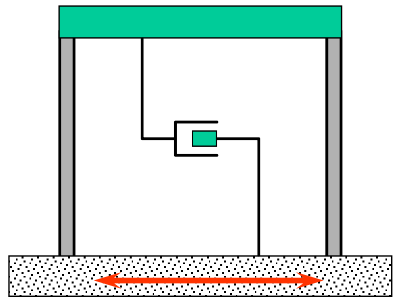
Structural Dynamics of Linear Elastic Single-Degree-of-Freedom (SDOF) Systems



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SDOF Dynamics 3 - 1

This set of slides covers the fundamental concepts of structural dynamics of linear elastic single-degree-of-freedom (SDOF) structures. A separate topic covers the analysis of linear elastic multiple-degree-of-freedom (MDOF) systems. A separate topic also addresses inelastic behavior of structures. Proficiency in earthquake engineering requires a thorough understanding of each of these topics.

Structural Dynamics

- Equations of motion for SDOF structures
- Structural frequency and period of vibration
- Behavior under dynamic load
- Dynamic magnification and resonance
- Effect of damping on behavior
- Linear elastic response spectra

This slide lists the scope of the present topic. In a sense, the majority of the material in the topic provides background on the very important subject of response spectra.

Importance in Relation to ASCE 7-05

- Ground motion maps provide ground accelerations in terms of *response spectrum* coordinates.
- Equivalent lateral force procedure gives base shear in terms of *design spectrum* and *period of vibration*.
- Response spectrum is based on *5% critical damping* in system.
- Modal superposition analysis uses design *response spectrum* as basic ground motion input.

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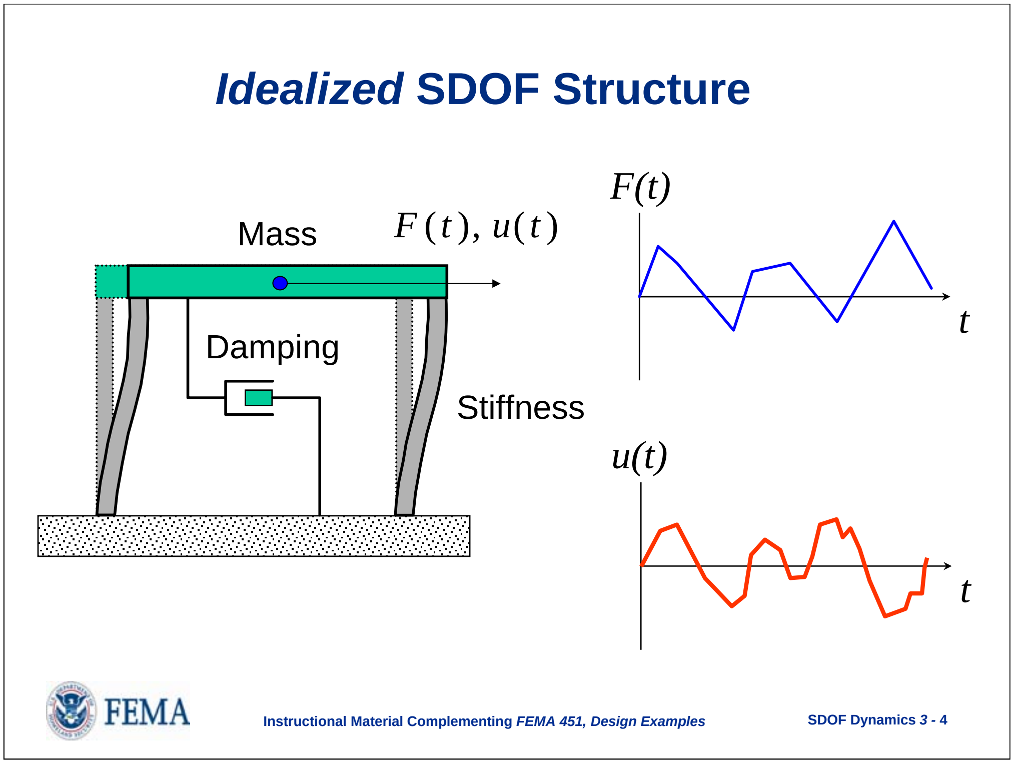
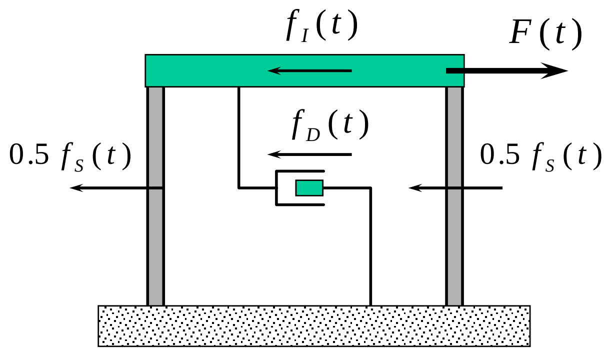
The relevance of the current topic to the ASCE 7-05 document is provided here. Detailed referencing to numbered sections in ASCE 7-05 is provided in many of the slides. Note that ASCE 7-05 is directly based on the 2003 *NEHRP Recommended Provisions for Seismic Regulations for New Buildings and Other Structures*, FEMA 450, which is available at no charge from the FEMA Publications Center, 1-800-480-2520 (order by FEMA publication number).



The simple frame is *idealized* as a SDOF mass-spring-dashpot model with a time-varying applied load. The function *u(t)* defines the displacement response of the system under the loading F(t). The properties of the structure can be completely defined by the mass, damping, and stiffness as shown.

The idealization assumes that all of the mass of the structure can be lumped into a single point and that all of the deformation in the frame occurs in the columns with the beam staying rigid. Represent damping as a simple viscous dashpot common as it allows for a linear dynamic analysis. Other types of damping models (e.g., friction damping) are more realistic but require nonlinear analysis.

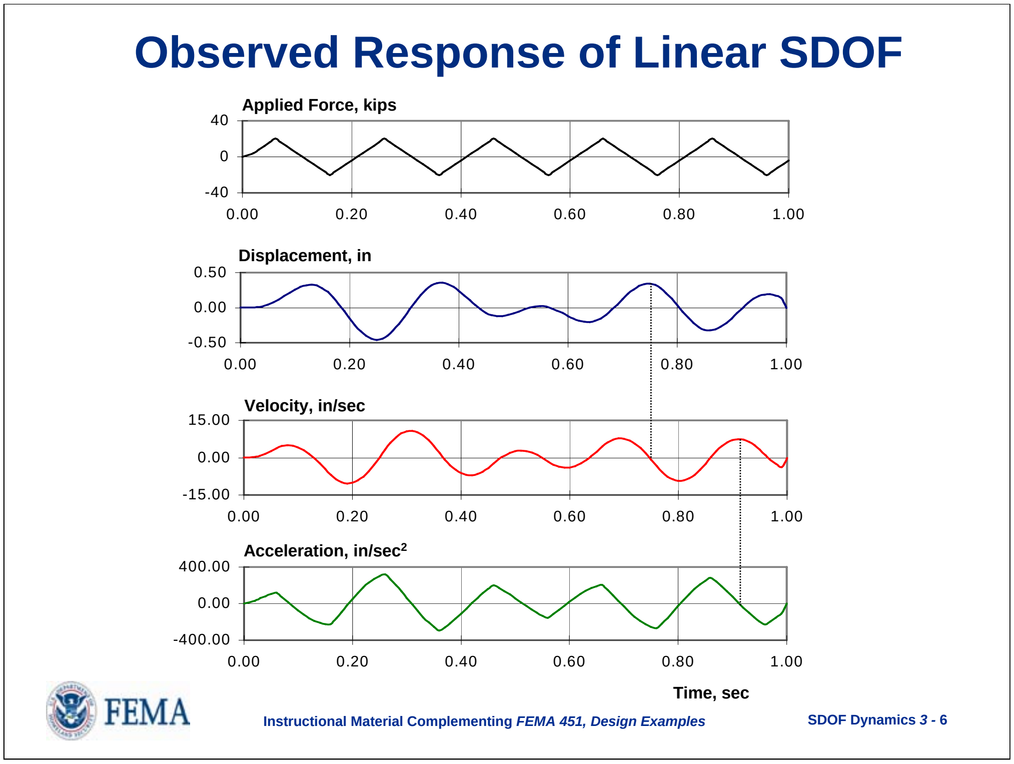
Equation of Dynamic Equilibrium



$$F(t) - f_I(t) - f_D(t) - f_S(t) = 0$$

$$f_I(t) + f_D(t) + f_S(t) = F(t)$$

Here the equations of motion are shown as a force-balance. At any point in time, the inertial, damping, and elastic resisting forces do not necessarily act in the same direction. However, at each point in time, dynamic equilibrium must be maintained.

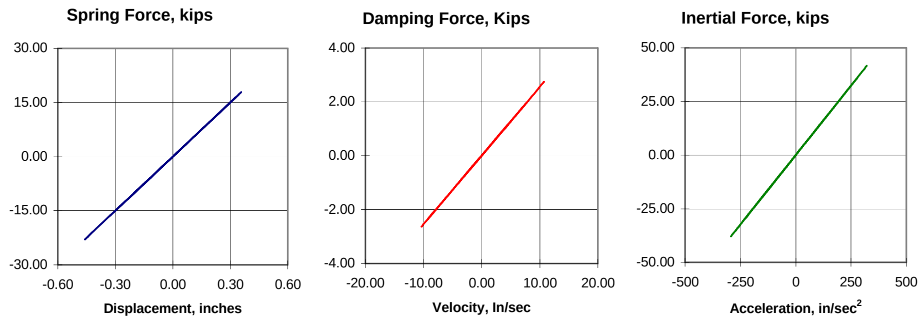


This slide (from NONLIN) shows a series of response histories for a SDOF system subjected to a saw-tooth loading. As a result of the loading, the mass will undergo displacement, velocity, and acceleration. Each of these quantities are measured with respect to the fixed base of the structure.

Note that although the loading is discontinuous, the response is relatively smooth. Also, the vertical lines show that velocity is zero when displacement is maximum and acceleration is zero when velocity is maximum.

NONLIN is an educational program for dynamic analysis of simple linear and nonlinear structures. Version 7 is included on the CD containing these instructional materials.

Observed Response of Linear SDOF
(Development of Equilibrium Equation)



Slope = *k*	**Slope = *c***	**Slope = *m***
= 50 kip/in	**= 0.254 kip-sec/in**	**= 0.130 kip-sec²/in**
$f_S(t) = k\ u(t)$	$f_D(t) = c\ \dot{u}(t)$	$f_I(t) = m\ \ddot{u}(t)$

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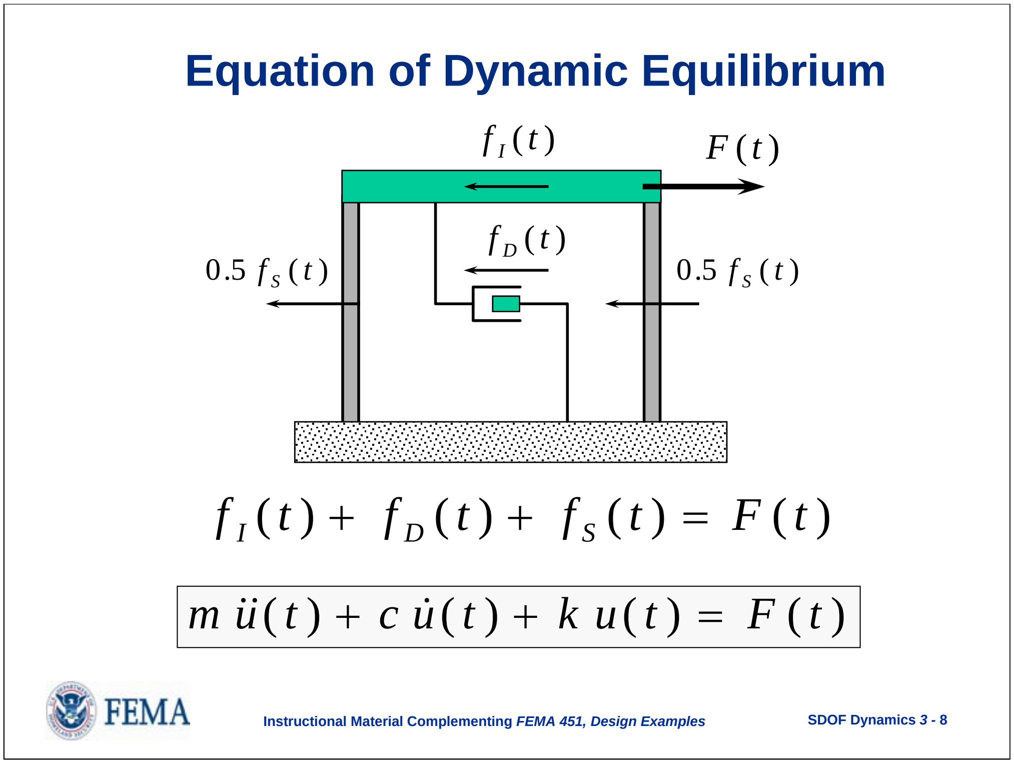
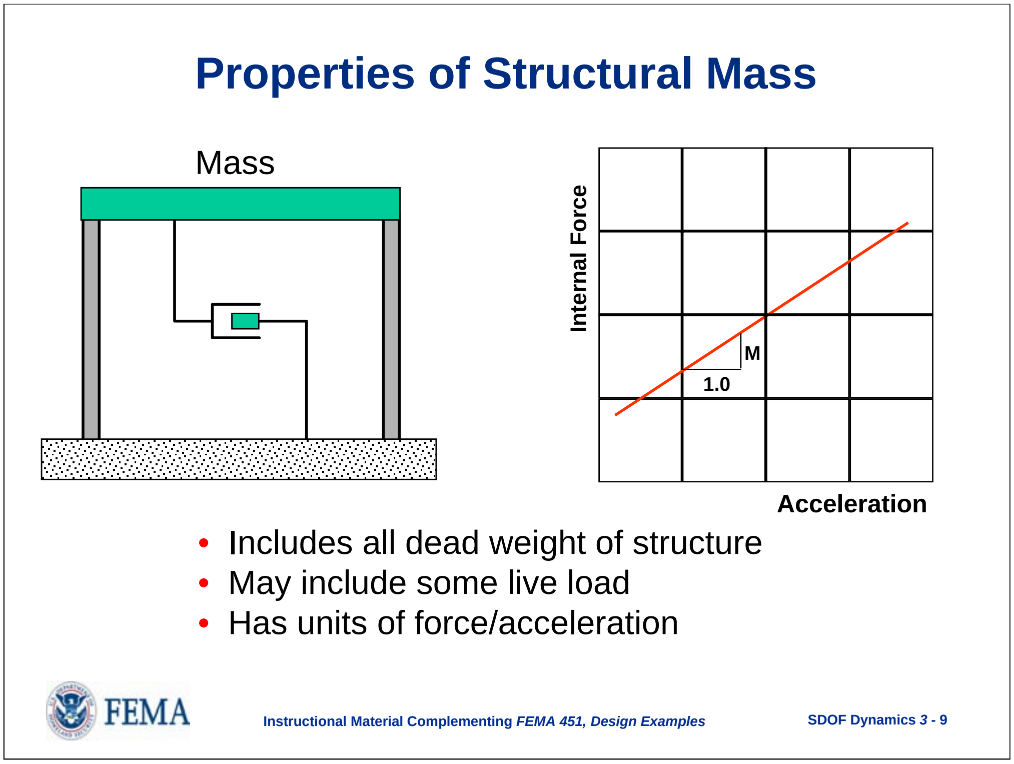
These X-Y curves are taken from the same analysis that produced the response histories of the previous slide. For a linear system, the resisting forces are proportional to the motion. The slope of the inertial-force vs acceleration curve is equal to the mass. Similar relationships exist for damping force vs velocity (slope = damping) and elastic force vs displacement (slope = stiffness).

The importance of understanding and correct use of units cannot be over emphasized.

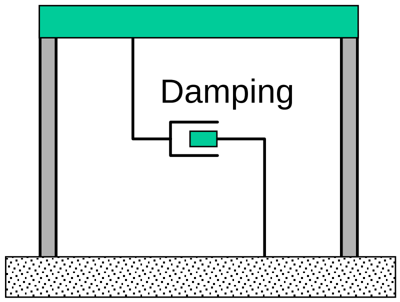


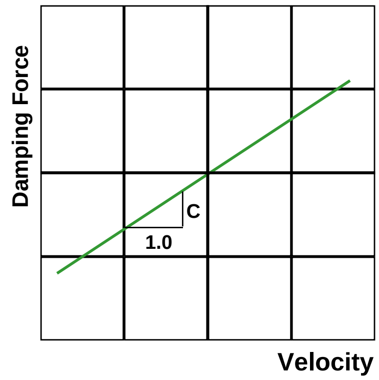
Here the equations of motion are shown in terms of the displacement, velocity, acceleration, and force relationships presented in the previous slide. Given the forcing function, *F(t),* the goal is to determine the response history of the system.



Mass is always assumed constant throughout the response. Section 12.7.2 of ASCE 7-05 defines this mass in terms of the “effective weight” of the structure. The effective weight includes 25% of the floor live load in areas used for storage, 10 psf partition allowance, operating weight of all permanent equipment, and 20% of the flat roof snow load when that load exceeds 30 psf.

Properties of Structural Damping





- In absence of dampers, is called *inherent damping*
- Usually represented by *linear* viscous dashpot
- Has units of force/velocity

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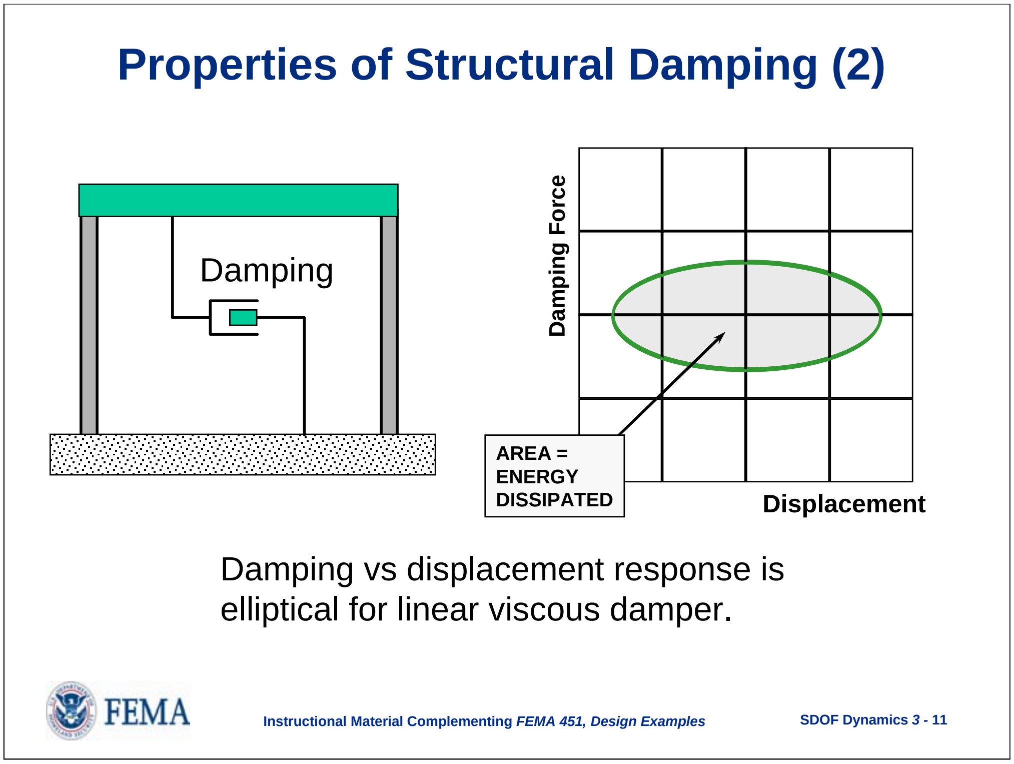
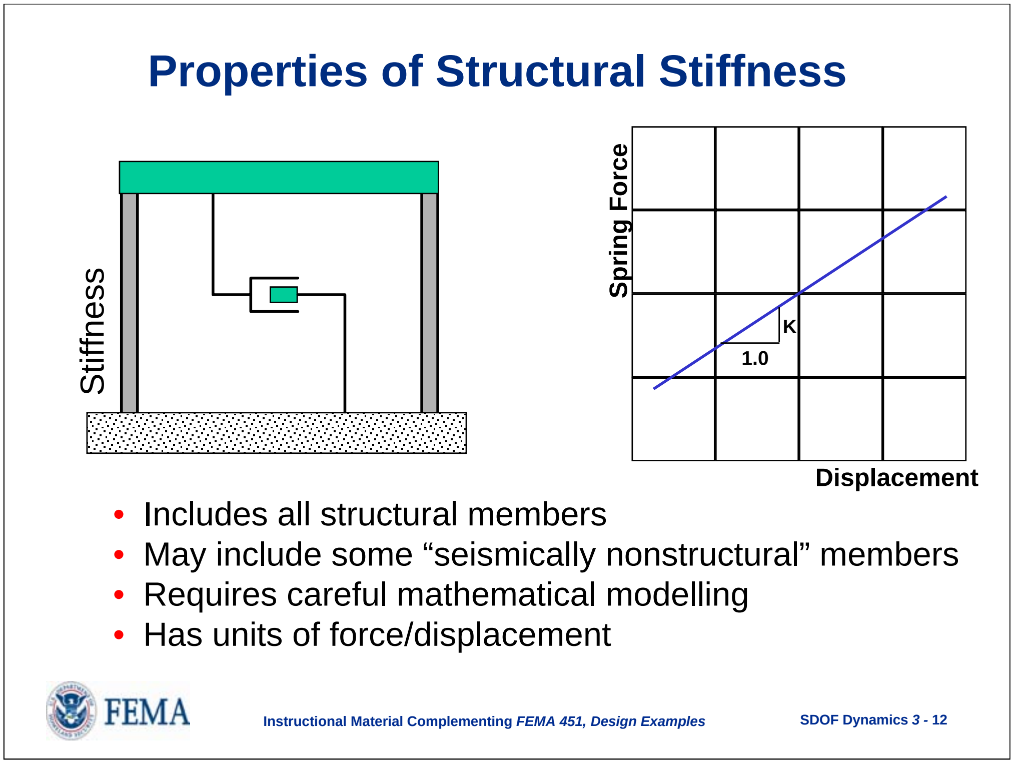
Except for the case of added damping, real structures do not have discrete dampers as shown. Real or *inherent* damping arises from friction in the material. For cracked concrete structures, damping is higher because of the rubbing together of jagged surfaces on either side of a crack.

In analysis, we use an equivalent viscous damper primarily because of the mathematical convenience. (Damping force is proportional to velocity.)



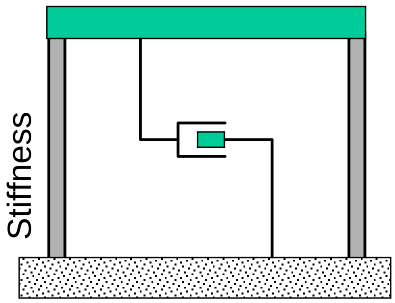
The force-displacement relationship for a linear viscous damper is an ellipse. The area within the ellipse is the *energy dissipated* by the damper. The greater the energy dissipated by damping, the lower the potential for damage in structures. This is the primary motivation for the use of added damping systems. Energy that is dissipated is irrecoverable.

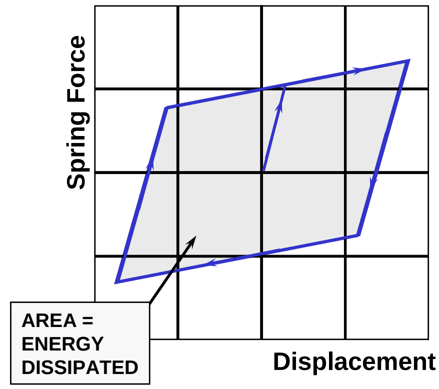


In this topic, it is assumed that the force-displacement relationship in the spring is linear elastic. Real structures, especially those designed according to current seismic code provisions, will not remain elastic and, hence, the force-deformation relationship is not linear. However, linear analysis is often (almost exclusively) used in practice. This apparent contradiction will be explained as this discussion progresses.

The modeling of the structure for stiffness has very significant uncertainties. Section 12.7.3 of ASCE 7-05 provides some guidelines for modeling the structure for stiffness.

