Transportation Economics and Decision Making

Lecture-10

Multinomial Logit Model

- The binomial logit model can be easily extended to accommodate choices among more than two alternatives
- Let us consider three alternatives in the choice set
- Probability that alternative 1 is chosen

$$P(1) = \frac{e^{v_1}}{e^{v_1} + e^{v_2} + e^{v_3}}$$

Multinomial Logit Model (2)

- If there are more alternatives than three then the probabilities can be expresses as follows

$$P(i) = \frac{e^{v_i}}{\sum_{i=1}^{I} e^{v_i}}$$

$$P(1) = \frac{e^{v_1}}{\sum_{i=1}^{I} e^{v_i}}$$

$$P(2) = \frac{e^{v_2}}{\sum_{i=1}^{I} e^{v_i}}$$

Multinomial Logit Model (3)

- The multinomial logit model has all the desirable properties of the binomial logit model
- In addition, it can be applied to any number of alternatives
- The probability of choosing an alternative depends on the relative utilities with all other alternatives

$$P(1) = \frac{e^{v_1}}{e^{v_1} + e^{v_2} + e^{v_3}}$$

$$= \frac{1}{1+e^{-(v_1-v_2)}+e^{-(v_1-v_3)}}$$

Example-1

- Consider travel to work and let there be three modes of choice set

Mode	V
Drive Alone	2.5
Carpool	2
Bus	1

Mode	V	exp(v)
Drive Alone	2.5	12.18249
Carpool	2	7.389056
Bus	1	2.718282
Total		22.28983

Mode	V	exp(v)	Probability
Drive Alone	2.5	12.18249	0.546549
Carpool	2	7.389056	0.331499
Bus	1	2.718282	0.121952
		22.2898 3	1

- As expected the mode with highest deterministic component of utility has the highest probability of being chosen

Incorporation of Attributes of Alternatives and Individuals

- Deterministic component of a mode's utility depends on the attribute of that mode (and not of other modes) and the individual making the choice
- Suppose deterministic component of the utility of mode j
- $V_j = -T_j - 5C_j/Y$

Mode	Time (T), Hours	Cost ©, $
Drive Alone	0.5	2
Carpool	0.75	1
Bus	1	0.75

			Y=15			Y=30		
Mode	Time (T), Hours	Cost ©, $	V	exp(v)	Prob	V	exp(v)	Prob
Drive Alone	0.5	2	-1.167	0.311	0.333	-0.833	0.435	0.375*
Carpool	0.75	1	-1.083	0.338	0.361*	-0.917	0.400	0.345
Bus	1	0.75	-1.250	0.287	0.306	-1.125	0.325	0.280
Total				0.936	1.000		1.159	1.000

Scenario MNL

- If the bus fare increase by $0.25, then the resulting probability choices are

Mode	Time (T), Hours	Cost ©, $	V	exp(v)	Prob	V	exp(v)	Prob
Drive Alone	0.5	2	-1.167	0.311	0.341	-0.833	0.435	0.379*
Carpool	0.75	1	-1.083	0.338	0.371*	-0.917	0.400	0.349
Bus	1	1	-1.333	0.264	0.289	-1.167	0.311	0.272
				0.913	1.000		1.146	1.000

- The outcomes are unaltered because of the fare increase in bus

Alternative Specific Constant

- In the logit model two modes have equal probabilities if they have equal travel time and cost
- In practice, however other factors such as comfort, reliability, and safety may cause one mode to have greater probability of being chosen than another
- The best way to account for these is to include variables representing them in the deterministic component of the utility functions

Alternative Specific Constant (2)

- However this is not possible often in practice, since many of these factors are difficult to measure and predict
- An alternative method can be implemented easily by adding a constant in the deterministic component of the utility function for all modes except one (reference case or base case)
- These constants are called alternative-specific constants

Alternative Specific Constant (3)

- The alternate specific constant for a given mode is the average amount that factors not included in the deterministic component of the utility function
  - As a contribution to the difference between the utilities of the given mode and base mode
- In other words, it is the average contribution of the error terms to the difference between two modes' utilities

Example: Alternative specific constants (1)

- Suppose that deterministic components of the utility functions are
  - $V_{DA} = 0.8 - T_{DA} - 5C_{DA}/Y$
  - $V_{CP} = 0.2 - T_{CP} - 5C_{CP}/Y$
  - $V_B = T_B - 5C_B/Y$

