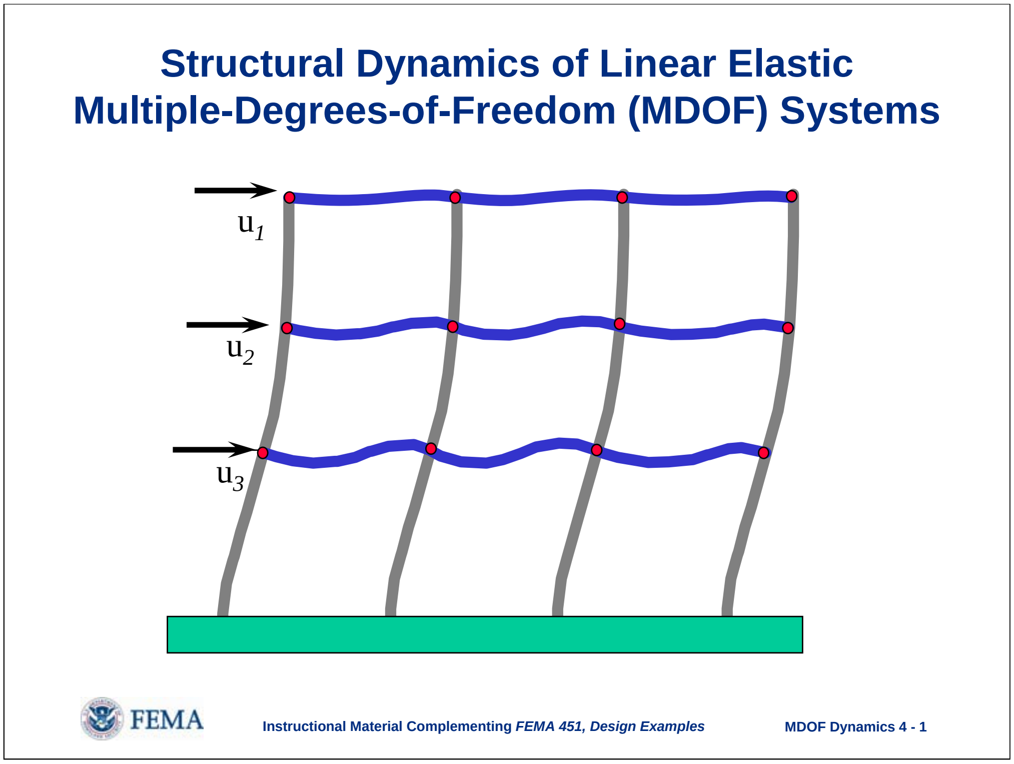

This topic covers the analysis of multiple-degrees-of-freedom (MDOF) elastic systems. The basic purpose of this series of slides is to provide background on the development of the code-based equivalent lateral force (ELF) procedure and modal superposition analysis. The topic is limited to two-dimensional systems.

Structural Dynamics of Elastic MDOF Systems

- Equations of motion for MDOF systems
- Uncoupling of equations through use of natural mode shapes
- Solution of uncoupled equations
- Recombination of computed response
- Modal response history analysis
- Modal response spectrum analysis
- Equivalent lateral force procedure

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Emphasis is placed on simple elastic systems. More complex three-dimensional systems and nonlinear analysis are advanced topics covered under Topic 15-5, Advanced Analysis.

Symbol Styles Used in this Topic

M U	Matrix or vector (column matrix)
m u	Element of matrix or vector or set (often shown with subscripts)
W g	Scalars

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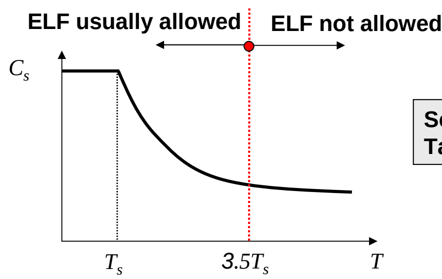
The notation indicated on the slide is used throughout.

Relevance to ASCE 7-05

ASCE 7-05 provides guidance for three specific analysis procedures:

- Equivalent lateral force (ELF) analysis
- Modal superposition analysis (MSA)
- Response history analysis (RHA)



See ASCE 7-05 Table 12.6-1

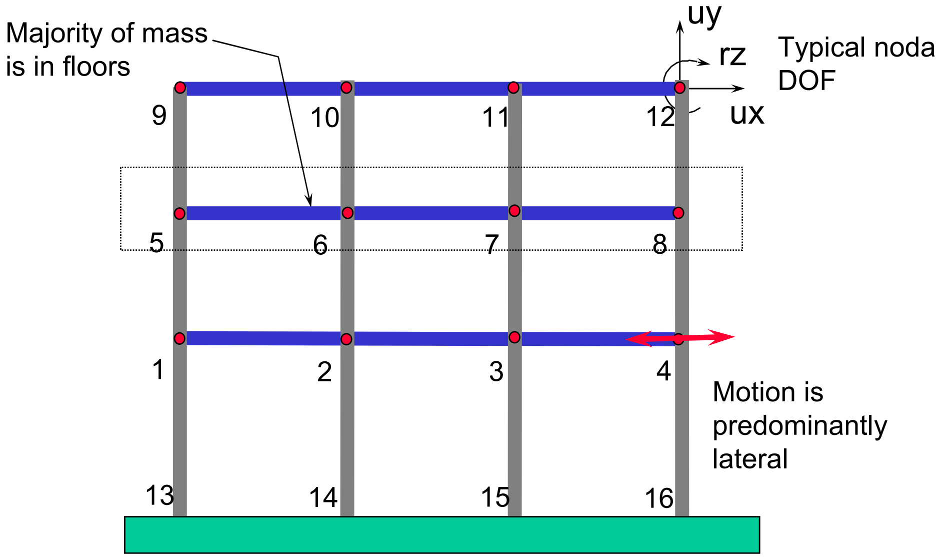
Table 12.6-1 of ASCE 7-05 provides the permitted analytical procedures for systems in different Seismic Design Categories (SDCs). Note that ASCE 7-05 is directly based on the 2003 *NEHRP Recommended Provisions for Seismic Regulations for New Buildings and Other Structures*, FEMA 450, which is available at no charge from the FEMA Publications Center, 1-800-480-2520 (order by FEMA publication number).

Use of the ELF procedure is allowed in the **vast majority** of cases. MSA or RHA is required only for longer period systems or for shorter period systems with certain configuration irregularities (e.g., torsional or soft/weak story irregularities).

Note that response history analysis is never specifically required.

More details will be provided in the topic on seismic load analysis.

Planar Frame with 36 Degrees of Freedom



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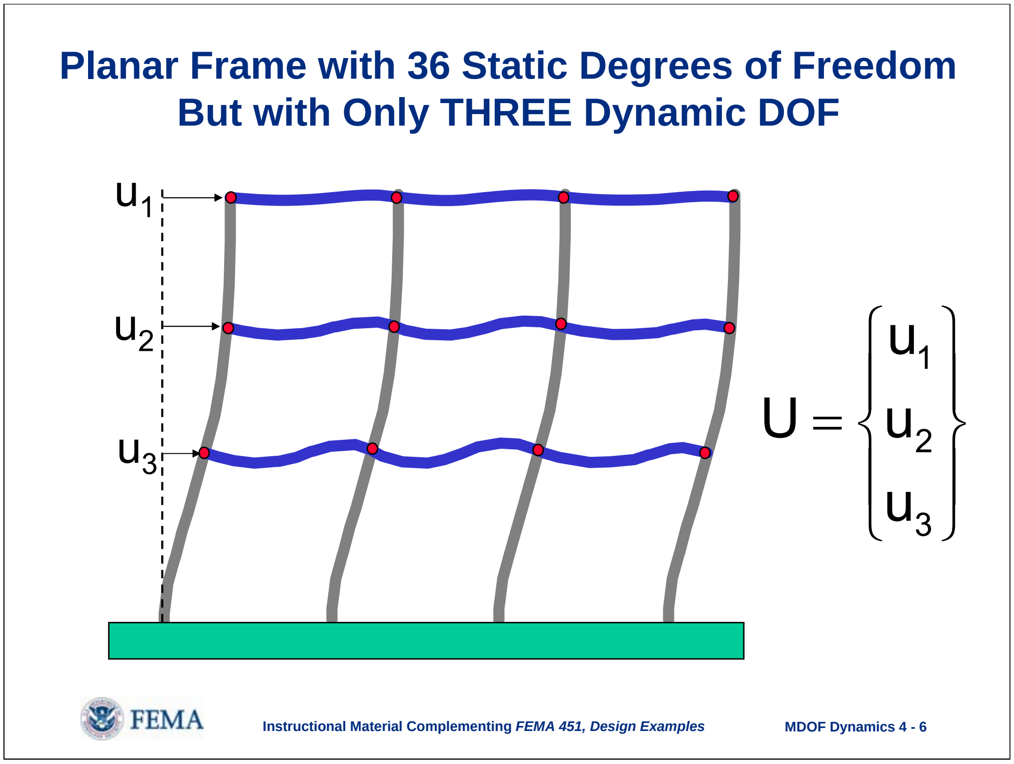
This slide shows that the number of degrees of freedom needed for a dynamic analysis may be less than the number required for static analysis. The principal assumptions that allow this are:

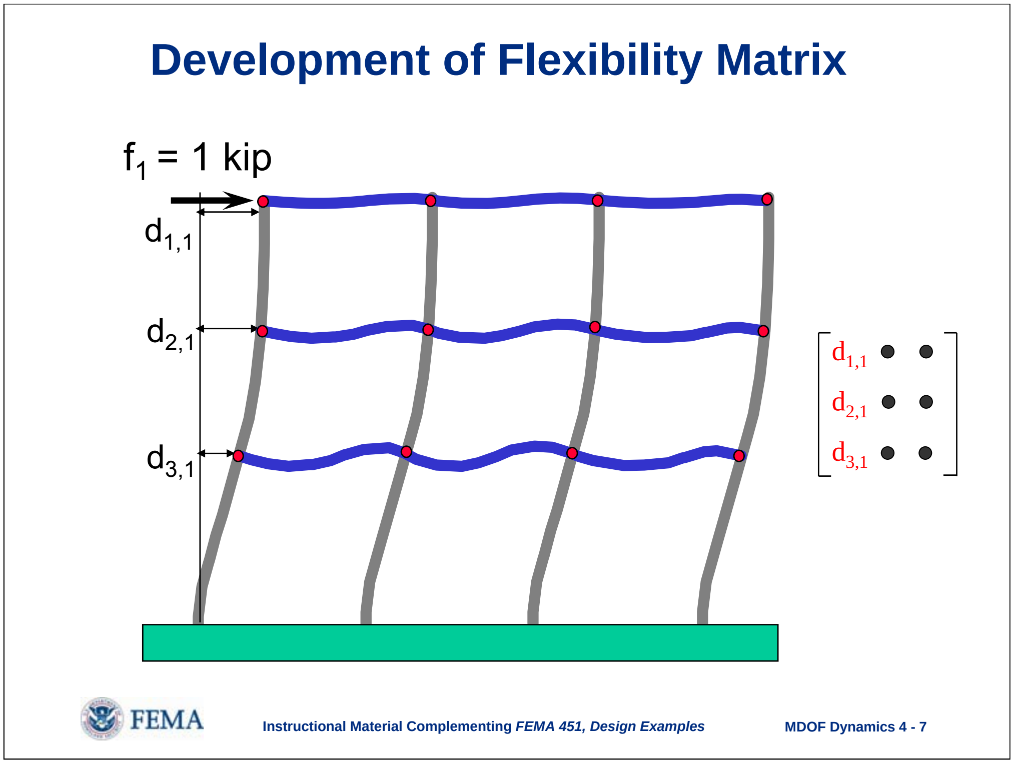
1. Vertical and rotational masses not required,
2. Horizontal mass may be lumped into the floors, and
3. Diaphragms (floors) are axially rigid.

However, all information must be retained in the reduced dynamic model.



The 36 static degrees of freedom may be reduced to only 3 lateral degrees of freedom for the dynamic analysis. This reduction is valid only if the dynamic forces are lateral forces. The three dynamic degrees of freedom are u_1, u_2 and u_3, the lateral story displacements.

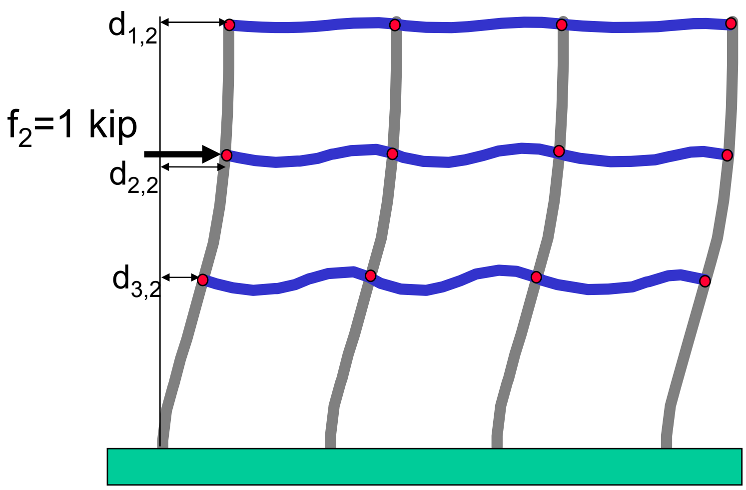
Note that these are the relative displacements and, as such, do not include the ground displacements.



An important concept of analysis of MDOF systems is the change of basis from "normal" Cartesian coordinates to modal coordinates. One way to explain the concept is to show that a flexibility matrix, as generated on this and the next two slides, is simply a column-wise collection of displaced *shapes*. The lateral deflection under any loading may be represented as a linear combination of the columns in the flexibility matrix. This is analogous to the mode shape matrix explained later.

The first column of the flexibility matrix is generated here. Note that a unit load has been used. It is also important to note that ALL 36 DOF ARE REQUIRED in the analysis from which the 3 displacements are obtained.

Development of Flexibility Matrix
(continued)



$$\begin{bmatrix} d_{1,1} & d_{1,2} & \bullet \\ d_{2,1} & d_{2,2} & \bullet \\ d_{3,1} & d_{3,2} & \bullet \end{bmatrix}$$

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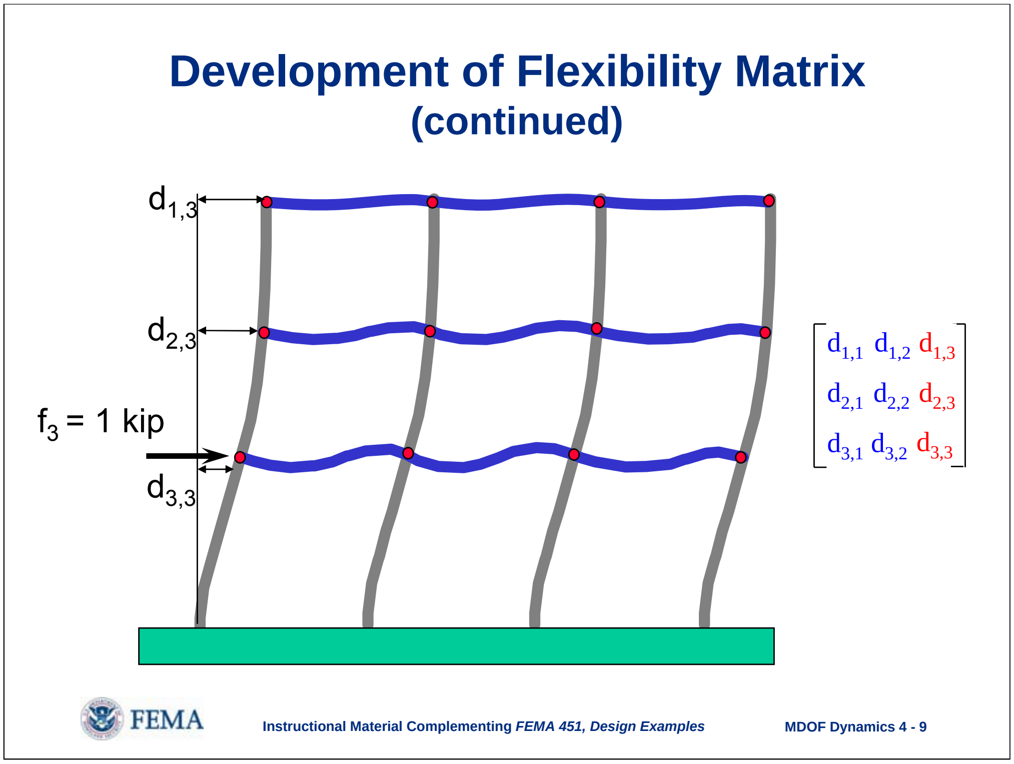
The unit load is applied at DOF 2 and the second column of the flexibility matrix is generated.



