CLOSED-FORM SETTLEMENT SOLUTION
FOR NORMALLY CONSOLIDATED SOILS

Assume a homogeneous, compressible soil layer of thickness H that will consolidate under the weight of a surface load of infinite lateral extent. If we subdivide the compressible layer into an infinite number of sublayers of height dz, the total settlement is given by the integral

$$\Delta H = C_{c\varepsilon} \int_{z_{top}}^{z_{bot}} \log \frac{\sigma'_{vf}(z)}{\sigma'_{vo}(z)} dz$$

This will be easier to integrate if we rewrite it as

$$\Delta H = C_{c\varepsilon} \int_{z_{top}}^{z_{bot}} \log \sigma'_{vf}(z)\, dz - C_{c\varepsilon} \int_{z_{top}}^{z_{bot}} \log \sigma'_{vo}(z)\, dz$$

If we let $x \equiv \sigma'_v(z)$, then $dx \equiv d\sigma'_v(z) = \gamma' dz$ since the *in situ* stress increases with depth in direct proportion to the buoyant unit weight of the soil (γ').

Substituting $\sigma'_v(z) \rightarrow x$ and $dz \rightarrow dx/\gamma'$, we can rewrite the settlement equation as

$$\Delta H = \frac{C_{c\varepsilon}}{\gamma'} \int_{\sigma'_{vf}{}^{top}}^{\sigma'_{vf}{}^{bot}} \log x\, dx - \frac{C_{c\varepsilon}}{\gamma'} \int_{\sigma'_{vo}{}^{top}}^{\sigma'_{vo}{}^{bot}} \log x\, dx$$

Now $\int \log x\, dx = x \log x - x$ so

$$\Delta H = \frac{C_{c\varepsilon}}{\gamma'} \left(\sigma'_{vf}{}^{bot} \log \sigma'_{vf}{}^{bot} - \sigma'_{vf}{}^{bot} - \sigma'_{vf}{}^{top} \log \sigma'_{vf}{}^{top} + \sigma'_{vf}{}^{top} \right)$$
$$- \frac{C_{c\varepsilon}}{\gamma'} \left(\sigma'_{vo}{}^{bot} \log \sigma'_{vo}{}^{bot} - \sigma'_{vo}{}^{bot} - \sigma'_{vo}{}^{top} \log \sigma'_{vo}{}^{top} + \sigma'_{vo}{}^{top} \right)$$

Since, $\sigma'_{vf} - \sigma'_{vo} = \Delta \sigma'_v$ everywhere, this reduces to

$$\Delta H = \frac{C_{c\varepsilon}}{\gamma'} \left[\left(\sigma'_{vo}{}^{top} \log \sigma'_{vo}{}^{top} - \sigma'_{vo}{}^{bot} \log \sigma'_{vo}{}^{bot} \right) - \left(\sigma'_{vf}{}^{top} \log \sigma'_{vf}{}^{top} - \sigma'_{vf}{}^{bot} \log \sigma'_{vf}{}^{bot} \right) \right]$$

which is the final solution for the consolidation settlement.

> The equation above can be used as long as the unit weight of the soil is constant throughout the compressible layer and the initial (σ'_{vo}) and final (σ'_{vf}) stresses both increase with depth at the rate $d\sigma/dz = \gamma'$. In practice, this means that the surface load has a large lateral extent (more like an engineered fill than a footing).

CLOSED-FORM SETTLEMENT SOLUTION
FOR OVERCONSOLIDATED SOILS

The settlement equation for an overconsolidated soil is as follows:

$$s = C_{r\varepsilon} H_o \log \frac{\sigma'_p}{\sigma'_{vo}} + C_{c\varepsilon} H_o \log \frac{\sigma'_{vf}}{\sigma'_p}$$

If we subdivide the compressible layer into sublayers of height H_i then the settlement in each sublayer is

$$s_i = C_{r\varepsilon} H_i \log \frac{\sigma'_{p,i}}{\sigma'_{vo,i}} + C_{c\varepsilon} H_i \log \frac{\sigma'_{vf,i}}{\sigma'_{p,i}}$$

In the limit, if we subdivide the compressible layer into an infinite number of sublayers of height dz, the settlement is given by the integral

$$\Delta H = C_{r\varepsilon} \int_{z_{top}}^{z_{bot}} \log \frac{\sigma'_p(z)}{\sigma'_{vo}(z)} dz + C_{c\varepsilon} \int_{z_{top}}^{z_{bot}} \log \frac{\sigma'_{vf}(z)}{\sigma'_p(z)} dz$$

