Response to Arbitrary Excitations

Thus far we have assumed that the inputs and outputs of single-degree-of-freedom (SDOF) systems have been harmonic functions of the form:

$$u(t) = Ae^{i\Omega t} \tag{60a}$$

$$p(t) = Be^{i\Omega t} \tag{60b}$$

This form is convenient for deriving transfer functions for SDOF systems, but is of limited practical value by itself because many actual inputs and outputs are more complex. We must combine the solutions we have developed for harmonic excitation with another tool, the Fourier Transform, to allow us to determine the response of SDOF systems to an arbitrary excitation. It is important to note that the linear (or equivalent linear) nature of most soil dynamics problems enables us to use this approach.

Fourier Transforms

The Fourier Transform is a way to decompose arbitrary displacement and/or force time histories into their harmonic components. Once decomposed, transfer functions can be applied to the individual harmonic components. Finally, an inverse Fourier Transform can be used to "reassemble" the individual harmonic responses to obtain the response time history.

Let's assume we have a displacement or force time history which we denote as x(t). We can express x(t) as a Fourier series:

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\Omega_0 t + b_n \sin n\Omega_0 t\right) \tag{61a}$$

where

$$a_0 = \frac{1}{T}\int_0^T x(t)\ dt \tag{61b}$$

$$a_n = \frac{2}{T}\int_0^T x(t)\ \cos\left(n\Omega_0 t\right)\ dt \tag{61c}$$

$$b_n = \frac{2}{T}\int_0^T x(t)\ \sin\left(n\Omega_0 t\right)\ dt \tag{61d}$$

or using complex exponentials:

$$x(t) = c_n \sum_{n=-\infty}^{\infty} e^{in\Omega_0 t} \tag{62}$$

In a Fourier series, each term is separated by $\Delta f = f_0$. As Δf decreases to 0, we reduce the Fourier series to an integral:

$$x(t) = \int_{-\infty}^{\infty} X(\Omega) e^{i\Omega t} d\Omega \tag{63a}$$

with continuous "Fourier coefficients" given by:

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)\ e^{-i\Omega t} dt \tag{63b}$$

Equation 63b is the continuous Fourier transform used to convert a time history x(t) to the frequency domain in which the signal is portrayed as a function of frequency. Note that $X(\Omega)$ is a complex-valued function. Similarly, Eq. 63a is the inverse Fourier transform enabling us to go from the *frequency domain* to the *time domain*.

We typically do not record continuous or analog signals with modern digital instrumentation. Instead we measure a continuous signal that is digitized and stored at a sampling interval Δt. Thus we must use a discrete Fourier transform (DFT):

$$X(k\Delta f) = \sum_{n=0}^{N-1} x(n\Delta t)\ e^{-i2\pi k\Delta f\Delta t} \tag{64a}$$

and the inverse discrete Fourier transform:

$$x(n\Delta t) = \frac{1}{N}\sum_{n=0}^{N-1} X(k\Delta f)\ e^{i2\pi k\Delta f\Delta t} \tag{64b}$$

where N is the total number of points in the time history and k and n are and frequency and time indices, respectively.

The fast Fourier transform (FFT or IFFT) is simply a numerically efficient version of the DFT. The algorithm used to calculate the FFT is most efficient when the number of points being transformed is equal to an integer power of 2 (i.e., 2^n where n is an integer).

Thus it is common to increase the increase the number of points to the next larger integer power of 2 by "padding" the end of the record with a sufficient number of zeros.

There are several features of the Fourier Transform that are helpful to know when using it as a tool to solve problems. Suppose that a force or displacement time history contains N points (where $N = 2^n$) with sampling interval Δt and that the total length of the time history is T. When the FFT algorithm is applied to that time history to transform it to the frequency domain, the result will be N values in the frequency domain separated by:

$$\Delta f = \frac{1}{T} \tag{65}$$

The Nyquist frequency is the maximum frequency that can be accurately resolved by a sampling interval Δt and is equal to:

$$f_{Nyquist} = \frac{1}{2\Delta t} \tag{66}$$

Finally, for real-valued time histories (the most common case) only the first N/2 + 1 points in the frequency domain are unique. The remaining points are complex conjugates of values in the first half of the record. One must be extremely careful in selecting the correct frequencies and mathematical operations for values in the second half of the frequency domain record to obtain correct results from the inverse FFT algorithm.

Frequency Domain Solutions

Using Fourier Transforms we can decompose a complex time history into its harmonic components. This enables us to apply the transfer functions we developed for harmonic motion. We will use a 3-step approach:

Use the FFT to calculate the amplitude and phase spectra corresponding to a given input time history. The input time history may be a force (active loading) or a ground motion (passive loading).

$$p(t) \rightarrow FFT \rightarrow P(\Omega)$$

Calculate the response of the SDOF system in the frequency domain using the transfer (frequency response) function:

$$U(\Omega) = H(\Omega)\, P(\Omega)$$

Use the inverse FFT to obtain the response of the SDOF system in the time domain.

$U(\Omega) \rightarrow IFFT \rightarrow u(t)$

Time Domain Solutions (Duhamel's Integral)

It is also possible to calculate the response of a SDOF to an arbitrary forcing function directly in the time domain using Duhamel's integral. Consider the following equation of motion for a SDOF system subjected to an arbitrary force p(t):

$$m\ddot{u} + c\dot{u} + ku = p(t) \tag{67}$$

The solution to Eq. 67 is given by Duhamel's integral:

$$u(t) = \frac{e^{-\omega_n \beta t}}{m\omega_d} \int_0^t e^{\omega_n \beta \tau}\, p(\tau) \sin(\omega_d (t-\tau)) d\tau \tag{68}$$

where τ is a dummy variable. Interested readers are referred to Weaver et al. (1990) or other texts on engineering vibrations for the derivation of Duhamel's integral. Equation 68 may also be viewed as a convolution integral in which the forcing function p(t) is convolved with the impulse response of a SDOF system.

Unfortunately, the application of Eq. 68 is limited to situations in which the parameters of the SDOF system (k, m, and c) are independent of frequency. In many machine-foundation vibration problems, the equivalent values of k and c are functions of the frequency of the excitation.