Coefficient of Consolidation by Inflection Point Method

By G. Mesri,[1] Member, ASCE, T. W. Feng,[2] and M. Shahien,[3] Association Members, ASCE

Abstract: The coefficient of consolidation in the Terzaghi theory of consolidation has been commonly determined by fitting the theory to observed compression with time in an incremental loading oedometer test. The Terzaghi theory is applicable to compression and swelling. A similar coefficient governed by horizontal permeability is used in the Barron formulation of consolidation with vertical drains. The most widely used graphical methods of Casagrande and Taylor, which fit the theory to observed data at 50 and 90% primary compression, respectively, are rather laborious. The inflection point method requires only a visual identification of the inflection point on the compression versus log time curve. The values of coefficients of consolidation by the simple inflection point method are quite similar to those from the Casagrande method.

According to the Terzaghi theory of one-dimensional consolidation, the relationship between average degree of consolidation, U, and time factor, T, in an incremental loading consolidation test is

$$U = 1 - \sum_{m=o}^{\infty} \frac{2}{M^2} \exp(-M^2 T) \quad (1)$$

where $T = c_v t/H^2$; c_v = coefficient of consolidation; H = maximum drainage distance; t = time elapsed from application of pressure increment; and $M = \pi(2m + 1)/2$. During primary consolidation, average void ratio and average degree of consolidation are related according to

$$\frac{\partial e}{\partial \log t} = (\Delta e)_p \frac{\partial U}{\partial \log t} \quad (2)$$

where $(\Delta e)_p$ = void ratio decrease at end of primary consolidation. The Terzaghi U versus log T curve according to (1) has an inflection point at which the sense of concavity of the curve changes. At the inflection point $\partial U/\partial \log t = \partial U/\partial \log T = 0.6868$, and from (2)

$$\frac{\partial e}{\partial \log t} = 0.6868(\Delta e)_p \quad (3)$$

Mesri and Godlewski (1977) used (3) (for empirical analysis 0.6868 was rounded up to 0.70) to interpret the shape of the compression versus log t curve in the transition from primary consolidation to secondary compression. An inflection point is observed only when $0.6868(\Delta e)_p$ is greater than the secondary compression index, $C_\alpha = \Delta e/\Delta \log t$. For a pressure increment on the compression curve with slope $C_c = \Delta e/\Delta \log \sigma'_v$:

$$\frac{\partial e}{\partial \log t} = 0.6868 C_c \log \left[1 + \frac{\Delta\sigma'_v}{\sigma'_{vo}}\right] \quad (4)$$

where σ'_{vo} = initial effective vertical pressure, and $\Delta\sigma'_v$ = pressure increment. When the pressure increment ratio, $\Delta\sigma'_v/\sigma'_{vo}$, is equal to unity, then

$$\frac{\partial e/\partial \log t}{C_c} = 0.21 \quad (5)$$

Since the highest value of C_α/C_c for geotechnical materials rarely exceeds 0.07 (Terzaghi et al. 1996), which is significantly less than 0.21, an inflection point is expected to be observed for all soils and pressure increment ratios of unity completely either in recompression or in the compression range. On the other hand, a small pressure increment ratio such as $\Delta\sigma'_v/\sigma'_{vo} = 0.2$ leads to $(\partial e/\partial \log t)/C_c = 0.054$. Therefore, if a $\Delta\sigma'_v/\sigma'_{vo} = 0.2$ is applied to a peat specimen with C_α/C_c equal or greater than 0.054, then an inflection point will not be observed.

When the pressure increment spans a preconsolidation pressure σ'_p, then the expression for $(\Delta e)_p$ is

$$(\Delta e)_p = C_c \left[\frac{C_r}{C_c} \log \frac{\sigma'_p}{\sigma'_{vo}} + \log \frac{\sigma'_{vf}}{\sigma'_p}\right] \quad (6)$$

where C_r = recompression index. In this case the presence or absence of an inflection point depends on magnitudes of $\Delta\sigma'_v/\sigma'_{vo}$, σ'_p/σ'_{vo}, as well as C_r/C_c, which has a rather narrow range of values. For high values of σ'_p/σ'_{vo}, e.g., 1.8, an inflection point may not be observed even for pressure increment ratios near unity. Also, for unloading, even a pressure decrement ratio of unity may not produce an inflection point in cases where $C_{\alpha s}/C_s$ has values as high as 0.3, where C_s = swelling index and $C_{\alpha s}$ = secondary swelling index (Mesri et al. 1978).