The unit load is applied at DOF 3 and the third column of the flexibility matrix is generated.

Concept of Linear Combination of Shapes
(Flexibility)

$$U = \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} \\ d_{2,1} & d_{2,2} & d_{2,3} \\ d_{3,1} & d_{3,2} & d_{3,3} \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}$$

$$U = \begin{Bmatrix} d_{1,1} \\ d_{2,1} \\ d_{3,1} \end{Bmatrix} f_1 + \begin{Bmatrix} d_{1,2} \\ d_{2,2} \\ d_{3,2} \end{Bmatrix} f_2 + \begin{Bmatrix} d_{1,3} \\ d_{2,3} \\ d_{3,3} \end{Bmatrix} f_3$$

$$D F = U \qquad K = D^{-1} \qquad K U = F$$

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For any general loading, F, the displaced shape U is a linear combination of the terms in the columns of the flexibility matrix. Hence, columns of the flexibility matrix are a *basis* for the mathematical representation of the displaced shape.

Note the relationship between flexibility and stiffness. Also note that flexibility as a basis for defining elastic properties is rarely used in modern linear structural analysis.

Static Condensation

$$\begin{bmatrix} K_{m,m} & K_{m,n} \\ K_{n,m} & K_{n,n} \end{bmatrix} \begin{Bmatrix} U_m \\ U_n \end{Bmatrix} = \begin{Bmatrix} F_m \\ \{0\} \end{Bmatrix} \begin{matrix} \text{DOF with mass} \\ \text{Massless DOF} \end{matrix}$$

$$① \quad K_{m,m}U_m + K_{m,n}U_n = F_m$$

$$② \quad K_{n,m}U_m + K_{n,n}U_n = \{0\}$$

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Static condensation is a mathematical procedure wherein all unloaded degrees of freedom are removed from the system of equilibrium equations. The resulting stiffness matrix (see next slide), although smaller than the original, retains all of the stiffness characteristics of the original system.

In the system shown earlier, the full stiffness matrix would be 36 by 36. Only 3 of the 36 DOF have mass (m = 3) and 33 are massless (n = 33). If the full 36 by 36 matrix were available (and properly partitioned), the 3 by 3 matrix could be determined through static condensation.

Note that there are other (more direct) ways to statically condense the non-dynamic degrees of freedom. Gaussian elimination is the most common approach.

Static Condensation
(continued)

Rearrange 2 $U_n = -K_{n,n}^{-1} K_{n,m} U_m$

Plug into 1 $K_{m,m} U_m - K_{m,n} K_{n,n}^{-1} K_{n,m} U_m = F_m$

Simplify $\left[K_{m,m} - K_{m,n} K_{n,n}^{-1} K_{n,m} \right] U_m = F_m$

$$\boxed{\hat{K} = K_{m,m} - K_{m,n} K_{n,n}^{-1} K_{n,m}}$$

Condensed stiffness matrix

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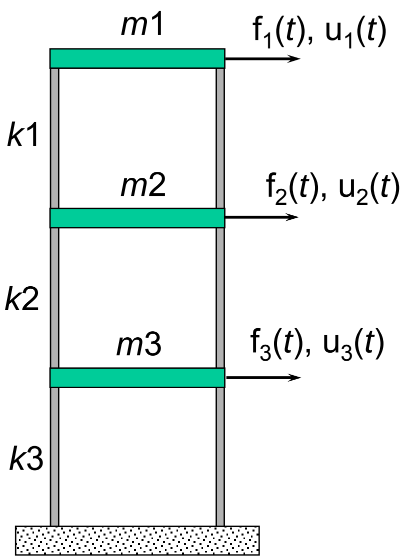
Derivation of static condensation (continued). For the current example, the inverse of K would be identical to the flexibility matrix.

Idealized Structural Property Matrices



$$K = \begin{bmatrix} k1 & -k1 & 0 \\ -k1 & k1+k2 & -k2 \\ 0 & -k2 & k2+k3 \end{bmatrix}$$

$$M = \begin{bmatrix} m1 & 0 & 0 \\ 0 & m2 & 0 \\ 0 & 0 & m3 \end{bmatrix}$$

$$F(t) = \begin{Bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{Bmatrix} \qquad U(t) = \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix}$$

Note: Damping to be shown later

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In the next several slides, a simple three-story frame will be utilized. The representation of the columns as very flexible with respect to the girders is a gross (and not very accurate) approximation. In general, K would be developed from a static condensation of a full stiffness matrix.

In this simple case, K may be determined by imposing a unit displacement at each DOF while restraining the remaining DOF. The forces required to hold the structure in the deformed position are the columns of the stiffness matrix.

The mass matrix is obtained by imposing a unit acceleration at each DOF while restraining the other DOF. The columns of the mass matrix are the (inertial) forces required to impose the unit acceleration. There are no inertial forces at the restrained DOF because they do not move. Hence, the lumped (diagonal) mass matrix is completely accurate for the structure shown.

The terminology used for load, *F(t),* and displacement, *U(t),* indicate that these quantities vary with time.

Coupled Equations of Motion for Undamped Forced Vibration

$$M\ddot{U}(t) + KU(t) = F(t)$$

$$\begin{bmatrix} m1 & 0 & 0 \\ 0 & m2 & 0 \\ 0 & 0 & m3 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \ddot{u}_3(t) \end{Bmatrix} + \begin{bmatrix} k1 & -k1 & 0 \\ -k1 & k1+k2 & -k2 \\ 0 & -k3 & k2+k3 \end{bmatrix} \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{Bmatrix}$$

DOF 1 $m1\ddot{u}_1(t) + k1u_1(t) - k1u_2(t) = f_1(t)$

DOF 2 $m2\ddot{u}_2(t) - k1u_1(t) + k1u_2(t) + k2u_2(t) - k2u_3(t) = f_2(t)$

DOF 3 $m3\ddot{u}_3(t) - k2u_2(t) + k2u_3(t) + k3u_3(t) = f_3(t)$

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This slide shows the MDOF equations of motion for an undamped system subjected to an independent time varying load at DOF 1, 2, and 3. The purpose of this slide is to illustrate the advantages of transforming from u_1, u_2, u_3 to modal coordinates.

When the matrix multiplication is carried out, note that each equation contains terms for displacements at two or more stories. Hence, these equations are “coupled” and cannot be solved independently.

There exist methods for solving the coupled equations of motion but, as will be shown later, this is inefficient in most cases. Instead, the equations will be uncoupled by changing coordinates.

Developing a Way To Solve the Equations of Motion

- This will be done by a transformation of coordinates from *normal coordinates* (displacements at the nodes) To *modal coordinates* (amplitudes of the natural Mode shapes).
- Because of the *orthogonality property* of the natural mode shapes, the equations of motion become uncoupled, allowing them to be solved as SDOF equations.
- After solving, we can transform back to the normal coordinates.

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Our solution approach is to impose a change of coordinates from the *normal coordinates* to the *modal coordinates*. The normal coordinates are simply the displacements at each of the three original DOF in the structure.

The modal coordinates are quite different. They are amplitudes (multipliers) on independent shapes, the linear combination of which forms a mathematical basis for the displaced shape of the system. Through the orthogonality properties of the mode shapes, the equations can be uncoupled, and this is the principal advantage of using these shapes.

Solutions for System in Undamped Free Vibration (Natural Mode Shapes and Frequencies)

$$M\ddot{U}(t)+KU(t)=\{0\}$$

Assume $U(t)=\phi\sin\omega t$ $\ddot{U}(t)=-\omega^2\phi\sin\omega t$

Then $K\phi-\omega^2 M\phi=\{0\}$ has three (n) solutions:

$$\phi_1=\begin{Bmatrix}\phi_{1,1}\\ \phi_{2,1}\\ \phi_{3,1}\end{Bmatrix},\quad \omega_1 \qquad \phi_2=\begin{Bmatrix}\phi_{1,2}\\ \phi_{2,2}\\ \phi_{3,2}\end{Bmatrix},\quad \omega_2 \qquad \phi_3=\begin{Bmatrix}\phi_{1,3}\\ \phi_{2,3}\\ \phi_{3,3}\end{Bmatrix},\quad \omega_3$$

Natural frequency

Natural mode shape

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When the loading term is removed, the equations represent the motion of an undamped structure in free vibration. Under undamped free vibration, the deflected shape of the structure stays the same, and only the amplitude changes with time.

Through the solution of an eigenvalue problem, the free vibration shapes and their natural frequencies are obtained.

Note that there will be n independent solutions to the problem where n is the number of dynamic degrees of freedom.

Solutions for System in Undamped Free Vibration (continued)

For a SINGLE Mode

$$K\Phi = M\Phi\Omega^2$$ For ALL Modes

Where: $\Phi = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \end{bmatrix}$

$$\Omega^2 = \begin{bmatrix} \omega_1^2 & & \\ & \omega_2^2 & \\ & & \omega_3^2 \end{bmatrix} \quad K\phi = \omega^2 M\phi$$

Note: Mode shape has arbitrary scale; usually $\phi_{1,i} = 1.0$ or $\Phi^T M \Phi = I$

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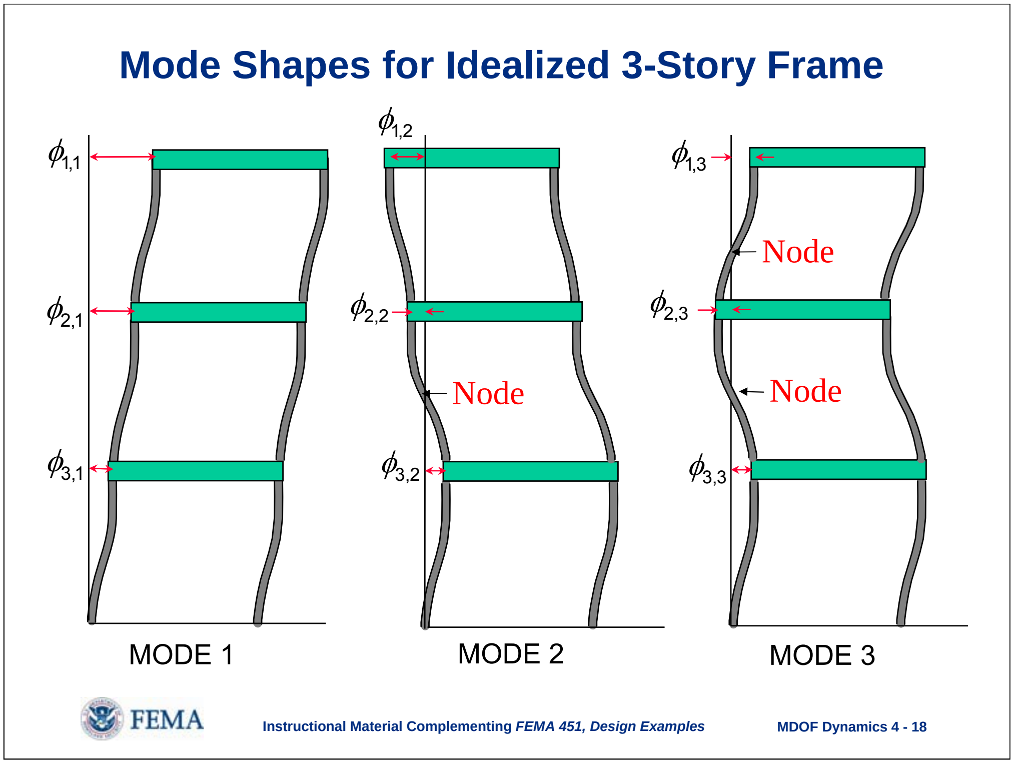
Note that the amplitude of the mode shapes is **arbitrary** because the shape appears on each side of the equation. The shapes may be independently scaled. The two scaling techniques shown have different advantages. These advantages will be described later when use is made of the shapes in uncoupling the equations of motion.



This plot shows idealized mode shapes for a three-story building. Note the relation between modes and "nodes," the number of zero crossings. Higher modes will always vibrate at a greater frequency than the lower modes.

As with any displaced shape, the mode shapes must be compatible with the boundary conditions.

Note that any kinematically admissible displaced shape may be obtained from a linear combination of the mode shapes.

Concept of Linear Combination of Mode Shapes (Transformation of Coordinates)

$$U = \Phi Y$$

Mode shape

$$U = \begin{bmatrix} \phi_{1,1} & \phi_{1,2} & \phi_{1,3} \\ \phi_{2,1} & \phi_{2,2} & \phi_{2,3} \\ \phi_{3,1} & \phi_{3,2} & \phi_{3,3} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \qquad U = \begin{Bmatrix} \phi_{1,1} \\ \phi_{2,1} \\ \phi_{3,1} \end{Bmatrix} y_1 + \begin{Bmatrix} \phi_{1,2} \\ \phi_{2,2} \\ \phi_{3,2} \end{Bmatrix} y_2 + \begin{Bmatrix} \phi_{1,3} \\ \phi_{2,3} \\ \phi_{3,3} \end{Bmatrix} y_3$$

Modal coordinate = amplitude of mode shape

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The mode shape matrix, like the flexibility matrix, is a mathematical basis for defining the displaced shape of the structure.

The mode shape matrix has the advantage that it is an orthogonal basis and, as such, will diagonalize the stiffness matrix.

Orthogonality Conditions

$$\Phi = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \end{bmatrix}$$

Generalized mass

$$\Phi^T M \Phi = \begin{bmatrix} m_1^* & & \\ & m_2^* & \\ & & m_3^* \end{bmatrix}$$

Generalized stiffness

$$\Phi^T K \Phi = \begin{bmatrix} k_1^* & & \\ & k_2^* & \\ & & k_3^* \end{bmatrix}$$

Generalized damping

$$\Phi^T C \Phi = \begin{bmatrix} c_1^* & & \\ & c_2^* & \\ & & c_3^* \end{bmatrix}$$

Generalized force

$$\Phi^T \mathrm{F}(t) = \begin{Bmatrix} \mathrm{f}_1^*(t) \\ \mathrm{f}_2^*(t) \\ \mathrm{f}_3^*(t) \end{Bmatrix}$$

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The orthogonality condition is an extremely important concept as it allows for the full uncoupling of the equations of motion.

The damping matrix (which is not involved in eigenvalue calculations) will be diagonalized as shown only under certain conditions. In general, C will be diagonalized if it satisfies the Caughey criterion: $CM^{-1}K = KM^{-1}C$

We will assume a diagonalizable C and show conditions that will enforce this assumption.

Development of Uncoupled Equations of Motion

MDOF equation of motion: $M\ddot{U} + C\dot{U} + KU = F(t)$

Transformation of coordinates: $U = \Phi Y$

Substitution: $M\Phi\ddot{Y} + C\Phi\dot{Y} + K\Phi Y = F(t)$

Premultiply by Φ^{T}: $\Phi^{T}M\Phi\ddot{Y} + \Phi^{T}C\Phi\dot{Y} + \Phi^{T}K\Phi Y = \Phi^{T}F(t)$

Using orthogonality conditions, uncoupled equations of motion are:

$$\begin{bmatrix} m_1^* & & \\ & m_2^* & \\ & & m_3^* \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{Bmatrix} + \begin{bmatrix} c_1^* & & \\ & c_2^* & \\ & & c_3^* \end{bmatrix} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{Bmatrix} + \begin{bmatrix} k_1^* & & \\ & k_2^* & \\ & & k_3^* \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} f_1^*(t) \\ f_2^*(t) \\ f_3^*(t) \end{Bmatrix}$$

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This slide shows how the original equations of motion are uncoupled by a change of coordinates and by use of the orthogonality conditions.