This will be easier to integrate if we rewrite it as

$$\Delta H = C_{r\varepsilon} \int_{z_{top}}^{z_{bot}} \log \sigma'_p(z)\, dz - C_{r\varepsilon} \int_{z_{top}}^{z_{bot}} \log \sigma'_{vo}(z)\, dz + C_{c\varepsilon} \int_{z_{top}}^{z_{bot}} \log \sigma'_{vf}(z)\, dz - C_{c\varepsilon} \int_{z_{top}}^{z_{bot}} \log \sigma'_p(z)\, dz$$

This can be consolidated slightly to

$$\Delta H = \left(C_{r\varepsilon} - C_{c\varepsilon}\right) \int_{z_{top}}^{z_{bot}} \log \sigma'_p(z)\, dz - C_{r\varepsilon} \int_{z_{top}}^{z_{bot}} \log \sigma'_{vo}(z)\, dz + C_{c\varepsilon} \int_{z_{top}}^{z_{bot}} \log \sigma'_{vf}(z)\, dz$$

If we assume that the overconsolidation margin is constant with depth within the layer and the difference between σ'_{vf} and σ'_{vo} is constant with depth (which assumes a surface load of infinite lateral extent), then all three stresses increase with depth in direct proportion to the unit weight of the soil. This lets us write

$$x \equiv \sigma'_v(z)$$

$$dx \equiv d\sigma'_v(z) = \gamma' dz$$

Substituting $\sigma'_v(z) \rightarrow x$ and $dz \rightarrow dx/\gamma'$, we can write the settlement equation as

$$\Delta H = \frac{C_{r\varepsilon} - C_{c\varepsilon}}{\gamma'} \int_{\sigma'^{\,top}_p}^{\sigma'^{\,bot}_p} \log x\, dx - \frac{C_{r\varepsilon}}{\gamma'} \int_{\sigma'^{\,top}_{vo}}^{\sigma'^{\,bot}_{vo}} \log x\, dx + \frac{C_{c\varepsilon}}{\gamma'} \int_{\sigma'^{\,top}_{vf}}^{\sigma'^{\,bot}_{vf}} \log x\, dx$$

Now $\int \log x\, dx = x \log x - x$ so the settlement equation can be written as

$$\Delta H = \frac{C_{r\varepsilon} - C_{c\varepsilon}}{\gamma'}\left(\sigma_p'^{\,bot}\log\sigma_p'^{\,bot} - \sigma_p'^{\,bot} - \sigma_p'^{\,top}\log\sigma_p'^{\,top} + \sigma_p'^{\,top}\right)$$
$$-\frac{C_{r\varepsilon}}{\gamma'}\left(\sigma_{vo}'^{\,bot}\log\sigma_{vo}'^{\,bot} - \sigma_{vo}'^{\,bot} - \sigma_{vo}'^{\,top}\log\sigma_{vo}'^{\,top} + \sigma_{vo}'^{\,top}\right)$$
$$+\frac{C_{c\varepsilon}}{\gamma'}\left(\sigma_{vf}'^{\,bot}\log\sigma_{vf}'^{\,bot} - \sigma_{vf}'^{\,bot} - \sigma_{vf}'^{\,top}\log\sigma_{vf}'^{\,top} + \sigma_{vf}'^{\,top}\right)$$

If we isolate only the terms that <u>don't</u> involve a logarithm:

$$\Delta H = \frac{C_{r\varepsilon} - C_{c\varepsilon}}{\gamma'}\left(-\sigma_p'^{\,bot} + \sigma_p'^{\,top}\right) - \frac{C_{r\varepsilon}}{\gamma'}\left(-\sigma_{vo}'^{\,bot} + \sigma_{vo}'^{\,top}\right) + \frac{C_{c\varepsilon}}{\gamma'}\left(-\sigma_{vf}'^{\,bot} + \sigma_{vf}'^{\,top}\right)$$

Rearranging:

$$\Delta H = \frac{C_{r\varepsilon}}{\gamma'}\left(-\sigma_p'^{\,bot} + \sigma_p'^{\,top} + \sigma_{vo}'^{\,bot} - \sigma_{vo}'^{\,top}\right) - \frac{C_{c\varepsilon}}{\gamma'}\left(-\sigma_p'^{\,bot} + \sigma_p'^{\,top} - \sigma_{vf}'^{\,bot} + \sigma_{vf}'^{\,top}\right)$$