Properties of Structural Stiffness (2)





- Is almost always nonlinear in real seismic response
- Nonlinearity is implicitly handled by codes
- Explicit modelling of nonlinear effects is possible

This is an idealized response of a simple inelastic structure. The area within the curve is the *inelastic hysteretic energy* dissipated by the yielding material. The larger hysteretic energy in relation to the damping energy, the greater the damage.

In this topic, it is assumed that the material does not yield. Nonlinear inelastic response is explicitly included in a separate topic.

Undamped Free Vibration

Equation of motion: $m\ddot{u}(t) + k\,u(t) = 0$

Initial conditions: $\dot{u}_0 \quad u_0$

Assume: $u(t) = A\sin(\omega t) + B\cos(\omega t)$

Solution: $A = \dfrac{\dot{u}_0}{\omega} \qquad B = u_0 \qquad \omega = \sqrt{\dfrac{k}{m}}$

$$u(t) = \frac{\dot{u}_0}{\omega}\sin(\omega t) + u_0\cos(\omega t)$$

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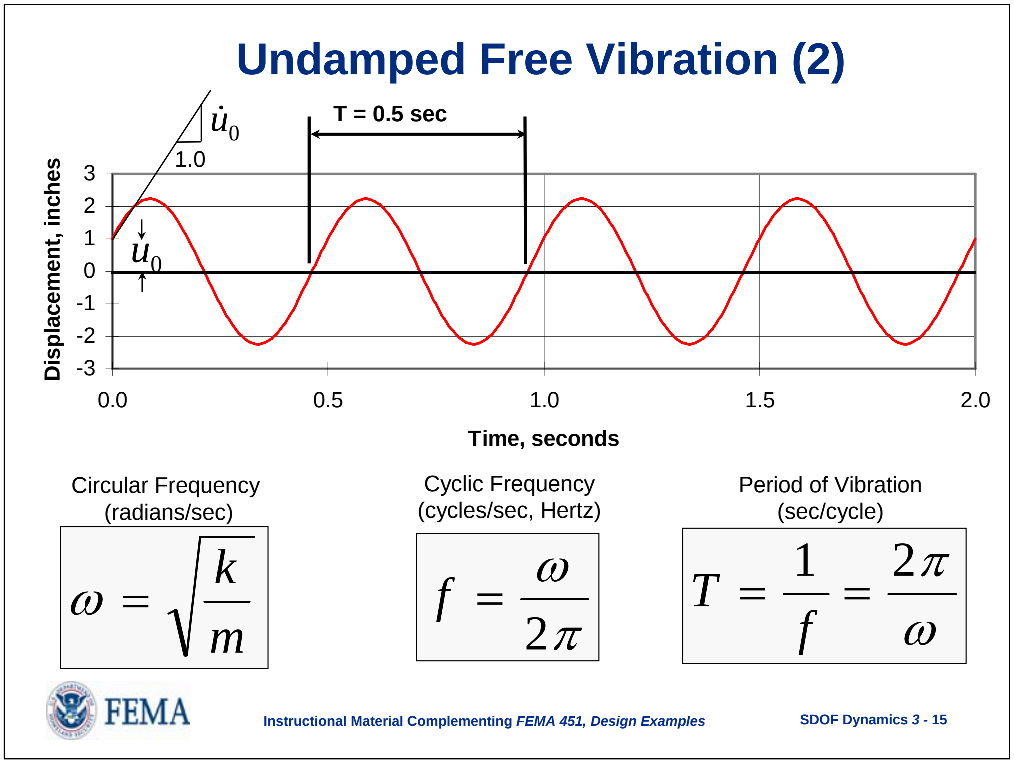
In this unit, we work through a hierarchy of increasingly difficult problems. The simplest problem to solve is undamped free vibration. Usually, this type of response is invoked by imposing a static displacement and then releasing the structure with zero initial velocity. The equation of motion is a second order differential equation with constant coefficients. The displacement term is treated as the primary unknown.

The assumed response is in terms of a sine wave and a cosine wave. It is easy to see that the cosine wave would be generated by imposing an initial displacement on the structure and then releasing. The sine wave would be imposed by initially “shoving” the structure with an initial velocity. The computed solution is a combination of the two effects.

The quantity ω is the circular frequency of free vibration of the structure (radians/sec). The higher the stiffness relative to mass, the higher the frequency. The higher the mass with respect to stiffness, the lower the frequency.



This slide shows a computed response history for a system with an initial displacement and velocity. Note that the slope of the initial response curve is equal to the initial velocity ($v = d_u/d_t$). If this term is zero, the free vibration response is a simple cosine wave. Note also that the undamped motion shown will continue forever if uninhibited. In real structures, damping will eventually reduce the free vibration response to zero.

The relationship between circular frequency, cyclic frequency, and period of vibration is emphasized. The period of vibration is probably the easiest to visualize and is therefore used in the development of seismic code provisions. The higher the mass relative to stiffness, the longer the period of vibration. The higher the stiffness relative to mass, the lower the period of vibration.

Approximate Periods of Vibration
(ASCE 7-05)

$$T_a = C_t h_n^x$$

C_t = 0.028, x = 0.8 for steel moment frames
C_t = 0.016, x = 0.9 for concrete moment frames
C_t = 0.030, x = 0.75 for eccentrically braced frames
C_t = 0.020, x = 0.75 for all other systems

Note: This applies ONLY to building structures!

$$T_a = 0.1N$$

For moment frames < 12 stories in height, minimum story height of 10 feet. N = number of stories.

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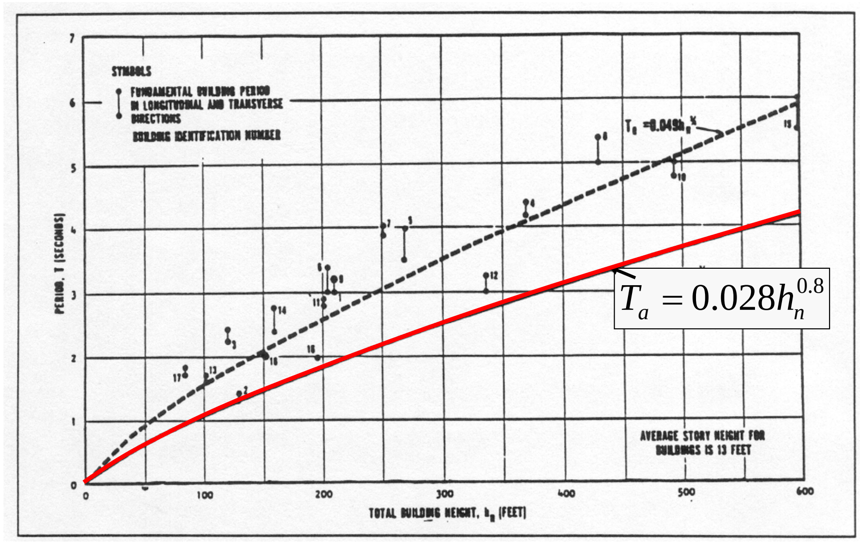
One of the first tasks in any seismic design project is to estimate the period of vibration of the structure. For preliminary design (and often for final design), an empirical period of vibration is used. Section 12.8.2 of ASCE 7-05 provides equations for estimating the period. These equations are listed here.

Empirical Data for Determination of Approximate Period for Steel Moment Frames



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T_a is based on curve-fitting of data obtained from measured response of California buildings after small earthquakes. As will be seen later, the smaller the period, the larger the earthquake force that must be designed for. Hence, a lower bound empirical relationship is used.

Because the empirical period formula is based on measured response of <u>buildings</u>, it should not be used to estimate the period for other types of structure (bridges, dams, towers).

Periods of Vibration of Common Structures

20-story moment resisting frame	T = 1.9 sec
10-story moment resisting frame	T = 1.1 sec
1-story moment resisting frame	T = 0.15 sec
20-story braced frame	T = 1.3 sec
10-story braced frame	T = 0.8 sec
1-story braced frame	T = 0.1 sec
Gravity dam	T = 0.2 sec
Suspension bridge	T = 20 sec

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This slide shows typical periods of vibration for several simple structures. Engineers should develop a “feel” for what an appropriate period of vibration is for simple building structures.

For building structures, the formula *T* = 0.1 in is the simplest “reality check.” The period for a 10-story building should be approximately 1 sec. If a computer analysis gives a period of 0.2 sec or 3.0 sec for a 10-story building, something is probably amiss in the analysis.

Adjustment Factor on Approximate Period
(Table 12.8-1 of ASCE 7-05)

$$T = T_a C_u \le T_{computed}$$

S_{D1}	C_u
> 0.40g	1.4
0.30g	1.4
0.20g	1.5
0.15g	1.6
< 0.1g	1.7

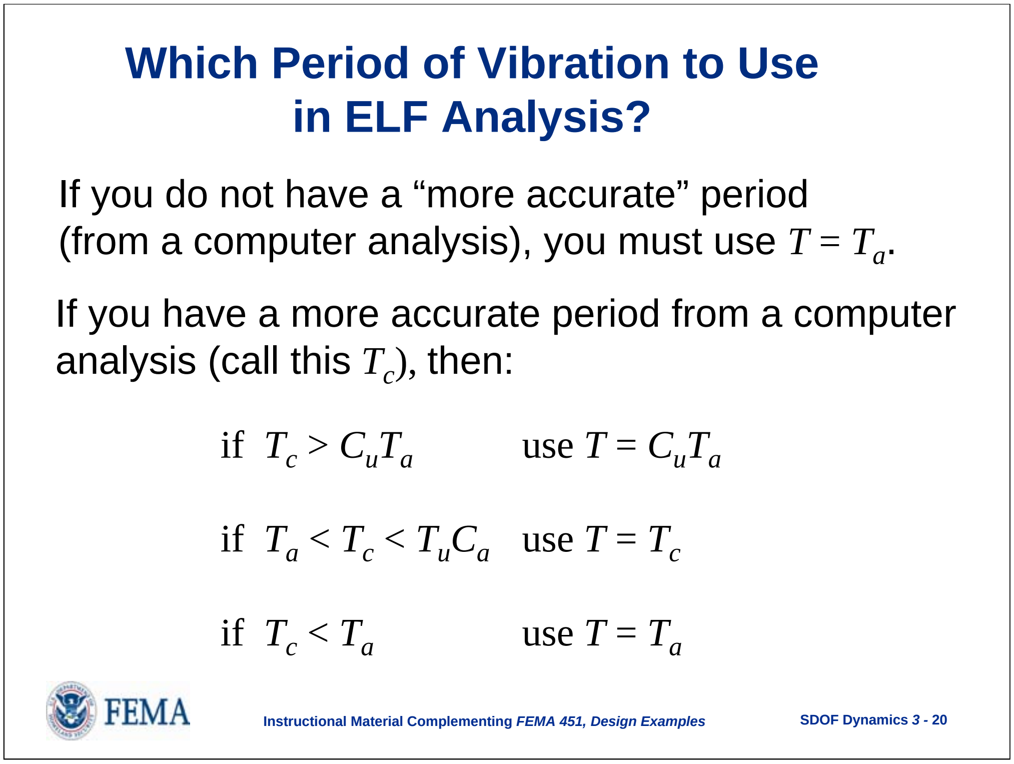
Applicable **ONLY** if $T_{computed}$ comes from a "properly substantiated analysis."

In some cases, it is appropriate to remove the "conservatism" from the empirical period formulas. This is done through use of the C_u coefficient. This conservatism arises from two sources:

1. The lower bound period was used in the development of the period formula.
2. This lower bound period is about 1/1.4 times the best-fit period.

The empirical formula was developed on the basis of data from California buildings. Buildings in other parts of the country (e.g., Chicago) where seismic forces are not so high will likely be larger than those for the same building in California.

It is important to note that the larger period cannot be used without the benefit of a "properly substantiated" analysis, which is likely performed on a computer.



This slide shows the limitations on the use of $C_u T_a$. ASCE-7-05 will not allow the use of a period larger than $C_u T_a$ regardless of what the computer analysis says. Similarly, the *NEHRP Recommended Provisions* does not require that you use a period less than T_a.

Damped Free Vibration

Equation of motion: $m\ddot{u}(t) + c\dot{u}(t) + k\,u(t) = 0$

Initial conditions: $u_0 \quad \dot{u}_0$

Assume: $u(t) = e^{st}$

Solution:

$$u(t) = e^{-\xi\omega t}\left[u_0 \cos(\omega_D t) + \frac{\dot{u}_0 + \xi\omega u_0}{\omega_D}\sin(\omega_D t)\right]$$

$$\xi = \frac{c}{2m\omega} = \frac{c}{c_c}$$

$$\omega_D = \omega\sqrt{1-\xi^2}$$

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This slide shows the equation of motion and the response in *damped* free vibration. Note the similarity with the undamped solution. In particular, note the exponential decay term that serves as a multiplier on the whole response.

Critical damping (c_c) is defined as the amount of damping that will produce no oscillation. See next slide.

The *damped circular frequency* is computed as shown. Note that in many practical cases (x < 0.10), it will be effectively the same as the undamped frequency. The exception is very highly damped systems.

Note that the damping ratio is often given in terms of % critical.

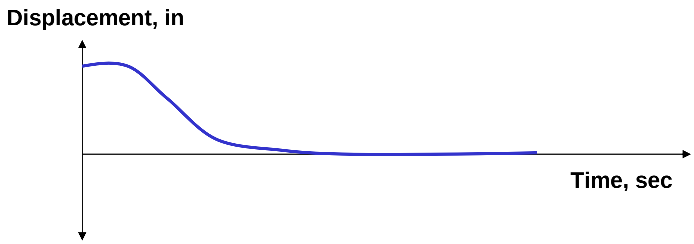
Damping in Structures

$$\xi = \frac{c}{2m\omega} = \frac{c}{c_c}$$

c_c is the c*ritical damping constant.*

ξ is expressed as a ratio (0.0 < ξ < 1.0) in computations.

Sometimes ξ is expressed as a% (0 < ξ < 100%).



Response of Critically Damped System, ξ=1.0 or 100% critical

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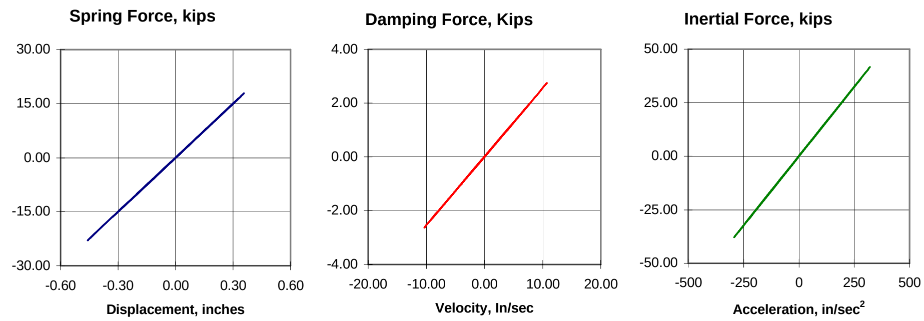
The concept of critical damping is defined here. A good example of a critically damped response can be found in heavy doors that are fitted with dampers to keep the door from slamming when closing.

Damping in Structures

True damping in structures is NOT viscous. However, for low damping values, viscous damping allows for linear equations and vastly simplifies the solution.



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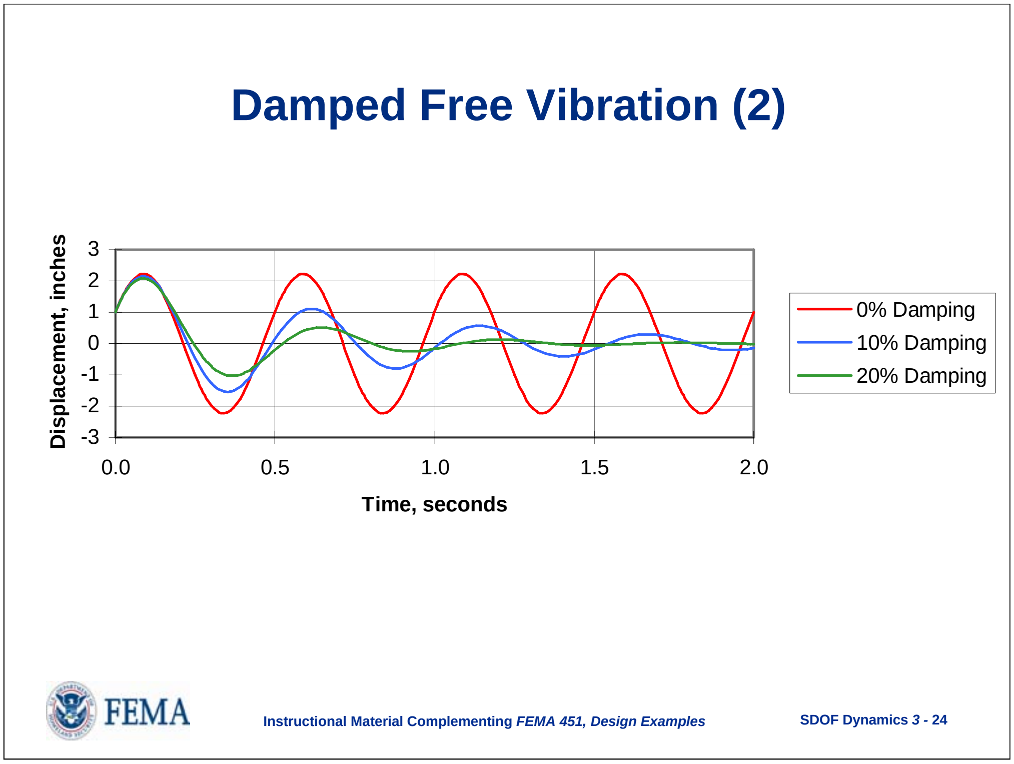
An earlier slide is repeated here to emphasize that damping in real structures is NOT viscous. It is frictional or hysteretic. Viscous damping is used simply because it linearizes the equations of motion. Use of viscous damping is acceptable for the modeling of inherent damping but should be used with <u>extreme caution</u> when representing added damping or energy loss associated with yielding in the primary structural system.



This slide shows some simple damped free vibration responses. When the damping is zero, the vibration goes on forever. When the damping is 20% critical, very few cycles are required for the free vibration to be effectively damped out. For 10% damping, peak is approximately ½ of the amplitude of the previous peak.

Damping in Structures (2)

Welded steel frame	$\xi = 0.010$
Bolted steel frame	$\xi = 0.020$
Uncracked prestressed concrete	$\xi = 0.015$
Uncracked reinforced concrete	$\xi = 0.020$
Cracked reinforced concrete	$\xi = 0.035$
Glued plywood shear wall	$\xi = 0.100$
Nailed plywood shear wall	$\xi = 0.150$
Damaged steel structure	$\xi = 0.050$
Damaged concrete structure	$\xi = 0.075$
Structure with added damping	$\xi = 0.250$

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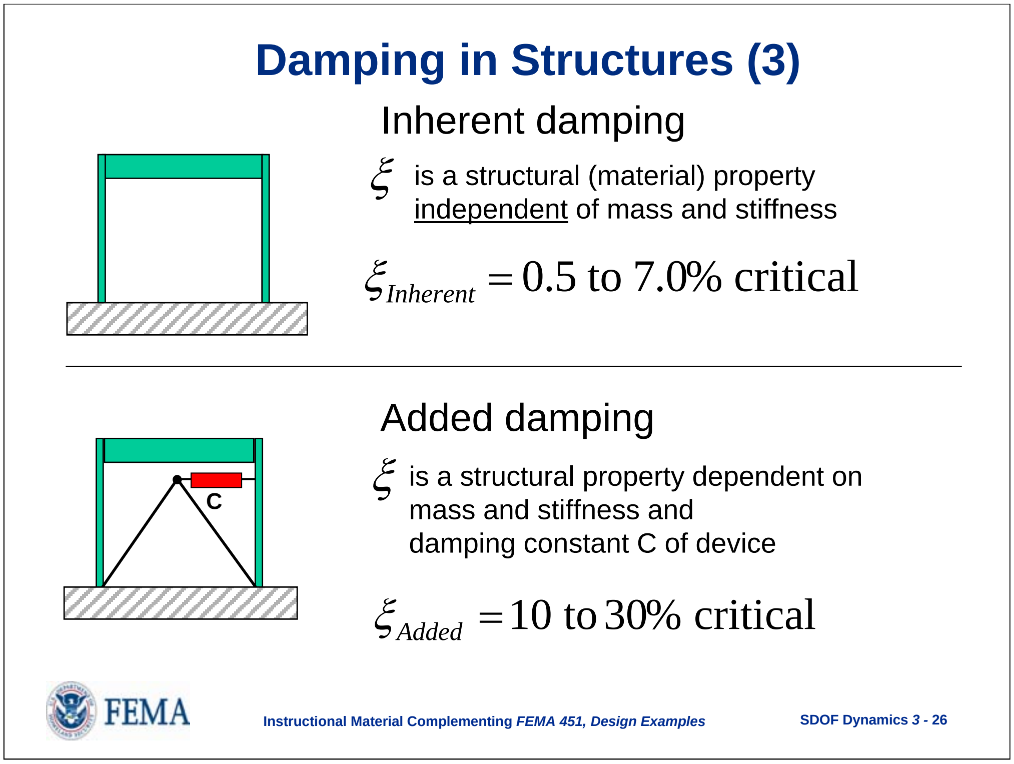
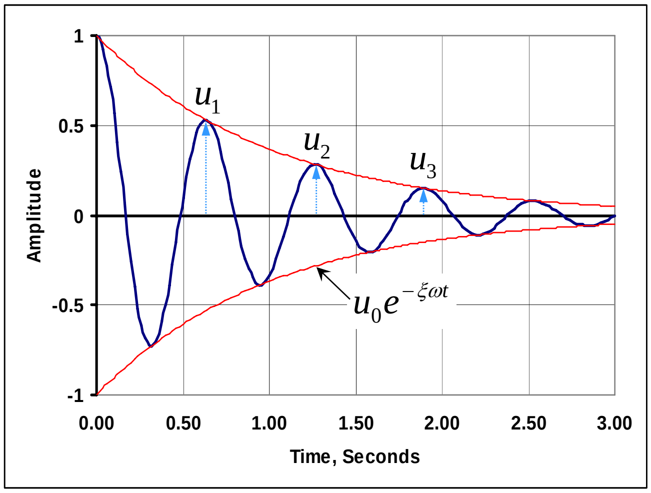
Some realistic damping values are listed for structures comprised of different materials. The values for undamaged steel and concrete (upper five lines of table) may be considered as working stress values.



The distinction between *inherent damping* and *added damping* should be clearly understood.

Measuring Damping from Free Vibration Test



For all damping values

$$\ln \frac{u_1}{u_2} = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

For <u>very low</u> damping values

$$\xi \cong \frac{u_1 - u_2}{2\pi u_2}$$

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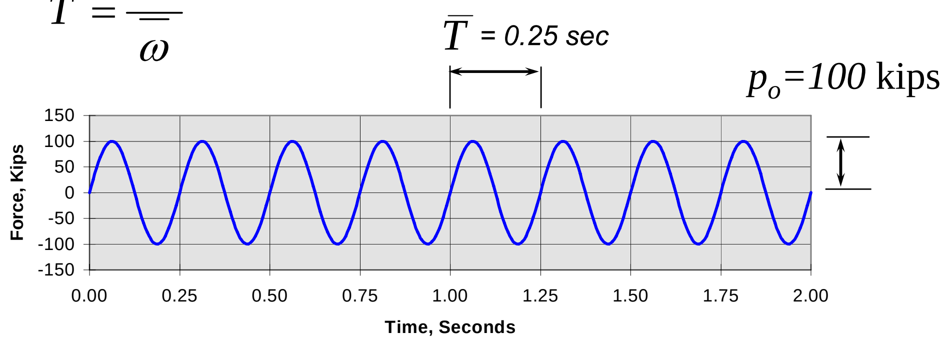
One of the simplest methods to measure damping is a free vibration test. The structure is subjected to an initial displacement and is suddenly released. Damping is determined from the formulas given. The second formula should be used only when the damping is expect to be less than about 10% critical.

Undamped Harmonic Loading

Equation of motion: $m\ddot{u}(t) + ku(t) = p_0 \sin(\bar{\omega} t)$

$\bar{\omega}$ = **frequency of the forcing function**

$$\bar{T} = \frac{2\pi}{\bar{\omega}}$$



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The next series of slides covers the response of undamped SDOF systems to simple harmonic loading. Note that the loading frequency is given by the omega term with the overbar. The loading period is designated in a similar fashion.