Development of Uncoupled Equations of Motion (Explicit Form)

Mode 1 $$m_1^*\ddot{y}_1 + c_1^*\dot{y}_1 + k_1^* y_1 = f_1^*(t)$$

Mode 2 $$m_2^*\ddot{y}_2 + c_2^*\dot{y}_2 + k_2^* y_2 = f_2^*(t)$$

Mode 3 $$m_3^*\ddot{y}_3 + c_3^*\dot{y}_3 + k_3^* y_3 = f_3^*(t)$$

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This slide shows the fully uncoupled equations of motion. Each equation has a single unknown, y, which is the amplitude of the related mode shape.

The * superscript in the mass, damping, stiffness, and load terms represents the fact that these are the "generalized" quantities for the system transformed into modal coordinates.

Development of Uncoupled Equations of Motion (Explicit Form)

Simplify by dividing through by m* and defining $\xi_i = \dfrac{c_i^*}{2 m_i^* \omega_i}$

Mode 1 $$\ddot{y}_1 + 2\xi_1\omega_1\dot{y}_1 + \omega_1^2 y_1 = f_1^*(t)/m_1^*$$

Mode 2 $$\ddot{y}_2 + 2\xi_2\omega_2\dot{y}_2 + \omega_2^2 y_2 = f_2^*(t)/m_2^*$$

Mode 3 $$\ddot{y}_3 + 2\xi_3\omega_3\dot{y}_3 + \omega_3^2 y_3 = f_3^*(t)/m_3^*$$

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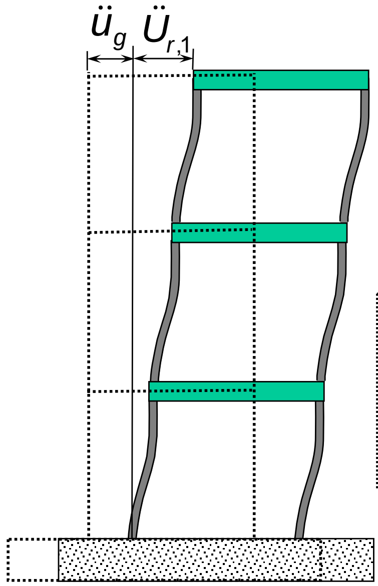
Here, each equation is simply divided by the generalized mass (never zero) for that mode. Using the definition for damping ratio (see the topic on SDOF dynamics), the generalized damping term in each equation has been eliminated. Now, the solution for each mode depends only on the damping ratio, the frequency, and the loading history.

Earthquake "Loading" for MDOF System



$$F_I(t) = M\begin{Bmatrix} \ddot{u}_g(t) + \ddot{u}_{r,1}(t) \\ \ddot{u}_g(t) + \ddot{u}_{r,2}(t) \\ \ddot{u}_g(t) + \ddot{u}_{r,3}(t) \end{Bmatrix} =$$

$$\boxed{M\begin{Bmatrix} 1.0 \\ 1.0 \\ 1.0 \end{Bmatrix}\ddot{u}_g(t)} + M\begin{Bmatrix} \ddot{u}_{r,1}(t) \\ \ddot{u}_{r,2}(t) \\ \ddot{u}_{r,3}(t) \end{Bmatrix}$$

Move to RHS as $F_{EFF}(t) = -\,M\,R\ddot{u}_g(t)$

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As with SDOF systems, it is necessary to develop effective earthquake forces when the loading arises from a ground acceleration history.

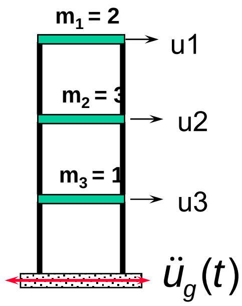
At each level of the structure, the inertial force is equal to the mass times the total acceleration. The total acceleration is equal to the ground acceleration (a scalar) and the appropriate relative acceleration (a component of a vector).

The inertial force is converted into the sum of two vectors through use of the "influence coefficient vector," R. R contains a value of one for each mass that develops an inertial force when the whole system is accelerated horizontally. In this case, all of the values are 1 because all masses are affected by the horizontal ground motion. When forming R, the whole structure is accelerated as a rigid body.

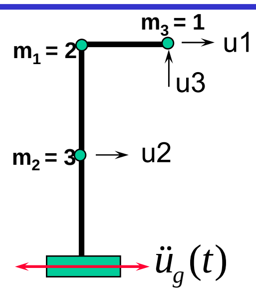
The relative acceleration part of the inertial force is retained on the left, and the ground acceleration part is moved to the right to form the effective earthquake load.

Modal Earthquake Loading

$$F^*(t) = -\Phi^T M R \ddot{u}_g(t)$$



$$M = \begin{bmatrix} 2 & & \\ & 3 & \\ & & 1 \end{bmatrix} \qquad R = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$



$$M = \begin{bmatrix} 2+1 & & \\ & 3 & \\ & & 1 \end{bmatrix} \qquad R = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

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The influence coefficient vector, R, is not always full of zeros. This is explained by the two examples taken from Clough and Penzien.

In the first case, inertial forces develop at each DOF when the structure is accelerated horizontally.

In the second case, the vertical mass at DOF 3 does not develop an inertial force when the whole structure is horizontally accelerated as a rigid body and, hence, the relater term in R is zero. This represents the fact that the ground motion part of the total acceleration at this DOF is identically zero.

Note, however, that this mass will develop inertial forces when the structure responds to the ground shaking because relative vertical accelerations will occur at that DOF.

Definition of Modal Participation Factor

For earthquakes: $f_i^*(t) = -\phi_i^T MR\ddot{u}_g(t)$

Typical modal equation:

$$\ddot{y}_i + 2\xi_i\omega_i\dot{y}_i + \omega_i^2 y_i = \frac{f_i^*(t)}{m_i^*} = -\boxed{\frac{\phi_i^T MR}{m_i^*}}\ddot{u}_g(t)$$

Modal participation factor p_i

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When the effective earthquake force is transformed into modal coordinates and the modal equation is divided by the generalized mass, a term called the "modal participation factor" emerges.

The modal participation factor often appears in computer output for standard structural dynamics programs (e.g., *SAP2000*). Note that the quantity is unitless but is affected by the way the modes are scaled.

If one compares the simplified equation for SFOF earthquake response to the one shown in this slide, it may be seen that the only difference is the presence of P in the modal equation. Hence, the solution for a single mode of the full system is simply P times the solution for the SDOF system (with the same ξ and ω).

Caution Regarding Modal Participation Factor

$$p_i = \frac{\phi_i^T MR}{m_i^*}$$

$$m_i^* = \boxed{\phi_i^T M \phi_i}$$

Its value is dependent on the (arbitrary) method used to scale the mode shapes.

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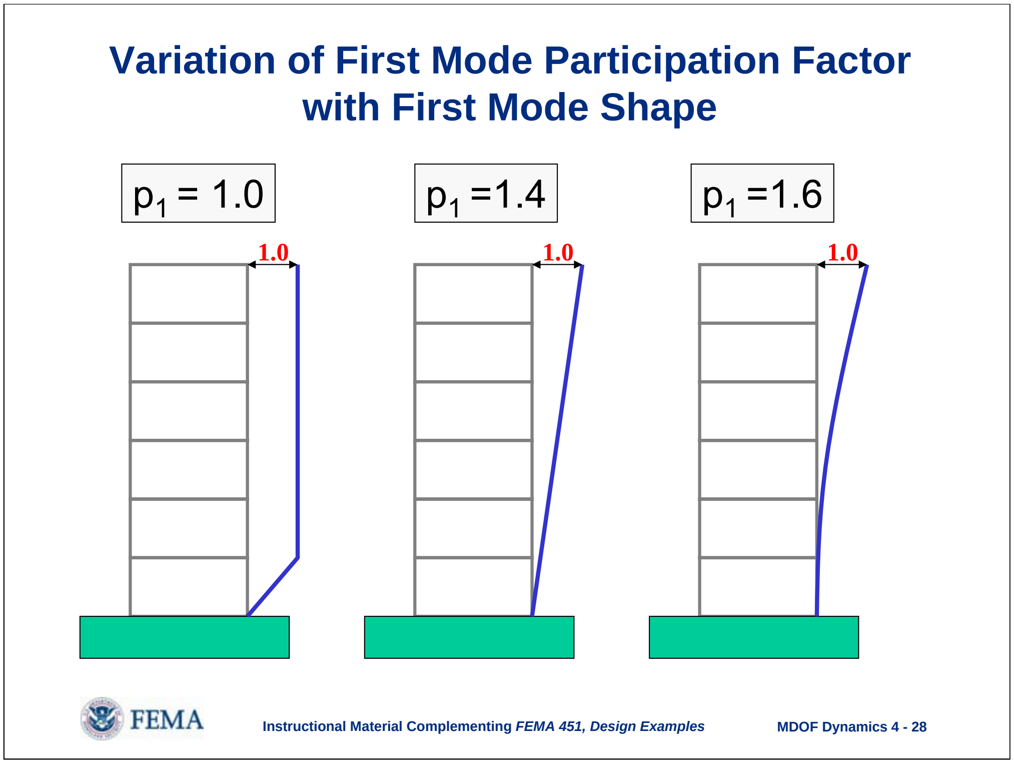
It is very important to note that the value of the modal participating factor is a function of how the modes are normalized. This is apparent because the modal scale factor is squared in the denominator but appears only once in the numerator.

However, the mode's computed response history (with an extra ϕ in the denominator), when multiplied by the corresponding mode shape, will produce an invariant response in that mode.



This slide shows what the modal participation factor would be for systems with three possible first mode shapes. Note that in each case, the mode shape has been normalized to have the maximum value equal to 1.0.

The first case, with P = 1, is impossible because the upper portion of the system is acting as a rigid body (and as such is not truly a MSOF). Thus, P = 1 is an lower bound for system normalized as shown. The P values for the next two systems are more realistic. And the values of P between 1.4 and 1.6 can be used as a reality check when evaluating the accuracy of the output from computer software.

Concept of Effective Modal Mass

For each Mode *I*, $\bar{m}_i = p_i^2 m_i^*$

- The sum of the effective modal mass for all modes is
  equal to the total structural mass.
- The value of effective modal mass is *independent* of mode shape scaling.
- Use enough modes in the analysis to provide a total effective mass not less than 90% of the total structural mass.

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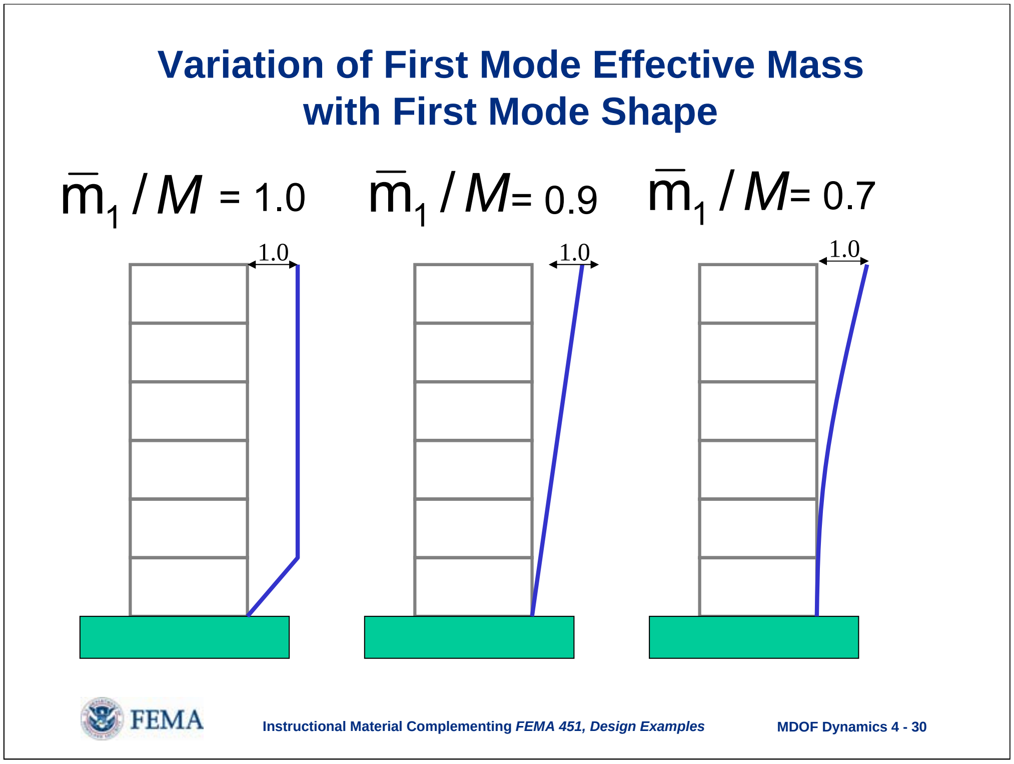
The concept of effective modal mass is important. It can be interpreted as the part of the total mass responding to the earthquake in each mode . Note the scaling problem disappears. The requirement that the number of modes needed to produce 90% of the effective mass is a code provision.



The effective modal mass for various first modes is given here. Again, it is assumed that each mode is normalized to give 1.0 at the top. This is important for the modal participation factor but not for effective modal mass because the latter does not depend on modal scaling.

The first mode shape is, of course, impossible because there is no mass left for the other modes.

Derivation of Effective Modal Mass

(continued)

For each mode:

$$\ddot{y}_i + 2\xi_i \omega_i \dot{y}_i + \omega_i^2 y_i = -p_i \ddot{u}_g$$

SDOF system:

$$\ddot{q}_i + 2\xi_i \omega_i \dot{q}_i + \omega_i^2 q_i = -\ddot{u}_g$$

Modal response history, $q_i(t)$ is obtained by first solving the SDOF system.

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The derivation for effective modal mass is given in the following slides.

Derivation of Effective Modal Mass
(continued)

From previous slide $y_i(t) = p_i \, q_i(t)$

Recall $u_i(t) = \phi_i \, y_i(t)$

Substitute $u_i(t) = p_i \, \phi_i \, q_i(t)$

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Derivation of effective modal mass continued.

Derivation of Effective Modal Mass
(continued)

Applied "static" forces required to produce $u_i(t)$:

$$V_i(t) = Ku_i(t) = P_i K\phi_i q_i(t)$$

Recall: $$K\phi_i = \omega_i^2 M\phi_i$$

Substitute:

$$V_i(t) = M\phi_i P_i \omega_i^2 q_i(t)$$

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Derivation of effective modal mass continued.

Derivation of Effective Modal Mass
(continued)

Total shear in mode: $\bar{V}_i = V_i^T R$

$$\bar{V}_i = (M\phi_i)^T R P_i \omega^2 q_i(t) = \phi_i^T M R P_i \boxed{\omega^2 q_i(t)}$$

"Acceleration" in mode

Define effective modal mass:

$$\bar{M}_i = \phi_i^T M R P_i$$

and

$$\bar{V}_i = \bar{M}_i \omega^2 q_i(t)$$

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Derivation of effective modal mass continued.

Derivation of Effective Modal Mass
(continued)

$$\bar{M}_i = \phi_i^T MRP_i = \left[\frac{\phi_i^T MR}{\phi_i^T M \phi_i} \right]_i \phi_i^T M \phi_i P_i$$

$$\bar{M}_i = P_i^2 m_i^*$$

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Derivation of effective modal mass continued.

