One Dimensional Wave Propagation

We will begin with an introduction to wave propagation theory to understand how wave propagation can be used to assess the geometry and material properties of a body. An appropriate place to begin is with one-dimensional wave propagation.

Derivation

When a uniform, homogeneous bar is loaded axially we can model the stress distribution throughout the beam by looking at a very small slice of the given bar (Figure 2.1). The stress increase along a length of the bar, dx, can be given by $\partial\sigma / \partial x$.

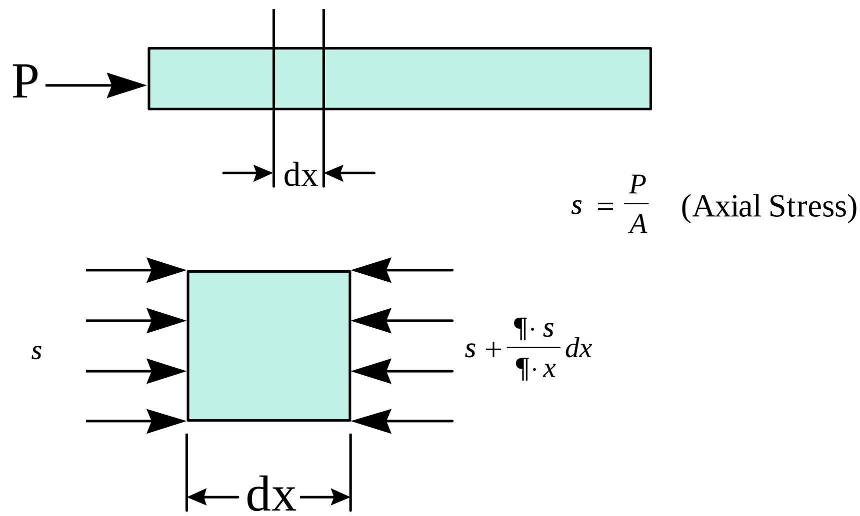

Figure 1 Normal Stresses Acting on a Differential Element of a Bar

Based on Newton's second law, we can write the equilibrium equation of the differential slice as follows:

$$-\sigma+\sigma+\frac{\partial\sigma}{\partial x}dx=\rho\cdot dx\frac{\partial^2 u}{\partial\cdot t^2} \qquad (1)$$

Where *u* is the displacement in the *x* direction, *t* is time, and *r* is the mass density of the bar. Canceling terms we arrive at:

$$\frac{\partial\sigma}{\partial\cdot x}=\rho\frac{\partial^2 u}{\partial\cdot t} \quad (2)$$

By assuming a linear relationship between stress and strain, an adequate assumption when analyzing wave propagation, we can use Young's Modulus to help simplify the equation. Recall that:

$$E=\frac{\sigma}{\varepsilon} \quad (3)$$

where *E* is Young's Modulus. Strain (ε) can be written as,

$$\varepsilon=\frac{\partial\cdot u}{\partial\cdot x} \quad (4)$$

Substituting this into Equation 3 we obtain,

$$\sigma=\frac{\partial\cdot u}{\partial\cdot x}E \quad (5)$$

Differentiating this equation with respect to *x*, we obtain:

$$\frac{\partial\sigma}{\partial\cdot x}=E\frac{\partial^2 u}{\partial\cdot x^2} \quad (6)$$

Substituting this equation into equation 2 yields,

$$\frac{\partial^2 u}{\partial\cdot t^2}=\frac{E}{\rho}\frac{\partial^2 u}{\partial\cdot x^2} \quad (7)$$

or

$$\frac{\partial^2 u}{\partial\cdot t^2}=V_b^2\frac{\partial^2 u}{\partial\cdot x^2} \quad (8)$$

where

$$V_b=\sqrt{\frac{E}{\rho}} \quad (9)$$

V_b is the velocity of the longitudinal stress wave propagation. Equation 8 is the *one dimensional wave equation*. This second order partial differential equation can be used to analyze one-dimensional motions of an elastic material.

Solution of the One Dimensional Wave Equation

The general solution of this equation can be written in the form of two independent variables,

$$\xi = V_b t + x \tag{10}$$

$$\eta = V_b t - x \tag{11}$$

By using these variables, the displacement, u, of the material is not only a function of time, t, and position, x; but also wave velocity, V_b. Using a solution developed by D'Alembert we are able to express the one-dimensional wave equation as follows:

$$u(\xi, \eta) = F(\xi) + G(\eta) \tag{12}$$

or

$$u(x, t) = F(V_b t + x) + G(V_b t - x) \tag{13}$$

F and G are functions of the boundary conditions of the problem. The function $F(V_b t+x)$ represents the wave front that propagates in the negative x direction, while the function $G(V_b t-x)$ represents the wave that travels in the positive x direction. This is shown in the accompanying worksheet.