Undamped Harmonic Loading (2)

Equation of motion: $m\,\ddot{u}(t) + k\,u(t) = p_0 \sin(\overline{\omega} t)$

Assume system is initially at rest:

Particular solution: $u(t) = C \sin(\overline{\omega} t)$

Complimentary solution: $u(t) = A\sin(\omega t) + B\cos(\omega t)$

Solution:

$$u(t) = \frac{p_0}{k} \frac{1}{1-(\overline{\omega}/\omega)^2} \left(\sin(\overline{\omega} t) - \frac{\overline{\omega}}{\omega} \sin(\omega t) \right)$$

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This slide sets up the equation of motion for undamped harmonic loading and gives the solution. We have assumed the system is initially at rest.

Undamped Harmonic Loading

Define $$\beta = \frac{\overline{\omega}}{\omega}$$

Loading frequency ($\overline{\omega}$)

Structure's natural frequency (ω)

$$u(t) = \frac{p_0}{k} \frac{1}{1-\beta^2}\left(\sin(\overline{\omega}t) - \beta \sin(\omega t)\right)$$

Static displacement: $\frac{p_0}{k}$

Dynamic magnifier: $\frac{1}{1-\beta^2}$

Steady state response (at loading frequency): $\sin(\overline{\omega}t)$

Transient response (at structure's frequency): $\beta \sin(\omega t)$

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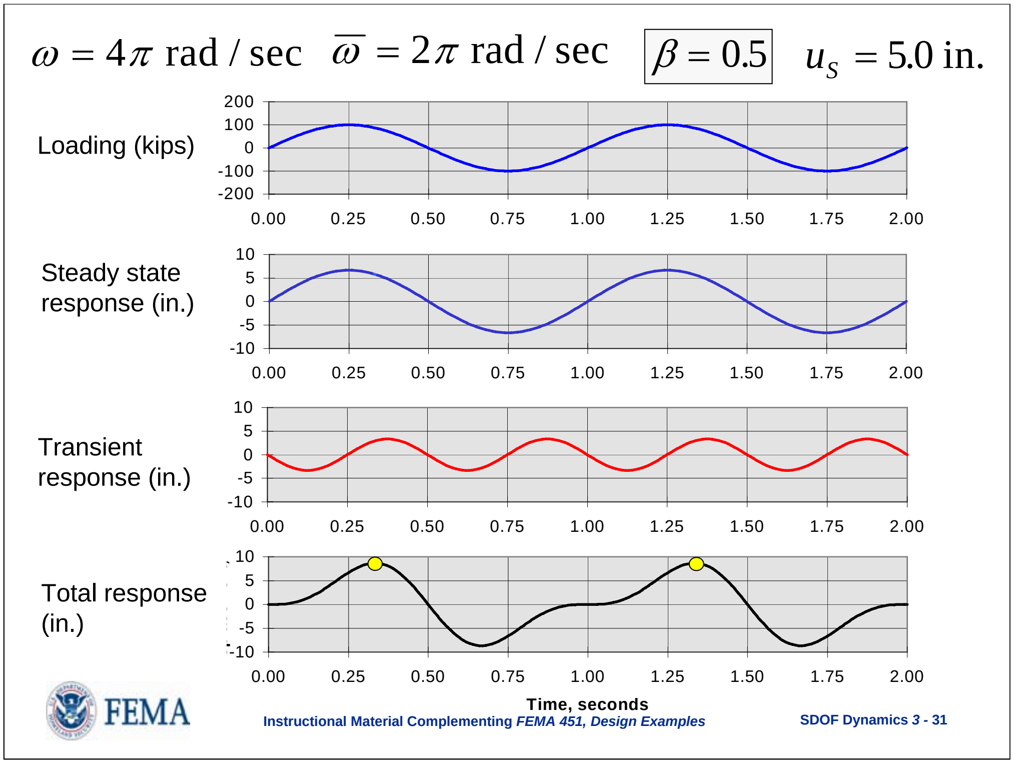
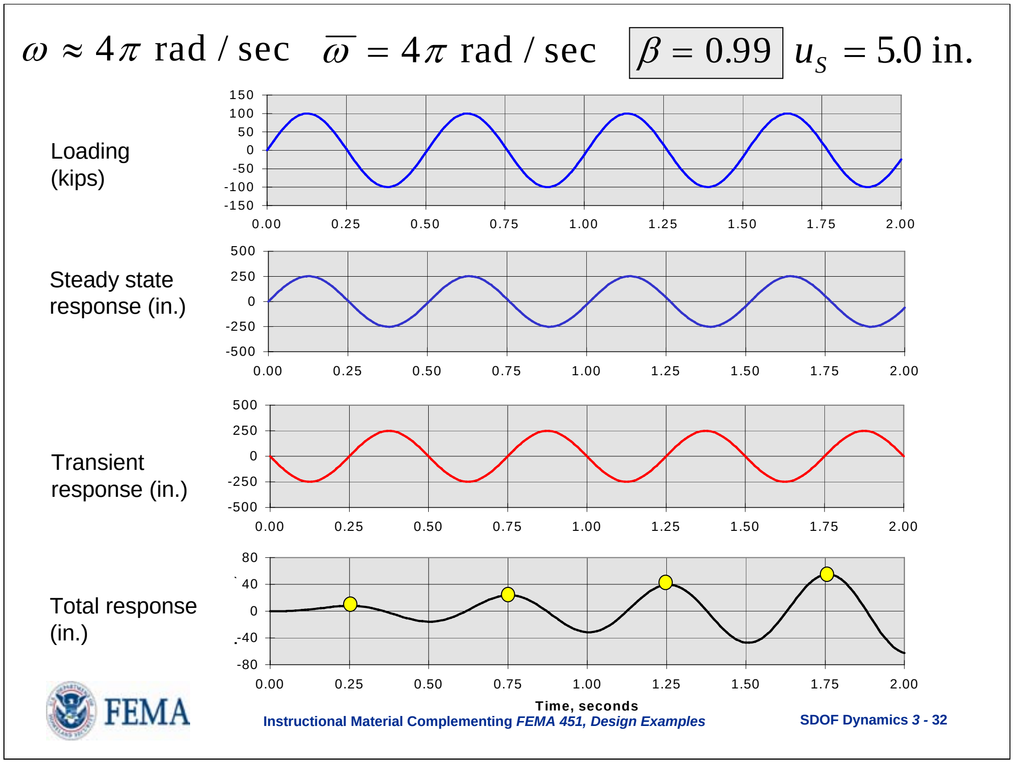
Here we break up the response into the steady state response (at the frequency of loading) and the transient response (at the structure's own natural frequency). Note that the term p_o/k is the "static" displacement. The *dynamic magnifier* shows how the dynamic effects may increase (or decrease) the response. This magnifier is a function of the frequency ratio *β*. Note that the magnifier goes to infinity if the frequency ratio *β* is 1.0. This defines the resonant condition.

In other words, the response is equal to the static response, times a multiplier, times the sum of two sine waves, one in phase with the load and the other in phase with the structure's undamped natural frequency.



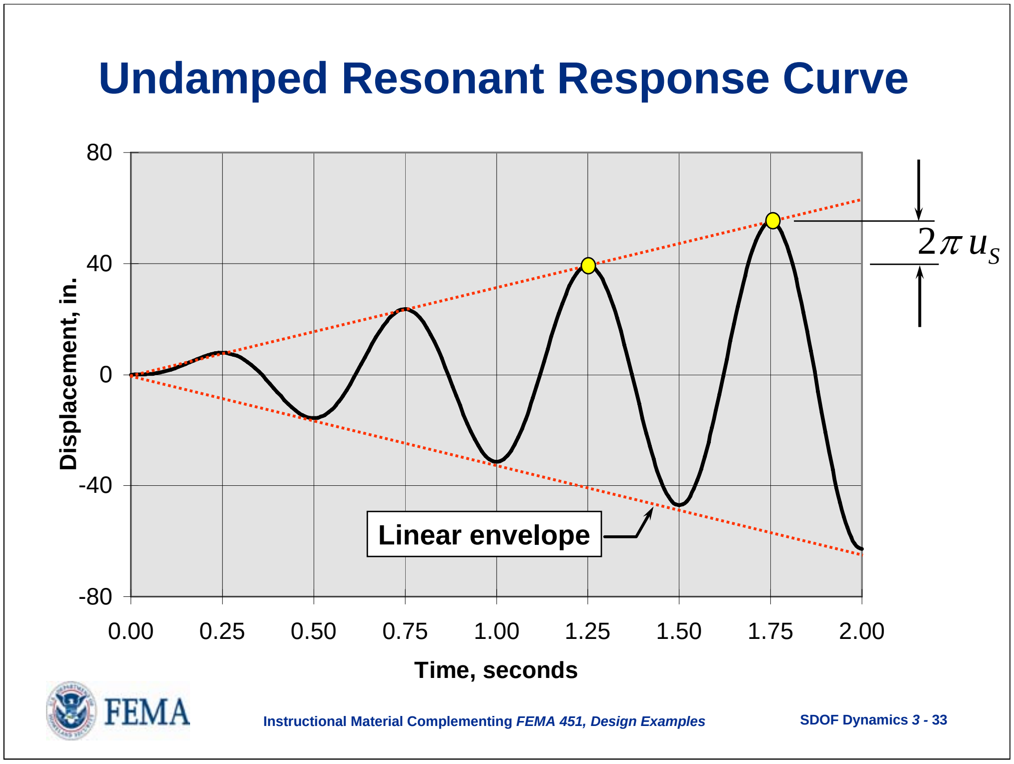
This is a time-history response of a structure with a natural frequency of 4 rad/sec (*f* = 2 Hz, *T* = 0.5 sec), and a loading frequency of 2 rad/sec (*f* = 1 Hz, *T* = 1 sec), giving a frequency ratio β of 0.5. The harmonic load amplitude is 100 kips. The static displacement is 5.0 inches. Note how the steady state response is at the frequency of loading, is in phase with the loading, and has an amplitude greater than the static displacement. The transient response is at the structure's own frequency. In real structures, damping would cause this component to disappear after a few cycles of vibration.



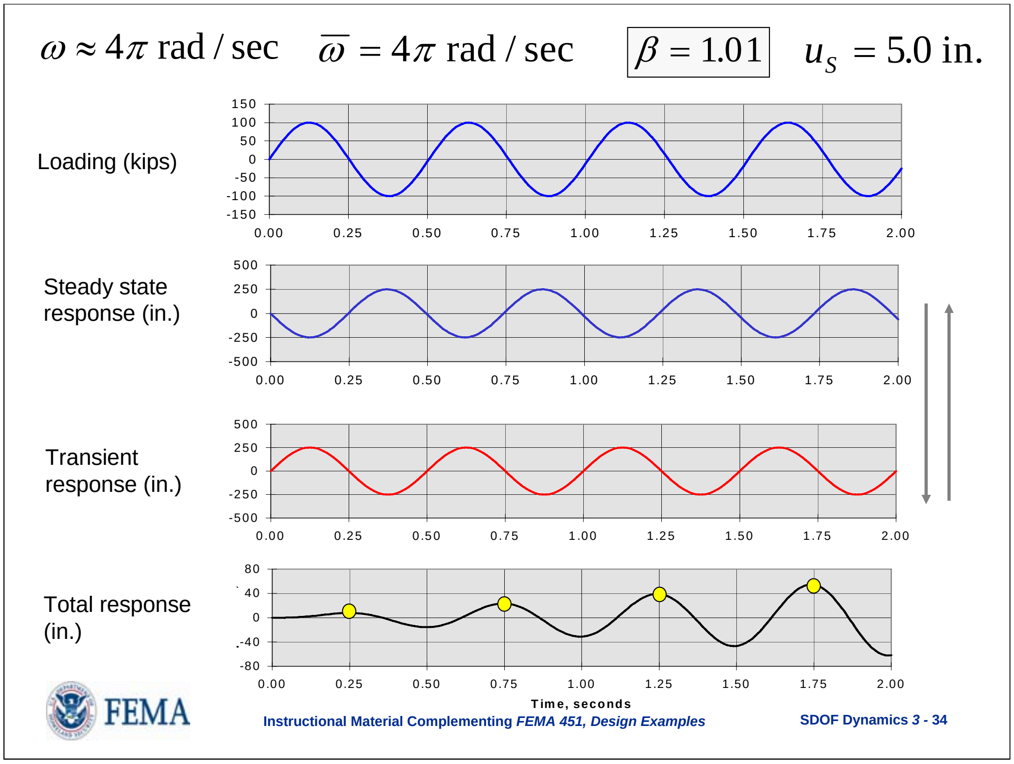
In this slide, ϖ has been increased to 4π rad/sec, and the structure is almost at resonance. The steady state response is still in phase with the loading, but note the huge magnification in response. The transient response is practically equal to and opposite the steady state response. The total response increases with time.

If one looks casually at the steady state and transient response curves, it appears that they should cancel out. Note, however, that the two responses are not exactly in phase due to the slight difference in the loading and natural frequencies. This can be seen most clearly at the time 1.75 sec into the response. The steady state response crosses the horizontal axis to the right of the vertical 1.75 sec line while the transient response crosses exactly at 1.75 sec.

In real structures, the observed increased amplitude could occur only to some limit and then yielding would occur. This yielding would introduce hysteretic energy dissipation (apparent damping), causing the transient response to disappear and leading to a constant, damped, steady state response.

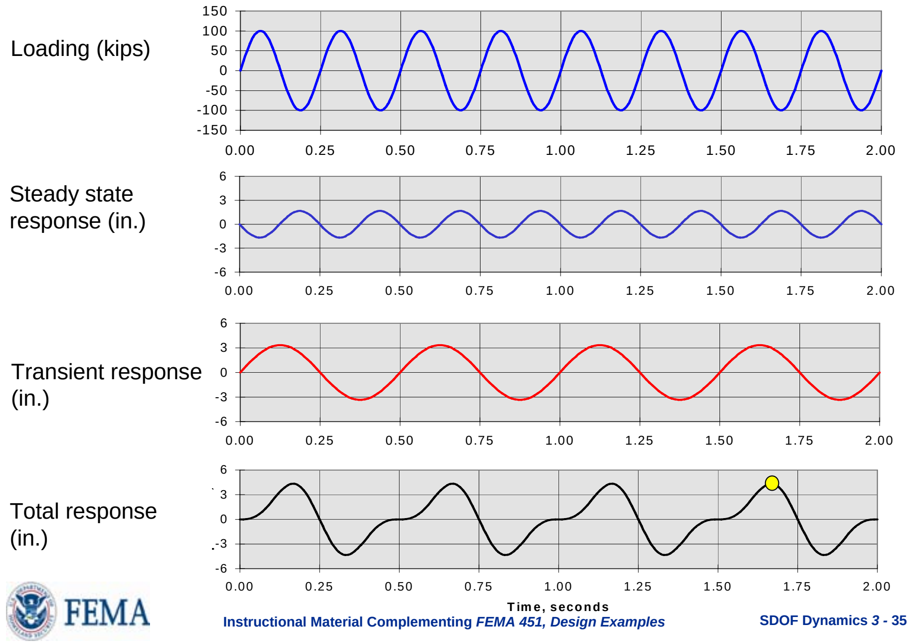


This is an enlarged view of the total response curve from the previous slide. Note that the response is bounded within a linear increasing envelope with the increase in displacement per cycle being 2π times the static displacement.

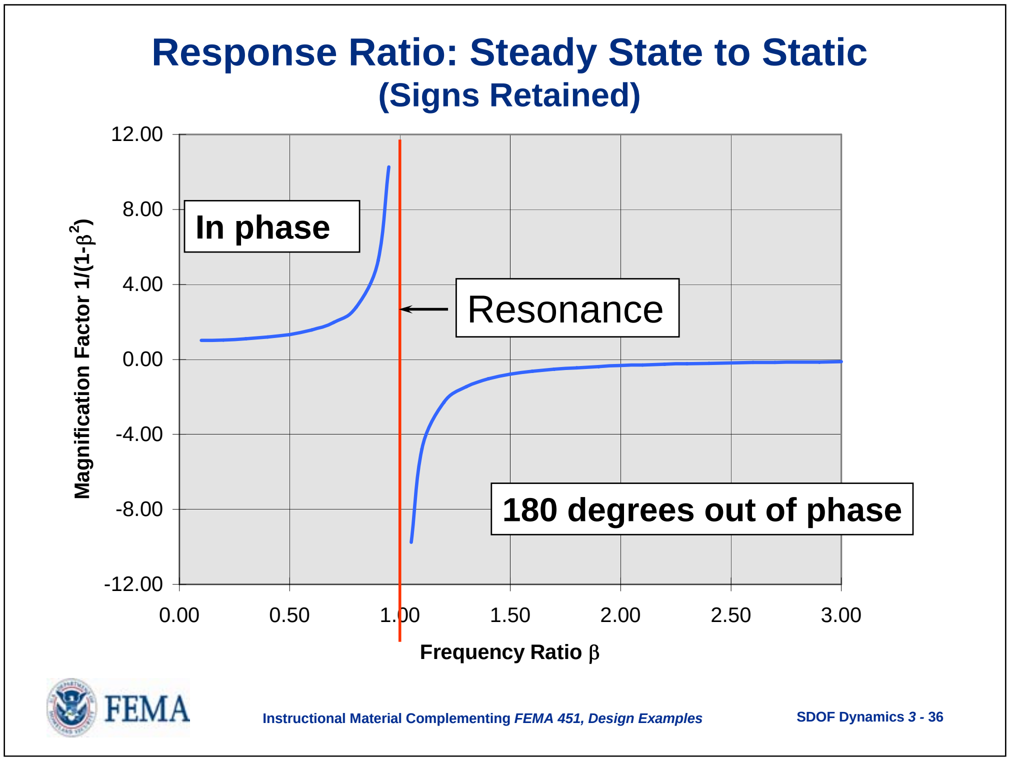


In this slide, the loading frequency has been slightly increased, but the structure is still nearly at resonance. Note, however, that the steady state response is 180 degrees out of phase with the loading and the transient response is in phase. The resulting total displacement is effectively identical to that shown two slides back.

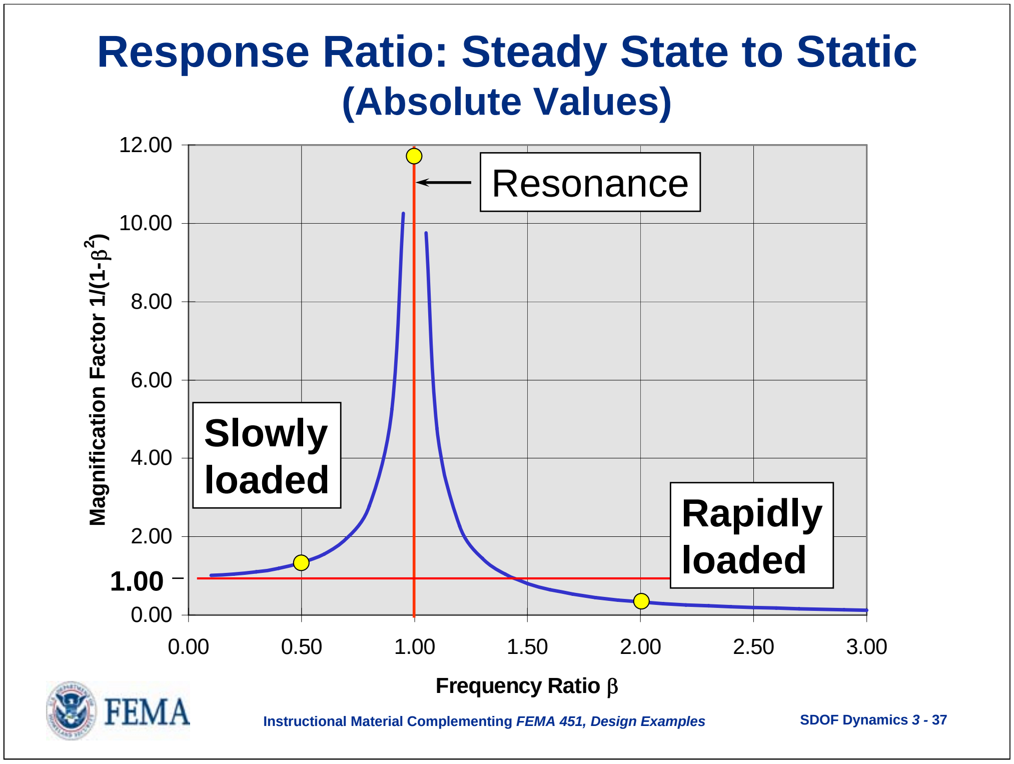
$\omega = 4\pi \text{ rad/sec} \quad \overline{\omega} = 8\pi \text{ rad/sec} \quad \boxed{\beta = 2.0} \quad u_S = 5.0 \text{ in.}$



The loading frequency is now twice the structure's frequency. The important point here is that the steady state response amplitude is now less than the static displacement.



This plot shows the ratio of the steady state response to the static displacement for the structure loaded at different frequencies. At low loading frequencies, the ratio is 1.0, indicating a nearly static response (as expected). At very high frequency loading, the structure effectively does not have time to respond to the loading so the displacement is small and approaches zero at very high frequency. The resonance phenomena is very clearly shown. The change in sign at resonance is associated with the in-phase/out-of-phase behavior that occurs through resonance.



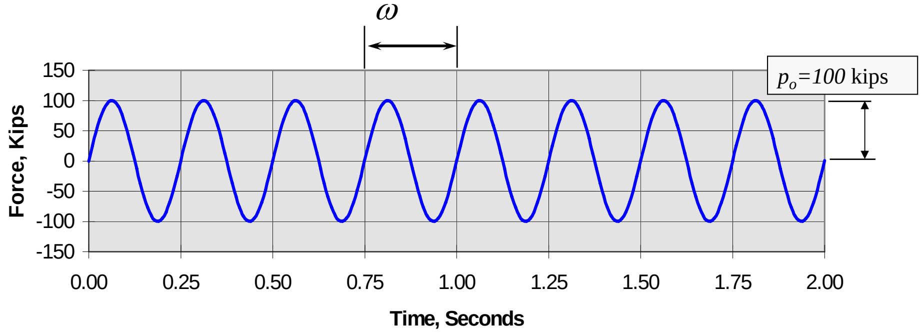
This is the same as the previous slide but absolute values are plotted. This clearly shows the resonance phenomena.

Damped Harmonic Loading

Equation of motion:

$$m\ddot{u}(t) + \boxed{c\dot{u}(t)} + k\,u(t) = p_0 \sin(\overline{\omega} t)$$

$$\overline{T} = \frac{2\pi}{\overline{\omega}} = 0.25\,\text{sec}$$



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We now introduce damping into the behavior. Note the addition of the appropriate term in the equation of motion.

Damped Harmonic Loading

Equation of motion:

$$m\ddot{u}(t) + c\dot{u}(t) + k\,u(t) = p_0 \sin(\overline{\omega} t)$$

Assume system is initially at rest

Particular solution: $u(t) = C\sin(\overline{\omega}t) + D\cos(\overline{\omega}t)$

Complimentary solution:

$$u(t) = e^{-\xi\omega t}\left[A\sin(\omega_D t) + B\cos(\omega_D t)\right]$$

$$\xi = \frac{c}{2m\omega}$$

Solution:

$$u(t) = e^{-\xi\omega t}\left[A\sin(\omega_D t) + B\cos(\omega_D t)\right] + C\sin(\overline{\omega}t) + D\cos(\overline{\omega}t)$$

$$\omega_D = \omega\sqrt{1-\xi^2}$$

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This slide shows how the solution to the differential equation is obtained. The transient response (as indicated by the *A* and *B* coefficients) will damp out and is excluded from further discussion.

