not shown here, but several observations from them should be mentioned. For each optimization there is an initial stage in which the spectrum of the structure changes rapidly and erratically. The number of iterations required to pass this stage is related to the size of the subspace in the sense that a larger subspace will generally take more iterations to smooth out. The erratic behavior appears to be an artifact of the search for the proper active set of design variables since the termination of that phase is generally associated with the adoption of an appropriate active set. The larger subspace only complicates the problem of selecting the active set by allowing more flexibility in the reassignment of material at each iteration. The oscillations noted for $n = 6$ (between iterations 10 and 15) are typical of this algorithm and can generally be cured by increasing the step-length parameter $r$.

Because of the erratic activity in the first few iterations, it is quite possible that a design variable might be removed from the active set prematurely. A conservative active strategy seems prudent during this stage. For example, all the design variables were returned to the active set after each iteration for the first 20 iterations in the case $n = 10$.

In general, one cannot decide a priori if the eigenvalues are important to the optimal design because mode shifting occurs during the course of optimization. To show that this is true, consider Fig. 4, showing the mode shapes of the initial design and those for the optimized design ($n = 8$). For the loading considered here one would expect the seventh mode to have a high participation in the objective function. However, when using a subspace size
of eight in the objective, the seventh mode of the initial design evolves into the ninth mode of the optimal design. Consequently, the apparently important seventh mode of the initial design eventually does not participate in the objective at all. This behavior is not unexpected since the algorithm tries to increase the eigenvalues of the most participatory modes the most. The deletion of a mode from consideration is an artifact of retaining modes according to the value of their associated eigenvalue (i.e., the sorting convention for eigenvalues). Based on this example it would appear that it is better if the objective contains all important modes. An alternative approach would be to track mode shapes by updating the basis using a Newton-Raphson scheme, and thereby ensuring the participation of the initially chosen basis.

Once a mode leaves the subspace the resources are allocated accordingly. While the mode is outside the subspace the value of its corresponding eigenvalue is guaranteed to be greater than that of all those in the subspace. The banished mode may return if another mode is pushed out. Such mode cycling is unlikely in the lower modes of planar structures but should be expected for three-dimensional structures. We have seen no evidence that the algorithm stalls because of mode cycling.

Optimization under Multiple Load Cases

As mentioned earlier, there is some flexibility in selecting the load weighting factors for optimization under multiple load cases. In practice it will not be feasible to determine the whole noninferior optimal set and specific weights must be chosen. The choice of weights might be based on subsidiary analyses or other information not included in the objective function. In this section we examine the effect of different choices of the load case weighting factors by optimizing the same two-story frame under load cases I and II.

To restrict the domain of possibilities, we select the weights to be proportional to selected eigenvalues in the subspace. The most obvious choice is equal weighting, \( \pi = (1, 1) \). In the equally weighted case it was noticed that the second buckling eigenvector turned out to be a sidesway mode at the optimum and the third buckling eigenvector turned out to be a torsional
TABLE 2. Properties of Optimized Designs for Setback Frame

<table>
<thead>
<tr>
<th>Property</th>
<th>Initial design (2)</th>
<th>Single load ( n = 10 ) (3)</th>
<th>Multiple load ( n = 5 ) (4)</th>
<th>Displacement constraint (5)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Width (in.)</td>
<td>Height (in.)</td>
<td>Width (in.)</td>
<td>Height (in.)</td>
</tr>
<tr>
<td>(a) 1st-Story Columns</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Width (in.)</td>
<td>5.50</td>
<td>6.92</td>
<td>5.62</td>
<td>1.80</td>
</tr>
<tr>
<td>Height (in.)</td>
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<td>7.50</td>
<td>7.50</td>
<td>19.02</td>
</tr>
<tr>
<td>(b) 2nd-Story Columns</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Width (in.)</td>
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<td>6.15</td>
<td>4.14</td>
<td>1.80</td>
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<td>Height (in.)</td>
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<td>7.50</td>
<td>7.50</td>
<td>11.87</td>
</tr>
<tr>
<td>(c) 1st-Story Girders</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Width (in.)</td>
<td>1.70</td>
<td>1.80</td>
<td>1.80</td>
<td>1.80</td>
</tr>
<tr>
<td>Height (in.)</td>
<td>9.80</td>
<td>7.50</td>
<td>7.50</td>
<td>7.50</td>
</tr>
<tr>
<td>(d) 2nd-Story Girders</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Width (in.)</td>
<td>1.60</td>
<td>1.80</td>
<td>4.11</td>
<td>5.46</td>
</tr>
<tr>
<td>Height (in.)</td>
<td>7.50</td>
<td>7.50</td>
<td>7.50</td>
<td>7.50</td>
</tr>
<tr>
<td>(e) Braces</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Width (in.)</td>
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<td>1.80</td>
<td>1.80</td>
<td>1.80</td>
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<tr>
<td>Height (in.)</td>
<td>7.50</td>
<td>7.50</td>
<td>7.50</td>
<td>7.50</td>
</tr>
</tbody>
</table>

Note: 1 in. = 25.4 mm.

