Foundation Vibrations

Consider the actively loaded foundation shown in Fig. 15.9 of Gazetas (1991). We would like to utilize the SDOF analogy to calculate the motion of the footing resulting from forces and moments applied to the machine/foundation system.

There are six degrees of freedom of the machine foundation system (i.e. translation and rotation about the x, y, and z axis as shown.

Vertical Motion of Surface Footings

Let's first consider vertical motion since it is uncoupled with the other degress of freedom. As shown in Fig. 15.10 of Gazetas (1991), the equation of motion of the machine/foundation system can be expressed as:

$$m\ddot{u}_z(t) + P_z(t) = F_z(t) \tag{1}$$

where $P_z(t)$ is the reaction of the soil supporting the foundation. Using the SDOF analogy, Pz(t) can be expressed as the sum of the spring and dashpot forces as follows:

$$P_z(t) = \kappa_z u_z(t) = \left(\overline{K}_z + i\Omega C_z\right) u_z(t) \tag{2}$$

Substituting Eq. 2 into Eq. 1 and setting $u_z(t) = u_z e^{i\Omega t}$ yields:

$$\left(\overline{K}_z - \Omega^2 m + i\Omega C_z\right) u_z = F_z \tag{3}$$

Thus we have the transfer function between the vertical displacement and force applied to the machine/foundation system:

$$H_z(\Omega) = \frac{u_z}{F_z} = \frac{1}{\left(\overline{K}_z - \Omega^2 m + i\Omega C_z\right)} = \frac{1}{\kappa_z - \Omega^2 m} \tag{4}$$

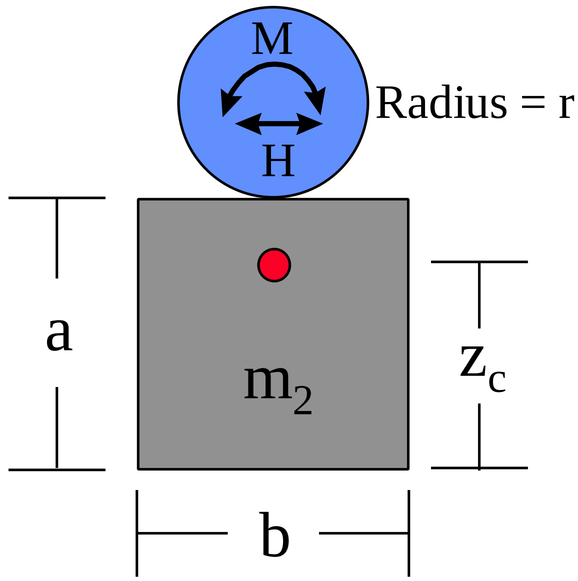
Gazetas (1991) presents tables and charts which allow $\overline{K}_z$ and C_z to be determined as a function of frequency. Figure 15.13 shows the geometry and notation used by Gazetas (1991) to define the problem.

Extension to Other Modes of Vibration

Let’s now extend this approach to include other modes of vibration. Since these other modes include rotational modes, let's review some basic definitions of the center of gravity and moments of inertia.

Center of Gravity

Consider the system shown at the right that is an idealization of a machine-foundation system composed of simple geometric shapes. Assume that the horizontal force and moment are applied at the centroid of the machine.



The expression to determine the centroid of the combined system is the average of the centroids of the individual components weighted by the masses of the individual components:

$$z_c = \frac{\left[m_2 \frac{a}{2} + m_1(a + r)\right]}{m_1 + m_2} \tag{5}$$

We use the parallel axis theorem to obtain the mass moment of inertia about the centroid of the system:

$$I_{system} = \sum_i \left(I_i + m_i d_i^2\right) \tag{6}$$

Thus, for example, the mass moment of inertia of the system shown above is:

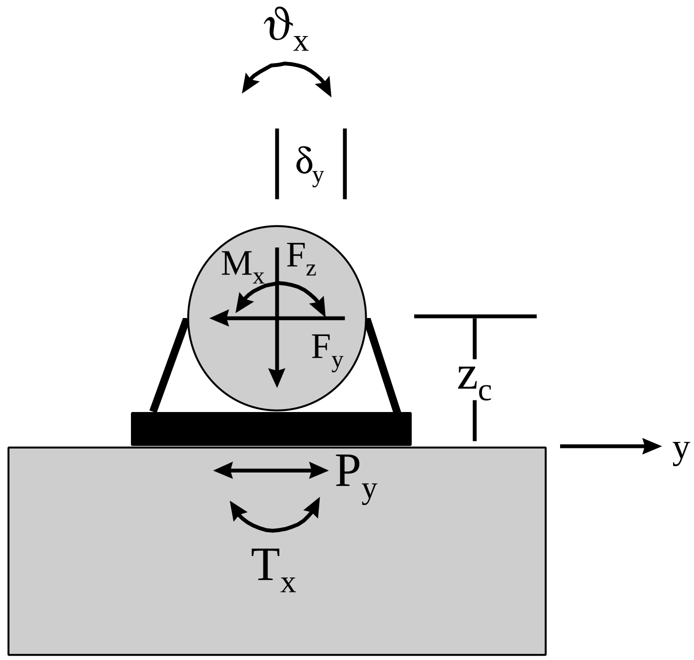
$$I_{system} = \frac{1}{12} m_2\left(a^2 + b^2\right) + m_2\left(z_c - \frac{a}{2}\right)^2 + \frac{1}{2} m_1 r^2 + m_1\left(a + r - z_c\right)^2 \tag{7}$$

Finally, it is necessary to calculate the equivalent moments and forces *at the centroid of the system*. Thus:

$$H' = H \tag{8a}$$

$$M' = M + H\left(a + r - z_c\right) \tag{8b}$$

The figure below shows the forces and moments, the soil reactions, and the resulting displacements and rotations for the six degrees of freedom of the system.



Note that the forces and moments and the resulting displacements and rotations are defined at the center of gravity of the system while the soil reactions are defined at the interface between the footing and soil.

Equations of Motion

Summing forces yields the equations of motion for the three translational degrees of freedom.

Vertical: $m\ddot{u}_z(t) + P_z(t) = F_z(t)$ (9a)

Horizontal (y): $m\ddot{\delta}_y(t) + P_y(t) = F_y(t)$ (9b)

Horizontal (x): $m\ddot{\delta}_x(t) + P_x(t) = F_x(t)$ (9c)

Summing moments about the center of gravity of the system yields the three equations for motion for rotational motion.

Rotation about x axis: $I_{ox}\ddot{\vartheta}_x(t) + T_x(t) - P_y(t)\,z_c = M_x(t)$ (9d)

Rotation about y axis: $I_{oy}\ddot{\vartheta}_y(t) + T_y(t) - P_x(t)\,z_c = M_y(t)$ (9e)

Torsion: $T_z(t) + J_z \ddot{\vartheta}_z(t) = M_z(t)$ (9f)

where I_{0x}, I_{0y}, and J_z are the mass polar moments of inertia about the center of gravity.

Soil Reactions

For vertical motion we have already seen that the soil reaction is given by:

$$P_z(t) = \kappa_z u_z(t) \tag{10a}$$

Horizontal motion is more complex because of the coupling between the horizontal translation and rocking (rotation) about the base. The soil reactions are given by:

$$P_y(t) = \kappa_y\left[\delta_y(t) - z_c\vartheta_x(t)\right] + \kappa_{yrx}\vartheta_x(t) \tag{10b}$$

The $\left[\delta_y(t) - z_c\vartheta_x(t)\right]$ term is the net horizontal translation that is obtained by subtracting the component of the horizontal motion due to rocking about the base from the total horizontal motion. The second term is the coupling that occurs when a horizontal force causes rocking about the base. The horizontal soil reaction in the x direction is defined in the same way:

$$P_x(t) = \kappa_x\left[\delta_x(t) - z_c\vartheta_y(t)\right] + \kappa_{xry}\vartheta_y(t) \tag{10c}$$