Development of a Modal Damping Matrix

In previous development, we have assumed:

$$\Phi^T C \Phi = \begin{bmatrix} c_1^* & & \\ & c_2^* & \\ & & c_3^* \end{bmatrix}$$

Two methods described herein:

- Rayleigh “proportional damping”
- Wilson “discrete modal damping”

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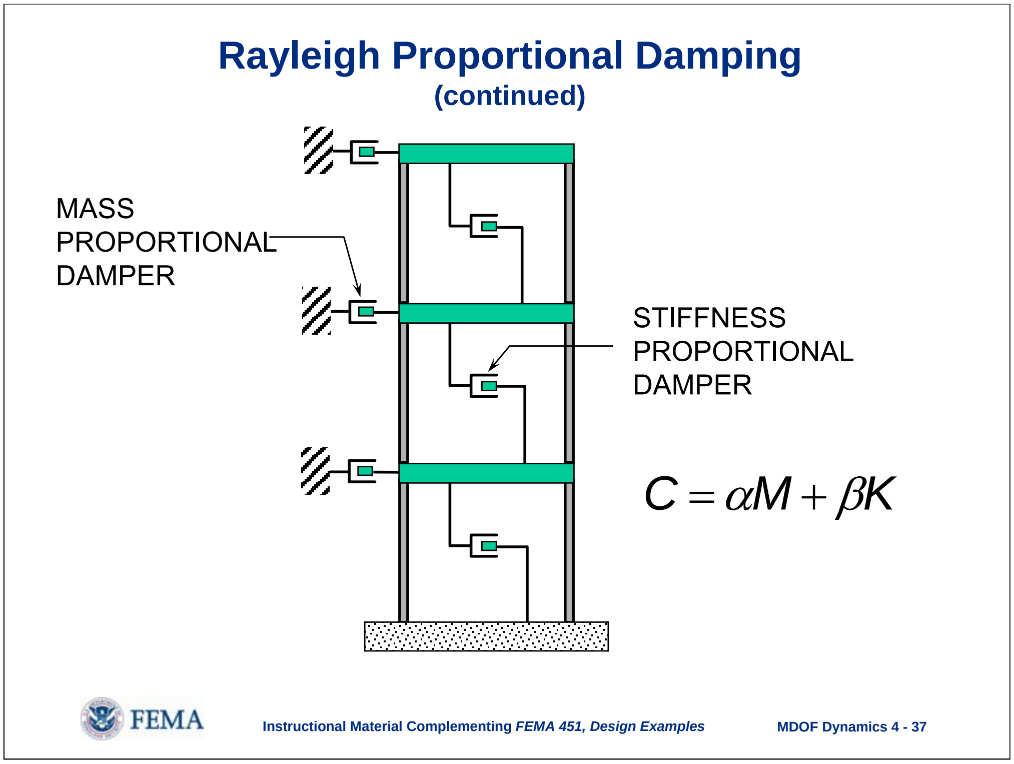
For structures without added dampers, the development of an explicit damping matrix, C, is not possible because discrete dampers are not attached to the dynamic DOF. However, some mathematical entity is required to represent natural damping.

An arbitrary damping matrix cannot be used because there would be no guarantee that the matrix would be diagonalized by the mode shapes.

The two types of damping shown herein allow for the uncoupling of the equations.



The mathematical model for Rayleigh damping consists of a series of mass and stiffness proportional dampers. Since the mass matrix and the stiffness matrix are diagonalized by the mode shapes, so then would be any linear combination of the two.

If the dampers (as shown) are given arbitrary (nonproportional) values, the damping matrix will not be uncoupled. A complex (imaginary number) eigenvalue problem would then be required.

It should be noted that the mass proportional component of Rayleigh damping will produce artificial reactions along the height of the structure. Hence, the elastic force plus the stiffness proportional damping force across a story will not be equal to the inertial shear across the story. This apparent equilibrium error can be a problem for highly damped structures (damping above 10% critical).

Rayleigh Proportional Damping
(continued)

$$C = \alpha M + \beta K$$

For modal equations to be uncoupled:

$$2\omega_n \xi_n = \phi_n^T C \phi_n$$

Assumes
$\Phi^T M \Phi = I$

Using orthogonality conditions:

$$2\omega_n \xi_n = \alpha + \beta \omega_n^2$$

$$\xi_n = \frac{1}{2\omega_n}\alpha + \frac{\omega_n}{2}\beta$$

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The development of the proportionality constants is shown here. For any system, the damping in any two modes is specified. By solving a 2 by 2 set of simultaneous equations, the alpha (mass proportional) and beta (stiffness proportional) terms are computed. Once these terms are known, the damping obtained in the other modes may be back calculated.

Rayleigh Proportional Damping
(continued)

Select damping value in two modes, ξ_m and ξ_n

Compute coefficients α and β:

$$\begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = 2\frac{\omega_m \omega_n}{\omega_n^2 - \omega_m^2} \begin{bmatrix} \omega_n & -\omega_m \\ -1/\omega_n & 1/\omega_m \end{bmatrix} \begin{Bmatrix} \xi_m \\ \xi_n \end{Bmatrix}$$

Form damping matrix $C = \alpha M + \beta K$

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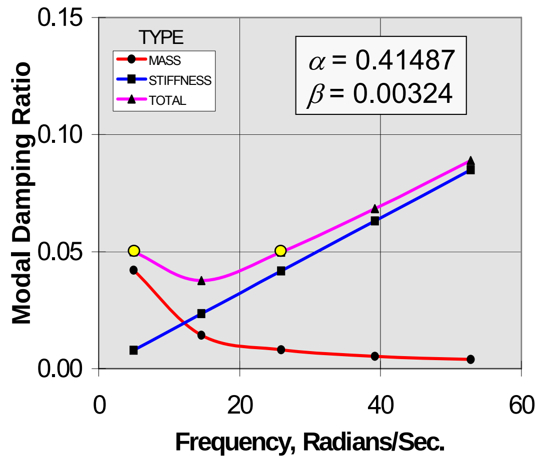
The mass and stiffness proportionality constants are determined as shown.

Rayleigh Proportional Damping (Example)

5% critical in Modes 1 and 3

Structural frequencies

Mode	ω
1	**4.94**
2	14.6
3	**25.9**
4	39.2
5	52.8



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This slide shows how the damping varies for a 5-DOF structure with 5% damping specified in Modes 1 and 3. The total damping (uppermost line) and the mass and stiffness proportional components are shown.

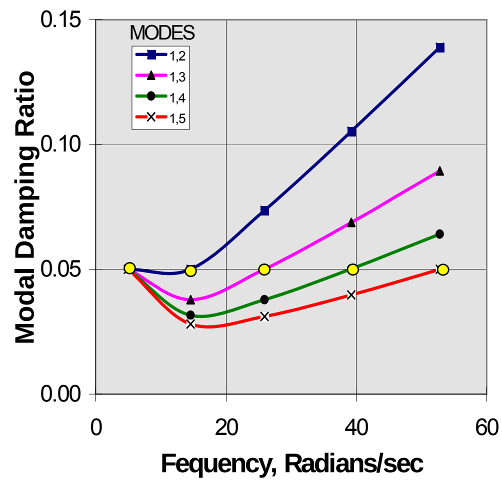
Note that Mode 2 will have less than 5% damping and Modes 4 and 5 will have greater than 5% damping.

Note that for stiffness proportional damping only (alpha = 0), the damping increases linearly with frequency, effectively damping out the higher modes.

Rayleigh Proportional Damping (Example)
5% Damping in Modes 1 & 2, 1 & 3, 1 & 4, or 1 & 5

Proportionality factors
(5% each indicated mode)

Modes	α	β
1 & 2	.36892	0.00513
1 & 3	.41487	0.00324
1 & 4	.43871	0.00227
1 & 5	.45174	0.00173



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This slide shows total damping for various combinations of modal damping. Note that if 5% is specified in Modes 1 and 2, Modes 4 and 5 will effectively be damped out. If 5% is specified in Modes 1 and 5, Modes 2, 3, and 4 may be underdamped.

Rayleigh damping is rarely used in large-scale linear structural analysis. It is very often used in nonlinear analysis where the full set of equations is simultaneously integrated. Interestingly, a proportional damping matrix is really not required in that case. For example, DRAIN-2D allows element-by-element stiffness proportional damping.

Wilson Damping

Directly specify modal damping values ξ_i^*

$$\Phi^T C \Phi = \begin{bmatrix} c_1^* & & \\ & c_2^* & \\ & & c_3^* \end{bmatrix} = \begin{bmatrix} 2m_1^*\omega_1\xi_1^* & & \\ & 2m_2^*\omega_2\boxed{\xi_2^*} & \\ & & 2m_3^*\omega_3\xi_3^* \end{bmatrix}$$

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In Wilson damping, the modal damping values are directly specified for the uncoupled equations of motion. In this case, an explicit damping matrix C need not be formed.

Formation of Explicit Damping Matrix From "Wilson" Modal Damping

(NOT Usually Required)

$$\Phi^T C \Phi = \begin{bmatrix} 2\xi_1\omega_1 & & & & \\ & 2\xi_2\omega_2 & & & \\ & & \bullet\bullet\bullet & & \\ & & & 2\xi_{n-1}\omega_{n-1} & \\ & & & & 2\xi_n\omega_n \end{bmatrix} = c$$

$$\left(\Phi^T\right)^{-1} c \Phi^{-1} = C$$

$$C = M\left[\sum_{i=1}^{n} 2\xi_i\omega_i\phi_i^T\phi_i\right]M$$

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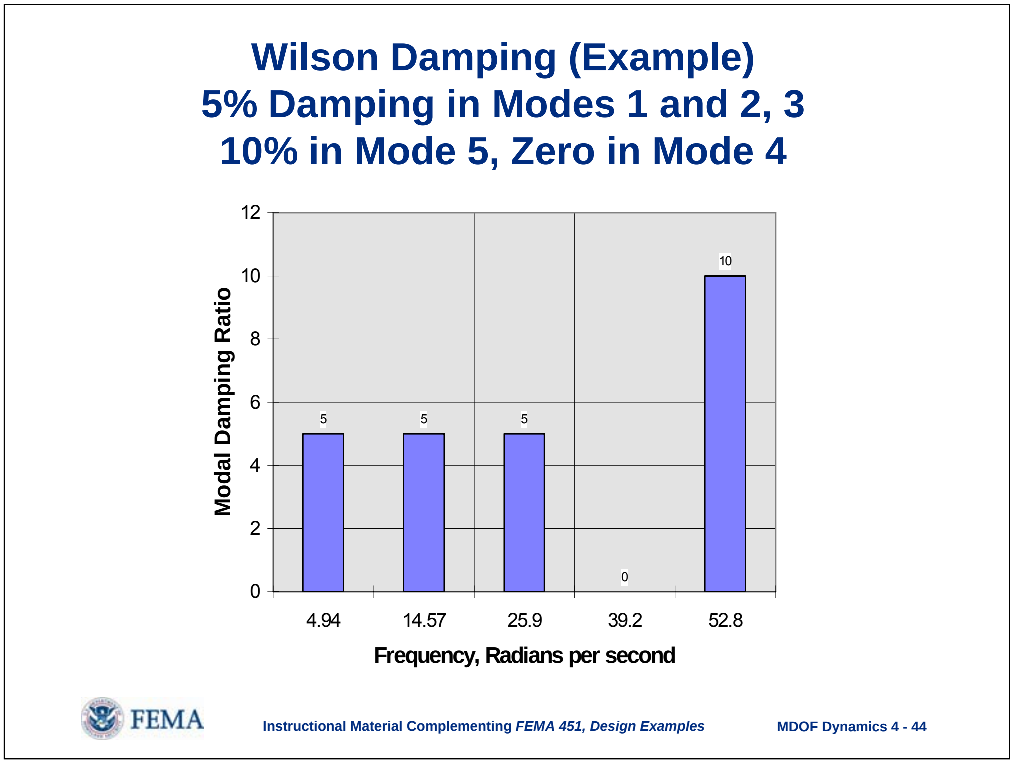
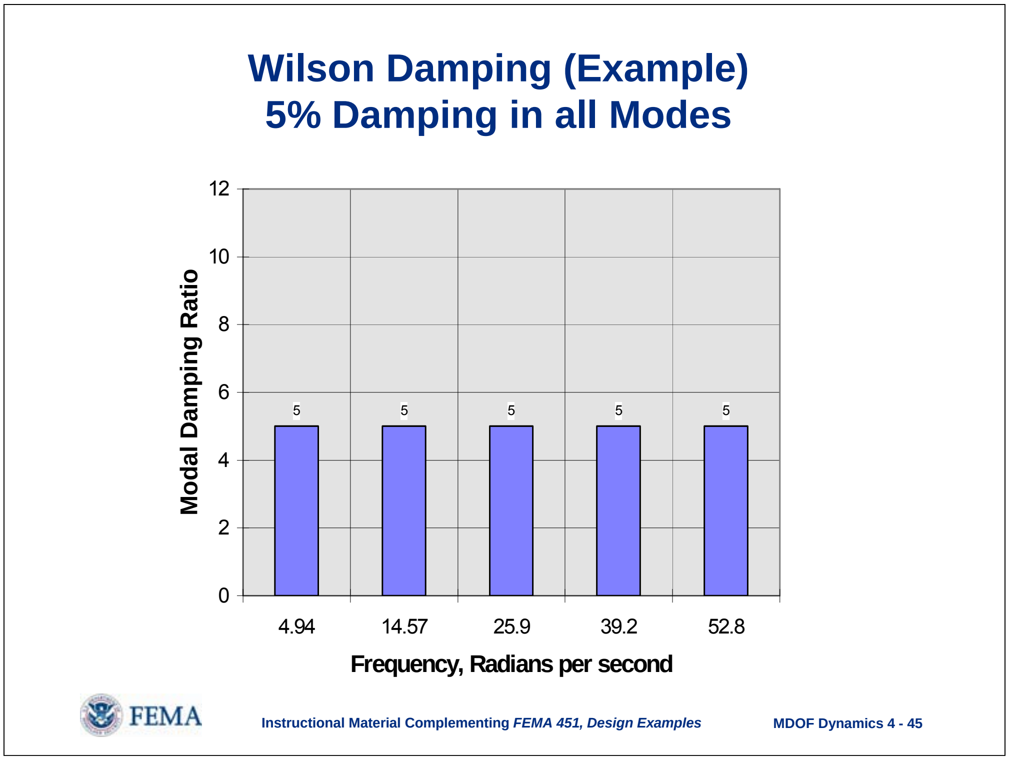
A Wilson damping matrix can be explicitly formed using the equation shown. In general, the damping matrix will be full, causing coupling across degrees of freedom that are not actually coupled. As with Rayleigh damping, this may lead to errors in highly damped structures (damping above about 10% critical).



In Wilson damping, only those modal DOF that are assigned damping will be damped. All other modes will be undamped. This illustration shows 5% damping in Modes 1, 2 and 3, zero damping in Mode 4, and 10% in Mode 5.



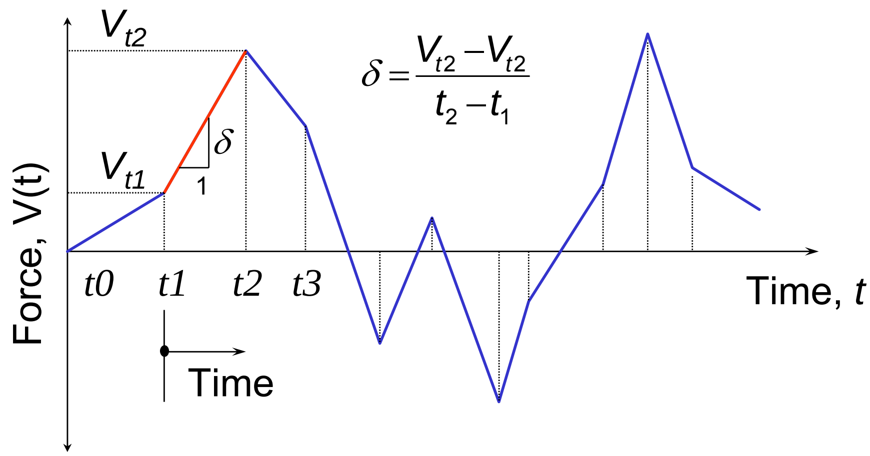
The same damping usually will be used in each mode as shown here. Instead of using high damping values in the higher modes, simply do not include the modes in the analysis. (This is explained later.)

Solution of MDOF Equations of Motion

- Explicit (step by step) integration of *coupled* equations
- Explicit integration of FULL SET of *uncoupled* equations
- Explicit integration of PARTIAL SET of *uncoupled* Equations (approximate)
- Modal response spectrum analysis (approximate)

This slide shows the scope of the remainder of the unit. In all cases, the information will be presented by example.