Damped Harmonic Loading

Transient response at structure's frequency (eventually damps out)

$$u(t) = \boxed{e^{-\xi\omega t}\left[A\sin(\omega_D t) + B\cos(\omega_D t)\right]} +$$

$$\boxed{C\sin(\overline{\omega}t) + D\cos(\overline{\omega}t)}$$

Steady state response, at loading frequency

$$C = \frac{p_o}{k}\frac{1-\beta^2}{(1-\beta^2)^2 + (2\xi\beta)^2} \qquad D = \frac{p_o}{k}\frac{-2\xi\beta}{(1-\beta^2)^2 + (2\xi\beta)^2}$$

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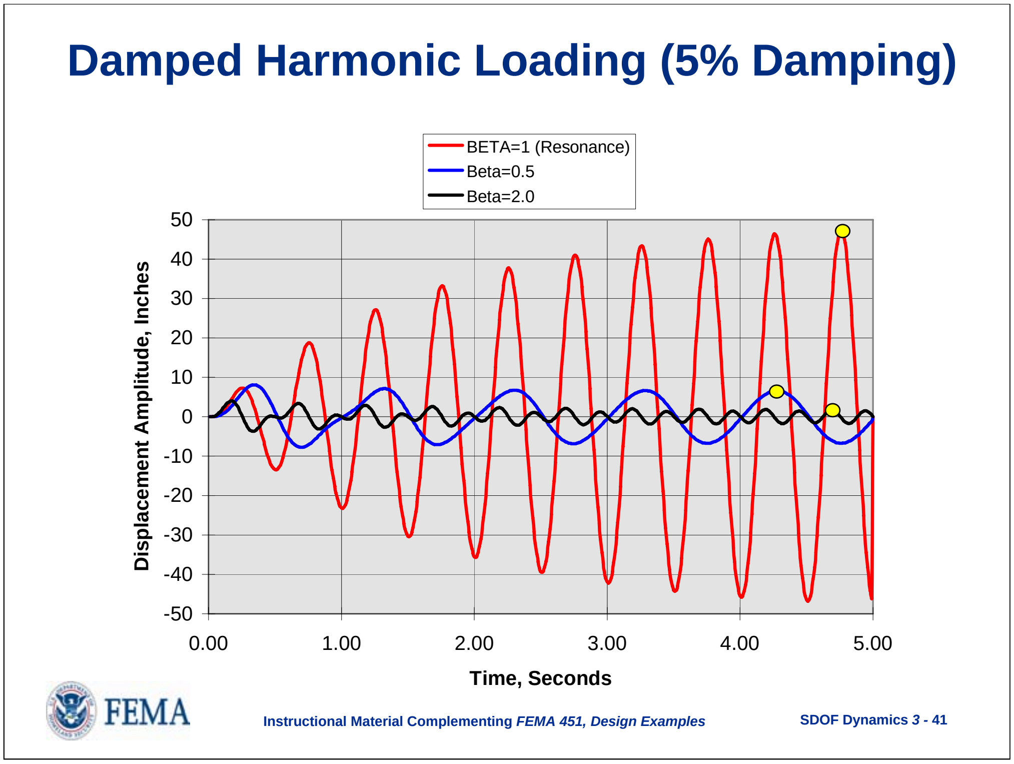
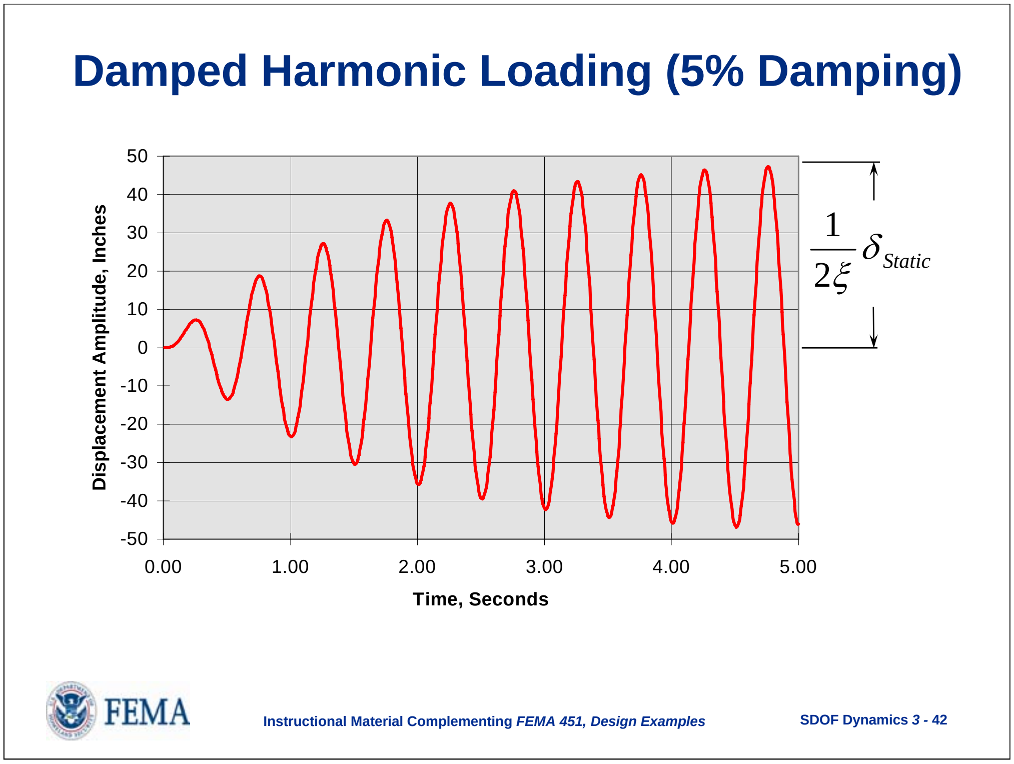
This slide shows the *C* and *D* coefficients of the steady state response. Note that there is a component in phase with the loading (the sine term) and a component out of phase with the loading (the cosine term). The actual phase difference between the loading and the response depends on the damping and frequency ratios.

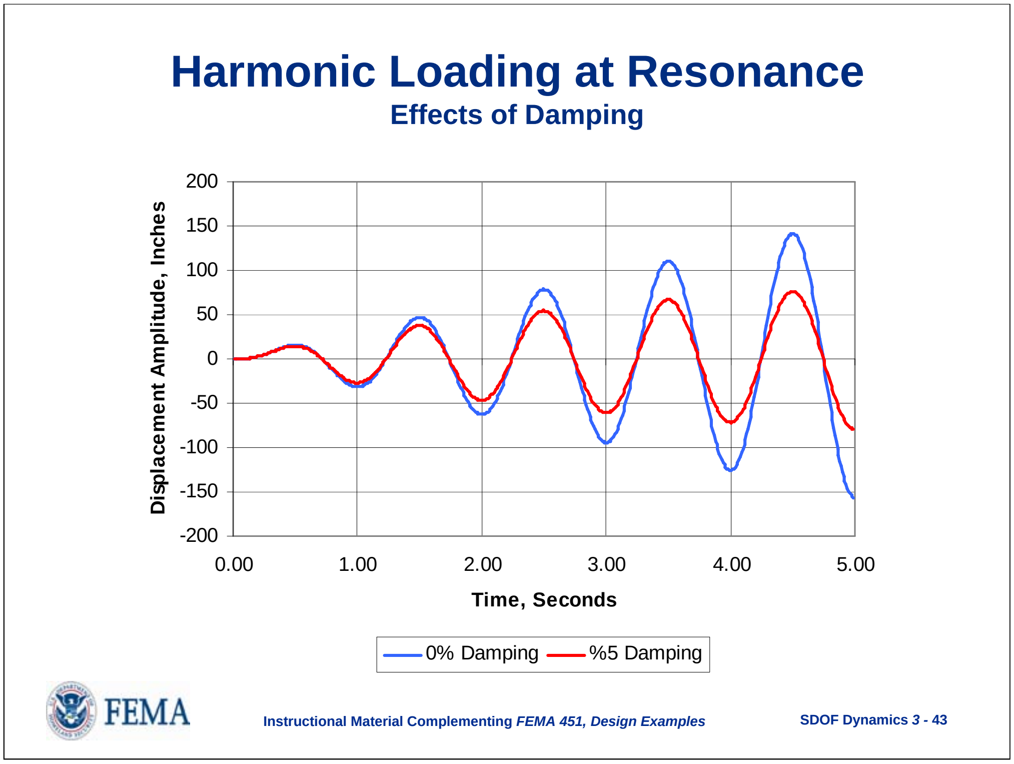
Note the exponential decay term causes the transient response to damp out in time.



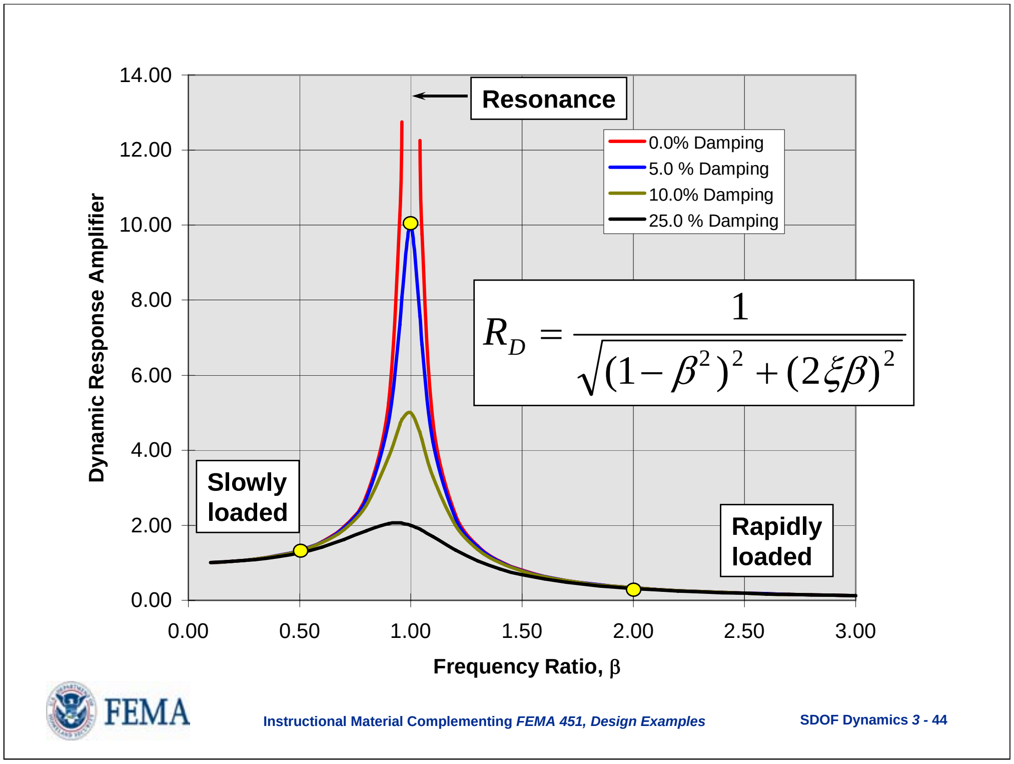
This plot shows the response of a structure at three different loading frequencies. Of significant interest is the resonant response, which is now limited. (The undamped response increases indefinitely.)



For viscously damped structures, the resonance amplitude will always be limited as shown.



A comparison of damped and undamped responses is shown here. The undamped response has a linear increasing envelope; the damped curve will reach a constant steady state response after a few cycles.



This plot shows the dynamic magnification for various damping ratios. For increased damping, the resonant response decreases significantly. Note that for slowly loaded structures, the dynamic amplification is 1.0 (effectively static). For high frequency loading, the magnifier is zero.

Note also that damping is most effective at or near resonance ($0.5 < \beta < 2.0$).

Summary Regarding Viscous Damping in Harmonically Loaded Systems

- For systems loaded at a frequency near their natural frequency, the dynamic response exceeds the static response. This is referred to as *dynamic amplification*.
- An undamped system, loaded at resonance, will have an unbounded increase in displacement over time.

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A summary of some of the previous points is provided.

Summary Regarding Viscous Damping in Harmonically Loaded Systems

- Damping is an effective means for *dissipating energy* in the system. Unlike strain energy, which is recoverable, dissipated energy is not recoverable.
- A damped system, loaded at resonance, will have a limited displacement over time with the limit being ($1/2\xi$) times the static displacement.
- Damping is most effective for systems loaded at or near resonance.

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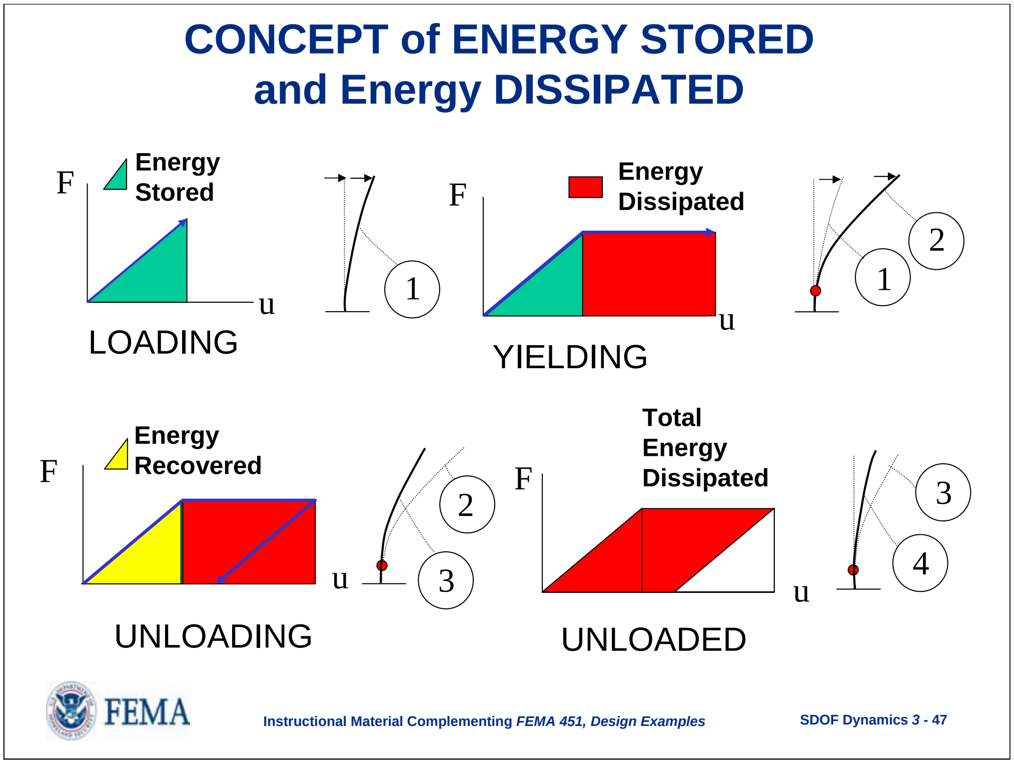
Summary continued.

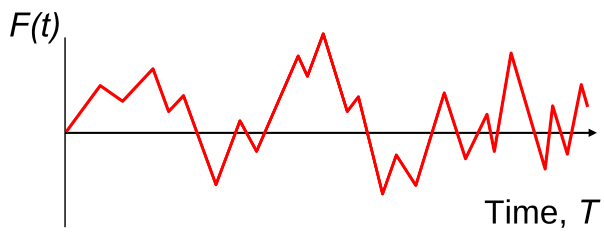


It is very important that the distinction between stored energy and dissipated energy be made clear. (Note that some texts use the term “absorbed” energy in lieu of stored energy.)

In the first diagram, the system remains elastic and all of the strain energy is stored. If the bar were released, all of the energy would be recovered.

In the second diagram, the applied deformation is greater than the elastic deformation and, hence, the system yields. The energy shown in green is stored, but the energy shown in red is dissipated. If the bar is unloaded, the stored energy is recovered, but the dissipated energy is lost. This is shown in Diagrams 3 and 4.

General Dynamic Loading



The discussion will now proceed to general dynamic loading. By general loading, it is meant that no simple mathematical function defines the entire loading history.

General Dynamic Loading Solution Techniques

- Fourier transform
- Duhamel integration
- Piecewise exact
- Newmark techniques

All techniques are carried out numerically.

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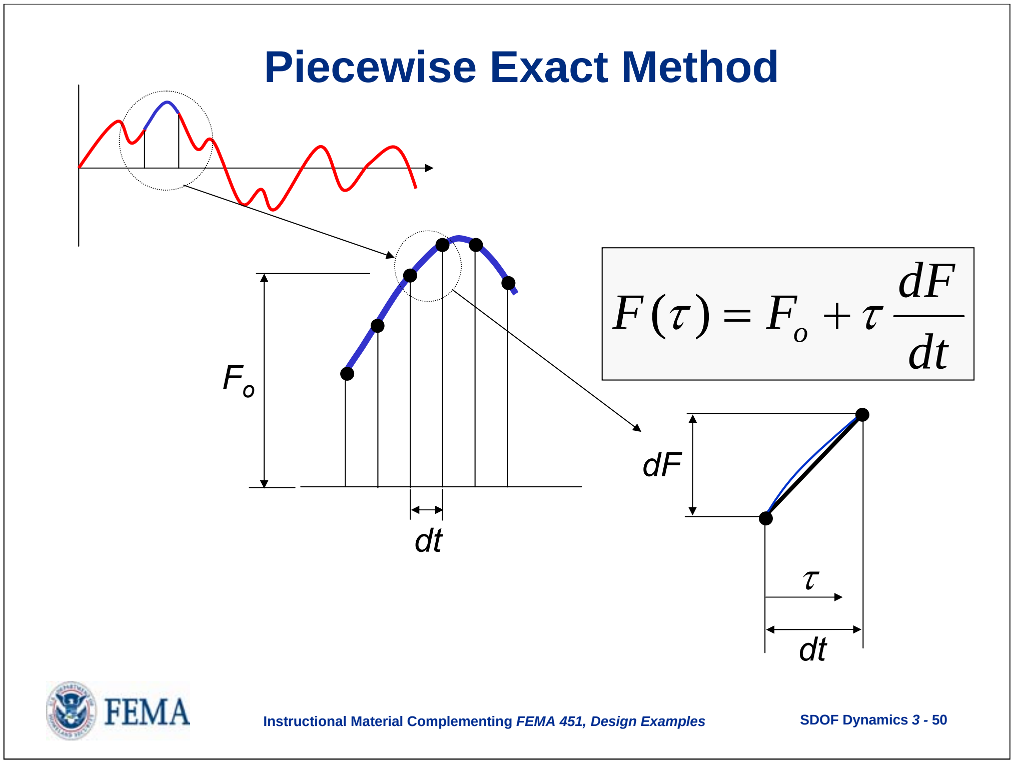
There are a variety of ways to solve the general loading problem and all are carried out numerically on the computer. The Fourier transform and Duhamel integral approaches are not particularly efficient (or easy to explain) and, hence, these are not covered here. Any text on structural dynamics will provide the required details.

The piecewise exact method is used primarily in the analysis of linear systems. The Newmark method is useful for both linear and nonlinear systems. Only the basic principles underlying of each of these approaches are presented.



In the piecewise exact method, the loading function is broken into a number of straight-line segments. In a sense, the name of the method is a misnomer because the method is not exact when the actual loading is smooth (like a sine wave) because the straight line load segments are only an approximation of the actual load. When the actual load is smooth, the accuracy of the method depends on the level of discretization when defining the loading function.

For earthquake loads, the load is almost always represented by a recorded accelerogram, which does consist of straight line segments. (There would be little use in trying to interpolate the ground motion with smooth curves.) Hence, for the earthquake problem, the piecewise exact method is truly exact.

Piecewise Exact Method

Initial conditions $u_{o,0} = 0 \quad \dot{u}_{o,0} = 0$

Determine "exact" solution for 1st time step

$$u_1 = u(\tau) \qquad \dot{u}_1 = \dot{u}(\tau) \qquad \ddot{u}_1 = \ddot{u}(\tau)$$

Establish new initial conditions

$$u_{o,1} = u(\tau) \qquad \dot{u}_{0,1} = \dot{u}(\tau)$$

LOOP

Obtain exact solution for next time step

$$u_2 = u(\tau) \qquad \dot{u}_2 = \dot{u}(\tau) \qquad \ddot{u}_2 = \ddot{u}(\tau)$$

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The basic idea of the piecewise exact method is to develop a solution for a straight line loading segment knowing the initial conditions. Given the initial conditions and the load segment, the solution at the end of the load step is determined and this is then used as the initial condition for the next step of the analysis. The analysis then proceeds step by step until all load segments have been processed.

Piecewise Exact Method

Advantages:

- Exact if load increment is linear
- Very computationally efficient

Disadvantages:

- Not generally applicable for inelastic behavior

Note: NONLIN uses the piecewise exact method for response spectrum calculations.

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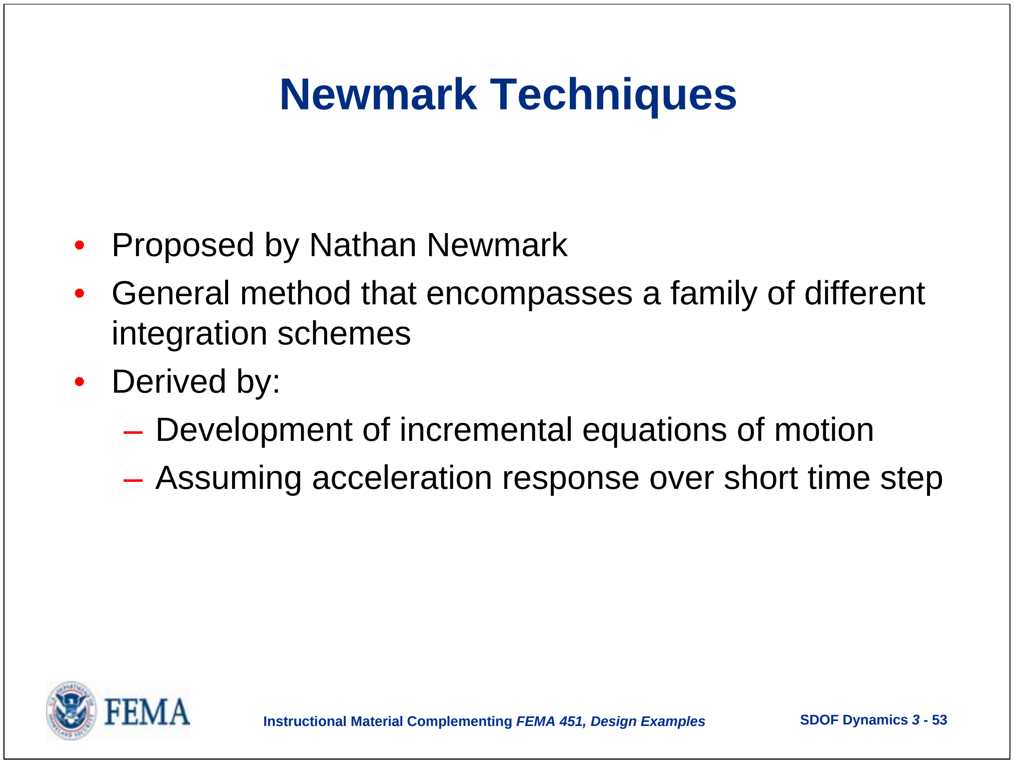
It should be noted that the piecewise exact method may be used for nonlinear analysis in certain circumstances. For example, the "fast nonlinear analysis" (FNA) method developed by Ed Wilson and used in SAP 2000 utilizes the piecewise exact method. In FNA, the nonlinearities are "right-hand sided," leaving only linear terms in the left-hand side of the equations of motion.



The Newmark method is one of the most popular methods for solving the general dynamic loading problem. It is applicable to both linear and nonlinear systems. It is equally applicable to both SDOF and MDOF systems.

The Newmark method is described in more detail in the topic on inelastic behavior of structures.

Newmark Method

Advantages:

- Works for inelastic response

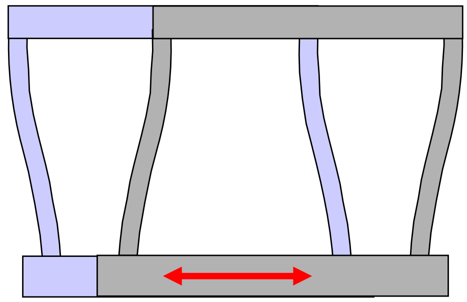
Disadvantages:

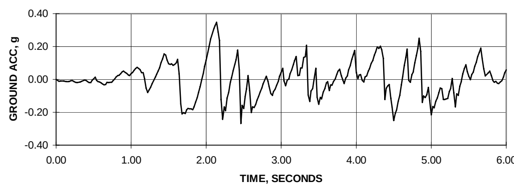
- Potential numerical error

Note: NONLIN uses the Newmark method for general response history calculations

The advantages and disadvantages of the Newmark method are listed. The principal advantage is that the method may be applied to inelastic systems. The method also may be used (without decoupling) for multiple-degree-of-freedom systems.

