Transfer (Frequency Response) Functions

To characterize the response of a SDOF system to forced vibrations it is useful to define a *transfer function* or *frequency response function* between the input and output of the system.

$$H(\Omega) = \frac{\text{Output}}{\text{Input}}$$
(44)

By normalizing the output of the system with respect to the input, we emphasize the characteristics and response of the system over the characteristics of the output or input.

Let's define a transfer function between the steady-state displacement output and force input of a SDOF system undergoing forced vibrations:

$$H(\Omega) = \frac{\text{Displacement}}{\text{Force}} = \frac{u_{ss}(t)}{Pe^{i\Omega t}}$$
(45a)

$$H(\Omega) = \frac{\frac{Pe^{i\Omega t}}{k - \Omega^2 m + i\Omega c}}{Pe^{i\Omega t}}$$
(45b)

$$H(\Omega) = \frac{1}{k - \Omega^2 m + i\Omega c}$$
(45c)

$$H(\Omega) = \frac{1}{k} \frac{1}{1 - \frac{\Omega^2}{\omega_n^2} + 2i\beta \frac{\Omega}{\omega_n}}$$
(45d)

Notice that the transfer function is a complex-valued quantity meaning the response of the SDOF system can be characterized by a magnitude and phase. Figure 6 shows the magnitude and phase plots for a SDOF system expressed as a function of the normalized frequency, Ω/ω_n .



Figure 6 Magnitude and Phase of Transfer Function

Equations 45c and 45d and Figure 6 can be used to provide insight into the parameters that control the response of a SDOF in different frequency ranges. Note in Equations 45c and d that when $\Omega \rightarrow 0$, the transfer function reduces to:

$$H(\Omega = 0) = \frac{1}{k}$$
(46)

Thus, the *stiffness* of the system controls the response at low frequency. As $\Omega \rightarrow \omega_n$, the transfer function reduces to:

$$H(\Omega = \omega_n) = \frac{1}{i\Omega c} = \frac{1}{2ik\beta}$$
(47)

and the response of the system is controlled to a large extent by the *damping* in the system. Finally, as Ω becomes large, the transfer function becomes:

$$H(\Omega \to \infty) \propto \frac{1}{-\Omega^2 m}$$
(48)

and the response of the system is largely controlled by the mass (the inertia) of the system.

Other Forms of the Transfer Function

The transfer function defined above was expressed in terms of the displacement. Other response quantities such as the velocity and acceleration of the mass can also be used to define a transfer function for various applications. The names associated with each of these transfer or frequency response functions are given in Table 1.

Table 1Transfer Functions Used in Vibration Analysis (after Inman, 1994)

Response Parameter	Transfer Function	Inverse Transfer Function
Displacement	Receptance	Dynamic Stiffness
Velocity	Mobility	Impedance
Acceleration	Inertance	Apparent Mass

Ground Displacement and Acceleration

Consider the situation in which the system vibrates because of motion introduced at the base of the system, not by a force applied to the mass. The system is shown at the right.

The forces exerted by the spring and the dashpot on the mass are functions of the relative displacement and velocity between the mass and the base of the system. The absolute acceleration of the mass is still used $_{\rm (F=ma)}$.

Thus, the equation of motion of the system becomes:

$$k \neq c$$

 $z(t)$

m

u(t)

 $m\ddot{u} + c(\dot{u} - \dot{z}) + k(u - z) = 0$ (49)

The right hand side is set equal to zero because there are no external forces applied to the mass. Rearranging yields:

$$m\ddot{u} + c\dot{u} + ku = c\dot{z} + kz \tag{50}$$

We can assume that the ground motion is given by a harmonic function:

$$z(t) = Ae^{i\Omega t}$$
(51)

and that the response of the mass is given by:

$$\mathbf{u}(\mathbf{t}) = \mathbf{B}\mathbf{e}^{\mathbf{i}\Omega\mathbf{t}} \tag{52}$$

After differentiating, substituting, and solving, we obtain the solution for the steady-state displacement of the mass:

$$u(t) = \frac{k + i\Omega c}{k - \Omega^2 m + i\Omega c} A e^{i\Omega t}$$
(53)

We can also define a transfer function between the displacement of the mass and the input ground displacement:

$$H(\Omega) = \frac{u(t)}{z(t)}$$
(54a)

$$H(\Omega) = \frac{\frac{k + i\Omega c}{k - \Omega^2 m + i\Omega c} A e^{i\Omega t}}{A e^{i\Omega t}}$$
(54b)

$$H(\Omega) = \frac{k + i\Omega c}{k - \Omega^2 m + i\Omega c}$$
(54c)

$$H(\Omega) = \frac{1 + 2i\beta \frac{\Omega}{\omega_{n}}}{1 - \frac{\Omega^{2}}{\omega_{n}^{2}} + 2i\beta \frac{\Omega}{\omega_{n}}}$$
(54d)

As before we can express the complex-valued transfer function in terms of the magnitude and phase:



Figure 7 Magnitude and Phase of Transfer Function for Ground Displacement

In many earthquake engineering problems, the ground motion input at the base of the system is specified in terms of a ground acceleration.

$$\ddot{z}(t) = Ce^{i\Omega t}$$
(55)

Furthermore, the response parameter of interest is the *relative* displacement between the base and the mass. The latter is important because it is the relative displacements that are proportional to the forces induced in the structure.

$$y(t) = u(t) - z(t)$$
 (56)

Recall that the equation of motion is:

 $m\ddot{u} + c(\dot{u} - \dot{z}) + k(u - z) = 0 \tag{49 again}$

After substituting Eq. 56 we obtain:

$$\mathbf{m}(\ddot{\mathbf{y}} + \ddot{\mathbf{z}}) + \mathbf{c}\dot{\mathbf{y}} + \mathbf{k}\mathbf{y} = \mathbf{0} \tag{55a}$$

or

$$m\ddot{y} + c\dot{y} + ky = -m\ddot{z} \tag{57b}$$

After solving for y(t) we obtain:

$$y(t) = \frac{-m}{k - \Omega^2 m + i\Omega c} C e^{i\Omega t}$$
(58)

Finally, we can define the transfer function between the ground acceleration and the relative displacement of the mass and base:

$$H(\Omega) = \frac{y(t)}{\ddot{z}(t)}$$
(59a)

$$H(\Omega) = \frac{-m}{k - \Omega \ m + i\Omega c}$$
(59b)



$$H(\Omega) = \frac{-\frac{1}{\omega_n^2}}{1 - \frac{\Omega^2}{\omega_n^2} + 2i\beta\frac{\Omega}{\omega_n}}$$



Figure 8 Magnitude and Phase of Transfer Function for Ground Acceleration