Computed Response for Piecewise Linear Loading



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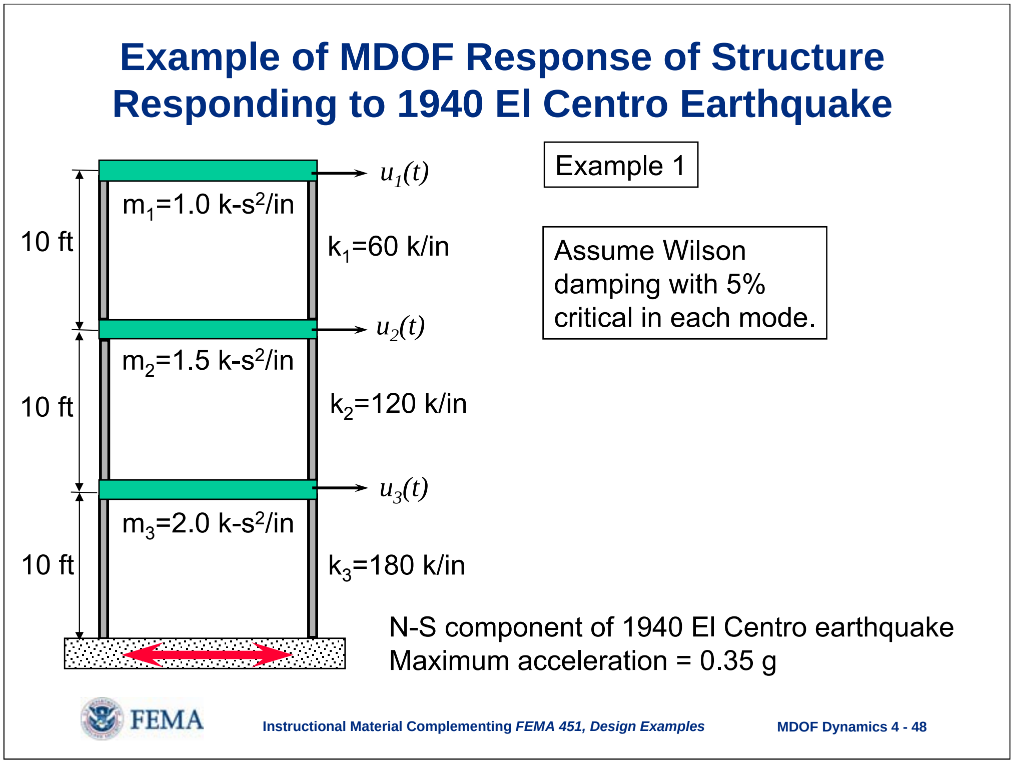
Once the equations have been uncoupled, the individual response history analyses are best carried out through the use of the piece-wise exact method covered in the single-degree-of-freedom topic.

The use of a Newmark method may introduce undesired effects (e.g., additional artificial damping in higher modes and period elongation). However, the Newmark method (or some variation thereof) must be used in nonlinear analyses. The exception is the Wilson FNA method as explained in *Three-Dimensional Analysis of Structures* by Wilson.

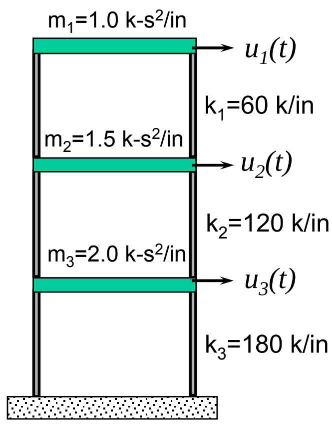


In the next several slides, this structure will be analyzed. This structure is taken from Example 26-3 of Clough and Penzien, 2nd Edition.

Note that the structure has nonuniform mass and stiffness. This is not a problem for the modal response history and modal response spectrum techniques but may pose problems for the ELF method. For comparison purposes, a similar example with uniform properties follows the example shown on this slide.

The second example is typically not included in a classroom presentation of this topic.

Example 1 (continued)



Form property matrices:

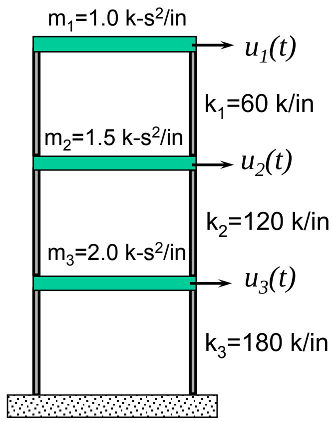
$$M = \begin{bmatrix} 1.0 & & \\ & 1.5 & \\ & & 2.0 \end{bmatrix} \text{kip} - \text{s}^2/\text{in}$$

$$K = \begin{bmatrix} 60 & -60 & 0 \\ -60 & 180 & -120 \\ 0 & -120 & 300 \end{bmatrix} \text{kip/in}$$

These are the mass and stiffness matrices of the structure. Note again that the stiffness matrix is highly idealized for this example.

The damping matrix is not needed because the damping in the various modes is assigned directly (Wilson damping).

Example 1 (continued)



Solve eigenvalue problem:

$$K\Phi = M\Phi\Omega^2$$

$$\Omega^2 = \begin{bmatrix} 21.0 & & \\ & 96.6 & \\ & & 212.4 \end{bmatrix} \sec^{-2}$$

$$\Phi = \begin{bmatrix} 1.000 & 1.000 & 1.000 \\ 0.644 & -0.601 & -2.57 \\ 0.300 & -0.676 & 2.47 \end{bmatrix}$$

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The mode shapes and frequencies were taken from Clough and Penzien and independently verified.

Note that the mode shapes have been scaled to produce a unit positive displacement at the top of the structure for each mode. It is often more convenient to scale such that $\phi_i^T M \phi_i = 1.0$.

Normalization of Modes Using $\Phi^T M \Phi = I$

$$\Phi = \begin{bmatrix} 0.749 & 0.638 & 0.208 \\ 0.478 & -0.384 & -0.534 \\ 0.223 & -0.431 & 0.514 \end{bmatrix} \quad \text{vs} \quad \begin{bmatrix} 1.000 & 1.000 & 1.000 \\ 0.644 & -0.601 & -2.57 \\ 0.300 & -0.676 & 2.47 \end{bmatrix}$$

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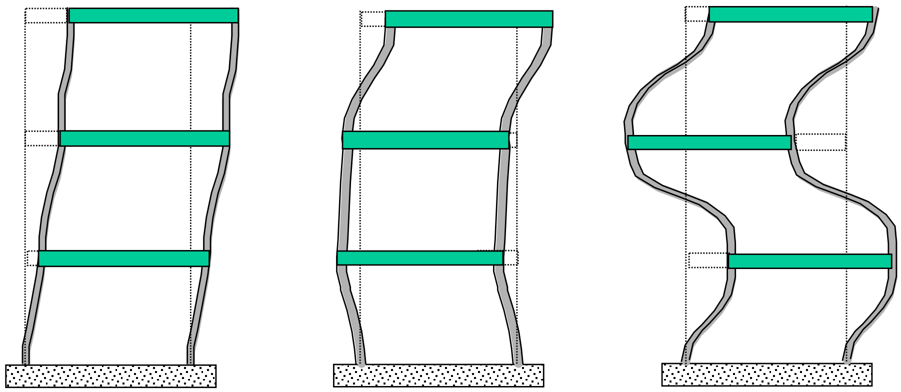
This slide demonstrates two alternatives for scaling the modes. The generalized mass matrix for the system with the modes as normalized to the left are 1.0 for each mode. The example uses the mode shapes shown on the right.

While the mode shapes have different scale factors, this will not affect the computed results.

Example 1 (continued)

Mode Shapes and Periods of Vibration



MODE 1	MODE 2	MODE 3
ω = 4.58 rad/sec	ω = 9.83 rad/sec	ω = 14.57 rad/sec
T = 1.37 sec	T = 0.639 sec	T = 0.431 sec

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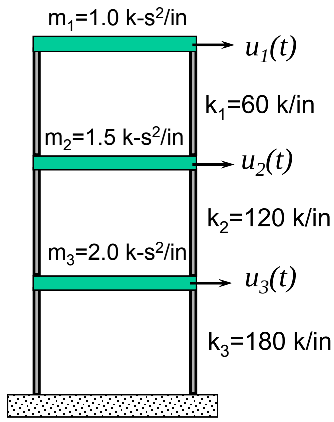
This slide shows the mode shapes approximately to scale. Note that the first mode period of vibration, 1.37 seconds, is more appropriate for a 10-story building than it is for a 3-story structure.

Example 1 (continued)



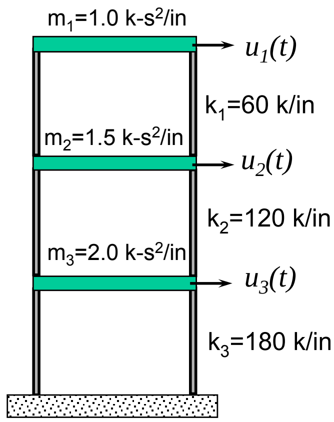
$$\omega_n = \begin{Bmatrix} 4.58 \\ 9.83 \\ 14.57 \end{Bmatrix} \text{rad/sec} \qquad T_n = \begin{Bmatrix} 1.37 \\ 0.639 \\ 0.431 \end{Bmatrix} \text{sec}$$

Compute Generalized Mass:

$$M^* = \Phi^T M \Phi = \begin{bmatrix} 1.801 & & \\ & 2.455 & \\ & & 23.10 \end{bmatrix} \text{kip} - \text{sec}^2/\text{in}$$

This slide shows the generalized mass matrix. These masses are appropriate for the *modes* and should not be confused with the story masses. The magnitude of the generalized mass terms is a function of the mode shape normalization technique used.

Example 1 (continued)



Compute generalized loading:

$$V^*(t) = -\Phi^T M R \ddot{v}_g(t)$$

$$V_n^* = -\begin{Bmatrix} 2.566 \\ -1.254 \\ 2.080 \end{Bmatrix} \ddot{v}_g(t)$$

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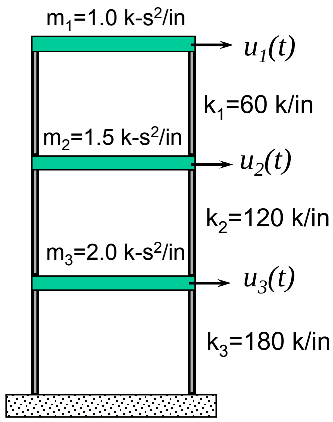
This slide shows how the effective earthquake force has been computed for each mode. The magnitude of the generalized effective earthquake force is also a function of the mode normalization technique used.

The values shown in the lower equation are NOT the modal participation factors. To get the participation factors, the equation for each mode must be divided by the generalized mass for that mode

Example 1 (continued)

Write uncoupled (modal) equations of motion:



$$\ddot{y}_1 + 2\xi_1\omega_1\dot{y}_1 + \omega_1^2 y_1 = V_1^*(t)/m_1^*$$

$$\ddot{y}_2 + 2\xi_2\omega_2\dot{y}_2 + \omega_2^2 y_2 = V_2^*(t)/m_2^*$$

$$\ddot{y}_3 + 2\xi_3\omega_3\dot{y}_3 + \omega_3^2 y_3 = V_3^*(t)/m_3^*$$

$$\ddot{y}_1 + 0.458\dot{y}_1 + 21.0 y_1 = -1.425\ddot{v}_g(t)$$

$$\ddot{y}_2 + 0.983\dot{y}_2 + 96.6 y_2 = 0.511\ddot{v}_g(t)$$

$$\ddot{y}_3 + 1.457\dot{y}_3 + 212.4 y_3 = -0.090\ddot{v}_g(t)$$

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Here, the full set of uncoupled equations has been formed. Note that, except for the formation of the generalized mass terms, the other generalized constants need not be computed as all that is needed are the modal frequencies and damping ratios. Note that Wilson damping has been assumed with 5% damping in each mode.

The values shown on the RHS of the last set of equations are the modal participation factors times -1.

Modal Participation Factors

Mode 1	1.425	1.911
Mode 2	−0.511	−0.799
Mode 3	0.090	0.435

Modal scaling $\phi_{i,1} = 1.0$ $\phi_i^T M \phi_i = 1.0$

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This slide shows the modal participation factors and how their magnitudes depend on mode shape scaling.

Modal Participation Factors (continued)

$$1.425\begin{Bmatrix}1.000\\0.644\\0.300\end{Bmatrix}=1.911\begin{Bmatrix}0.744\\0.480\\0.223\end{Bmatrix}$$

using $\phi_{1,1}=1$ using $\phi_1{}^T M\phi_1=1$

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The modal participation factor times the mode shape must be invariant. The slight discrepancies here are due to rounding error.

Effective Modal Mass

$$\bar{M}_n = P_n^{\,2} m_n^*$$

	$\bar{M}_n$	%	Accum%
Mode 1	3.66	81	81
Mode 2	0.64	14	95
Mode 3	0.20	5	100%
	4.50	100%	

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Note that the percentage of effective mass decreases with each mode. The sum of the effective mass for each mode must be the total mass. Most analysts use enough modes to account for no less than 90% of the total mass.

Example 1 (continued)

Solving modal equation via NONLIN:

For Mode 1:

$$\ddot{y}_1 + 2\xi_1\omega_1\dot{y}_1 + \omega_1^2 y_1 = V_1^*(t)/m_1^*$$

$$1.00\ddot{y}_1 + 0.458\dot{y}_1 + 21.0y_1 = -1.425\ddot{v}_g(t)$$

M = 1.00 kip-sec^2/in

C = 0.458 kip-sec/in

K_1 = 21.0 kips/inch

Scale ground acceleration by factor 1.425

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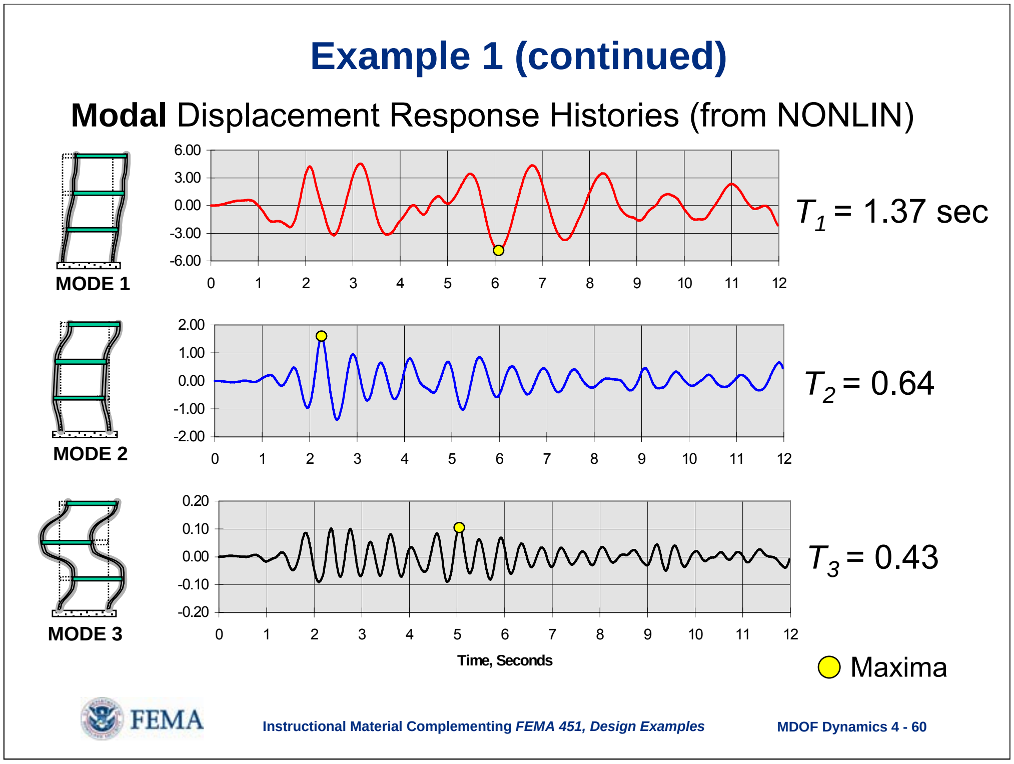
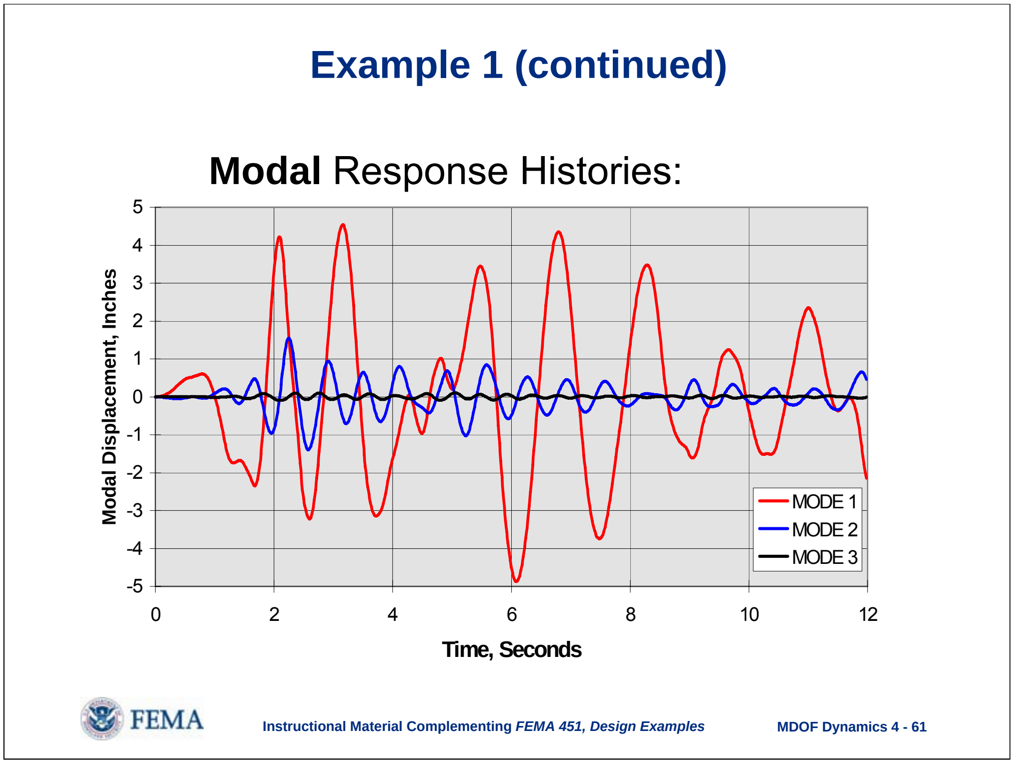
This slide shows how a single modal equation might be analyzed via NONLIN. After each modal time history response has been computed, they can be combined through the use of EXCEL and the NONLIN.XL1 files that may be written for each mode.