We assume that the disturbance moves unchanged in shape from x_o to x_1. The equation for this is:

$$F(V_b t_0 + x_0) = F(V_b t_1 + x_1) \tag{14}$$

Simplifying yields:

$$x_1 = x_0 - V_b (t_1 - t_0) \tag{15}$$

This shows that as time increases, the wave moves in the negative *x* direction by a distance equal to the bar velocity multiplied by the time interval (t_1 - t_0).

Experimental Example

An example using the one-dimensional wave equation to examine wave propagation in a bar is given in the following problem.

Given: A homogeneous, elastic, freely supported, steel bar has a length of 8.95 ft. (as shown below). A stress wave is induced on one end of the bar using an instrumented hammer and recorded on the opposite end using an accelerometer. The time it takes the wave to reach the opposite end of the steel bar is 530 x 10^{-6} seconds. The unit weight of the steel bar is 490 pcf. Find the Young's Modulus of the steel bar.

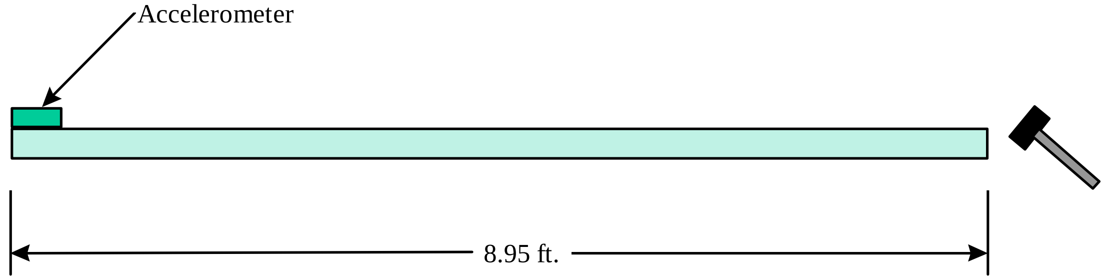

Figure 2 Steel Bar used in Experimental Example

Reflection and Transmission of Waves at an Interface

When a wave meets an interface between two materials of differing properties, a portion of the wave is transmitted through the interface, while the rest of the wave is reflected away from the interface, as shown below:

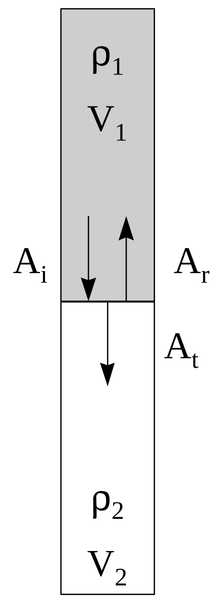

Figure 3 Reflection and Transmission at an Interface Between Different Materials

The general solution for one-dimensional wave propagation in the two materials is the D'Alembert solution:

$$u(x,t) = F(V_b t + x) + G(V_b t - x) \tag{16}$$

Let's assume a harmonic solution of the form:

$$u(x,t) = Ae^{ik(V_b t + x)} + Be^{ik(V_b t - x)} \tag{17}$$

Note that this is a particular form of the general solution in which:

$$F(\xi) = Ae^{ik\xi} \text{ and } G(\eta) = Ae^{ik\eta} \tag{18}$$

The constant k in Eqs. 17 and 18 is called the *wavenumber* and is defined as:

$$k = \frac{\omega}{V_b} \tag{19}$$

Substituting Eq. 19 into eq. 17 results in an alternate expression for the solution of the one-dimensional wave equation:

$$u(x,t) = Ae^{i(\omega t + kx)} + Be^{i(\omega t - kx)} \tag{20}$$

For the situation shown in the figure above, the incident wave can be represented by the expression representing a wave traveling to the right (downward) through Material 1,

$$u_1(x,t) = A_i e^{i(\omega t - k_1 x)} \tag{21}$$

Initially, there is no wave traveling in the negative *x* direction and thus no "F" term in this equation. (Note: k_1 is the wavenumber of the incident wave traveling in Material 1.)