In general, however, because for most clays and silts $C_\alpha/C_c = 0.04 \pm 0.01$ and values of σ'_p/σ'_{vo} are in the range of 1.2 to 1.6, an inflection point is expected and has been observed for IL oedometer tests subjected to load increment ratios in the range of 0.5–1.0.

Robinson (1997) has recommended using the elapsed time at inflection point; t_i, for computing the coefficient of consolidation from IL oedometer tests as follows:

$$c_v = \frac{T_{vi} H^2}{t_i} \quad (7)$$

where $T_{vi} = 0.405$ is the time factor at the inflection point. The advantage of this approach is that it does not require the definition of the beginning and end of the primary consolidation stage that are required for other widely used methods (Casagrande and Fadum 1940; Taylor 1948). Another advantage of this approach is that the inflection point at the average degree of consolidation of 70% is within the midrange of the compression curve and is least affected by initial compression and secondary compression. Therefore, when the inflection point is carefully identified, the computed c_v should be as reliable as the c_v from the Casagrande procedure.

Values of c_v and c_{vs} were determined from deformation versus log t curves of oedometer tests on undisturbed specimens of soft clay, silt deposits, and natural shales that had been previously carried out as a part of investigations of compressibility of soft clay and silt deposits, and swelling of shales

[1]Prof. of Civ. Engrg., Univ. of Illinois at Urbana-Champaign, IL 61801-2352.
[2]Assoc. Prof. of Civ. Engrg., Chung Yuan Univ., Chung-li, Taiwan, ROC.
[3]Post-doctoral Res. Assoc., Univ. of Illinois at Urbana-Champaign, IL.
Note. Discussion open until January 1, 2000. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this technical note was submitted for review and possible publication on April 9, 1998. This technical note is part of the ***Journal of Geotechnical and Geoenvironmental Engineering***, Vol. 125, No. 8, August, 1999. ©ASCE, ISSN 1090-0241/99/0008-0716–0718/$8.00 + $.50 per page. Technical Note No. 18087.

TABLE 1. Soft Clay Deposits

Soft clay deposits (1)	Location (2)	Wn (%) (3)	Wl (%) (4)	Wp (%) (5)	CF (%) (6)	σ'_p/σ'_{vo} (7)	C_α/C_c (8)
Batiscan	Quebec, Canada	71–88	49	22	80	1.6–1.7	0.03
Berthierville	Quebec, Canada	57–63	46	24	36	1.3–1.4	0.044
Boston Blue	Boston, USA	24–29	33	18	40	3.2	0.026
Broadback	Quebec, Canada	42–48	36	25	67	2.6–3.2	0.04
Brown Mexico City	Mexico City, Mexico	311–340	361	91	15	1.4	0.046
Chicago Blue	Chicago, USA	23	24	16	20	—	0.033
La Grande	Quebec, Canada	55–58	64	26	52	1.8–2	0.052
Ottawa	Quebec, Canada	80–91	68	29	75	1.7	0.03
Louiseville	Quebec, Canada	64–71	65	28	72	2.6–2.9	0.03
Olga	Quebec, Canada	85–94	67	29	67	2.2–2.5	0.033
Organic Poulding	Ohio, USA	84	70	45	53	—	0.048
Pancone (Pisa)	Pisa, Italy	55–65	75–90	34–36	73	1.65–2.02	0.034
San Francisco Bay Mud	San Francisco, USA	86–97	89	37	42	1.2	0.05
Singapore Marine	Singapore	38–79	54–86	19–32	54	1.4	0.032
St. Alban	Quebec, Canada	48–74	31–42	18–22	42	2.1–3.0	0.024
St. Espirit	Quebec, Canada	74–91	75	27	76	3.4–3.7	0.038
St. Hilaire	Quebec, Canada	62–71	55	23	76	1.4–1.5	0.031
Vasby	Vasby, Sweden	114–122	122	41	67	1.5–1.9	0.055