Development of Effective Earthquake Force





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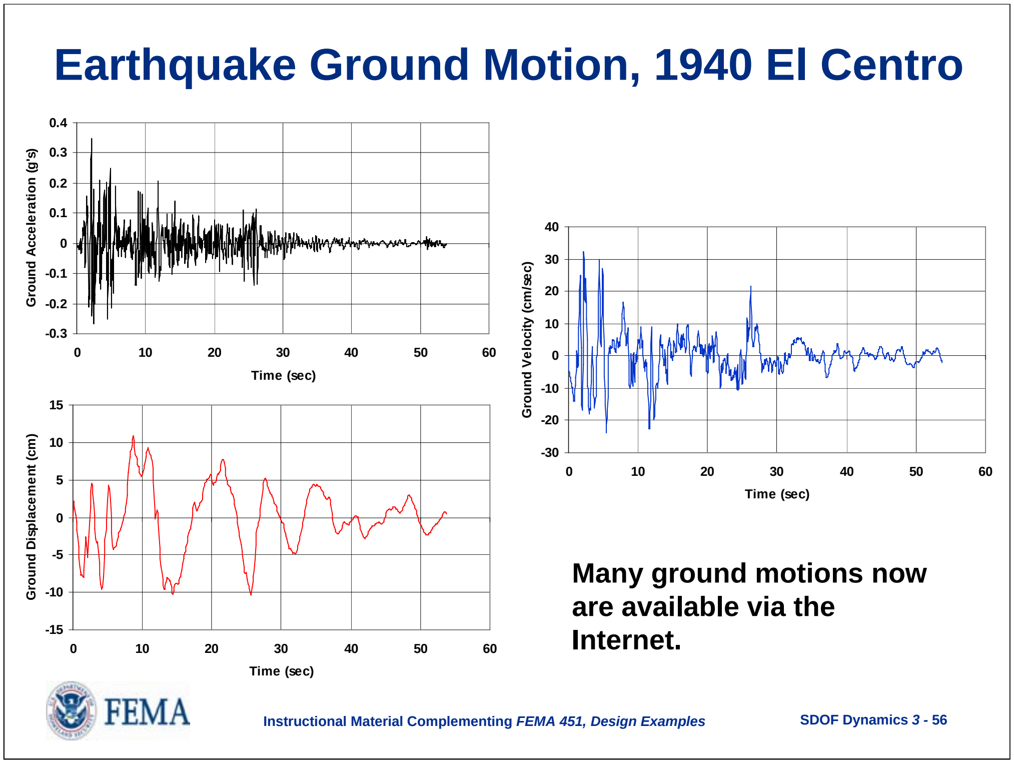
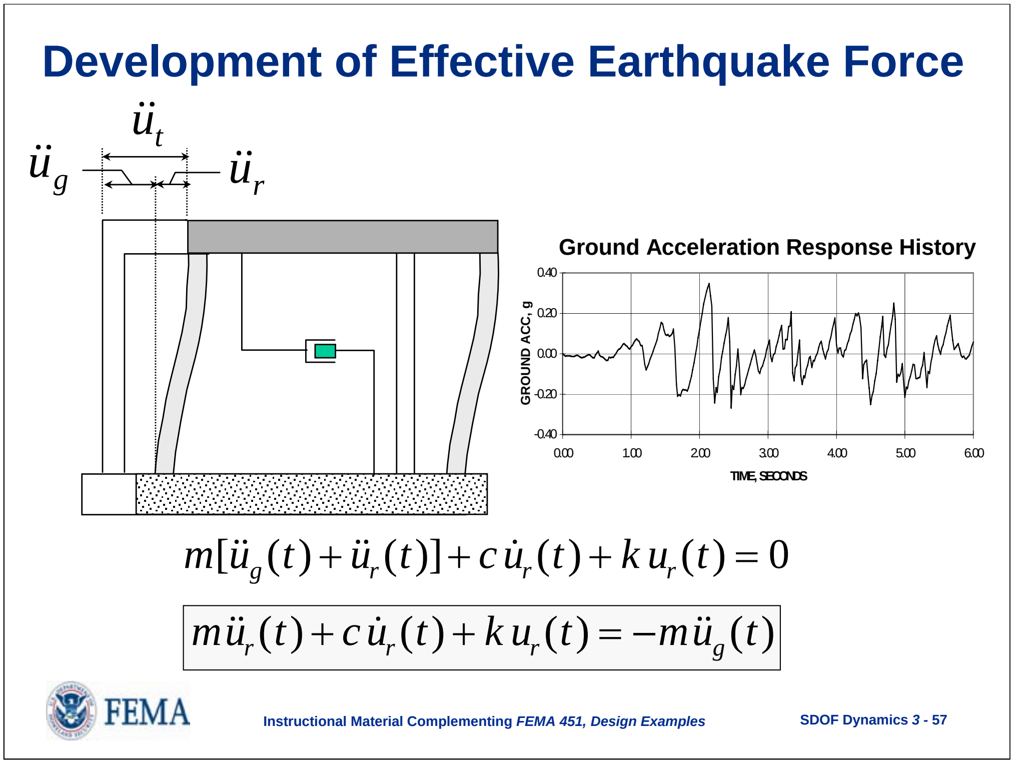
In an earthquake, no actual force is applied to the building. Instead, the ground moves back and forth (and up and down) and this movement induces inertial forces that then deform the structure. It is the displacements in the structure, relative to the moving base, that impose deformations on the structure. Through the elastic properties, these deformations cause elastic forces to develop in the individual members and connections.



Earthquake ground motions usually are imposed through the use of the ground acceleration record or *accelerogram*. Some programs (like Abaqus) may require instead that the ground displacement records be used as input.



In this slide, it is assumed that the ground acceleration record is used as input. The total acceleration at the center of mass is equal to the ground acceleration plus the acceleration of the center of mass relative to the moving base. The inertial force developed at the center of mass is equal to the mass times the total acceleration.

The damping force in the system is a function of the velocity of the top of the structure relative to the moving base. Similarly, the spring force is a function of the displacement at the top of the structure relative to the moving base. The equilibrium equation with the zero on the response history spectrum (RHS) represents the state of the system at any point in time. The zero on the RHS reflects the fact that there is no applied load.

If that part of the total inertial force due to the ground acceleration is moved to the right-hand side (the lower equation), all of the forces on the left-hand side are in terms of the relative acceleration, velocity, and displacement. This equation is essentially the same as that for an applied load (see Slide 8) but the “effective earthquake force” is simply the negative of the mass times the ground acceleration. The equation is then solved for the response history of the relative displacement.

“Simplified” form of Equation of Motion:

$$m\ddot{u}_r(t) + c\dot{u}_r(t) + ku_r(t) = -m\ddot{u}_g(t)$$

Divide through by ***m:***

$$\ddot{u}_r(t) + \frac{c}{m}\dot{u}_r(t) + \frac{k}{m}u_r(t) = -\ddot{u}_g(t)$$

Make substitutions:

$$\frac{c}{m} = 2\xi\omega \qquad \frac{k}{m} = \omega^2$$

Simplified form:

$$\boxed{\ddot{u}_r(t) + 2\xi\omega\dot{u}_r(t) + \omega^2 u_r(t) = -\ddot{u}_g(t)}$$

In preparation for the development of response spectra, it is convenient to simplify the equation of motion by dividing through by the mass. When the substitutions are made as indicated, it may be seen that the response is uniquely defined by the damping ratio, the undamped circular frequency of vibration, and the ground acceleration record.

For a given ground motion, the response history $u_r(t)$ is function of the structure's frequency ω and damping ratio ξ.

Structural frequency

$$\ddot{u}_r(t) + 2\xi\omega\dot{u}_r(t) + \omega^2 u_r(t) = -\ddot{u}_g(t)$$

Damping ratio

Ground motion acceleration history

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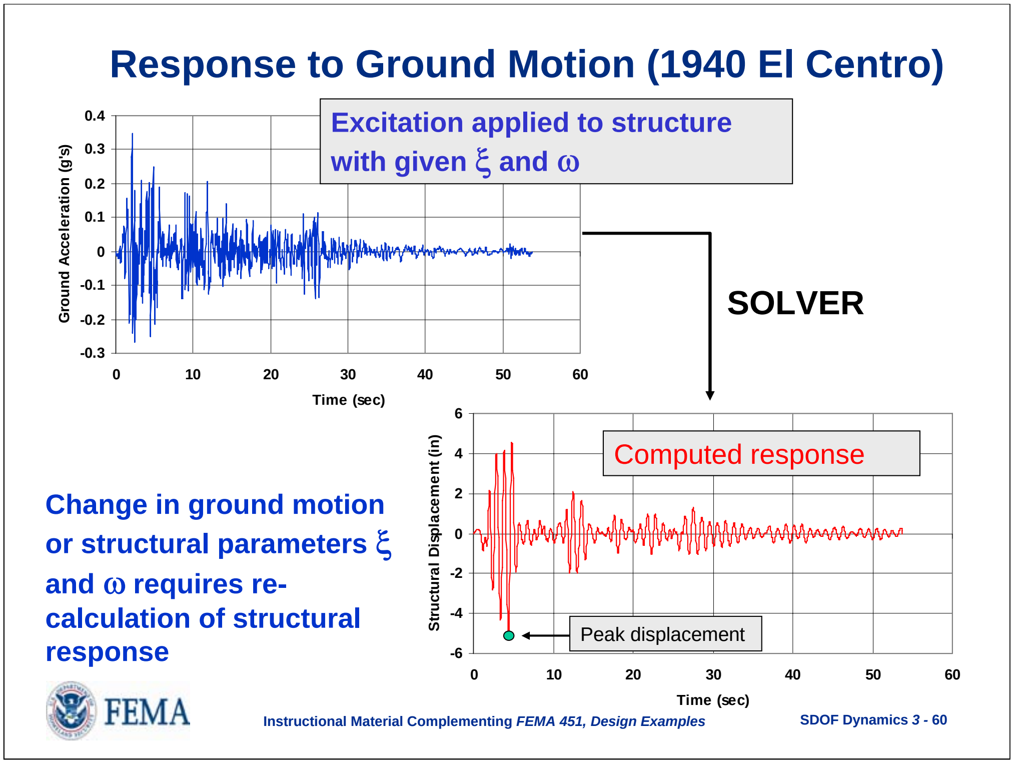
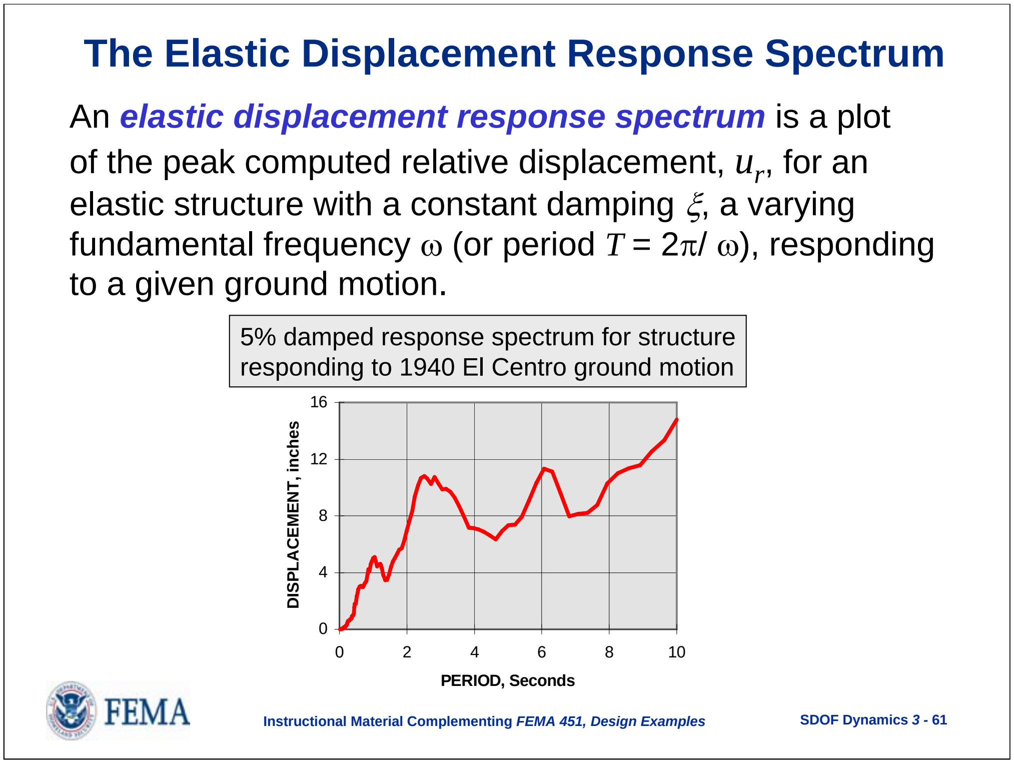
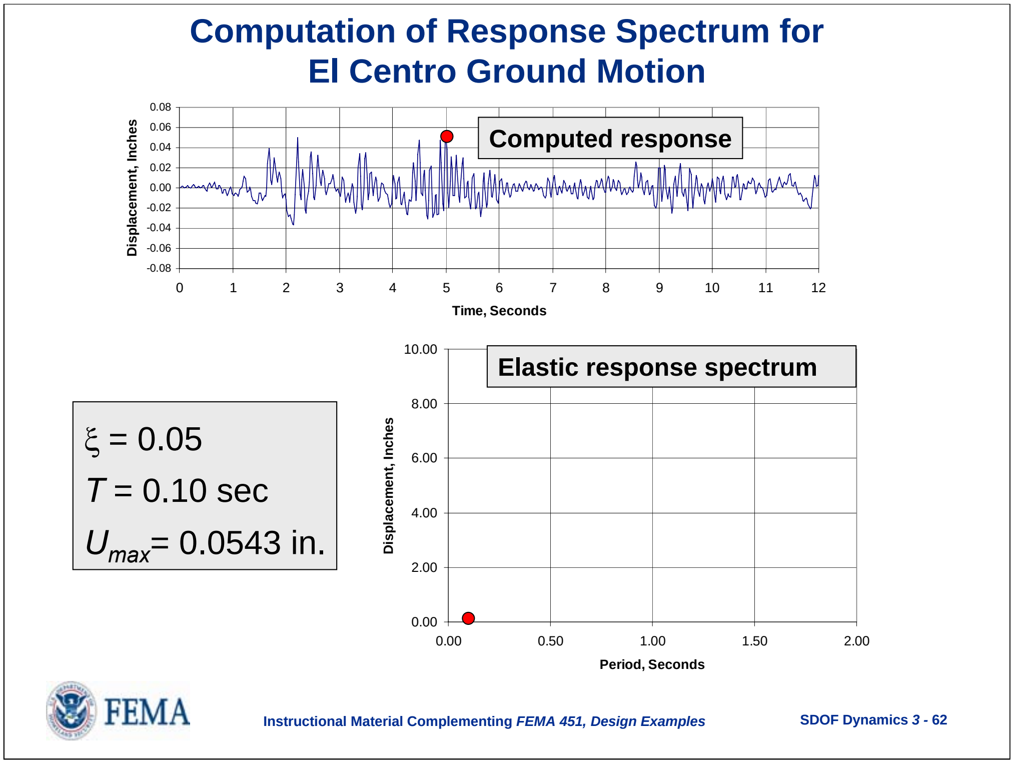
This restates the point made in the previous slide. A response spectrum is created for a particular ground motion and for a structure with a constant level of damping. The spectrum is obtained by repeatedly solving the equilibrium equations for structures with varying frequencies of vibration and then plotting the peak displacement obtained for that frequency versus the frequency for which the displacement was obtained.



The next several slides treat the development of the 5% damped response spectrum for the 1940 El Centro ground motion record. The “solver” indicated in the slide is a routine, such as the Newmark method, that takes the ground motion record, the damping ratio, and the system frequency as input and reports as output only the maximum absolute value of the relative displacement that occurred over the duration of the ground motion. It is important to note that by taking the absolute value, the sign of the peak response is lost. The time at which the peak response occurred is also lost (simply because it is not recorded).



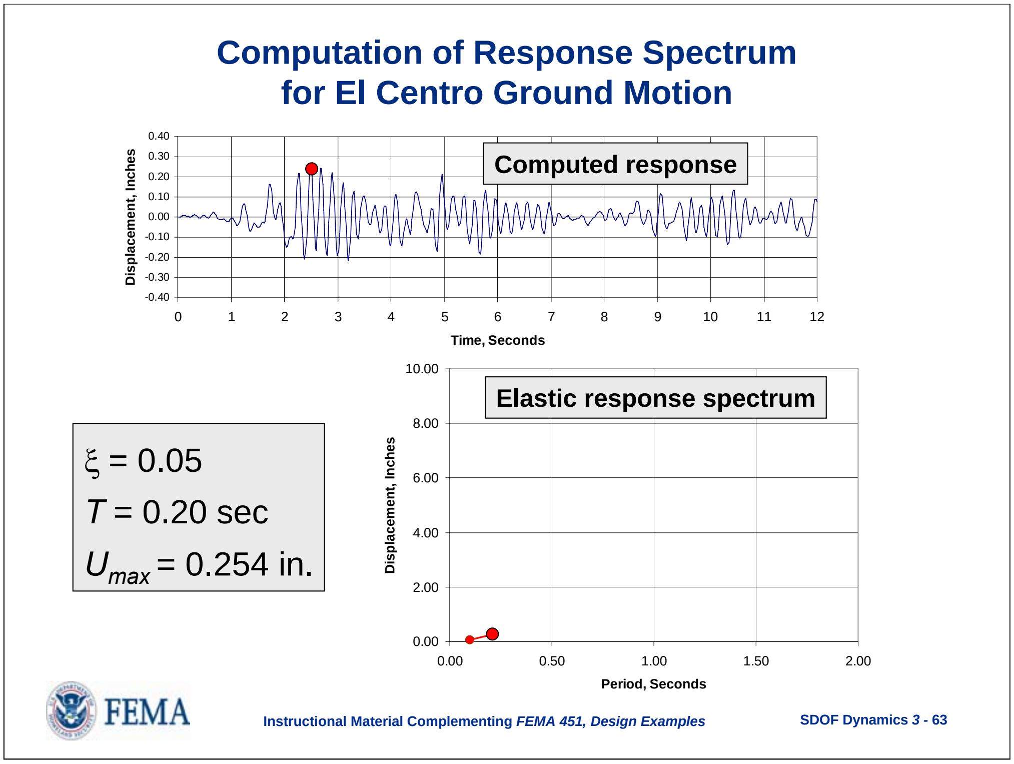
This slide is a restatement of the previous point.



Here, the first point in the response spectrum is computed. For this and all subsequent steps, the ground motion record is the same and the damping ratio is set as 5% critical. Only the frequency of vibration, represented by period *T*, is changed.

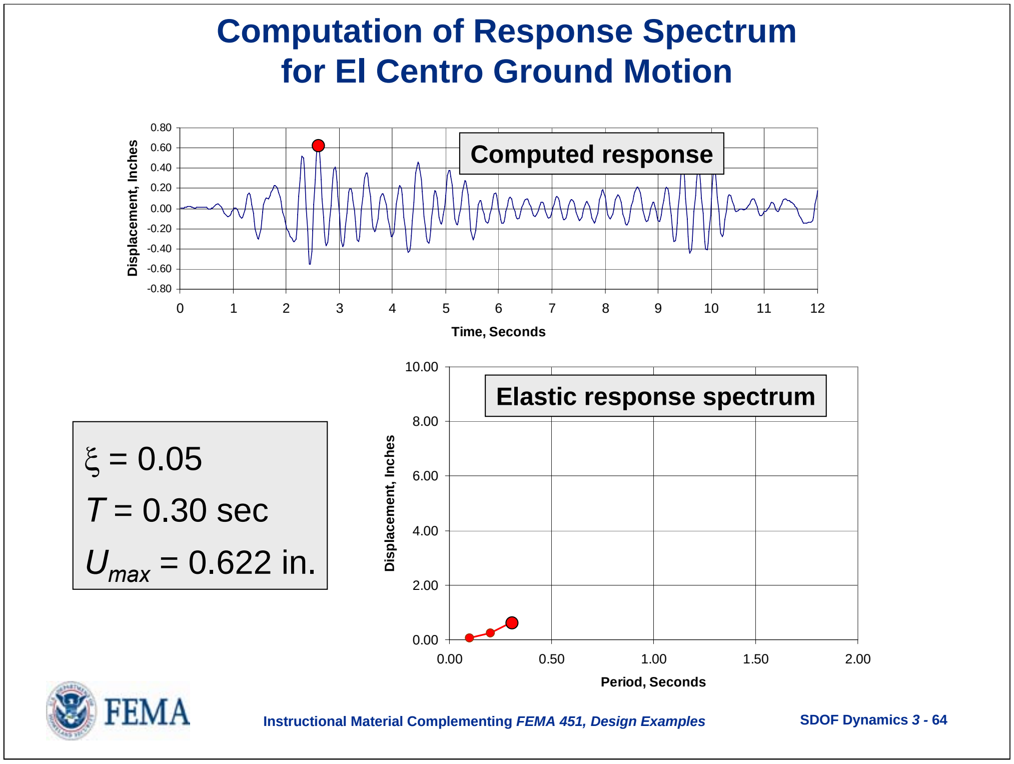
When *T* = 0.10 sec (circular frequency = 62.8 radians/sec), the peak computed relative displacement was 0.0543 inches. The response history from which the peak was obtained is shown at the top of the slide. This peak occurred at about 5 sec into the response, but this time is not recorded. Note the high frequency content of the response.

The first point on the displacement response spectrum is simply the displacement (0.0543 inches) plotted against the structural period (0.1 sec) for which the displacement was obtained.

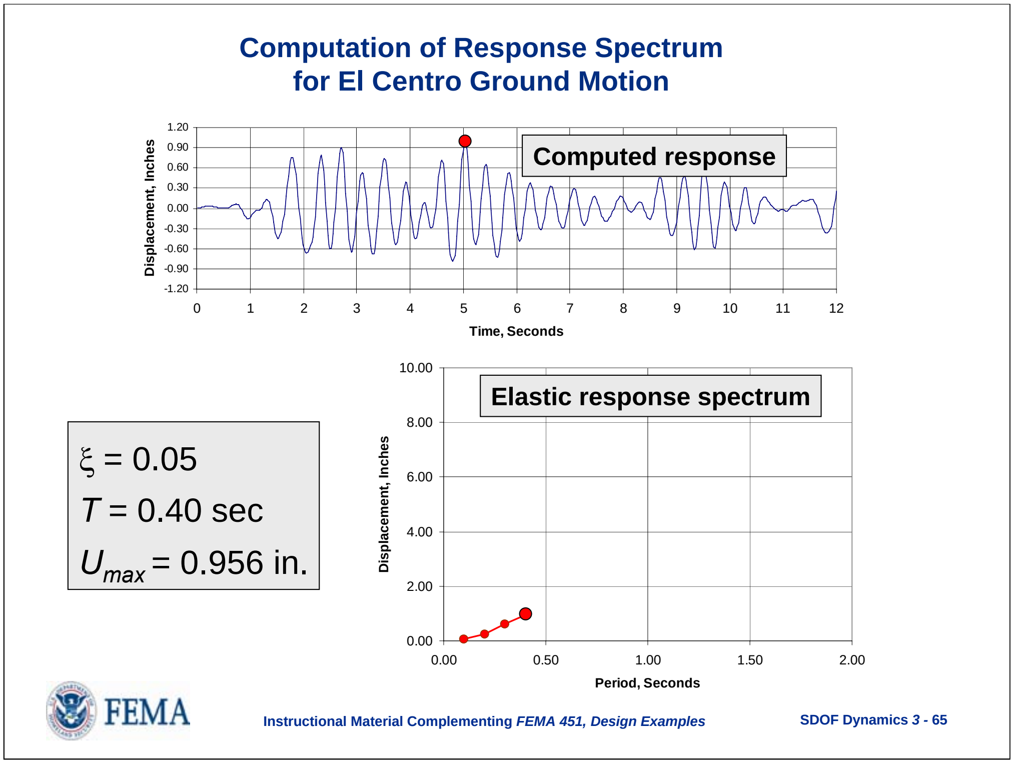


Here the whole procedure is repeated, but the system period is changed to 0.2 sec. The computed displacement history is shown at the top of the slide, which shows that the peak displacement was 0.254 inches. This peak occurred at about 2.5 sec into the response but, as before, this time is not recorded. Note that the response history is somewhat smoother than that in the previous slide.