Mode	Time (T), Hours	Cost ©, $	Without constants			With constants		
			V	exp(v)	Prob	V	exp(v)	Prob
Drive Alone	0.5	2	-0.833	0.435	0.379	-0.033	0.967	0.547
Carpool	0.75	1	-0.917	0.400	0.349	-0.717	0.488	0.276
Bus	1	1	-1.167	0.311	0.272	-1.167	0.311	0.176
Total				1.146	1.000		1.767	1.000

- In this bus is the base mode
- The alternative specific constants for drive alone and carpool are 0.8 and 0.2 respectively
- The signs and magnitudes of these constants indicate that on the average, factors other than travel time and cost that affect mode choice tend to
  - favor drive alone over carpool and bus
  - favor carpool over bus

Example: Alternative specific constants (2)

- Any mode can be chosen as the base case when alternate specific constants are introduced into the model.
- The choice probabilities will be the same, regardless of the base, if the difference between the values of the alternative specific constants for any two alternatives are the same for all choices of base
- Let us see an example
- Suppose that deterministic components of the utility functions are
  - $V_{DA} = T_{DA} - 5C_{DA}/Y$
  - $V_{CP} = -0.6 - T_{CP} - 5C_{CP}/Y$
  - $V_B = -0.8 - T_B - 5C_B/Y$

Example: Alternative specific constants (3)

- The difference between alternative specific constants for drive alone and carpool is 0.6
- The difference between constants for drive alone and bus is 0.8
- The difference between constants for carpool and bus is 0.2
- The above was exactly the same as we dealt with the first example using alternative specific constants

Mode	Time (T), Hours	Cost ©, $	Without constants			With constants		
			V	exp(v)	Prob	V	exp(v)	Prob
Drive Alone	0.5	2	-0.833	0.435	0.379	-0.833	0.435	0.547
Carpool	0.75	1	-0.917	0.400	0.349	-1.517	0.219	0.276
Bus	1	1	-1.167	0.311	0.272	-1.967	0.140	0.176
				1.146	1.000		0.794	1.000

Independence from Irrelevant Alternatives

- One of the most important properties of multinomial logit model is independence from irrelevant alternatives.
- The IIA property states that for any individual, the ratio of probabilities of choosing two alternatives is independent of the availability or attributes of any other alternatives

Independence from Irrelevant Alternatives

- For example, in a multinomial logit model of choice between drive-alone, carpool, and bus, the probabilities of choosing drive alone and carpool are

$$P(DA) = \frac{e^{v_{DA}}}{e^{v_{DA}} + e^{v_{CP}} + e^{v_B}}$$

$$P(CP) = \frac{e^{v_{CP}}}{e^{v_{DA}} + e^{v_{CP}} + e^{v_B}}$$

- The ratio of probabilities is

$$\frac{P(DA)}{P(CP)} = \frac{e^{v_{DA}}}{e^{v_{CP}}} = e^{v_{DA} - v_{CP}}$$

- This ratio is independent of the availability and attributes of bus

IIA

- The IIA property limits the response to transportation changes that can be predicted by the multinomial logit model.
- Example: if the available modes are da, cp, and b, a MNLmodel predicts that the proportion of non-bus travellers choosing carpool is independent of the quality of bus service, i.e. p(cp)/(p(da)+p(cp))
- The improvement in bus service would not be predicted to draw travellers from carpool and drive alone
- This is an important consequence of IIA

The Red Bus Blue Bus Paradox

- Suppose the modes available for travel home and work are
  - Drive alone and
  - A bus that is painted red (called as red bus or RB)
  - Assume that $V_{DA}=V_{RB}$
  - The binomial logit model suggest that P(DA) = P(RB)= 0.5
- Suppose a competing bus is introduced
  - That is painted blue or called Blue Bus (BB)
  - On the same route as RB
  - All the attributes of RB and BB are exactly the same
  - The only difference is color

- If the color does not affect the mode choice
  - Then initiation of a new bus should cause the existing bus riders to drive evenly between the RB and BB
  - The addition of BB to the choice sets should have no effect on travellers who choose to drive alone and bus
  - Therefore, the choice probabilities following the initiation of BB should result as
  - P(DA) = 0.5
  - P(RB) = 0.25
  - P(BB) = 0.25