TABLE 2. Clay Shales

Shale deposits (1)	Location (2)	Wn (%) (3)	Wl (%) (4)	Wp (%) (5)	Ws (%) (6)	CF (%) (7)	C_α/C_c (8)
Bearpaw	Ft. Peck Dam, Montana	15.8	288	44	10	88	0.029
Claggett	Benton, Montana	11.7	157	31	11	71	0.024
Comanche	Proctor Dam, Texas	11.5	62	32	22	68	0.023
Cucaracha	Panama Canal, Panama	18.4	111	42	11	63	0.022
Denver	Denver, Colorado	30.5	121	37	11	67	0.022
Patapsco	Washington, D.C.	21.6	77	25	11	59	0.017
Pierre	Limon, Colorado	24.3	82	30	13	42	0.030

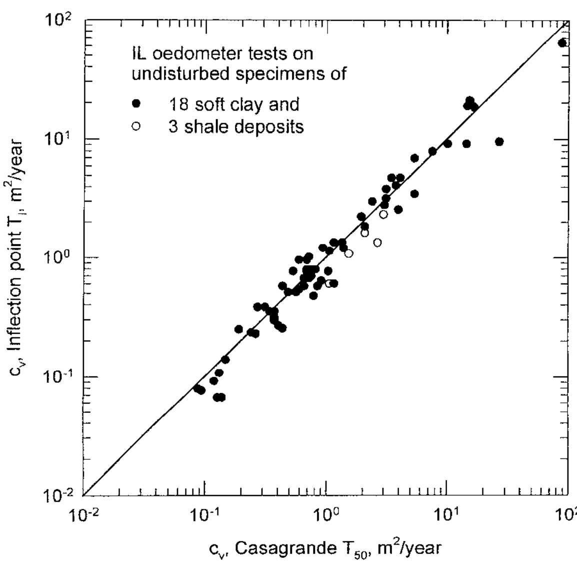

FIG. 1. Coefficient of Consolidation from Inflection Point Compared with that by Casagrande Procedure

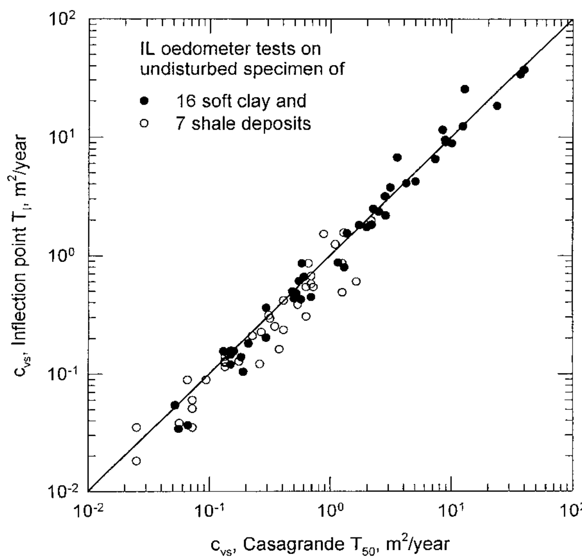

FIG. 2. Coefficient of Expansion from Inflection Method Compared with c_{vs} by Casagrande Procedure

(Choi 1982; Cepeda-Diaz 1986; Feng 1991; Hayat 1992; Ali 1993). The relevant properties of these materials are summarized in Tables 1 and 2, and the values of c_v for loading and c_{vs} for unloading computed using t_{50} according to the Casagrande procedure and using t_i at the inflection point are plotted in Figs. 1 and 2, respectively. As can be seen, the values of c_v by the simple inflection point method are quite similar to those from the more laborious (and widely used) Casagrande method.

It should be mentioned that other methods are also available for determining c_v from consolidation test data or field settlement or porewater pressure observations (e.g., Asaoka 1978; Sridharan et al. 1987). Furthermore, settlement analysis in situations where subsurface conditions have been carefully defined can be carried out using more refined and realistic formulations of time rate of consolidation that use variable permeability and compressibility without a need for a constant coefficient of consolidation (Mesri and Choi 1985; Mesri et al. 1995).

APPENDIX. REFERENCES

Ali, S. (1993). ''Engineering properties of undisturbed samples of Boston blue clay,'' PhD thesis, University of Illinois at Urbana-Champaign, Ill.

Asaoka, A. (1978). ''Observational procedure of settlement prediction.'' *Soils and Found.*, 18(4), 87–101.