Once the wave encounters the interface between Material 1 and Material 2, reflected and transmitted waves are generated. The total displacement in Material 1 is the sum of the incident and reflected wave:

$$u_1(x,t) = A_i e^{i(\omega t - k_1 x)} + A_r e^{i(\omega t + k_1 x)} \tag{22}$$

The displacement in Material B is the displacement produced by the transmitted wave,

$$u_2(x,t) = A_t e^{i(\omega t - k_2 x)} \tag{23}$$

To determine the amplitude of the reflected and transmitted waves we must use continuity to solve for F^A and G^B . The continuity of displacements at the interface (x = 0) implies the following:

$$u_1(0,t) = u_2(0,t) \tag{24a}$$

or

$$A_i + A_r = A_t \tag{24b}$$

and the continuity of stresses implies:

$$E_1 \frac{\partial u_1(0,t)}{\partial x} = E_2 \frac{\partial u_2(0,t)}{\partial x} \tag{25a}$$

where E denotes the Young's modulus of the material. Eq. 25a may be expressed as:

$$-ik_1A_iE_1 + ik_1A_rE_1 = -ik_2A_tE_2 \tag{25b}$$

Rearranging these expressions to solve for the amplitudes of the reflected and transmitted waves in terms of the amplitude of the incident wave yields. In terms of the particle displacements for the incident, reflected, and transmitted waves:

$$A_r = \frac{1-\dfrac{\rho_2 V_2}{\rho_1 V_1}}{1+\dfrac{\rho_2 V_2}{\rho_1 V_1}} A_i \tag{26a}$$

and

$$A_t = \frac{2}{1+\dfrac{\rho_2 V_2}{\rho_1 V_1}} A_i \tag{26b}$$

The product of the mass density, ρ, and the velocity, V, of a given material is the mechanical impedance. Notice that the expressions for the reflection and transmission coefficients are functions of the ratio of the mechanical impedances of Materials 1 and 2.

In terms of stresses associated with the incident, reflected, and transmitted waves:

$$\sigma_r = \frac{\dfrac{\rho_2 V_2}{\rho_1 V_1}-1}{\dfrac{\rho_2 V_2}{\rho_1 V_1}+1} \sigma_i \tag{27a}$$

$$\sigma_t = \frac{2\dfrac{\rho_2 V_2}{\rho_1 V_1}}{\dfrac{\rho_2 V_2}{\rho_1 V_1}+1} \sigma_i \tag{27b}$$

Standing Waves

Assume a solution to the one-dimensional wave equation of the form:

$$u(x,t) = U(x)\, e^{i\omega t} \tag{28}$$

Substituting Eq. 28 into the one-dimensional wave equation results in an ordinary differential equation:

$$\frac{d^2\, U(x)}{dx^2} + \frac{\omega^2}{V_b^2} U(x) = 0 \tag{29}$$

The solution to Eq. 29 is given by:

$$U(x) = A\cos\left(\frac{\omega x}{V_b}\right) + B\sin\left(\frac{\omega x}{V_b}\right) \tag{30}$$

where A and B are constants which depend on the boundary conditions. Assume a bar with a fixed end at x = 0 and a free end at x = L. The boundary conditions expressed mathematically are:

$$U(x = 0) = 0 \tag{31a}$$

and

$$\left.\frac{dU(x)}{dx}\right|_{x=L} = 0 \tag{31b}$$

Applying the first boundary condition results in:

$$A = 0 \tag{32a}$$

and the second boundary condition results in:

$$B\frac{\omega}{V_b}\cos\left(\frac{\omega L}{V_b}\right) = 0 \tag{32b}$$

Except for the trivial solution (B = 0), Eq. 32b is true only if:

$$\frac{\omega_n L}{V_b} = \frac{\pi}{2} + n\pi \qquad \text{for } n = 0, 1, 2, \ldots \tag{33}$$

After rearranging this equation in terms of the frequency (as opposed to the circular frequency):

$$f_n = \frac{V_b(2n+1)}{4L} \qquad \text{for } n = 0, 1, 2, \ldots \tag{34}$$

These expressions yield the natural frequencies of longitudinal vibration for the bar. The corresponding mode shapes are obtained by substituting the expression for the circular natural frequency back into Eq. 30:

$$U_n(x) = B\sin\left(\frac{\omega_n x}{V_b}\right) \tag{35}$$

where B is an arbitrary constant which scales the displacements. Similar expressions can be easily derived for other boundary conditions.