- The RB and BB are identical in all alternatives in all alternatives relevant to mode choice, $V_{RB} = V_{BB}$
- In addition, $V_{DA} = V_{RB}$, by assumption
- Therefore, $V_{DA} = V_{RB} = V_{BB} = 1/3$
- The RB/BB provides an important illustration of the possible consequence of IIA (but an extreme example)

Effects of IIA

- Consider an individual who has a choice between drive alone, carpool, bus and rail.
- Let the deterministic component of logit utility function be
  - $V_{DA}=0.8-T_{DA}-0.25C_{DA}$
  - $V_{CP}=0.2-T_{CP}-0.25C_{CP}$
  - $V_{DA}=-0.2-T_{B}-0.25C_{B}$
  - $V_{LR}=-T_{LR}-0.25C_{LR}$
- Units of T and C are travel time in hours and travel cost in hours

Effects of IIA

Mode	Time	Cost	Coefficient		
			Constant	Time	Cost
Drive Alone	0.50	2.00	0.80	-1.00	-0.25
Carpool	0.75	1.00	0.20	-1.00	-0.25
Bus	1.20	0.50	-0.20	-1.00	-0.25
Light Rail	1.00	0.75	0.00	-1.00	-0.25

Mode	Time	Cost	Coefficient			U	exp(U)	Prob.
			Constant	Time	Cost			
Drive Alone	0.50	2.00	0.80	-1.00	-0.25	-0.20	0.82	0.46
Carpool	0.75	1.00	0.20	-1.00	-0.25	-0.80	0.45	0.25
Bus	1.20	0.50	-0.20	-1.00	-0.25	-1.53	0.22	0.12
Light Rail	1.00	0.75	0.00	-1.00	-0.25	-1.19	0.30	0.17
							1.79	1.00

Effects of IIA

- Cost of light rail increases by $0.5, the revised probability

Mode	Time	Cost	Coefficient			U	exp(U)	Prob.
			Constant	Time	Cost			
Drive Alone	0.50	2.00	0.80	-1.00	-0.25	-0.20	0.82	0.47
Carpool	0.75	1.00	0.20	-1.00	-0.25	-0.80	0.45	0.26
Bus	1.20	0.50	-0.20	-1.00	-0.25	-1.53	0.22	0.12
Light Rail	1.00	1.25	0.00	-1.00	-0.25	-1.31	0.27	0.15
							1.75	1.00

- Before and after changes

Mode	Before	After	% Change
Drive Alone	0.4572	0.4666	0.0093
Carpool	0.2509	0.2561	0.0051
Bus	0.1215	0.1240	0.0025
Light Rail	0.1703	0.1534	-0.0169
			0.0000

Effects of IIA

- Notice that probability of choosing each mode other than light rail is predicted to increase.
- This is a consequence of IIA property.
- P(DA) before / P(DA) after = 1.020422
- P(CP) before / P(CP) after = 1.020422
- P(B) before / P(B) after = 1.020422
- P(DA)/P(CP) = 1.822 (before)
- P(DA)/P(CP) = 1.822 (after)
- P(DA)/P(B) = 3.76 (before)
- P(DA)/P(B) = 3.76 (after)

Effects of IIA

- In aggregate terms

"the riders who stop using light rail when its cost increases are predicted to distribute themselves among the remaining modes in proportion to the initial probabilities of choosing the remaining modes."

Effects of IIA

- However, such a result is possible but not realistic (if light rail and bus operate in different corridors so that bus is not a feasible alternative for light rail travellers)
- The observation is not consistent with expectations that if bus is not an alternative to light rail.
- One of the ways users have solved this problem is analyzing just transit as one more in the beginning and then disaggregating further transit sub-modes.