Since we've assumed that the overconsolidation margin is constant with depth within the layer and the difference between σ_{vf}' and σ_{vo}' is constant with depth, then all three stresses increase with depth by the same amount, $\gamma' \Delta z$, so

$$\sigma_p'^{\,top} - \sigma_p'^{\,bot} = \sigma_{vo}'^{\,top} - \sigma_{vo}'^{\,bot} = \sigma_{vf}'^{\,top} - \sigma_{vf}'^{\,bot}$$

which means that all of those terms that <u>don't</u> involve a logarithm cancel each other out, leaving

$$\Delta H = \frac{C_{r\varepsilon} - C_{c\varepsilon}}{\gamma'}\left(\sigma_p'^{\,bot}\log\sigma_p'^{\,bot} - \sigma_p'^{\,top}\log\sigma_p'^{\,top}\right)$$
$$-\frac{C_{r\varepsilon}}{\gamma'}\left(\sigma_{vo}'^{\,bot}\log\sigma_{vo}'^{\,bot} - \sigma_{vo}'^{\,top}\log\sigma_{vo}'^{\,top}\right)$$
$$+\frac{C_{c\varepsilon}}{\gamma'}\left(\sigma_{vf}'^{\,bot}\log\sigma_{vf}'^{\,bot} - \sigma_{vf}'^{\,top}\log\sigma_{vf}'^{\,top}\right)$$

as the closed-form equation for settlement of overconsolidated soils with a constant OCM within each layer and a constant $\Delta\sigma_v$ added at all depths due to a broad surface load.

What happens if the preconsolidation stress increases with depth at a different rate than the *in situ* stress? In other words, what happens if the OCM is not constant within the soil layer?

Let's back up to the equation

$$\Delta H = \left(C_{r\varepsilon} - C_{c\varepsilon}\right)\int_{z_{top}}^{z_{bot}} \log\sigma_p'(z)\, dz - C_{r\varepsilon}\int_{z_{top}}^{z_{bot}} \log\sigma_{vo}'(z)\, dz + C_{c\varepsilon}\int_{z_{top}}^{z_{bot}} \log\sigma_{vf}'(z)\, dz$$

If σ'_{vo} changes with depth as $s_o \Delta z$ and σ'_{vf} changes with depth as $s_f \Delta z$ and σ'_p changes with depth as $s_p \Delta z$ then we can make the following substitutions:

$$\sigma'_{vo}(z) \rightarrow x_o \text{ and } dz \rightarrow dx_o / s_o$$

$$\sigma'_{vf}(z) \rightarrow x_f \text{ and } dz \rightarrow dx_f / s_f$$

$$\sigma'_p(z) \rightarrow x_p \text{ and } dz \rightarrow dx_p / s_p$$

Now the settlement equation can be written as

$$\Delta H = \frac{C_{r\varepsilon} - C_{c\varepsilon}}{s_p} \int_{\sigma_p'^{top}}^{\sigma_p'^{bot}} \log x_p \, dx_p - \frac{C_{r\varepsilon}}{s_o} \int_{\sigma_{vo}'^{top}}^{\sigma_{vo}'^{bot}} \log x_o \, dx_o + \frac{C_{c\varepsilon}}{s_f} \int_{\sigma_{vf}'^{top}}^{\sigma_{vf}'^{bot}} \log x_f \, dx_f$$

As before, $\int \log x \, dx = x \log x - x$ so the settlement equation can be written as

$$\begin{aligned}\Delta H = \frac{C_{r\varepsilon} - C_{c\varepsilon}}{s_p} &\left(\sigma_p'^{bot} \log \sigma_p'^{bot} - \sigma_p'^{bot} - \sigma_p'^{top} \log \sigma_p'^{top} + \sigma_p'^{top} \right) \\ -\frac{C_{r\varepsilon}}{s_o} &\left(\sigma_{vo}'^{bot} \log \sigma_{vo}'^{bot} - \sigma_{vo}'^{bot} - \sigma_{vo}'^{top} \log \sigma_{vo}'^{top} + \sigma_{vo}'^{top} \right) \\ +\frac{C_{c\varepsilon}}{s_f} &\left(\sigma_{vf}'^{bot} \log \sigma_{vf}'^{bot} - \sigma_{vf}'^{bot} - \sigma_{vf}'^{top} \log \sigma_{vf}'^{top} + \sigma_{vf}'^{top} \right)\end{aligned}$$