FIG. 5. Response of Two-Story Frame with Various Weighting Functions

mode. To improve the performance of the design under both load cases I and II, which are of sidesway and torsional type, it was decided to put more importance on mode 2 and 3 by choosing \( \pi = (2, 3) \). The case with \( \pi = (1, 2) \) was also examined. The properties of the optimal designs are given in Table 2.

The nonlinear response of the initial and optimized designs are shown in Fig. 5. Two types of load-deformation curves are presented to aid the evaluation of the various designs: one with top displacement in x-direction and one with top displacement in z-direction. The one with top displacement in x-direction is analyzed under loading case I. Similarly, the one with top
displacement in z-direction is analyzed under load case II.

The most important observation about the responses of the various optimal designs is that they represent a significant improvement over the initial design for both of the loading cases. As can be seen, the design with the second- and third-mode weightings lead to the most robust design.

To demonstrate the effect of the number of design variables on the performance of the optimized design, the case with equal weighting of the two load cases was repeated with eight design variables $(dv = 8)$. As opposed to the previous cases, the girders at the two-story levels were given separate design variables. The results are shown in Table 2 and Fig. 5. The performance under load case I is greatly enhanced by the additional freedom, but the performance under load case II is barely improved over the initial design. Note that the girders in the second story are completely passive at the optimum.

**SETBACK FRAME EXAMPLE**

It is particularly difficult to identify design improvements for complex, irregular structures. The two-story setback frame shown in Fig. 6, taken from Cheng and Truman (1985), and redesigned to meet the ATC 3-06 (Applied Technology Council 1984) earthquake design recommendations, provides an example of an irregular structure in which the stability problems are not obvious. The structure is optimized and evaluated for two loading conditions in a subsequent development.

The initial design was found by traditional means to satisfy the earthquake provisions of ATC 3-06. A modal analysis procedure was employed to determine earthquake loads. The following seismic coefficients were used: (1) Effective peak acceleration $(A_a = 4)$; (2) effective peak velocity-related acceleration $(A_v = 0.4)$; (3) soil-profile characteristics of site $(S_2 = 1.2)$; (4) reduction factor to account for effects of inelastic behavior $(R = 4.5)$; (5) seismic category C; and (6) seismicity index of four. The loads on the structure included a dead load of 80 psf (3.83 kPa) and a live load of 40 psf (1.92 kPa). The critical load effect for seismic forces was determined to be 100% of the force in one direction plus 30% of the force in the perpendicular direction. Eight different load combinations were considered in accordance with ATC 3-06. Members were checked for the worst loading case and were redesigned as necessary. This procedure of analysis and redesign was carried out for several iterations until all the requirements were satisfied. The mem-

![FIG. 6. Topology of Setback Frame](image)
TABLE 3. Properties of Optimized Two-Story Frame for Different Subspace Sizes

<table>
<thead>
<tr>
<th>Property</th>
<th>Initial design</th>
<th>( n = 6 )</th>
<th>( n = 8 )</th>
<th>( n = 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( (1) )</td>
<td>( (2) )</td>
<td>( (3) )</td>
<td>( (4) )</td>
</tr>
<tr>
<td>Width (in.)</td>
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<td>7.76</td>
<td>7.61</td>
<td>8.28</td>
</tr>
<tr>
<td>Height (in.)</td>
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<td>5.51</td>
<td>5.34</td>
<td>5.04</td>
</tr>
<tr>
<td>( (a) ) First-Story Columns</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Width (in.)</td>
<td>10.09</td>
<td>5.88</td>
<td>5.96</td>
<td>6.38</td>
</tr>
<tr>
<td>Height (in.)</td>
<td>3.00</td>
<td>4.35</td>
<td>4.24</td>
<td>4.08</td>
</tr>
<tr>
<td>( (b) ) Second-Story Columns</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Width (in.)</td>
<td>9.69</td>
<td>7.92</td>
<td>8.48</td>
<td>8.09</td>
</tr>
<tr>
<td>Height (in.)</td>
<td>3.00</td>
<td>3.00</td>
<td>3.00</td>
<td>3.00</td>
</tr>
<tr>
<td>( (c) ) Girders</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: 1 in. = 25.4 mm.

ber properties of the final design were checked against the AISC (1980) specifications and all the requirements were satisfied. The properties of the initial design are given in Table 3.