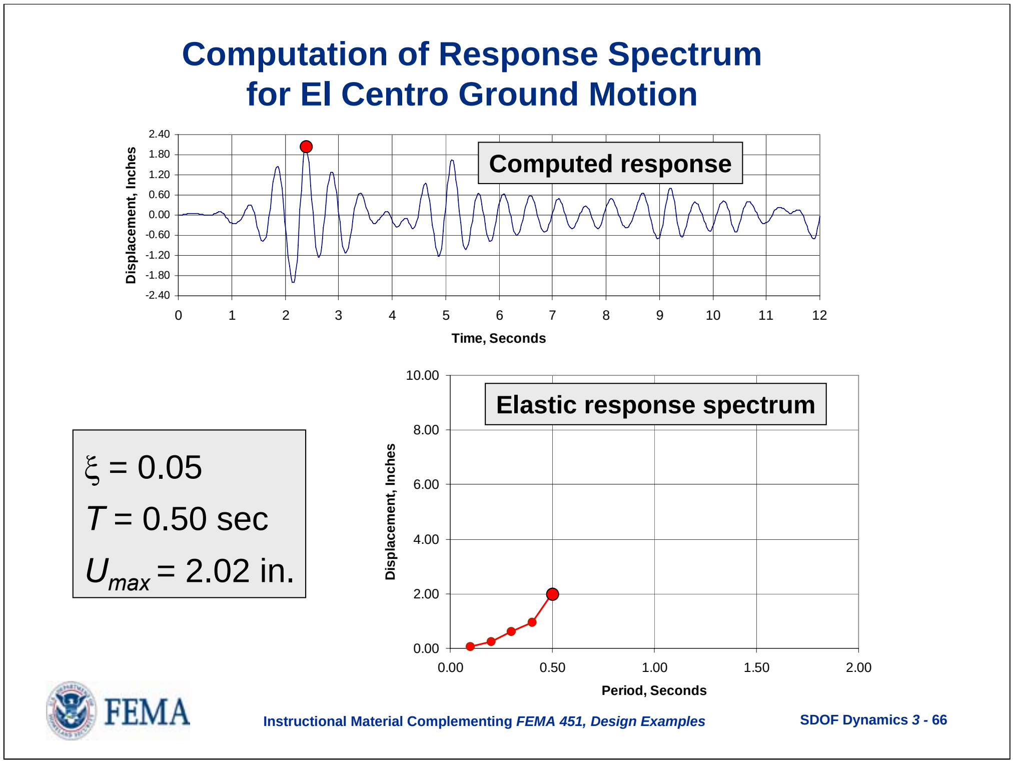
The second point on the response spectrum is the peak displacement (0.254 inch) plotted against the system period, which was 0.2 sec.



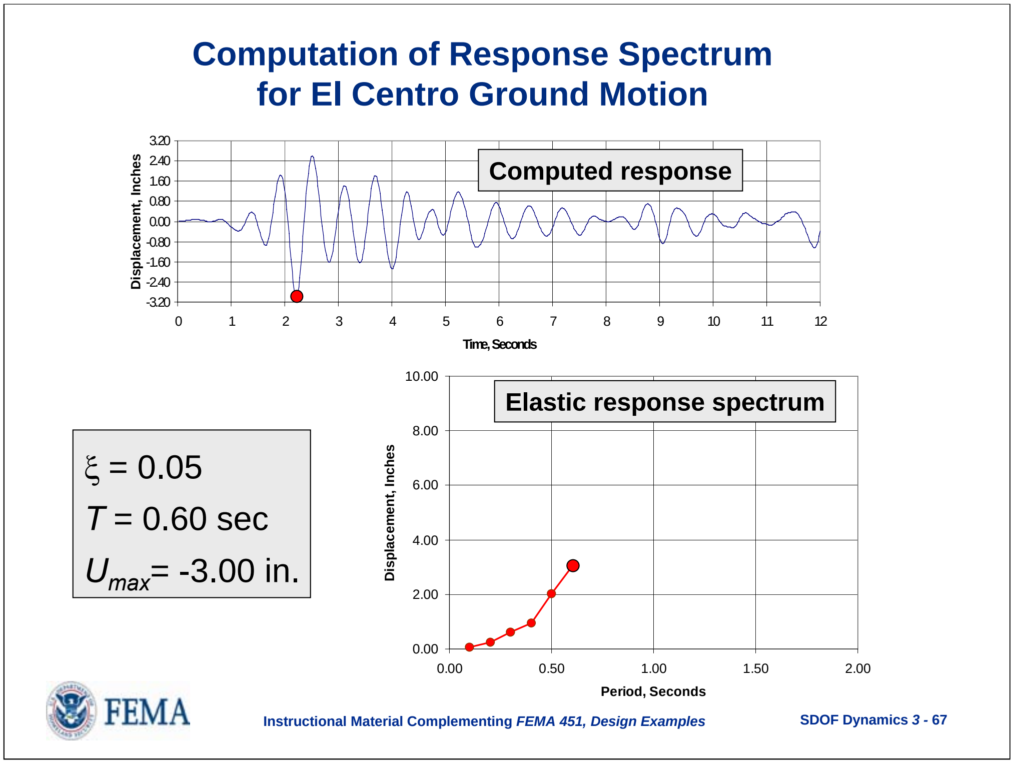
The third point on the response spectrum is the peak displacement (0.622 inch) plotted against the system period, which was 0.3 sec. Again, the response is somewhat “smoother” than before.



The fourth point on the response spectrum is the peak displacement (0.956 inch) plotted against the system period, which was 0.40 sec.

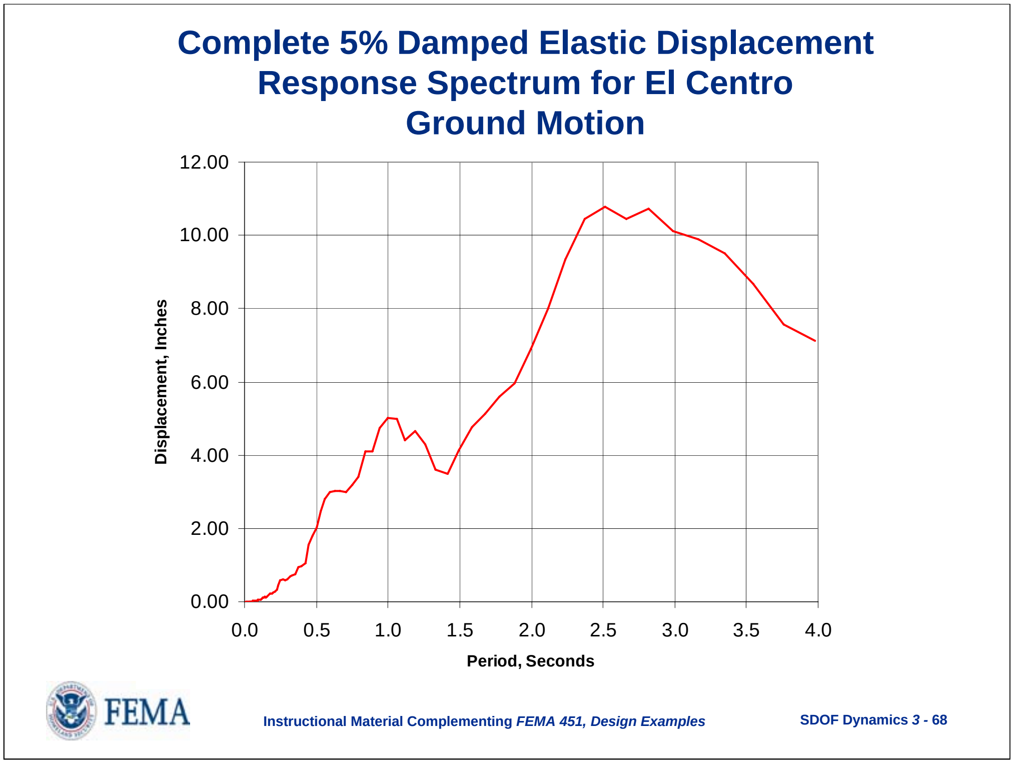


The next point on the response spectrum is the peak displacement (2.02 inches) plotted against the system period, which was 0.50 sec.



The next point on the response spectrum is the peak displacement (3.03 inches) plotted against the system period, which was 0.60 sec. Note that only the absolute value of the displacement is recorded.

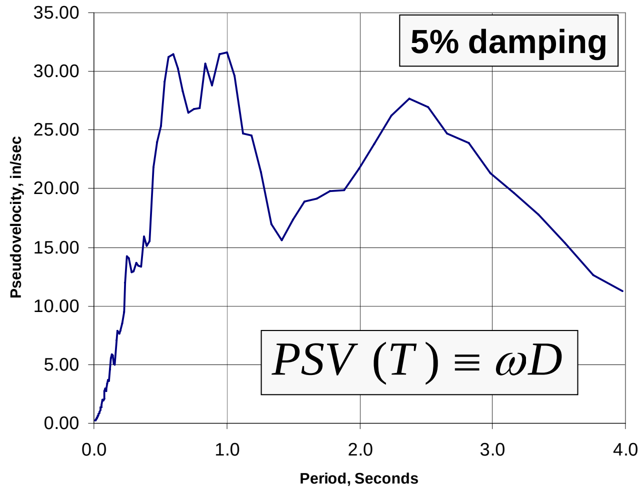
The complete spectrum is obtained by repeating the process for all remaining periods in the range of 0.7 through 2.0 sec. For this response spectrum, 2/0.1 or 20 individual points are calculated, requiring 20 full response history analyses. A real response spectrum would likely be run at a period resolution of about 0.01 sec, requiring 200 response history analyses.



This is the full 5% damped elastic displacement response spectrum for the 1940 El Centro ground motion. Note that the spectrum was run for periods up to 4.0 sec. This spectrum was generated using NONLIN.

Note also that the displacement is nearly zero when T is near zero. This is expected because the relative displacement of a very stiff structure (with T near zero) should be very small. The displacement then generally increases with period, although this trend is not consistent. The reductions in displacement at certain periods indicate that the ground motion has little energy at these periods. As shown later, a different earthquake will have an *entirely different* response spectrum.

Development of *Pseudovelocity Response Spectrum*



$$PSV\ (T) \equiv \omega D$$

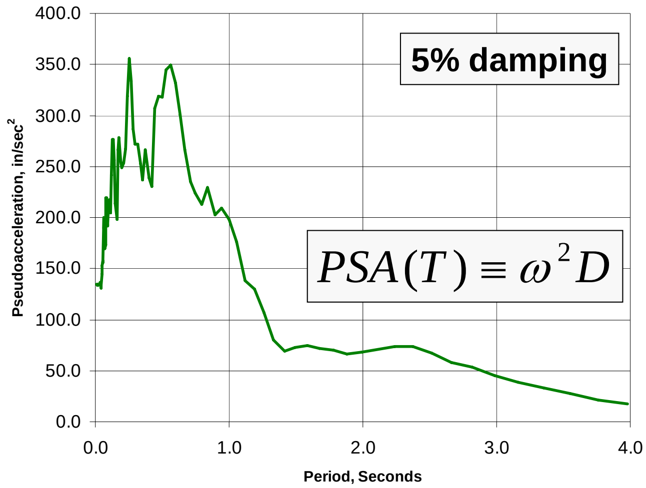
If desired, an elastic (relative) velocity response spectrum could be obtained in the same way as the displacement spectrum. The only difference in the procedure would be that the peak velocity computed at each period would be recorded and plotted.

Instead of doing this, the velocity spectrum is obtained in an *approximate* manner by assuming that the displacement response is harmonic and, hence, that the velocity at each (circular) frequency is equal to the frequency times the displacement. This comes from the rules for differentiating a harmonic function.

Because the velocity spectrum so obtained is not exact, it is called the *pseudovelocity* response spectrum.

Note that it appears that the pseudovelocity at low (near zero) periods is also near zero (but not exactly zero).

Development of *Pseudoacceleration Response Spectrum*



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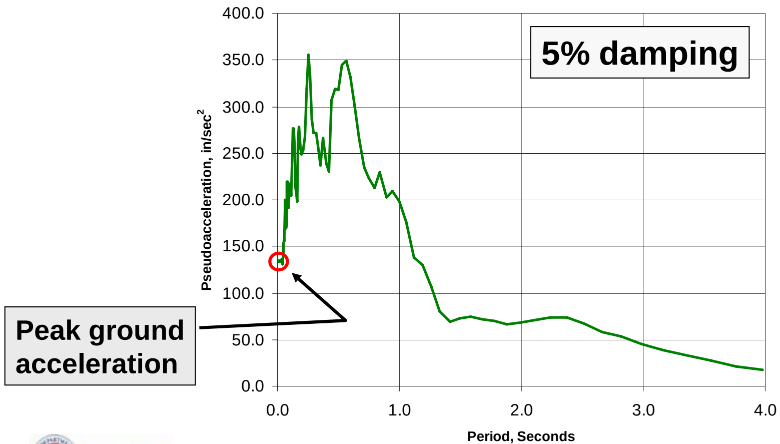
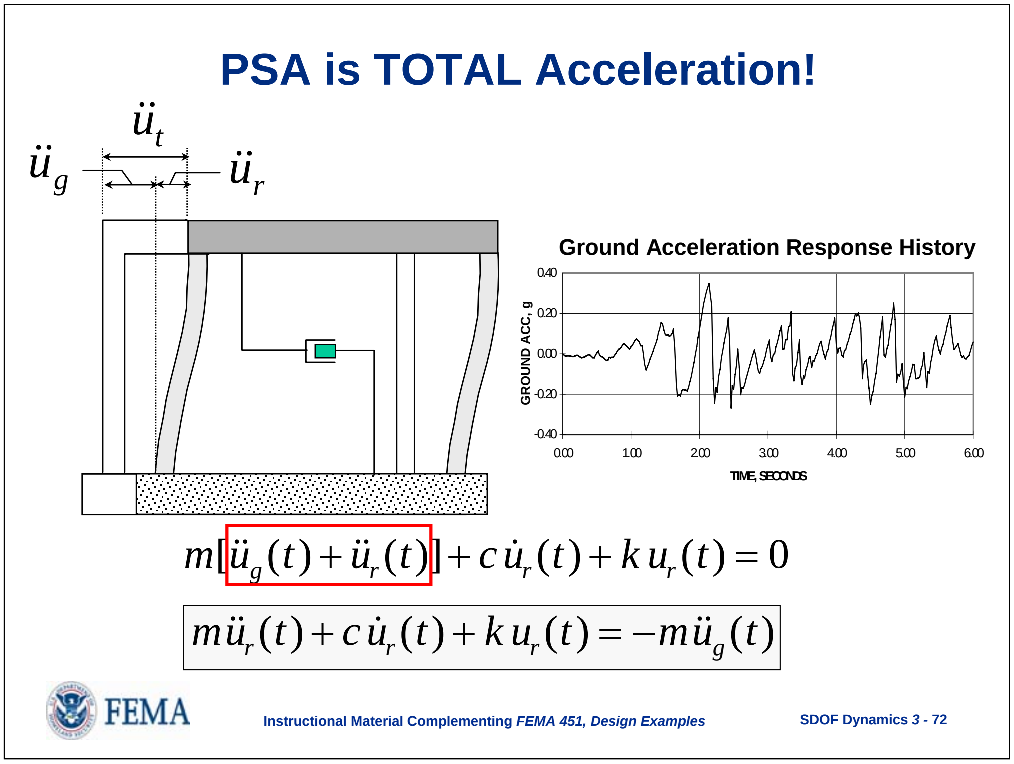
The pseudoacceleration spectrum is obtained from the displacement spectrum by multiplying by the circular frequencies squared. Note that the acceleration at a near zero period is not near zero (as was the case for velocity and displacement). In fact, the pseudoacceleration represents the *total acceleration* in the system while the pseudovelocity and the displacement are relative quantities.

Note About the Pseudoacceleration Response Spectrum

The pseudoacceleration response spectrum represents the **total acceleration** of the system, not the relative acceleration. It is nearly identical to the true total acceleration response spectrum for lightly damped structures.



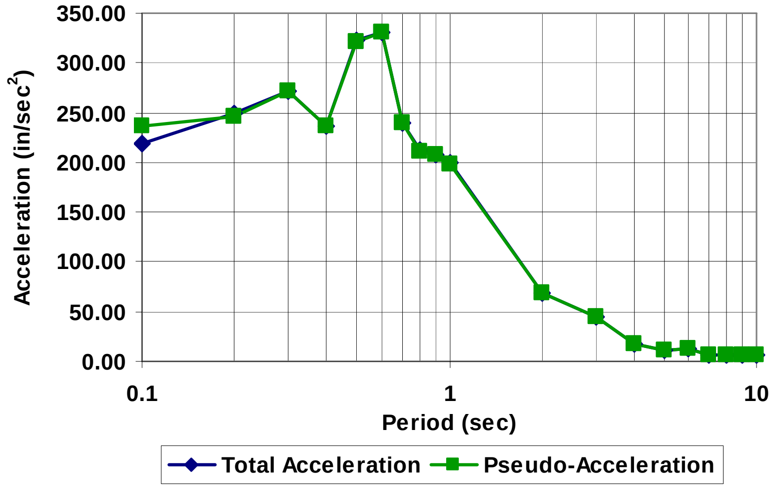
For very rigid systems (with near zero periods of vibration), the relative acceleration will be nearly zero and, hence, the pseudoacceleration, which is the total acceleration, will be equal to the peak ground acceleration.



This slide explains why the pseudoacceleration is equal to the total acceleration. The relative displacement is multiplied by omega to get pseudovelocity. The pseudovelocity then is multiplied by omega to get the total acceleration.

Difference Between Pseudo-Acceleration and Total Acceleration

(System with 5% Damping)



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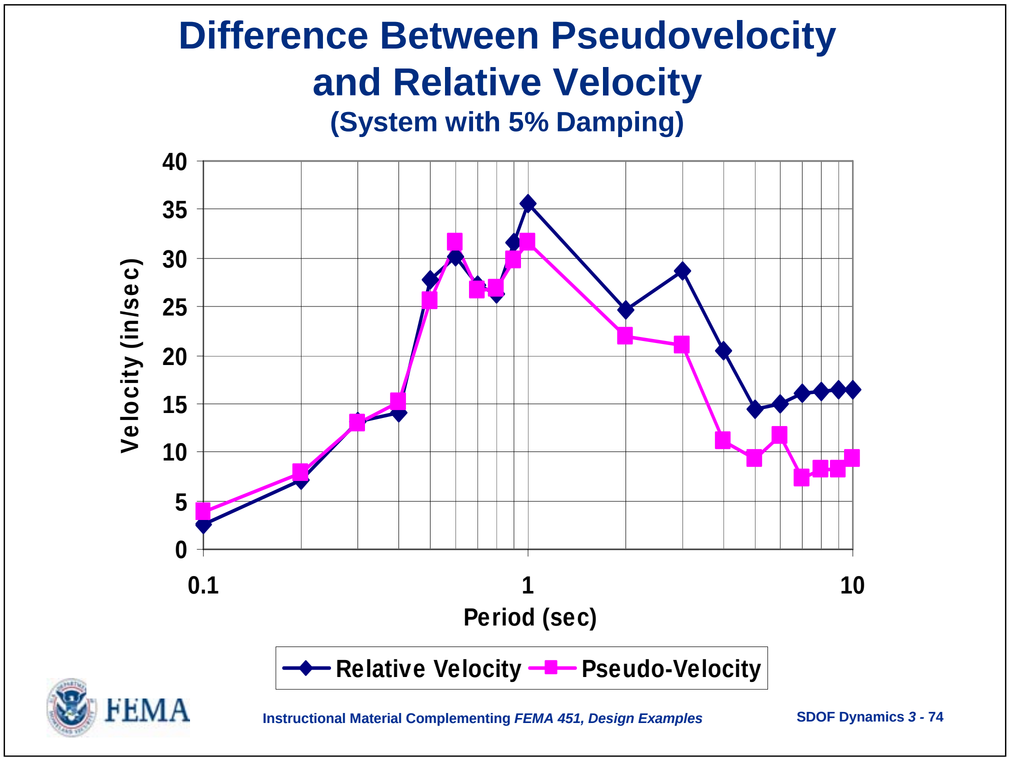
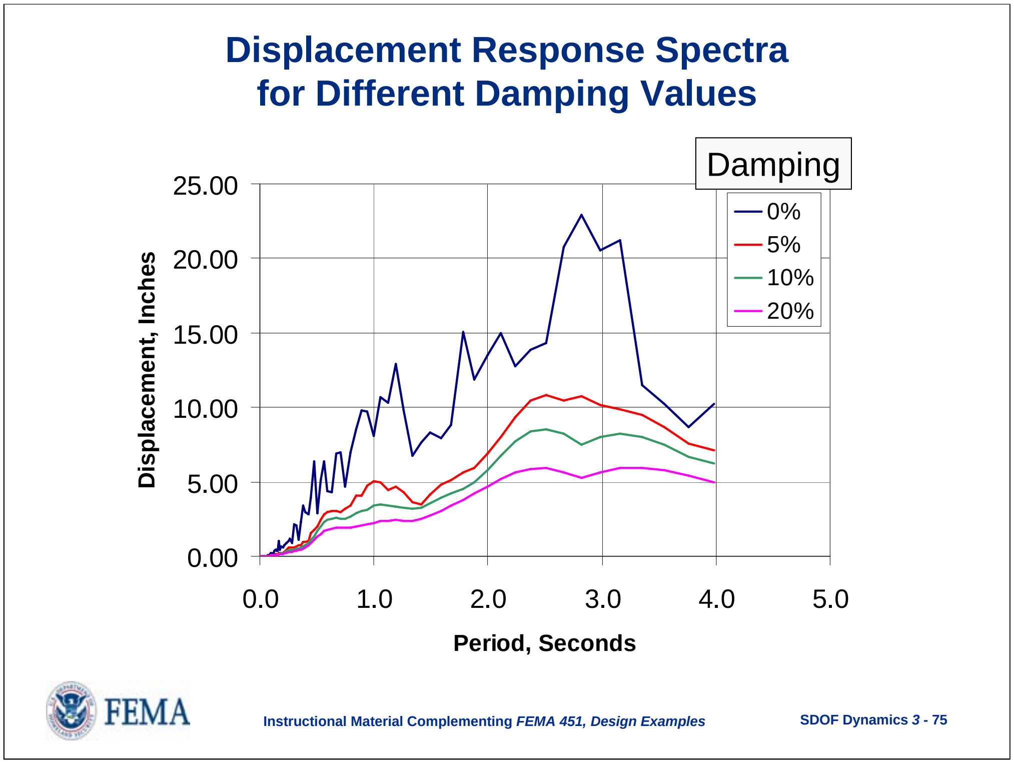
This plot shows total acceleration and pseudoacceleration for a 5% damped system subject to the El Centro ground motion. Note the similarity in the two quantities. The difference in the two quantities is only apparent at low periods.

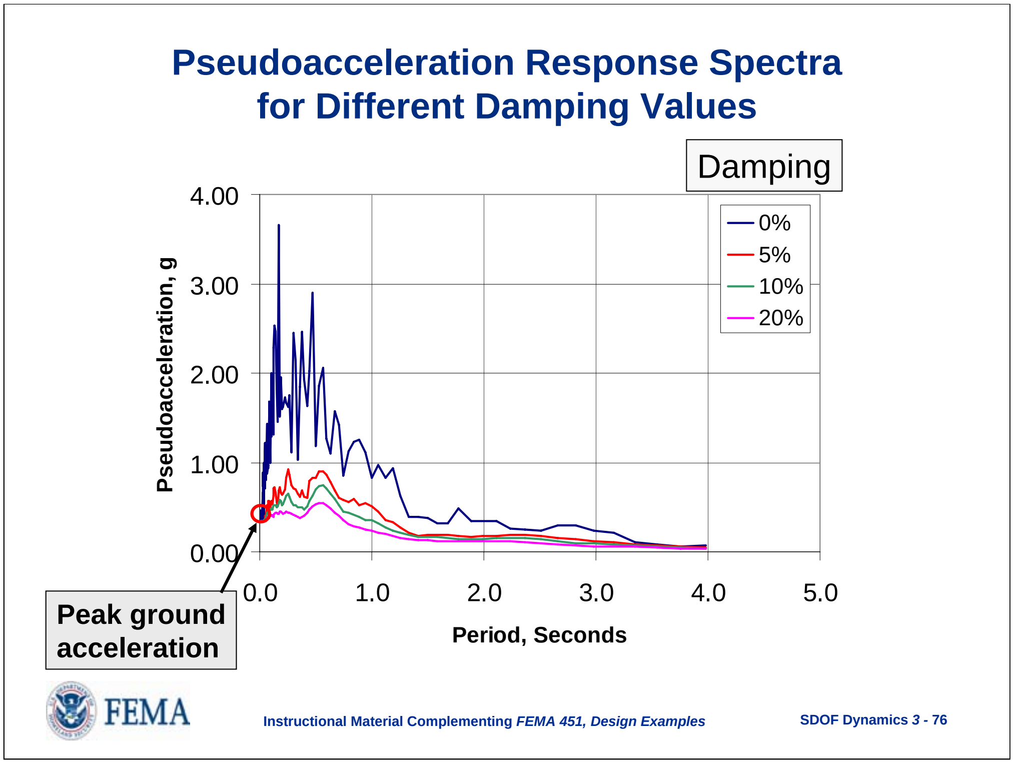
The difference can be much greater when the damping is set to 10%, 20%, or 30% critical, and the differences can appear in a wider range of periods.