Casagrande, A., and Fadum, R. E. (1940). ''Notes on soil testing for engineering purposes.'' *Publication 268*, Graduate School of Engineering, Harvard University, Cambridge, Mass.

Cepeda-Diaz, A. F. (1987). ''An experimental investigation of the engineering behavior of natural shales,'' PhD thesis, University of Illinois at Urbana-Champaign, Ill.

Choi, Y. K. (1982). ''Consolidation behavior of natural clays,'' PhD thesis, University of Illinois at Urbana-Champaign, Ill.

Feng, T. W. (1991). ''Compressibility and permeability of natural soft clays, and surcharging to reduce settlements,'' PhD thesis, University of Illinois at Urbana-Champaign, Ill.

Hayat, T. M. (1992). ''The coefficient of earth pressure at rest,'' PhD thesis, University of Illinois at Urbana-Champaign, Ill.

Mesri, G., and Choi, Y. K. (1985). ''Settlement analysis of embankments on soft clays.'' *J. Geotech. Engrg.*, ASCE, 111(4), 441–464.

Mesri, G., and Godlewski, P. M. (1977). ''Time- and stress-compressibility interrelationship.'' *J. Geotech. Engrg.*, ASCE, 103(5), 417–430.

Mesri, G., Lo, D. O. K., and Feng, T. W. (1994). ''Settlement of embankments on soft clays.'' *Proc., of Settlement '94*, Geotech. Spec. Publ., ASCE, Reston, Va., 40(1), 8–76.

Mesri, G., Ulrich, C. R., and Choi, Y. K. (1978). ''The rate of swelling of overconsolidated clays subjected to unloading.'' *Géotechnique*, 28(3), 281–307.

Robinson, R. G. (1997). ''Consolidation analysis by an inflection point method.'' *Géotechnique*, 47(1), 199–200.

Sridharan, A., Murthy, N. S., Prakash, K. (1987). ''Rectangular hyberbola method of consolidation analysis.'' *Géotechnique*, 37(3), 355–368.

Taylor, D. W. (1948). *Fundamentals of soil mechanics.* Wiley, New York.

Terzaghi, K., Peck, R. B., and Mesri, G. (1996). *Soil mechanics in engineering practice*, 3rd Ed., Wiley, New York.

Coefficient of Consolidation by Inflection Point Method[a]

Discussion by A. K. Parkin[4]

While the Casagrande and Taylor Methods for evaluating c_v have enjoyed wide acceptance for a very long time, several alternative procedures have emerged in more recent times offering significant advantages and eliminating some of the hand labor and error sources in the older theories. The inflection point method is one of these, with the particular advantages, as stated by the authors, that

> it does not require the definition of the beginning and end of the primary consolidation stage that are required for other widely used methods (Casagrande and Fadum 1940; Taylor 1948). Another advantage of this approach is that the inflection point at the average degree of consolidation of 70% is within the midrange of the compression curve and is least affected by initial compression and secondary compression.

The discusser would add that, as presented by Robinson (1997), who evaluates the *gradient* of the Casagrande plot,

$$M = dU/d(\log T)$$

it is also a type of *differential method*, which can provide enhanced sensitivity if not negated by too much experimental scatter. These advantages are real and desirable and should produce a value of c_v that is *more accurate* than the one ob-