This slide shows the computed response histories for each mode. These are the modal amplitudes, not the system displacements. The yellow dot shown in each history points out the location of the maximum (positive or negative) displacement response for that mode.

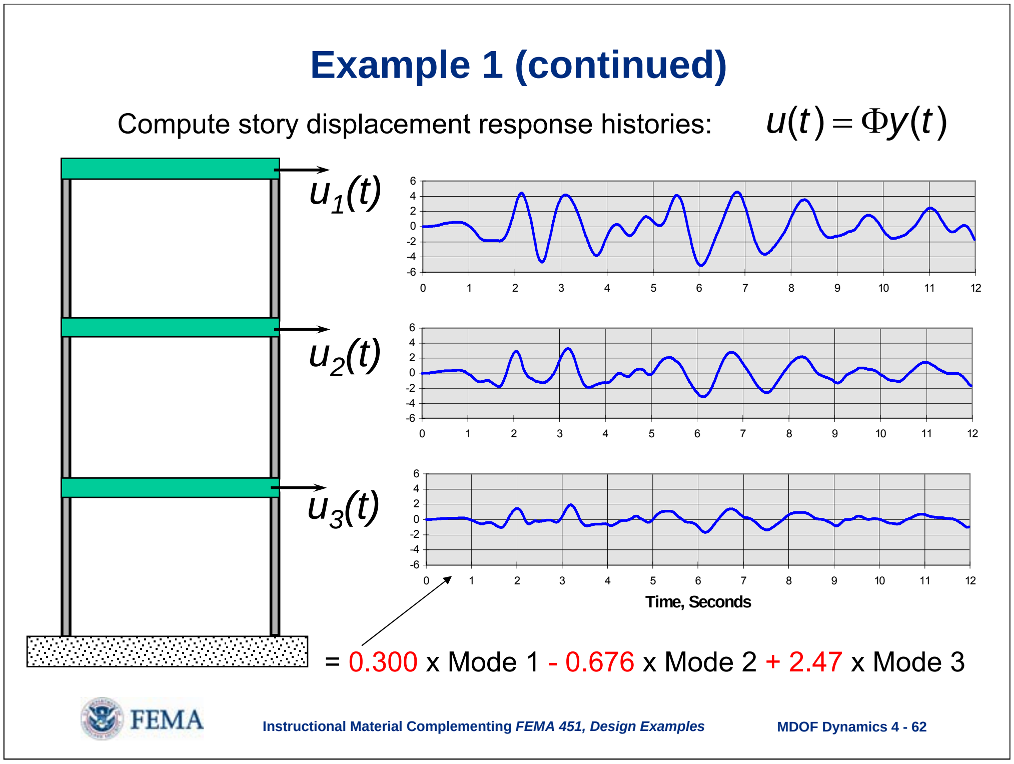
Note that the plots are drawn to different vertical scales.

Note that these maxima generally do not occur at the same times. Also note that the magnitudes of the modal response quantities is a function of the mode normalization technique used.



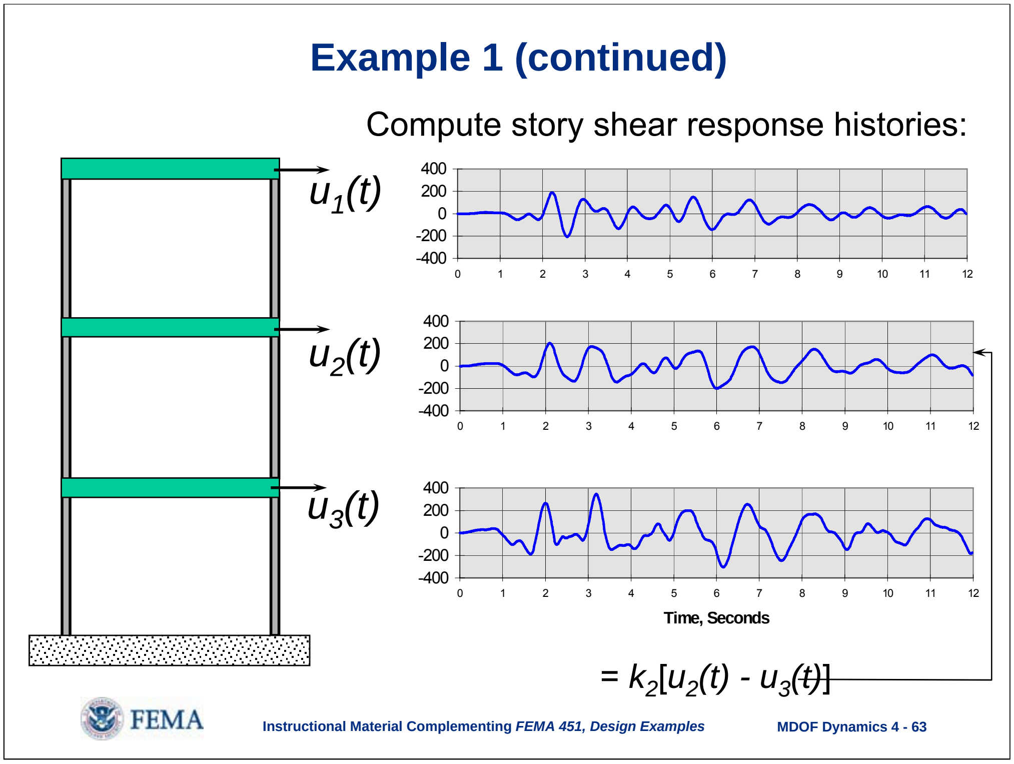
This is the same as the previous slide, but the modal displacements are drawn to the same scale. Note that the displacements in Mode 3 are very small when compared to Modes 1 and 2. This would seem to indicate that Mode 3 could be eliminated from the analysis with little introduction of error.

Note also the apparent difference in the frequency content of the modal responses.



By pre-multiplying the modal time histories by the mode shape matrix, the time histories of the displacements at each story are obtained.

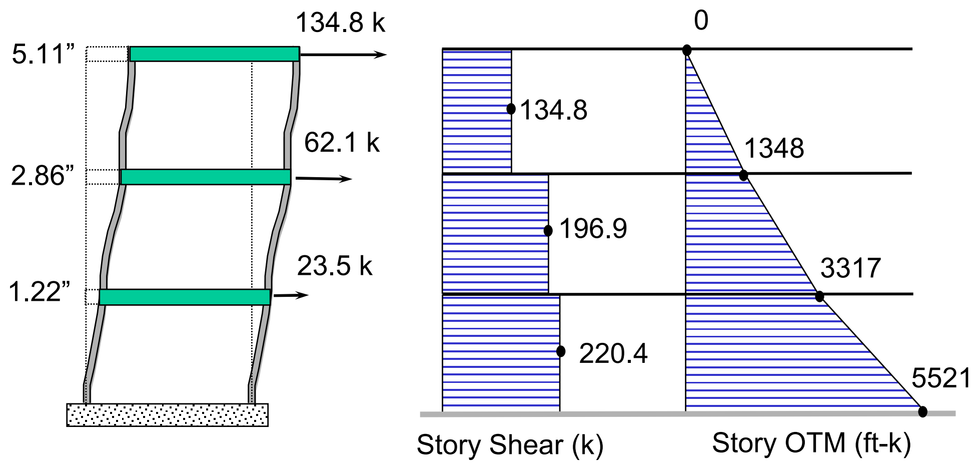
Essentially, we are transforming coordinates from modal space back to the original story SDOF. The equation at the bottom of the slide shows how the time history of displacements at the first story was obtained. The factors in the equation are the amplitudes of the 1st, 2nd,and 3rd modes at DOF U_3 (see Slide 50).



This slide shows how story shear response histories are obtained.

Example 1 (continued)

Displacements and forces at time of maximum displacements (t = 6.04 sec)



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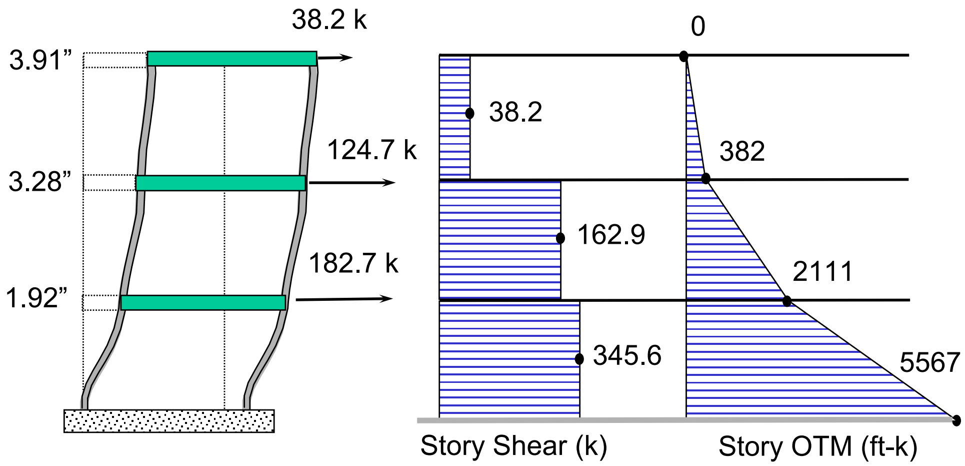
This is a summary “snap shot” of the response of the structure at the time of maximum displacement.

Note that the maximum displacement, shear, and overturning moments do not necessarily occur at the same point in time. (The following slide is needed to prove the point.)

Example 1 (continued)

Displacements and forces at time of maximum shear
(t = 3.18 sec)



This is a summary “snap shot” of the response of the structure at the time of maximum first-story shear.

Note that the maximum displacement, shear, and overturning moments do not necessarily occur at the same point in time. (The previous slide is needed to prove the point.)

Note that the inertial force pattern is not even close to “upper triangular.”

Modal Response Response Spectrum Method

- Instead of solving the time history problem for each mode, use a response spectrum to compute the **maximum** response in each mode.
- These maxima are generally **nonconcurrent**.
- Combine the maximum modal responses using some statistical technique, such as square root of the sum of the squares (SRSS) or complete quadratic combination (CQC).
- The technique is **approximate**.
- It is the basis for the equivalent lateral force (ELF) method.

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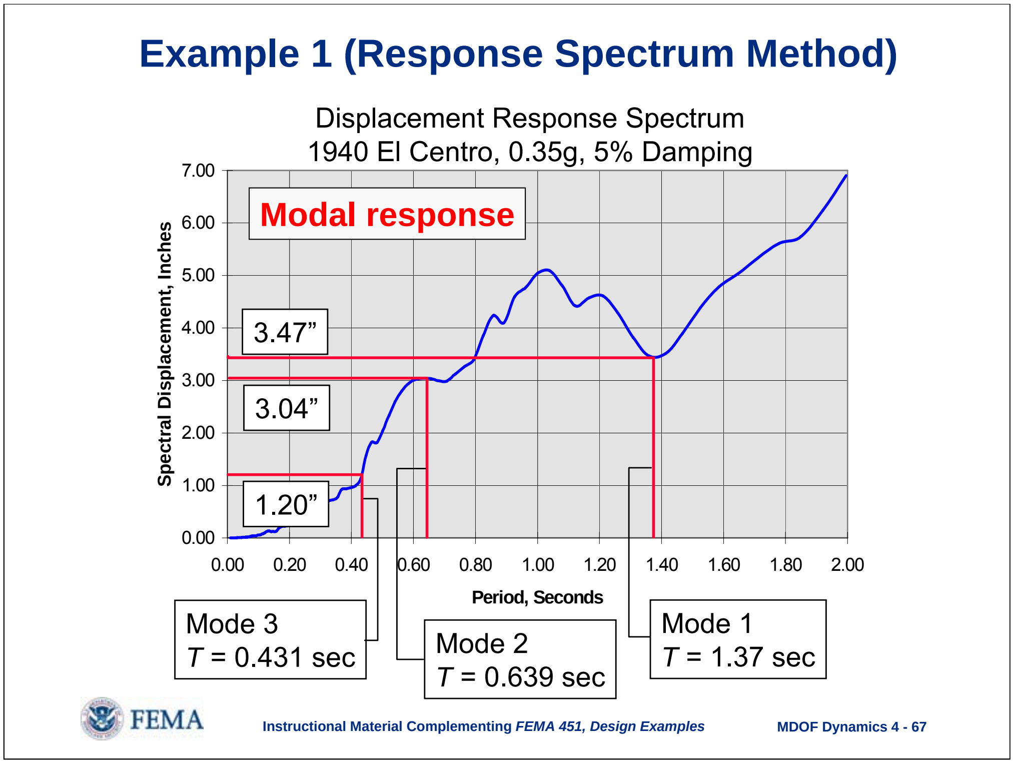
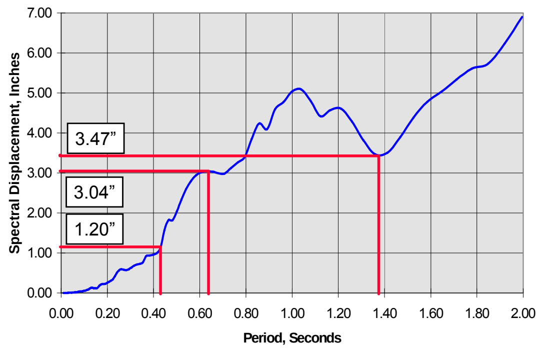
The next several slides introduce the modal response spectrum approach. Here, the individual modal displacements are obtained from a response spectrum and then the modal quantities are statistically combined. The method is approximate but generally accurate enough for design.



This slide shows the coordinates of the 5% damped El Centro displacement response spectrum at periods corresponding to Modes 1, 2 and 3 of the example structure.

In the response spectrum approach, these modal maxima are used in lieu of the full response histories obtained from direct integration of the uncoupled equations of motion.