This plot shows relative velocity and pseudovelocity for a 5% damped system subject to the El Centro ground motion. Here, the differences are much more apparent than for pseudoacceleration, and the larger differences occur at the higher periods. The differences will be greater for systems with larger amounts of damping.



The higher the damping, the lower the relative displacement. At a period of 2 sec, for example, going from zero to 5% damping reduces the displacement amplitude by a factor of two. While higher damping produces further decreases in displacement, there is a diminishing return. The % reduction in displacement by going from 5 to 10% damping is much less that that for 0 to 5% damping.



Damping has a similar effect on pseudoacceleration. Note, however, that the pseudoacceleration at a (near) zero period is the same for all damping values. This value is always equal to the peak ground acceleration for the ground motion in question.

Damping Is Effective in Reducing the Response for (Almost) Any Given Period of Vibration

- An earthquake record can be considered to be the combination of a large number of harmonic components.
- Any SDOF structure will be in near resonance with one of these harmonic components.
- Damping is most effective at or near resonance.
- Hence, a response spectrum will show reductions due to damping at all period ranges (except $T = 0$).

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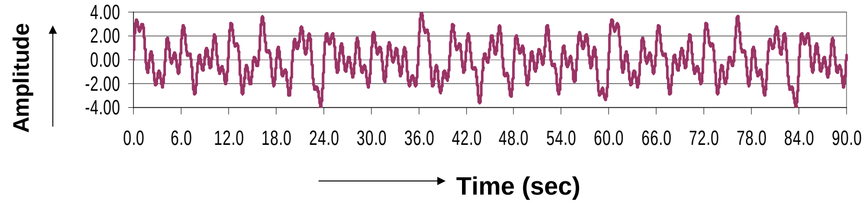
Damping is generally effective at all periods (except at $T = 0$). The reason for this is that ground motions consist of a large number of harmonics, each at a different frequency. When a response spectrum analysis is run for a particular period, there will be a near resonant response at that period. Damping is most effective at resonance and, hence, damping will be effective over the full range of periods for which the response spectrum is generated.

Damping Is Effective in Reducing the Response for Any Given Period of Vibration



- Example of an artificially generated wave to resemble a real time ground motion accelerogram.
- Generated wave obtained by combining five different harmonic signals, each having equal amplitude of 1.0.

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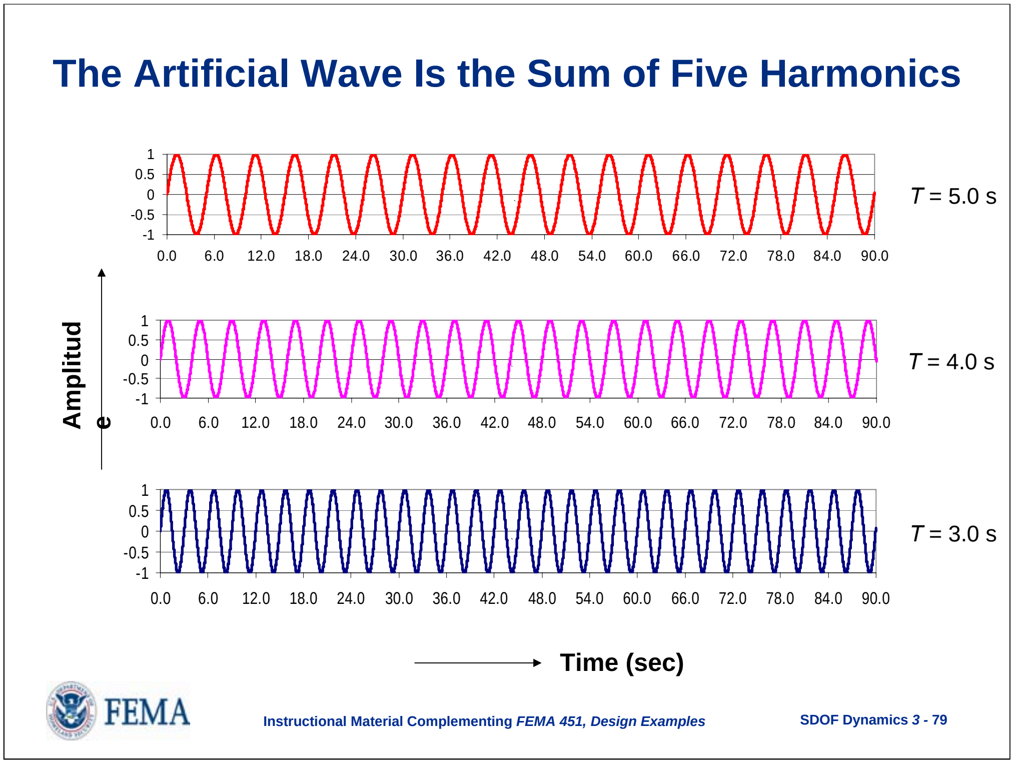
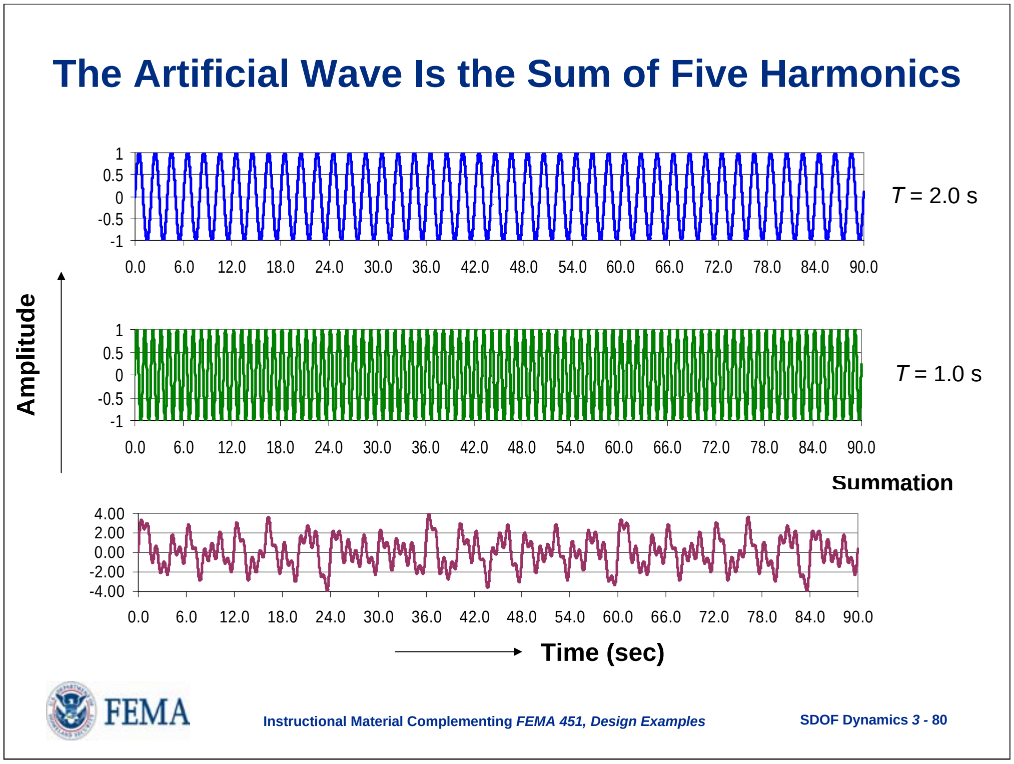
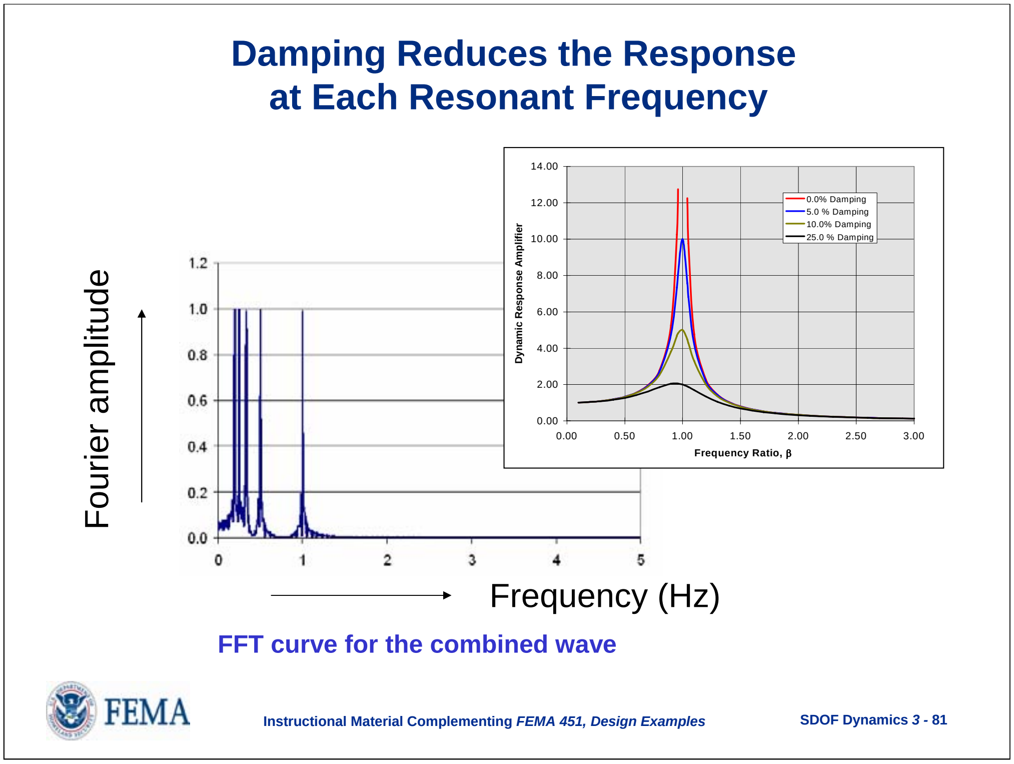
To demonstrate the point made in the previous slide, an “artificial” ground motion is made up from the sum of five simple harmonics.



Each of the harmonics has an amplitude of 1.0. The first three of the harmonics with *T* = 5, 4, and 3 sec are shown.



The remaining two harmonics (at T = 2 and 1 sec) and the sum are shown.



The Fourier amplitude spectrum of the artificial ground motion is shown at the left. This spectrum shows the five discrete harmonics that are in the artificial motion. If the response spectrum is run at intervals of 0.2 sec, there will be resonant response at each of these frequencies. Damping will be very effective in reducing the response at each of the frequencies.

Use of an Elastic Response Spectrum

Example Structure

K = 500 k/in

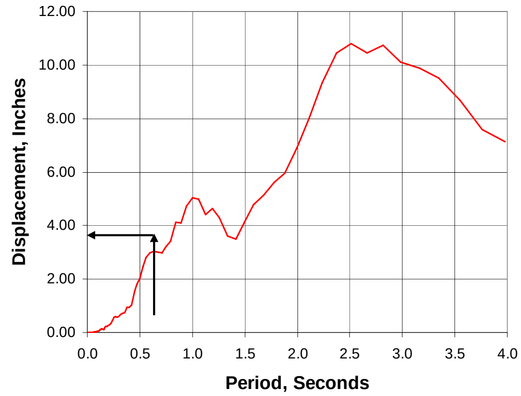
W = 2,000 k

M = 2000/386.4 = 5.18 k-sec^2/in

ω = (K/M)$^{0.5}$ =9.82 rad/sec

T = 2π/ω = 0.64 sec

5% critical damping



At T = 0.64 sec, displacement = 3.03 in.

This is a simple example of the use of an elastic displacement response spectrum. If the system is assumed to have 5% damping (matching the spectrum) and the system period is known, the peak displacement may be easily computed. Note that the sign of the displacement (positive or negative) and the time that the displacement occurred is not known as this information was discarded when the spectrum was generated.

Use of an Elastic Response Spectrum

<u>Example Structure</u>

K = 500 k/in

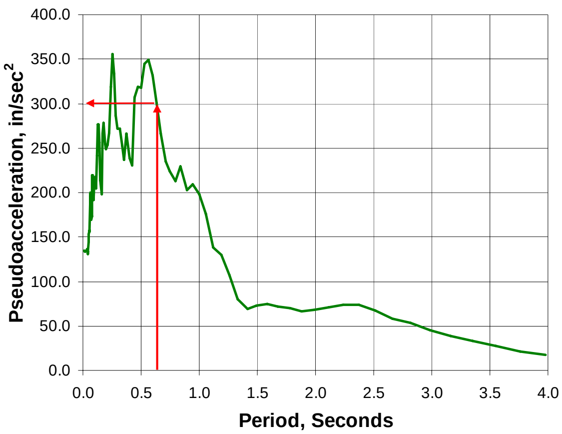
W = 2,000 k

M = 2000/386.4 = 5.18 k-sec^2/in

$\omega = (K/M)^{0.5}$ =9.82 rad/sec

$T = 2\pi/\omega$ = 0.64 sec

5% critical damping



At T = 0.64 sec, pseudoacceleration = 301 in./sec^2

Base shear = M x PSA = 5.18(301) = 1559 kips

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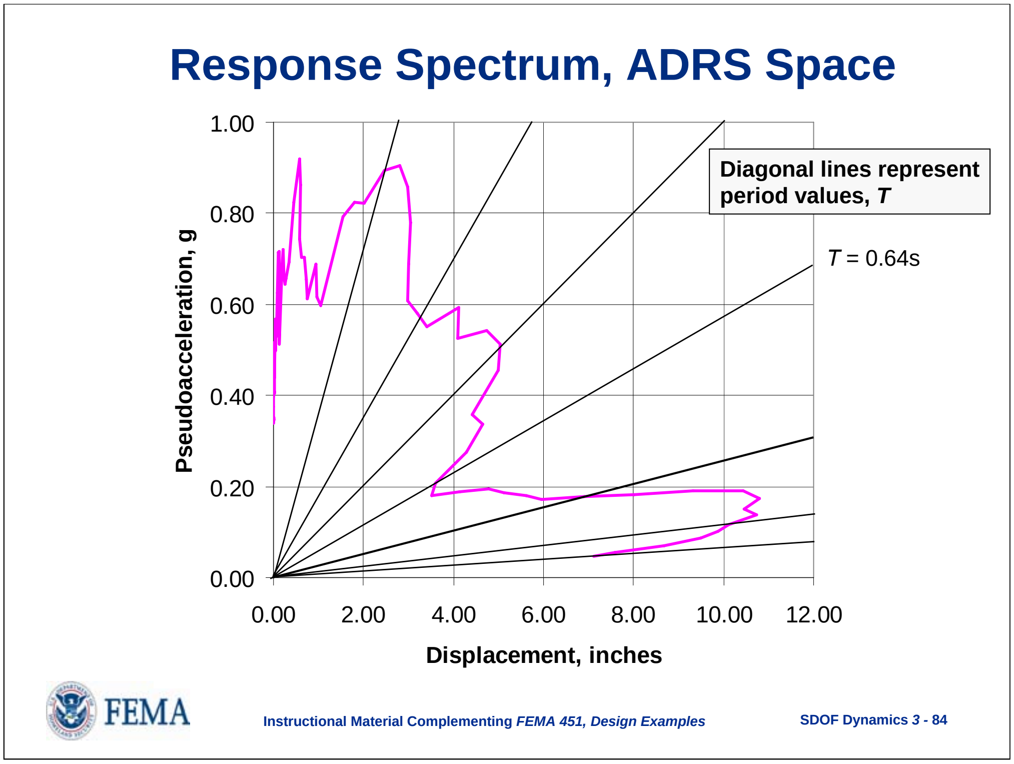
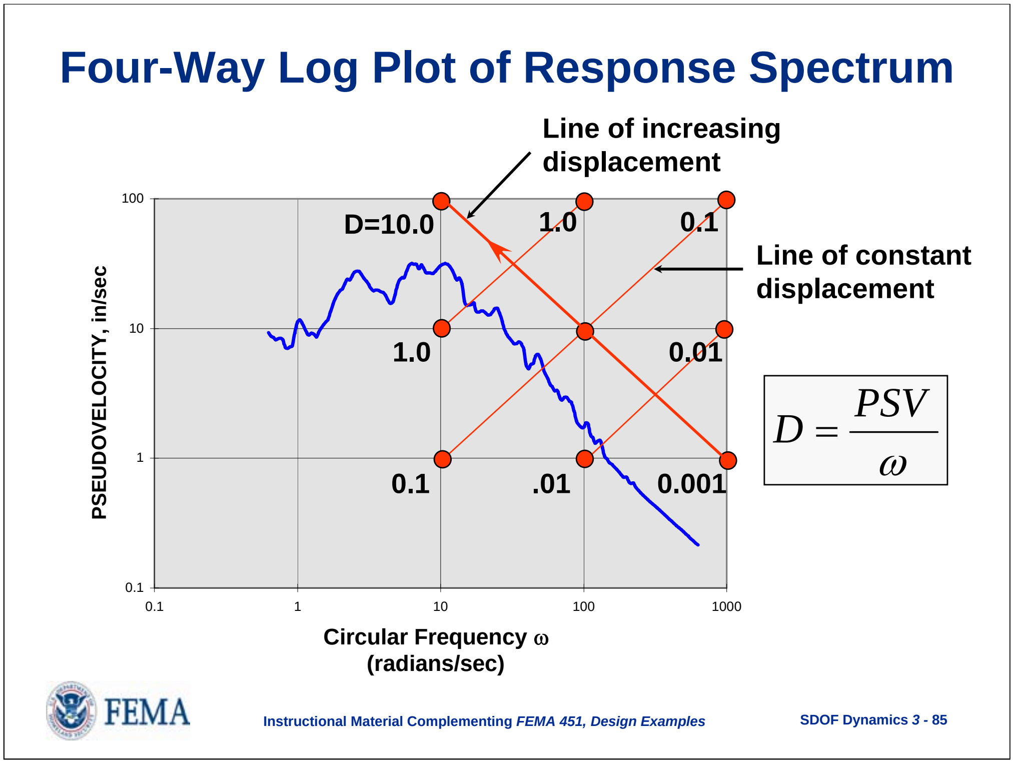
This is a simple example of the use of an elastic pseudoacceleration response spectrum. If the system is assumed to have 5% damping (matching the spectrum) and the system period and mass are known, the peak base shear may be easily computed. Note that the sign of the shear (positive or negative) and the time that the shear occurred is not known as this information (related to pseudoacceleration) was discarded when the spectrum was generated.



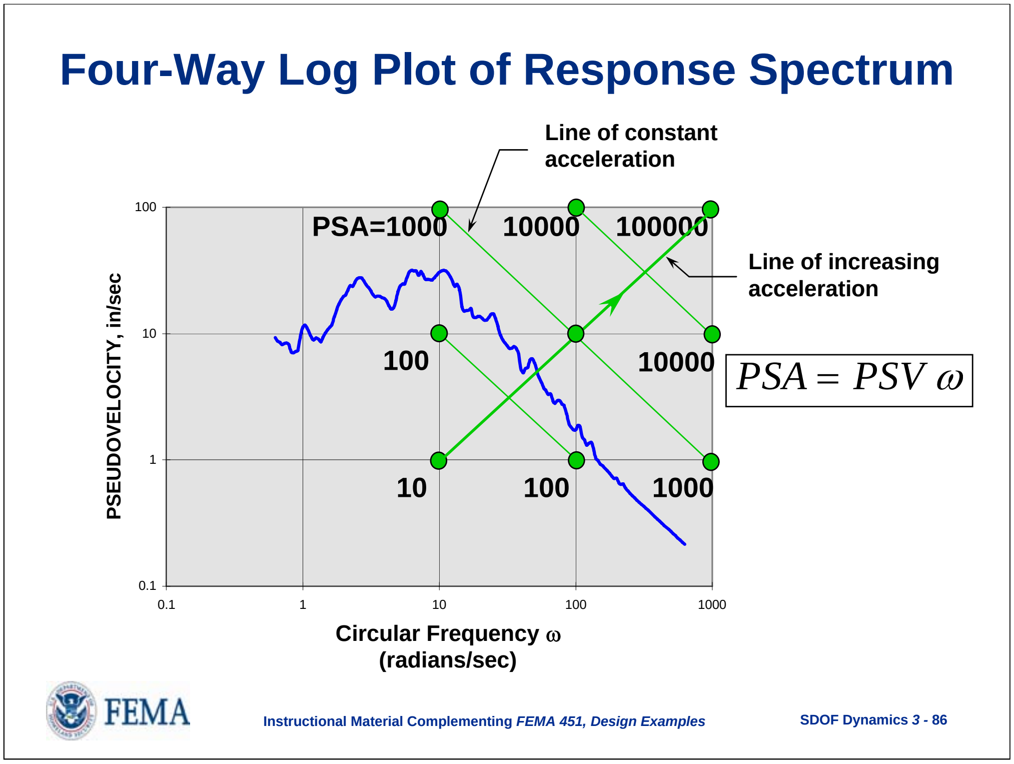
Another type of spectrum plot is the acceleration-displacement response spectrum (ADRS), which is also called a *demand spectrum*. Here, displacement is plotted on the x-axis and pseudoacceleration is plotted on the y-axis. Periods of vibration are represented as radial lines.

This kind of spectrum is most commonly used in association with “capacity spectra” developed from nonlinear static pushover analysis. A demand spectrum is also useful in assessing stiffness and damping requirements of base-isolated systems.



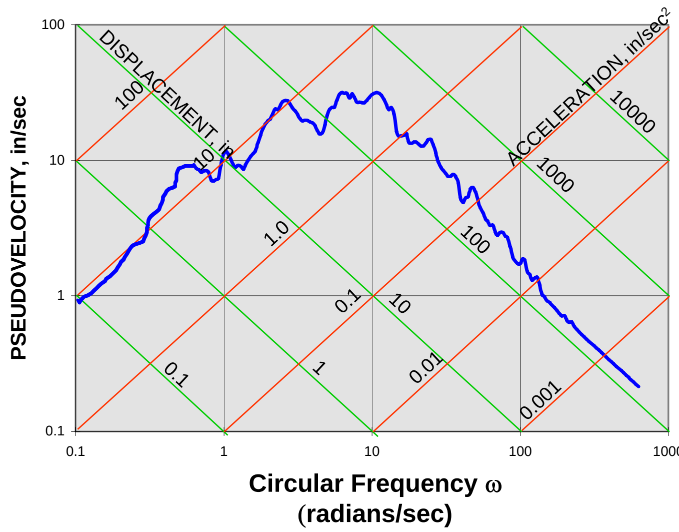
Response spectra often are plotted on four-way log paper. This type of spectrum is often called a "tripartite spectrum" because the displacement, pseudovelocity, and pseudoacceleration are all shown on the same plot.

On the plot, pseudovelocity is plotted on the vertical axis. Lines of constant and logarithmically increasing displacement are generated as shown. The use of circular frequency on the horizontal axis is rarely used in practice but is convenient for illustrating the development of the plot.



Lines of constant and logarithmically increasing pseudoacceleration are obtained in a similar manner.