This is as far as we can simplify this equation because we <u>cannot</u> write

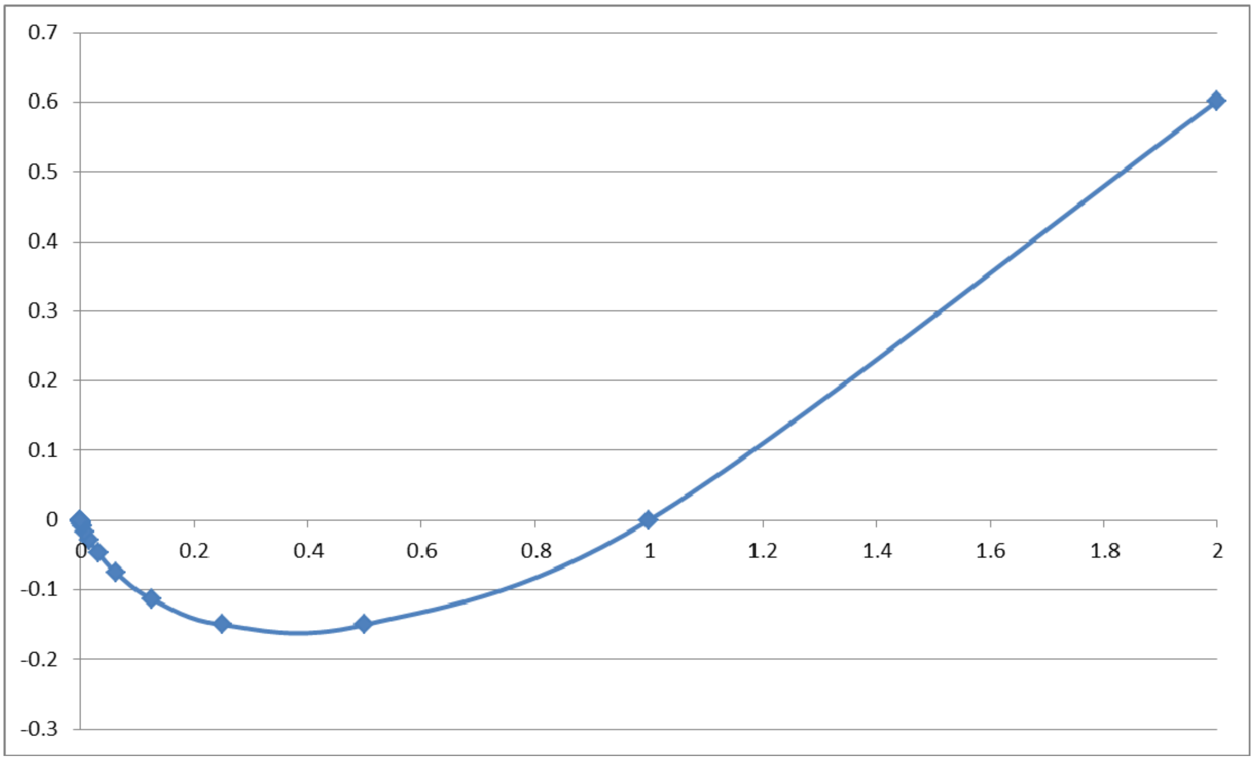
$$\sigma_p'^{top} - \sigma_p'^{bot} = \sigma_{vo}'^{top} - \sigma_{vo}'^{bot} = \sigma_{vf}'^{top} - \sigma_{vf}'^{bot}$$

as we did before to get half of the terms to cancel out. So the calculations become a bit more complicated, but still doable in closed form *as long as the stresses change linearly with depth*.

A NOTE FOR SOIL LAYERS AT THE SURFACE

If the soil layer being consolidated starts at the ground surface, $\sigma_p'^{top} = 0$, which poses a problem because log(0) is undefined.

If you plot x log(x) as x approaches zero, you find that the equation does not approach zero monotonically. Instead, it crosses the x-axis at x = 1, then dips below the axis before it asymptotically approaches zero again.



Therefore, assuming $\sigma_p'^{top} = 1$ provides the correct value of $\sigma_p'^{top} \log \sigma_p'^{top}$ while all other "small" values only approximate the correct answer.