Example 1 (continued)



Modal Equations of Motion

$$\ddot{y}_1 + 0.458\dot{y}_1 + 21.0y_1 = -1.425\ddot{v}_g(t)$$

$$\ddot{y}_2 + 0.983\dot{y}_2 + 96.6y_2 = 0.511\ddot{v}_g(t)$$

$$\ddot{y}_3 + 1.457\dot{y}_3 + 212.4y_3 = -0.090\ddot{v}_g(t)$$

Modal Maxima

$$\bar{y}_1 = 1.425 * 3.47 = 4.94"$$

$$\bar{y}_2 = 0.511 * 3.04 = 1.55"$$

$$\bar{y}_3 = 0.090 * 1.20 = 0.108"$$

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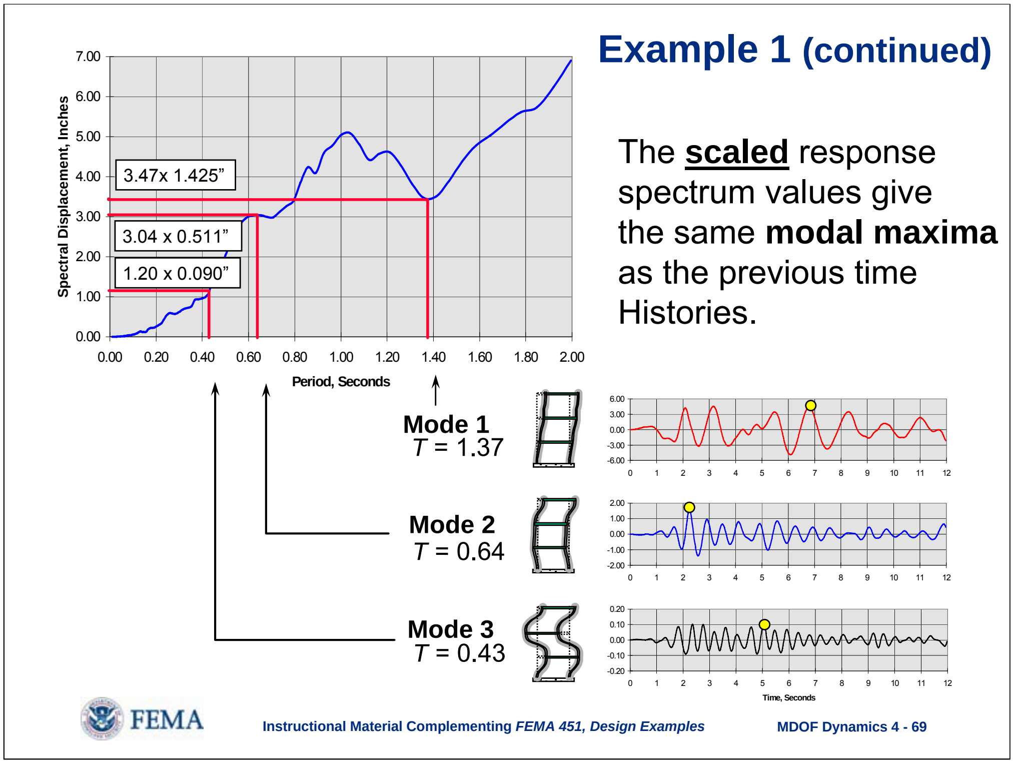
Before the response spectrum coordinates can be used, they need to be scaled by the appropriate factors shown on the right hand side of the individual modal equations of motion. Recall that these scale factors are the modal participation factors. There is no need to retain the signs because the signs will be lost when the modes are combined.

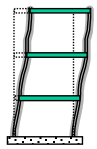


This slide shows that the scaled response spectrum coordinates are the same as the maxima obtained from the response history calculations. It also emphasizes that the modal maxima occur at different points in time.

Example 1 (continued)

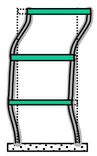
Computing **Nonconcurrent** Story Displacements

Mode 1



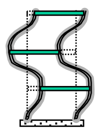
$$\begin{Bmatrix} 1.000 \\ 0.644 \\ 0.300 \end{Bmatrix} 4.940 = \begin{Bmatrix} 4.940 \\ 3.181 \\ 1.482 \end{Bmatrix}$$

Mode 2



$$\begin{Bmatrix} 1.000 \\ -0.601 \\ -0.676 \end{Bmatrix} 1.550 = \begin{Bmatrix} 1.550 \\ -0.931 \\ -1.048 \end{Bmatrix}$$

Mode 3



$$\begin{Bmatrix} 1.000 \\ -2.570 \\ 2.470 \end{Bmatrix} 0.108 = \begin{Bmatrix} 0.108 \\ -0.278 \\ 0.267 \end{Bmatrix}$$

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The modal displacements are obtained by multiplying the mode shapes by the values from the response spectrum.

Example 1 (continued)

Modal Combination Techniques (for Displacement)

Sum of Absolute Values:

$$\begin{Bmatrix} 4.940+1.550+0.108 \\ 3.181+0.931+0.278 \\ 1.482+1.048+0.267 \end{Bmatrix} = \begin{Bmatrix} 6.60 \\ 4.39 \\ 2.80 \end{Bmatrix}$$

At time of maximum displacement

"Exact"

$$\begin{Bmatrix} 5.15 \\ 2.86 \\ 1.22 \end{Bmatrix}$$

Square Root of the Sum of the Squares:

$$\begin{Bmatrix} \sqrt{4.940^2+1.550^2+0.108^2} \\ \sqrt{3.181^2+0.931^2+0.278^2} \\ \sqrt{1.482^2+1.048^2+0.267^2} \end{Bmatrix} = \begin{Bmatrix} 5.18 \\ 3.33 \\ 1.84 \end{Bmatrix}$$

Envelope of story displacement

$$\begin{Bmatrix} 5.15 \\ 3.18 \\ 1.93 \end{Bmatrix}$$

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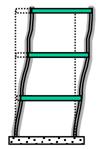
This slide shows two of the most common modal combination techniques, sum of absolute values and square root of the sum of the squares (SRSS). For very complicated structures with closely spaced modes, the complete quadratic combination (CQC) is preferred. The CQC method reduces to the SRSS method when the modes are not closely spaced. *SAP2000*, ETABS, RAMFRAME, and most commercial programs use the CQC approach.

Note the similarity between SRSS and the response history results. The comparison is particularly good when the response spectrum values are compared with the ENVELOPE values from the response history.

Example 1 (continued)

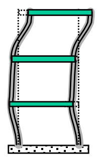
Computing Interstory Drifts

Mode 1



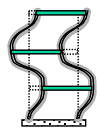
$$\begin{Bmatrix} 4.940-3.181 \\ 3.181-1.482 \\ 1.482-0 \end{Bmatrix} = \begin{Bmatrix} 1.759 \\ 1.699 \\ 1.482 \end{Bmatrix}$$

Mode 2



$$\begin{Bmatrix} 1.550-(-0.931) \\ -0.931-(-1.048) \\ -1.048-0 \end{Bmatrix} = \begin{Bmatrix} 2.481 \\ 0.117 \\ -1.048 \end{Bmatrix}$$

Mode 3



$$\begin{Bmatrix} 0.108-(-0.278) \\ -0.278-0.267 \\ 0.267-0 \end{Bmatrix} = \begin{Bmatrix} 0.386 \\ -0.545 \\ 0.267 \end{Bmatrix}$$

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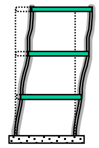
In this slide, modal interstory drifts are obtained from the modal displacements. These will be used to compute modal story shears.

Note that the interstory drifts are NOT obtained from the SRSSed displacements shown in the previous slide.

Example 1 (continued)

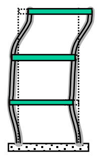
Computing Interstory Shears (Using Drift)

Mode 1



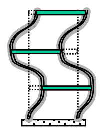
$$\begin{Bmatrix} 1.759(60) \\ 1.699(120) \\ 1.482(180) \end{Bmatrix} = \begin{Bmatrix} 105.5 \\ 203.9 \\ 266.8 \end{Bmatrix}$$

Mode 2



$$\begin{Bmatrix} 2.481(60) \\ 0.117(120) \\ -1.048(180) \end{Bmatrix} = \begin{Bmatrix} 148.9 \\ 14.0 \\ -188.6 \end{Bmatrix}$$

Mode 3



$$\begin{Bmatrix} 0.386(60) \\ -0.545(120) \\ 0.267(180) \end{Bmatrix} = \begin{Bmatrix} 23.2 \\ -65.4 \\ 48.1 \end{Bmatrix}$$

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Computation of modal story shears from modal interstory drift. To obtain the shears the drifts re multiplied by story stiffness.

Example 1 (continued)

Computing Interstory Shears: SRSS Combination

$$\begin{Bmatrix} \sqrt{106^2 + 149^2 + 23.2^2} \\ \sqrt{204^2 + 14^2 + 65.4^2} \\ \sqrt{267^2 + 189^2 + 48.1^2} \end{Bmatrix} = \begin{Bmatrix} 220 \\ 215 \\ 331 \end{Bmatrix}$$

"Exact" $\begin{Bmatrix} 38.2 \\ 163 \\ 346 \end{Bmatrix}$

At time of max. shear

"Exact" $\begin{Bmatrix} 135 \\ 197 \\ 220 \end{Bmatrix}$

At time of max. displacement

"Exact" $\begin{Bmatrix} 207 \\ 203 \\ 346 \end{Bmatrix}$

Envelope = maximum per story

Calculation of total story shear using SRSS. Note that the story shears were not obtained from the SRSS response of the modal displacements.

Note the remarkable similarity in results from the exact envelope values and from the response spectrum approach. This degree of correlation is somewhat unusual.

Caution:

Do NOT compute story shears from the story drifts derived from the SRSS of the story displacements.

Calculate the story shears in each mode (using modal drifts) and then SRSS the results.

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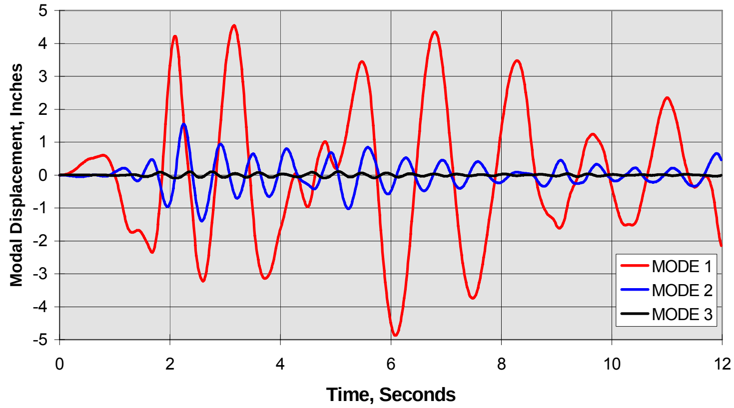
Self explanatory.

Using Less than Full (Possible) Number of Natural Modes

Modal Response Histories:



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Recall that the third mode produces insignificant displacement. This mode may be eliminated from both the direct analysis (integration of all equations) and the modal response spectrum approach with little error.

Using Less than Full Number of Natural Modes

Time-History for **Mode** 1

$$y(t) = \begin{bmatrix} y_1(t1) & y_1(t2) & y_1(t3) & y_1(t4) & y_1(t5) & y_1(t6) & y_1(t7) & y_1(t8) & .... & y_1(tn) \\ y_2(t1) & y_2(t2) & y_2(t3) & y_2(t4) & y_2(t5) & y_2(t6) & y_2(t7) & y_2(t8) & .... & y_2(tn) \\ y_3(t1) & y_3(t2) & y_3(t3) & y_3(t4) & y_3(t5) & y_3(t6) & y_3(t7) & y_3(t8) & .... & y_3(tn) \end{bmatrix}$$

Transformation:

$$\underbrace{\underset{3 \text{ x nt}}{u(t)} = \underset{3 \text{ x } 3}{\begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \end{bmatrix}} \underset{3 \text{ x nt}}{y(t)}}_{3 \text{ x nt}}$$

Time History for **DOF** 1

$$u(t) = \begin{bmatrix} u_1(t1) & u_1(t2) & u_1(t3) & u_1(t4) & u_1(t5) & u_1(t6) & u_1(t7) & u_1(t8) & .... & u_1(tn) \\ u_2(t1) & u_2(t2) & u_2(t3) & u_2(t4) & u_2(t5) & u_2(t6) & u_2(t7) & u_2(t8) & .... & u_2(tn) \\ u_3(t1) & u_3(t2) & u_3(t3) & u_3(t4) & u_3(t5) & u_3(t6) & u_3(t7) & u_3(t8) & .... & u_3(tn) \end{bmatrix}$$

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This slide shows the full modal response history analysis results (in matrix form) and the resulting multiplication to put the response back into story displacement DOF.

Using Less than Full Number of Natural Modes

Time History for **Mode** 1

$$y(t) = \begin{bmatrix} y_1(t1) & y_1(t2) & y_1(t3) & y_1(t4) & y_1(t5) & y_1(t6) & y_1(t7) & y_1(t8) & .... & y_1(tn) \\ y_2(t1) & y_2(t2) & y_2(t3) & y_2(t4) & y_2(t5) & y_2(t6) & y_2(t7) & y_2(t8) & .... & y_2(tn) \end{bmatrix}$$

NOTE: Mode 3 **NOT** Analyzed

Transformation:

$$\underbrace{\underset{3 \times nt}{u(t)} = \underset{3 \times 2}{\begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix}} \underset{2 \times nt}{y(t)}}_{3 \times nt}$$

Time history for **DOF** 1

$$u(t) = \begin{bmatrix} u_1(t1) & u_1(t2) & u_1(t3) & u_1(t4) & u_1(t5) & u_1(t6) & u_1(t7) & u_1(t8) & .... & u_1(tn) \\ u_2(t1) & u_2(t2) & u_2(t3) & u_2(t4) & u_2(t5) & u_2(t6) & u_2(t7) & u_2(t8) & .... & u_2(tn) \\ u_3(t1) & u_3(t2) & u_3(t3) & u_3(t4) & u_3(t5) & u_3(t6) & u_3(t7) & u_3(t8) & .... & u_3(tn) \end{bmatrix}$$

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If only two of three modes are integrated, the transformation still yields displacement response histories at each story. For large structures with hundreds of possible modes of vibration, it has been shown that only a few modes are required to obtain an accurate analysis. ASCE 7 requires that enough modes be included to represent at least 90% of the effective mass of the structure.