Four-Way Log Plot of Response Spectrum



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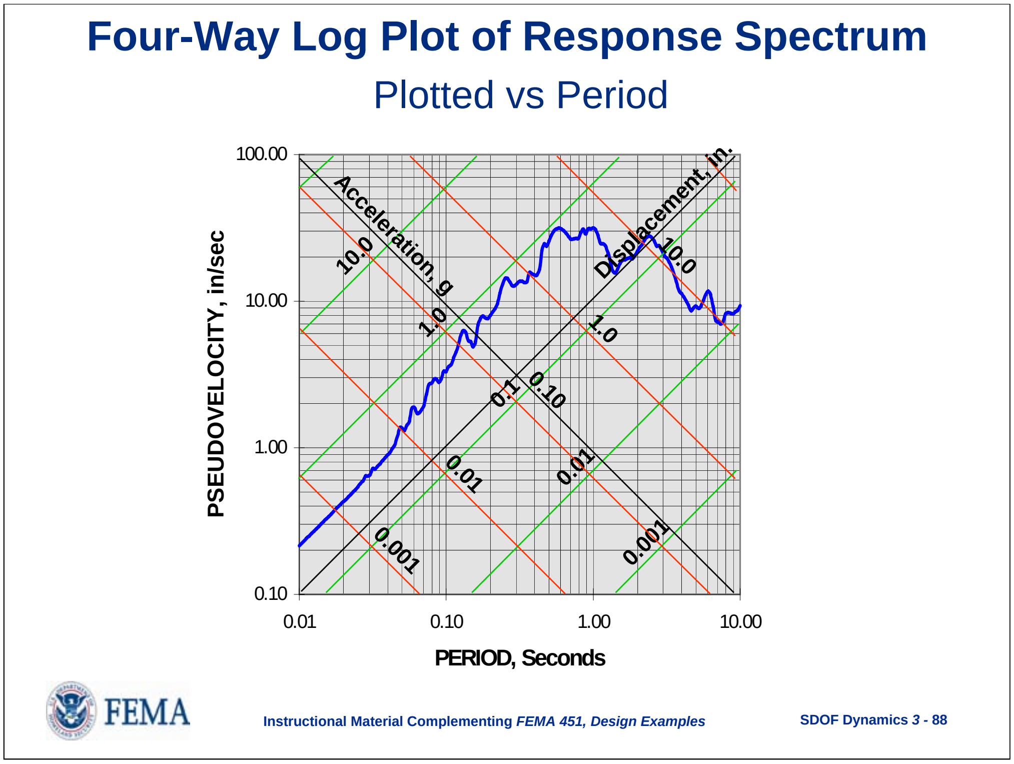
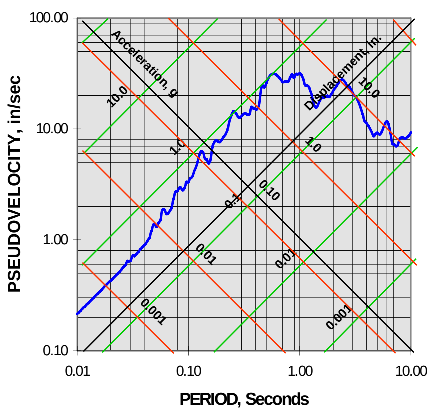
This is a completed spectrum for the 5% damped 1940 El Centro earthquake with maximum acceleration = 0.35g.



Response spectra usually are plotted versus structural period or structural cyclic frequency. This is the same spectrum as shown in the previous slide, but it is plotted versus period.

Development of an Elastic Response Spectrum

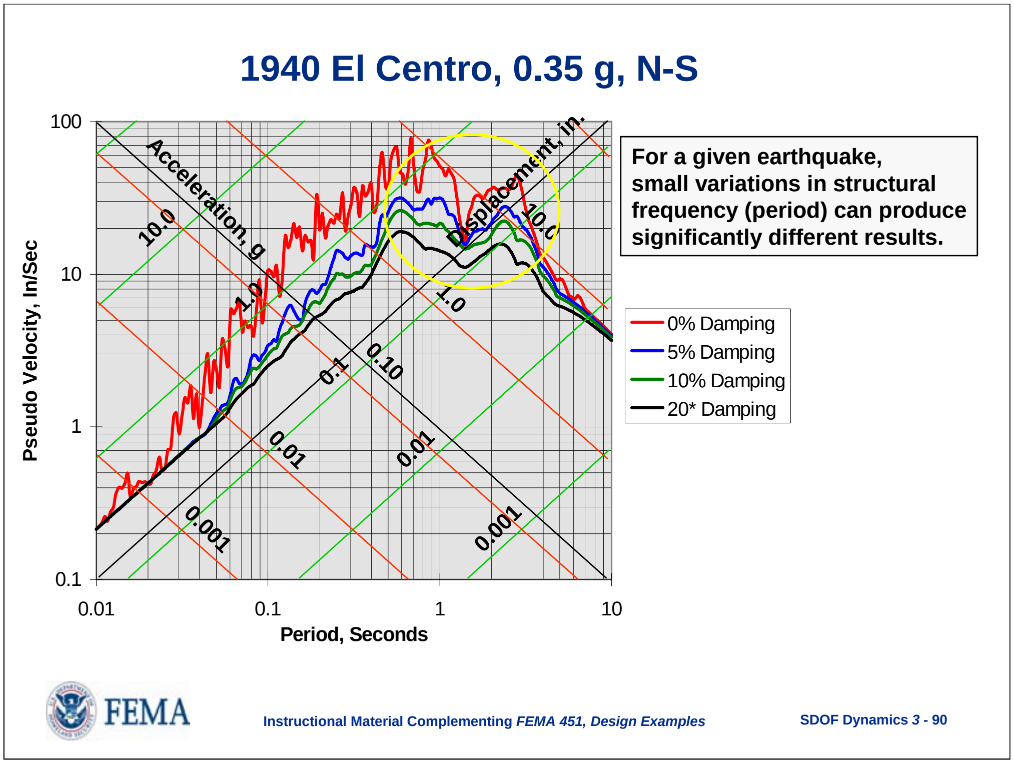


Problems with Current Spectrum:

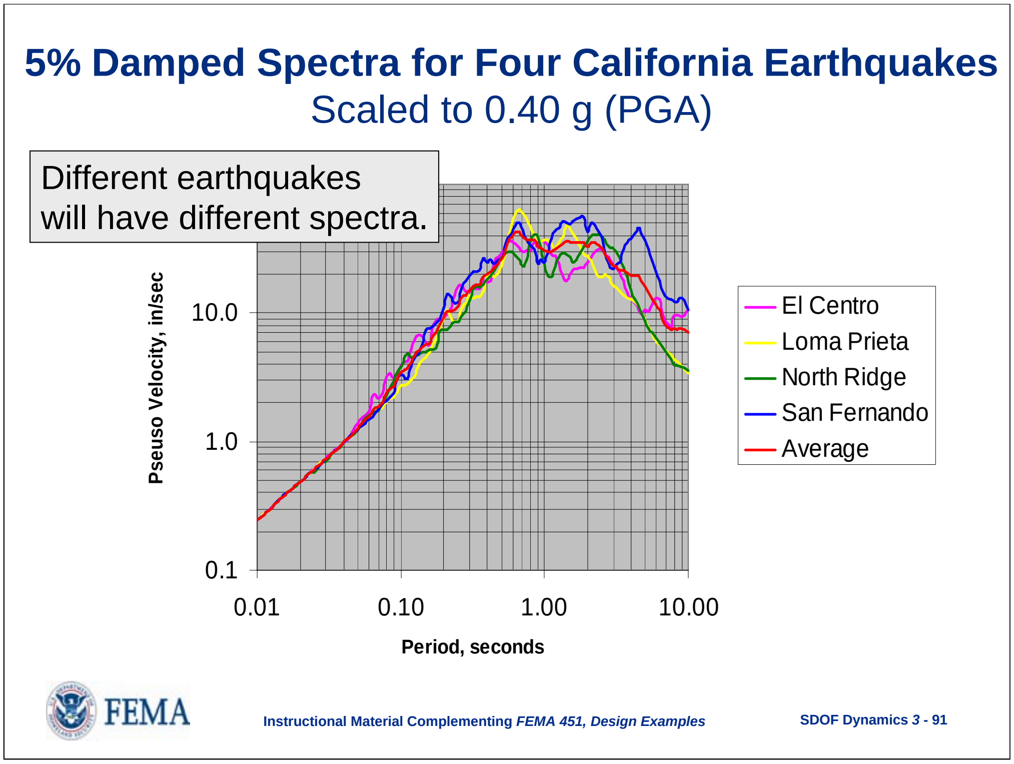
For a given earthquake, small variations in structural frequency (period) can produce significantly different results.

It is for a single earthquake; other earthquakes will have different Characteristics.

The use of a single earthquake spectrum in structural design is not recommended for the reasons shown on this slide. The same site experiencing different earthquakes (or different components of the same earthquake) often will have dissimilar spectra.



Note the significant changes (for any given damping value) in the 1.5 sec period range.



The spectra are scaled to 0.4 g with 5% damping. Note the differences.

Smoothed Elastic Response Spectra (Elastic DESIGN Response Spectra)

- Newmark-Hall spectrum
- ASCE 7 spectrum

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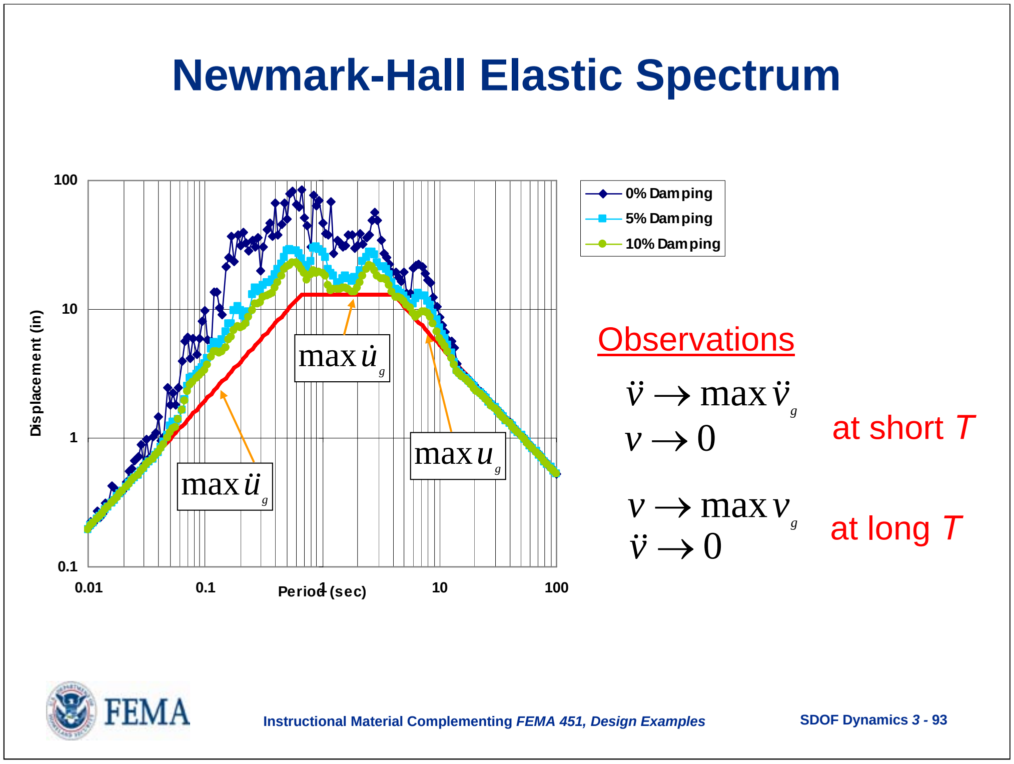
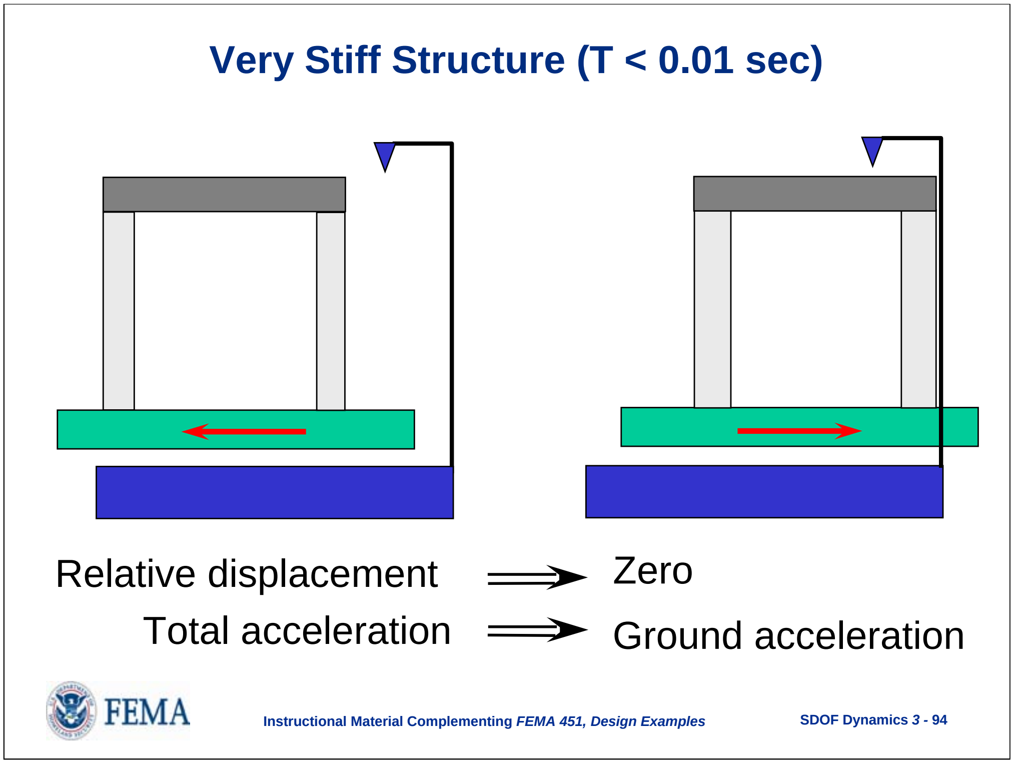
Because real ground motion spectra are difficult to work with in a design office, a variety of empirical spectra have been generated. One of the earliest of these empirical spectra was developed by Nathan Newmark. The next several slides describe this in detail.

The spectrum used by ASCE 7-05 is simpler than the Newmark spectrum, but explanation of the background of the ASCE 7 spectrum is more difficult. Certain key aspects of the ASCE 7 spectrum are presented in the topic on seismic load analysis.

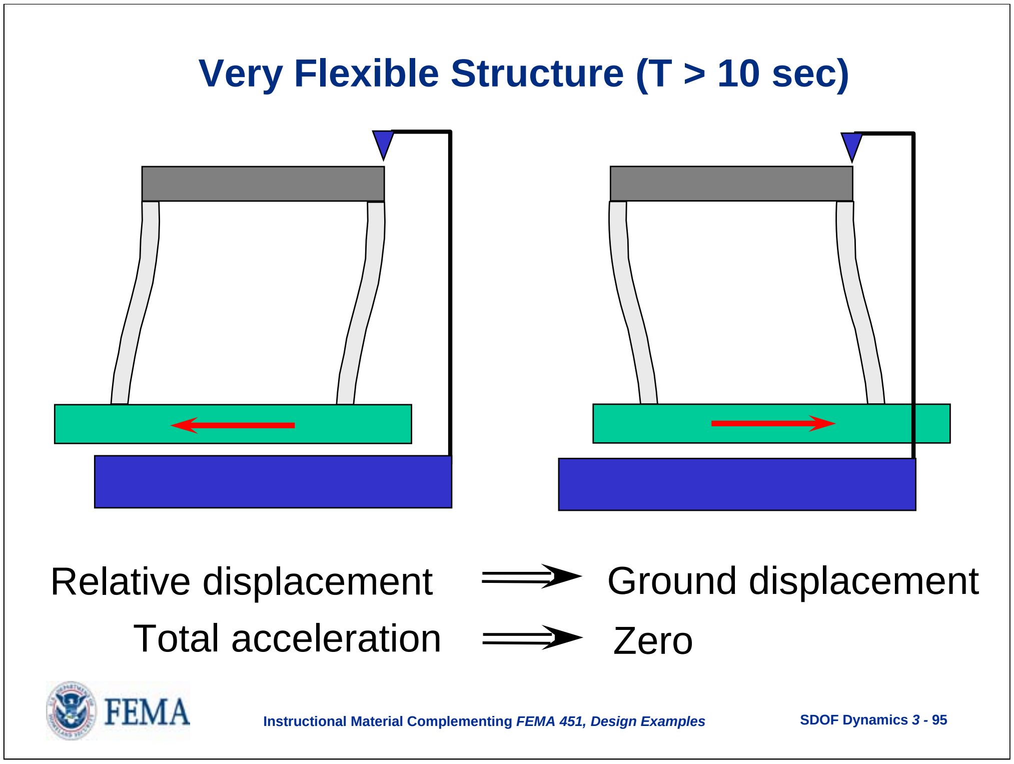


The Newmark spectrum is based on the following observations:

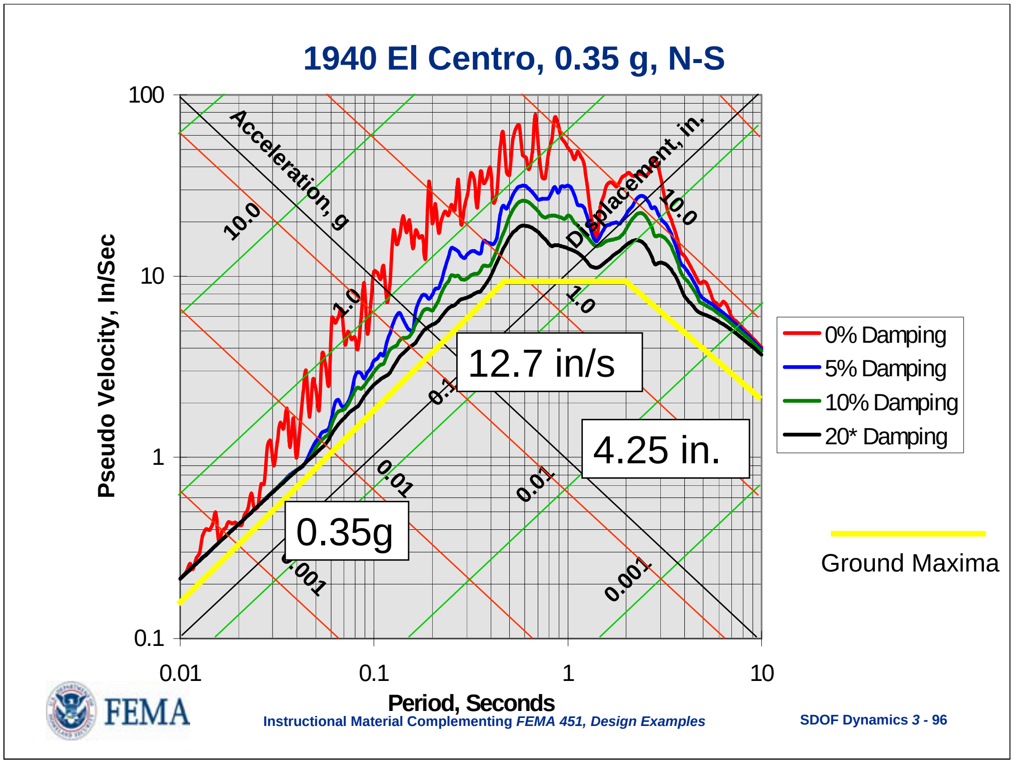
- The pseudoacceleration at very low periods is exactly equal to the peak ground acceleration.
- The relative displacement at very long periods is exactly equal to the peak ground displacement.
- At intermediate periods, the displacement, pseudovelocity, and pseudoacceleration are equal to the ground values times some empirical constant.



For very low period (high frequency) buildings, the maximum relative displacement will be zero. The maximum acceleration will approach the ground acceleration.



For very high period (low frequency) buildings, the maximum relative displacement will be equal to the maximum ground displacement. The maximum total acceleration will approach zero.



The yellow line shows the maximum recorded ground displacement, velocity, and acceleration from the 1940 El Centro earthquake. These lines clearly form a lower bound to the elastic response spectra. Note how the building response displacements, velocities, and accelerations are amplifications of the ground values. Note also how the amplifications decrease with increased damping.

Newmark's Spectrum Amplification Factors for Horizontal Elastic Response

Damping % Critical	One Sigma (84.1%) a_a	One Sigma (84.1%) a_v	One Sigma (84.1%) a_d	Median (50%) a_a	Median (50%) a_v	Median (50%) a_d
.05	5.10	3.84	3.04	3.68	2.59	2.01
1	4.38	3.38	2.73	3.21	2.31	1.82
2	3.66	2.92	2.42	2.74	2.03	1.63
3	3.24	2.64	2.24	2.46	1.86	1.52
5	**2.71**	**2.30**	**2.01**	2.12	1.65	1.39
7	2.36	2.08	1.85	1.89	1.51	1.29
10	1.99	1.84	1.69	1.64	1.37	1.20
20	1.26	1.37	1.38	1.17	1.08	1.01

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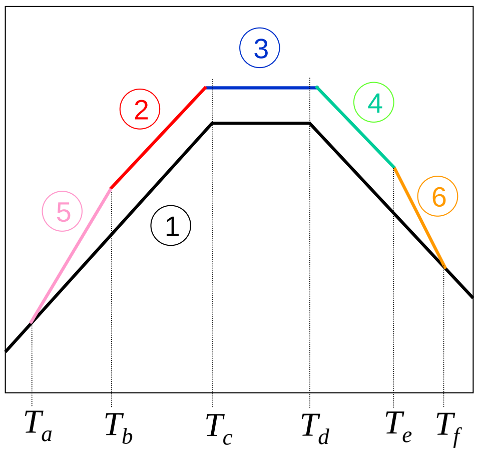
Newmark has developed a series of amplification factors to be used in the development of design spectra. These are based on the average of dozens of spectra recorded on firm soil sites for the western United States. Values are shown for the median and median plus one standard deviation.

Newmark-Hall Elastic Spectrum



1) Draw the lines corresponding to max $\ddot{v}_g, \dot{v}_g, v_g$

2) Draw line $\alpha_A \max \ddot{v}_g$ from T_b to T_c

3) Draw line $\alpha_V \max \dot{v}_g$ from T_c to T_d

4) Draw line $\alpha_D \max v_g$ from T_d to T_e

5) Draw connecting line from T_a to T_b

6) Draw connecting line from T_e to T_f

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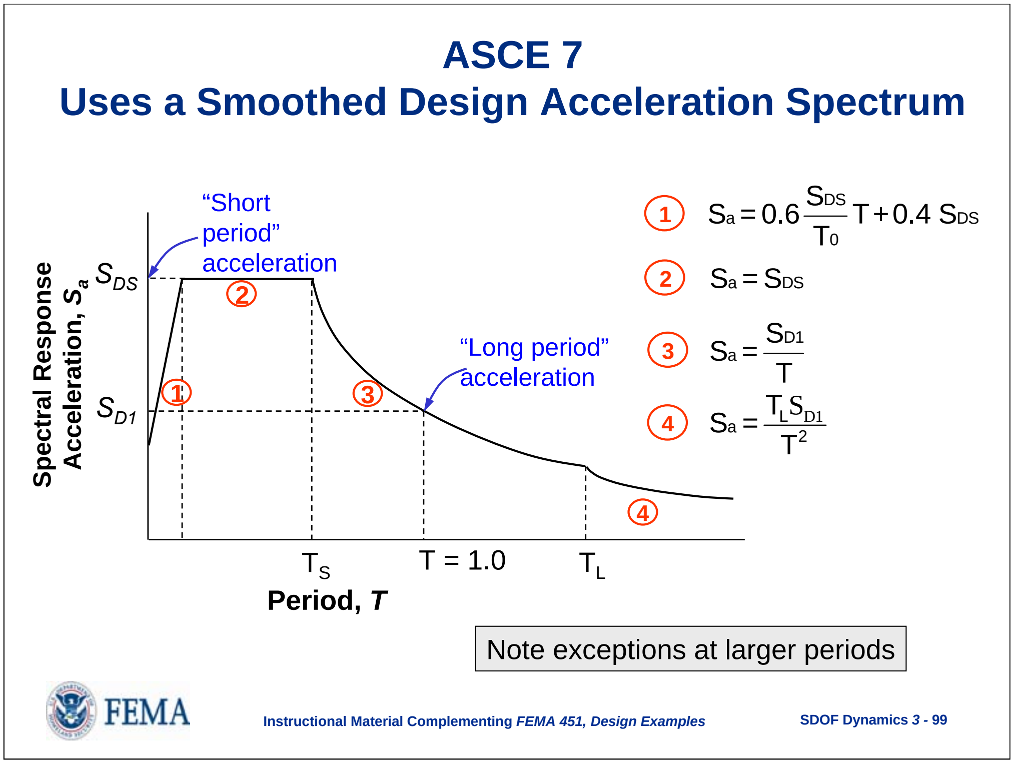
These are the steps in the development of the Newmark spectrum. Note that actual values are not present.



This plot shows the basic relationships used for the ASCE 7 spectrum. Note that the vertical axis is pseudoacceleration. The spectrum is derived from a series of maps giving spectral acceleration values for "short period" (T = 0.2 sec) or "long period" (T = 1 sec) buildings. Note that the part of the spectrum to the right of T_L (Curve 4) was introduced in the 2003 *NEHRP Recommended Provisions* and in ASCE 7-05.

The maps are based on very stiff soils. For design purposes, the acceleration spectra is not reduced to the ground acceleration at low periods (Line 1 on the plot). Damping is assumed to be 5% critical.

The ASCE 7 Response Spectrum

is a uniform hazard spectrum based on probabilistic and deterministic seismic hazard analysis.

This slide notes that the ASCE 7 spectrum is a "uniform hazard spectrum." This concept is covered in detail in the topic on seismic hazard analysis. The main purpose of this side is a transition into the hazards topic.