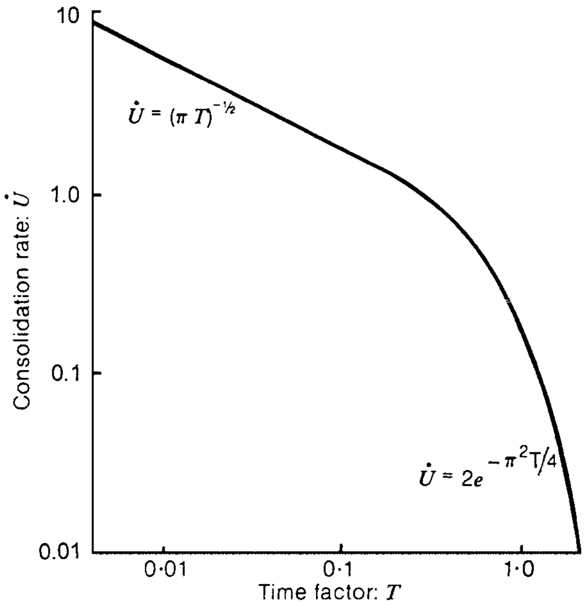

FIG. 3. Terzaghi Solution as Velocity-Time Plot

tainable from traditional methods. However, the above quote applies equally and in full to the velocity method of the discusser (Parkin 1978), which is also a differential procedure. The principal difference is that the velocity method is not locked in to any mathematical form for the *settlement-time* relationship and is therefore able to reveal variations from the basic theory of (1) that are smothered in the usual root time and log time plots. There is also a benefit in that the velocity solution—as in Fig. 3 (which is used as an overlay on a laboratory plot of log $\dot{S}$ v. log t, to obtain a direct relationship between t and T)—has a recognizable geometric form that can compensate for some experimental scatter, besides indicating very clearly any experimental situation to which the Terzaghi theory does not apply (as can happen with the root time procedure, in particular).

There are, of course, opinions that accuracy in the evaluation of c_v is not warranted because of the limited accuracy of prediction that is achievable in settlement calculations. However, this is essentially a problem in the determination of H (length of drainage path) and should not, in the discusser's opinion, discourage seeking the best possible value of c_v.

APPENDIX. REFERENCE

Parkin, A. K. (1978). ''Coefficient of consolidation by the velocity method.'' *Géotechnique*, London, 28(4), 472–474.

Closure by G. Mesri,[5] T. W. Feng,[6] and M. Shahien[7]

The discusser has introduced yet another method for computing the Terzaghi coefficient of consolidation from the odometer compression measurements. In preparing a response the writers have examined Parkin (1978).

A main reason for plotting the oedometer compression versus time data in a compression versus log time or compression

[a]August 1999, Vol. 125, No. 8, by G. Mesri, T. W. Feng, and M. Shahien (Paper 18087).

[4]Snowy Mountains Engineering Corp., P.O. Box 559, Carnegie, Victoria 3163, Australia.

[5]Prof. of Civ. Engrg., Univ. of Illinois at Urbana-Champaign, IL 61801-2352.

[6]Assoc. Prof. of Civ. Engrg., Chung Yuan Univ., Chung-li, Taiwan, ROC.

[7]Postdoctoral Res. Assoc., Univ. of Illinois at Urbana-Champaign, IL.

versus square root time is to define the compression corresponding to the end-of-primary (EOP) consolidation, S_{100}, using the Casagrande method or the Taylor method, respectively. This is required for determining end-of-primary (EOP) e versus log σ'_v for computing the magnitude of settlement. This objective is not achieved by the velocity method.

Most users may consider an additional work the computation of settlement rate and visually matching of log $\dot{S} - \log t$ and log $\dot{U} - \log T$. Furthermore, compression readings are usually made at elapsed times that are suitable for square root time or log time plots, but the (S, t) readings may not be suitable for accurate computation of compression rate $\Delta S/\Delta t$ (e.g., Δt too large, etc.).

The general shape of the $S - \log t$ as well as the beginning of $S - \sqrt{t}$ and the end of $S - \log t$ clearly show the departure of the measurements from the Terzaghi consolidation theory. Therefore, it may not be appropriate to state that they smother the observed variations from the theory.

The theory of consolidation, which has been developed by Mesri and coworkers (e.g., Mesri and Rokhsar 1974; Mesri and Choi 1985; Mesri et al. 1994) for settlement analysis in the ILLICON procedure, uses direct information for each sublayer on permeability as a function of void ratio and compressibility in terms of void ratio versus log σ'_v. Therefore, the writers do not routinely calculate the Terzaghi coefficient of consolidation for all oedometer tests that are performed for various purposes. The values of c_v were calculated for the technical note mainly because the writers think the inflection point method is quite simple and elegant. Therefore it was decided to check and, if appropriate, endorse it by means of data on natural soils. The inflection point method requires no work other than carefully noting the inflection point on a $S - \log t$ curve that is plotted to obtain $S_p = S_{100}$ for determining the EOP e versus log σ'_v relationship for settlement analysis.

APPENDIX. REFERENCES

Mesri, G., and Rokhsar, A. (1974). ''Theory of consolidation for clays.'' *J. Geotech. Engrg Div.*, ASCE, 100(8), 889–904.

Parkin, A. K. (1978). ''Coefficient of consolidation by the velocity method.'' *Géotechnique*, London, 28(4), 472–474.