The soil reaction moments for rocking about the x and y axis are given by:

$$T_x(t) = \kappa_{rx}\,\vartheta_x(t) + \kappa_{yrx}\left[\delta_y(t) - z_c\vartheta_x(t)\right] \tag{10d}$$

$$T_y(t) = \kappa_{ry}\,\vartheta_y(t) + \kappa_{xry}\left[\delta_x(t) - z_c\vartheta_y(t)\right] \tag{10e}$$

Finally, the moment reaction in torsion is:

$$T_z(t) = \kappa_{rz}\,\vartheta_z(t) \tag{10f}$$

The soil reactions in Eq. 10 are substituted into the equations of motion in Eq. 9. Furthermore, we assume that the forces, moments, displacements, and rotations are harmonic functions time:

$$F_x(t) = F_x e^{i\Omega t} \qquad F_y(t) = F_y e^{i\Omega t} \qquad F_z(t) = F_z e^{i\Omega t} \tag{11a}$$

$$M_x(t) = M_x e^{i\Omega t} \qquad M_y(t) = M_y e^{i\Omega t} \qquad M_z(t) = M_z e^{i\Omega t} \tag{11b}$$

$$\delta_x(t) = \delta_x e^{i\Omega t} \qquad \delta_y(t) = \delta_y e^{i\Omega t} \qquad u_z(t) = u_z e^{i\Omega t} \tag{11c}$$

$$\vartheta_x(t) = \vartheta_x e^{i\Omega t} \qquad \vartheta_y(t) = \vartheta_y e^{i\Omega t} \qquad \vartheta_z(t) = \vartheta_z e^{i\Omega t} \tag{11d}$$

After substituting the expressions in Eq.11 and their time derivatives, we obtain:

$$-\Omega^2 m u_z + \kappa_z u_z = F_z \tag{12a}$$

$$-\Omega^2 m\delta_x + \kappa_x(\delta_x - z_c\vartheta_y) + \kappa_{xry}\vartheta_y = F_x \tag{12b}$$

$$-\Omega^2 m\delta_y + \kappa_y(\delta_y - z_c\vartheta_x) + \kappa_{yrx}\vartheta_x = F_y \tag{12c}$$

$$-\Omega^2 J_z\vartheta_z + \kappa_{rz}\vartheta_z = M_z \tag{12d}$$

$$-\Omega^2 I_{ox}\vartheta_x + \kappa_{rx}\vartheta_x + \kappa_{yrx}(\delta_y - z_c\vartheta_{x)} - \kappa_y(\delta_y - z_c\vartheta_x)z_c + \kappa_{yrx}\vartheta_x z_c = M_x \tag{12e}$$

$$-\Omega^2 I_{oy}\vartheta_y + \kappa_{ry}\vartheta_y + \kappa_{xry}(\delta_x - z_c\vartheta_{y)} - \kappa_x(\delta_x - z_c\vartheta_y)z_c + \kappa_{xry}\vartheta_y z_c = M_y \tag{12f}$$

Finally, it is convenient to express this *system of equations* in matrix form:

$$\begin{bmatrix} F_z \\ F_x \\ F_y \\ M_z \\ M_x \\ M_y \end{bmatrix} = \begin{bmatrix} \kappa_z - \Omega^2 m & 0 & 0 & 0 & 0 & 0 \\ 0 & \kappa_x - \Omega^2 m & 0 & 0 & 0 & \kappa_{xry} - \kappa_x z_c \\ 0 & 0 & \kappa_y - \Omega^2 m & 0 & \kappa_{yrx} - \kappa_y z_c & 0 \\ 0 & 0 & 0 & \kappa_{rz} - \Omega^2 J_z & 0 & 0 \\ 0 & 0 & \kappa_{yrx} - \kappa_y z_c & 0 & \kappa_{rx} - \Omega^2 I_{ox} - 2\kappa_{yrx} z_c + \kappa_y z_c^2 & 0 \\ 0 & \kappa_{xry} - \kappa_x z_c & 0 & 0 & 0 & \kappa_{ry} - \Omega^2 I_{oy} - 2\kappa_{xry} z_c + \kappa_x z_c^2 \end{bmatrix} \begin{bmatrix} u_z \\ \delta_x \\ \delta_y \\ \vartheta_z \\ \vartheta_x \\ \vartheta_y \end{bmatrix}$$