Two particular cases are examined for the setback structure: (1) A single load case with 10 eigenvectors in the subspace; and (2) two load cases with equal weighting and five eigenvectors in the subspace. The two load cases are shown in Fig. 7. For both of these cases, 10 design variables were used: the height and width of the cross section for each of the first-story columns, the second-story columns, the first-story girders, the second-story girders, and the braces. Properties of the optimal designs are given in Table 3. The nonlinear response of the initial design and the optimized design for a single loading condition are shown in Fig. 8. Evidently the optimized design is stiffer and stronger than the initial design for that loading condition.

To illustrate the intrinsic merit of the design procedure proposed here, the setback structure was also designed with a single displacement constraint.

![Fig. 7. Load Cases for Setback Structure](image-url)
procedure. The volume of the structure was minimized with a top displacement constraint of 1.2 in. (30.5 mm), under a combination of load cases I and II, yielding a structure of the same volume as those optimized for stability. The properties of the displacement constrained design are also given in Table 3.

The nonlinear response of the initial design, the optimized design for multiple loading conditions, and the displacement constrained design are shown in Fig. 9. As with the previous structure, the structures optimized for stability behave better than the initial design for both loading conditions. In contrast, one can observe that the displacement constrained design is stiffer and stronger than the initial design under load case I, but is quite inferior under load case II, even though load case II was represented in the design process. Furthermore, the response of the structures under load case I indicates that the displacement constrained structure is generally more brittle than the structure optimized for stability. This observation illustrates the fallacy of using displacement limitations for the mitigation of stability problems, a philosophy implicit in many modern design codes.

**FIG. 8.** Response of Setback Structure for One Load Case and Subspace Size of Ten

**FIG. 9.** Comparison of Displacement Constraint Optimization and Stability-Based Optimization
SUMMARY AND CONCLUSIONS

We have presented an optimization-based design procedure that efficiently produces structural designs with better strength and stability characteristics than alternative structures of the same weight. The method establishes a rational framework in which to address the stability design of structures that might suffer multiple loading conditions.

While the proposed procedure searches among structures with a specific weight, the magnitude of the weight is not sacred. An estimate of a suitable target weight for the structure can be obtained from the preliminary design. Once optimized, the structure can be checked for compliance with other criteria. If those criteria are violated, then the target weight can be revised and the structure redesigned.

The process of structural design is complex and many factors must be considered. We do not claim to consider any factors other than the stability of the structure. However, the optimization problem could be augmented with additional constraints (e.g., stress limits) with no more difficulty than a traditional code-based optimization approach. However, many benefits accrue from formulating the problem with a single equality constraint, not the least of which is the avoidance of many local minima (there is some evidence that the present approach is unencumbered by multiple local minima). Consequently, we prefer to view the proposed method as a means of generating guidance as to where material should be placed to enhance the overall strength and stability of a structure rather than the ultimate design procedure.

ACKNOWLEDGMENTS

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APPENDIX I. REFERENCES

APPENDIX II. NOTATION

The following symbols are used in this paper:

\[ \begin{align*}
A(x) &= \text{gradient of area function;} \\
a(x) &= \text{area;} \\
B(x) &= \text{mode weighting matrix;} \\
c(x) &= \text{objective weighting parameters;} \\
D &= \text{iteration quotient matrix;} \\
G &= \text{geometric stiffness matrices;} \\
h(x) &= \text{constraint;} \\
I &= \text{moment of inertia;} \\
F_f, f &= \text{force matrix, force vector for load case } i; \\
K &= \text{stiffness matrix;} \\
L &= \text{member length;} \\
l(x, \lambda) &= \text{Lagrangian of objective function;} \\
Q &= \text{gradient of objective function;} \\
q &= \text{gradient of constraint;} \\
r &= \text{step-length parameter;} \\
r &= \text{effective load;} \\
v &= \text{effective displacement;} \\
w &= \text{specific weight;} \\
x &= \text{design variables;} \\
\end{align*} \]
\[ \begin{align*}
\lambda &= \text{Lagrange multiplier;} \\
\mu &= \text{buckling eigenvalue;} \\
\xi &= \text{scaling parameter;} \\
\pi &= \text{set of mode numbers;} \\
\rho &= \text{density;} \\
\phi &= \text{buckling eigenvector; and} \\
\omega_o &= \text{structure weight.}
\end{align*} \]