Avoid IIA property

- It is possible to avoid unrealistic consequences of IIA by adding additional variables in the deterministic components of utility function
  - Light rail travellers mainly do not have cars, therefore unlikely to use drive alone
  - The remaining options are carpool and bus
  - If carpooling is difficult, then the users are going to use bus
- The effect can be accommodated within a MNL model by including automobile ownership in the utility function

Example-Avoiding Unrealistic Consequences of IIA

- Let the deterministic component of logit utility function be
  - V_{DA}=-2.84-T_{DA}-0.25C_{DA}+4.5A
  - V_{CP}=-2.17-T_{CP}-0.25C_{CP} + 3.5A
  - V_{DA}=-0.2-T_{B}-0.25C_{B}
  - V_{LR}=-T_{LR}-0.25C_{LR}

Mode	Time	Cost	Coefficient			
			Constant	Time	Cost	A
Drive Alone	0.50	2.00	-2.84	-1.00	-0.25	4.50
Carpool	0.75	1.00	-2.17	-1.00	-0.25	3.50
Bus	1.20	0.50	-0.20	-1.00	-0.25	0.00
Light Rail	1.00	1.25	0.00	-1.00	-0.25	0.00

Avoiding IIA

Mode	Time	Cost	Coefficient				0 Cars		1 Car		2 Cars	
			Constant	Time	Cost	A	U	exp(U)	U	exp(U)	U	exp(U)
Drive Alone	0.50	2.00	-2.84	-1.00	-0.25	4.50	-3.84	0.02	0.66	1.93	5.16	174.16
Carpool	0.75	1.00	-2.17	-1.00	-0.25	3.50	-3.17	0.04	0.33	1.39	3.83	46.06
Bus	1.20	0.50	-0.20	-1.00	-0.25	0.00	-1.53	0.22	-1.53	0.22	-1.53	0.22
Light Rail	1.00	1.25	0.00	-1.00	-0.25	0.00	-1.31	0.27	-1.31	0.27	-1.31	0.27
								0.55		3.81		220.71

Mode	Probability		
	0 Cars	1 Car	2 Cars
Drive Alone	0.0391	0.5075	0.7891
Carpool	0.0763	0.3648	0.2087
Bus	0.3955	0.0571	0.0010
Light Rail	0.4891	0.0706	0.0012
	1	1	1

Avoiding IIA

- Share of mode
  - Assuming 25% 0 cars, 50% 1 car, and 25% 2 cars

Mode	Aggregate Share
Drive Alone	0.461
Carpool	0.254
Bus	0.128
Light Rail	0.158

  - Exactly equal to the previous case without considering auto ownership

Increased cost of light rail

- Suppose light rail cost increased by $0.5 then

Mode	Time	Cost	Coefficient				0 Cars		1 Car		2 Cars	
			Constant	Time	Cost	A	U	exp(U)	U	exp(U)	U	exp(U)
Drive Alone	0.50	2.00	-2.84	-1.00	-0.25	4.50	-3.84	0.02	0.66	1.93	5.16	174.16
Carpool	0.75	1.00	-2.17	-1.00	-0.25	3.50	-3.17	0.04	0.33	1.39	3.83	46.06
Bus	1.20	0.50	-0.20	-1.00	-0.25	0.00	-1.53	0.22	-1.53	0.22	-1.53	0.22
Light Rail	1.00	1.75	0.00	-1.00	-0.25	0.00	-1.44	0.24	-1.44	0.24	-1.44	0.24
								0.52		3.78		220.68

Mode	Probability		
	0 Cars	1 Car	2 Cars
Drive Alone	0.0414	0.5117	0.7892
Carpool	0.0810	0.3679	0.2087
Bus	0.4196	0.0576	0.0010
Light Rail	0.4580	0.0628	0.0011
	1	1	1

Mode	Change in Prob		
	0 Cars	1 Car	2 Cars
Drive Alone	0.0024	0.0042	0.0001
Carpool	0.0047	0.0031	0.0000
Bus	0.0241	0.0005	0.0000
Light Rail	-0.0312	-0.0078	-0.0001

Mode	BeforeAggregate Share	After Aggregate Share	Change
Drive Alone	0.461	0.464	0.003
Carpool	0.254	0.256	0.003
Bus	0.128	0.134	0.006
Light Rail	0.158	0.146	-0.012

Introduction of New Mode

- Consider two modes. DA and CP. We can get the probabilities as follows

Mode	Time	Cost	U	exp(U)	Prob
Drive Alone	0.5	2	-1	0.367879	0.46257
Carpool	0.6	1	-0.85	0.427415	0.53743
				0.795294	1

- If a new bus system is introduced, the revised probabilities will be as follows

Mode	Time	Cost	U	exp(U)	Prob
Drive Alone	0.5	2	-1	0.367879	0.311225
Carpool	0.6	1	-0.85	0.427415	0.361592
Bus	0.8	0.6	-0.95	0.386741	0.327182
				1.182035	1