Using Less than Full Number of Natural Modes

(Modal Response Spectrum Technique)

Sum of absolute values:

$$\begin{Bmatrix} 4.940+1.550+0.108 \\ 3.181+0.931+0.278 \\ 1.482+1.048+0.267 \end{Bmatrix} = \begin{Bmatrix} 6.60 \\ 4.39 \\ 2.80 \end{Bmatrix} \qquad \begin{Bmatrix} 6.49 \\ 4.112 \\ 2.53 \end{Bmatrix}$$

Square root of the sum of the squares:

$$\begin{Bmatrix} \sqrt{4.940^2+1.550^2+0.108^2} \\ \sqrt{3.181^2+0.931^2+0.278^2} \\ \sqrt{1.482^2+1.048^2+0.267^2} \end{Bmatrix} = \begin{Bmatrix} 5.18 \\ 3.33 \\ 1.84 \end{Bmatrix} \qquad \begin{Bmatrix} 5.18 \\ 3.31 \\ 1.82 \end{Bmatrix}$$

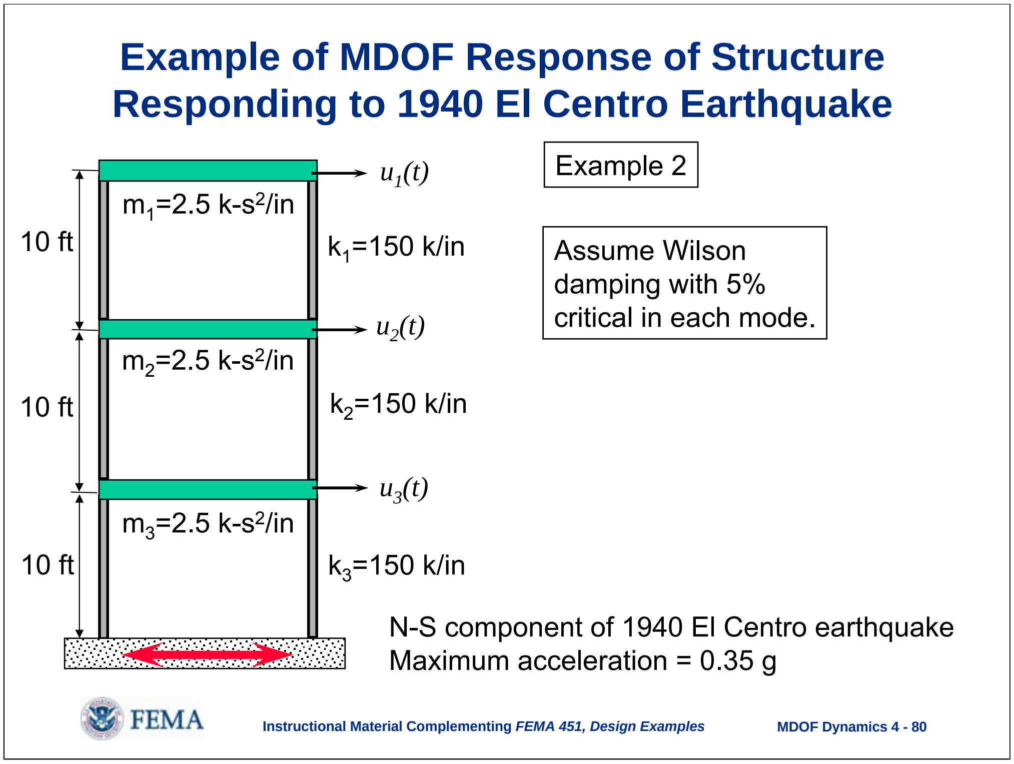
3 modes 2 modes

At time of maximum displacement

"Exact":

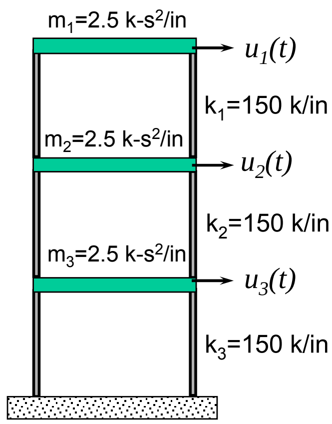
$$\begin{Bmatrix} 5.15 \\ 2.86 \\ 1.22 \end{Bmatrix}$$

Higher modes may also be eliminated from modal response spectrum analysis as shown here.



The next several slides present an example similar to the previous one but the story stiffnesses and masses are uniform.

Example 2 (continued)



Form property matrices:

$$M = \begin{bmatrix} 2.5 & & \\ & 2.5 & \\ & & 2.5 \end{bmatrix} \text{kip} - \text{s}^2/\text{in}$$

$$K = \begin{bmatrix} 150 & -150 & 0 \\ -150 & 300 & -150 \\ 0 & -150 & 300 \end{bmatrix} \text{kip/in}$$

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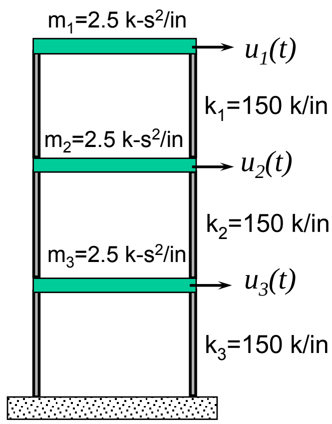
These are the mass and stiffness matrices of the structure. Note again that the stiffness matrix is highly idealized for this example.

The damping matrix is not needed because the damping in the various modes is assigned directly (Wilson damping).

Example 2 (continued)



Solve = eigenvalue problem:

$$K\Phi = M\Phi\Omega^2$$

$$\Omega^2 = \begin{bmatrix} 11.9 & & \\ & 93.3 & \\ & & 194.8 \end{bmatrix} \text{sec}^{-2}$$

$$\Phi = \begin{bmatrix} 1.000 & 1.000 & 1.000 \\ 0.802 & -0.555 & -2.247 \\ 0.445 & -1.247 & 1.802 \end{bmatrix}$$

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The mode shapes and frequencies were computed using Mathcad.

Note that the mode shapes have been scaled to produce a unit positive displacement at the top of the structure for each mode. It is often more convenient to scale such that $\phi_i^T M \phi_i = 1.0$.

Normalization of Modes Using $\Phi^T M \Phi = I$

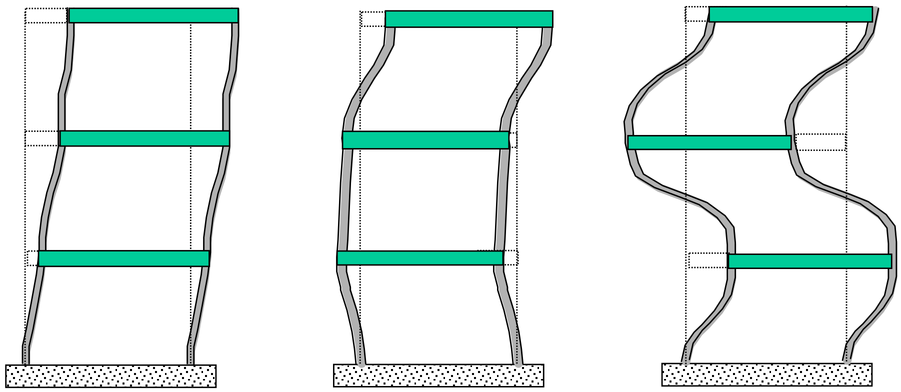
$$\Phi = \begin{bmatrix} 0.466 & 0.373 & 0.207 \\ 0.373 & -0.207 & -0.465 \\ 0.207 & -0.465 & 0.373 \end{bmatrix} \text{ vs } \begin{bmatrix} 1.000 & 1.000 & 1.000 \\ 0.802 & -0.555 & -2.247 \\ 0.445 & -1.247 & 1.802 \end{bmatrix}$$

This slide demonstrates two alternatives for scaling the modes. The generalized mass matrix for the system with the modes as normalized to the left are 1.0 for each mode. The example uses the mode shapes shown on the right.

While the mode shapes have different scale factors, this will not affect the computed results.

Example 2 (continued)

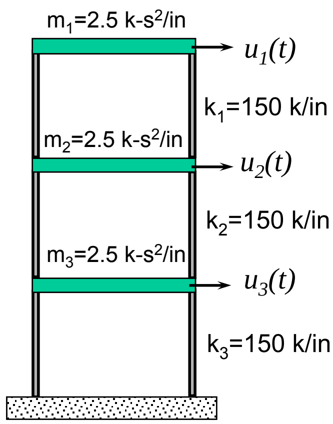
Mode Shapes and Periods of Vibration



Mode 1	Mode 2	Mode 3
ω = 3.44 rad/sec	ω = 9.66 rad/sec	ω = 13.96 rad/sec
T = 1.82 sec	T = 0.65 sec	T = 0.45 sec

This slide shows the mode shapes approximately to scale. Note that the first mode period of vibration, 1.82 seconds, is more appropriate for a 10-story building than it is for a 3-story structure.

Example 2 (continued)



$$\omega_n = \begin{Bmatrix} 3.44 \\ 9.66 \\ 13.96 \end{Bmatrix} \text{rad/sec} \qquad T_n = \begin{Bmatrix} 1.82 \\ 0.65 \\ 0.45 \end{Bmatrix} \text{sec}$$

Compute generalized mass:

$$M^* = \Phi^T M \Phi = \begin{bmatrix} 4.603 & & \\ & 7.158 & \\ & & 23.241 \end{bmatrix} \text{kip} - \text{sec}^2/\text{in}$$

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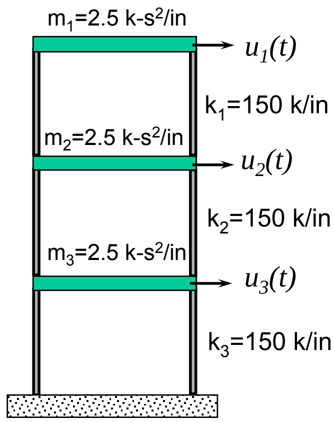
This slide shows the generalized mass matrix. These masses are appropriate for the *modes* and should not be confused with the story masses. The magnitude of the generalized mass terms is a function of the mode shape normalization technique used.

Example 2 (continued)



Compute generalized loading:

$$V^*(t) = -\Phi^T M R \ddot{v}_g(t)$$

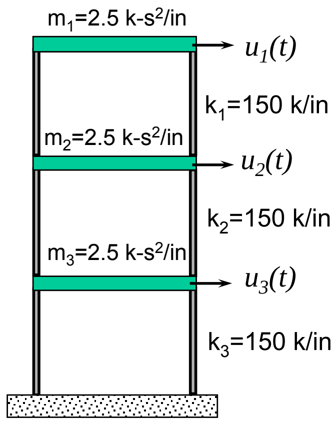
$$V_n^* = \begin{Bmatrix} -5.617 \\ 2.005 \\ -1.388 \end{Bmatrix} \ddot{v}_g(t)$$

This slide shows how the effective earthquake force has been computed for each mode. The magnitude of the generalized effective earthquake force is also a function of the mode normalization technique used.

The values shown in the lower equation are NOT the modal participation factors. To get the participation factors, the equation for each mode must be divided by the generalized mass for that mode

Example 2 (continued)

Write uncoupled (modal) equations of motion:



$$\ddot{y}_1 + 2\xi_1\omega_1\dot{y}_1 + \omega_1^2 y_1 = V_1^*(t)/m_1^*$$
$$\ddot{y}_2 + 2\xi_2\omega_2\dot{y}_2 + \omega_2^2 y_2 = V_2^*(t)/m_2^*$$
$$\ddot{y}_3 + 2\xi_3\omega_3\dot{y}_3 + \omega_3^2 y_3 = V_3^*(t)/m_3^*$$

$$\ddot{y}_1 + 0.345\dot{y}_1 + 11.88 y_1 = -1.22\ddot{v}_g(t)$$
$$\ddot{y}_2 + 0.966\dot{y}_2 + 93.29 y_2 = 0.280\ddot{v}_g(t)$$
$$\ddot{y}_3 + 1.395\dot{y}_3 + 194.83 y_3 = -0.06\ddot{v}_g(t)$$

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Here, the full set of uncoupled equations has been formed. Note that except for the formation of the generalized mass terms, the other generalized constants need not be computed as all that is needed are the modal frequencies and damping ratios. Note that Wilson damping has been assumed with 5% damping in each mode.

The values shown on the RHS of the last set of equations are the modal participation factors times -1.

Modal Participation Factors

$$\begin{array}{ll} \textit{Mode} & 1 \\ \textit{Mode} & 2 \\ \textit{Mode} & 3 \end{array} \begin{bmatrix} -1.22 \\ 0.28 \\ -0.060 \end{bmatrix} \begin{bmatrix} -2.615 \\ 0.748 \\ -0.287 \end{bmatrix}$$

Modal scaling $\phi_{i,1} = 1.0$ $\phi_i^T M \phi_i = 1.0$

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This slide shows the modal participation factors and how their magnitudes depend on mode shape scaling.

Effective Modal Mass

$$\overline{M_n} = P_n^{\,2} m_n$$

	$\overline{M}_n$	%	Accum%
Mode 1	6.856	91.40	91.40
Mode 2	0.562	7.50	98.90
Mode 3	0.083	1.10	100.0
	7.50	100%	

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Note that the percentage of effective mass decreases with each mode. The sum of the effective mass for each mode must be the total mass. Most analysts use enough modes to account for no less than 90% of the total mass.

Example 2 (continued)

Solving modal equation via NONLIN:

For Mode 1:

$$\ddot{y}_1 + 2\xi_1\omega_1\dot{y}_1 + \omega_1^2 y_1 = V_1^*(t)/m_1^*$$

$$1.00\ddot{y}_1 + 0.345\dot{y}_1 + 11.88y_1 = -1.22\ddot{v}_g(t)$$

M = 1.00 kip-sec^2/in

C = 0.345 kip-sec/in

K_1 = 11.88 kips/inch

Scale ground acceleration by factor 1.22

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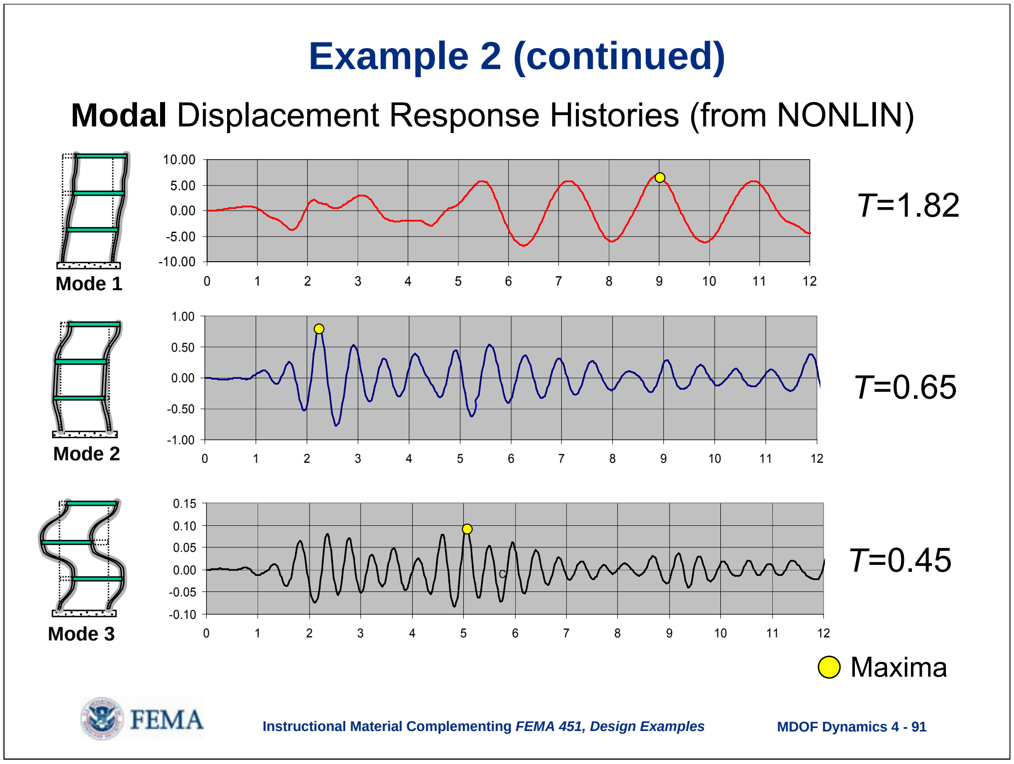
This slide shows how a single modal equation might be analyzed via NONLIN. After each modal time history response has been computed, they can be combined through the use of EXCEL and the NONLIN.XL1 files that may be written for each mode



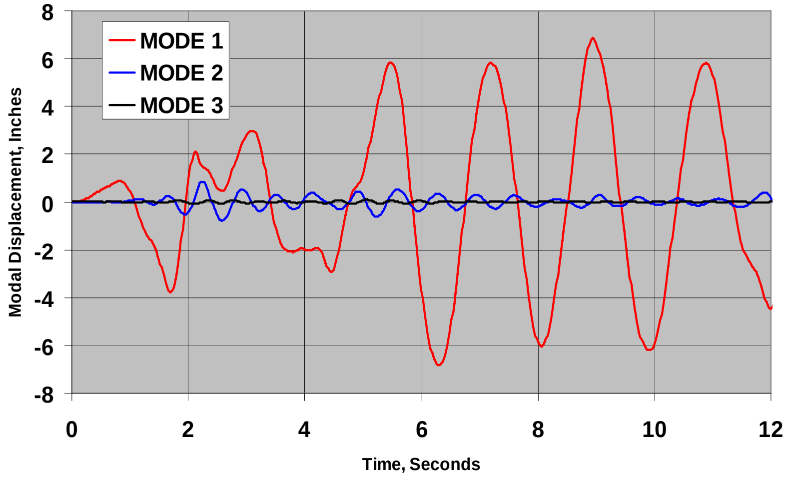
This slide shows the computed response histories for each mode. These are the modal amplitudes, not the system displacements. The yellow dot shown in each history points out the location of the maximum (positive or negative) displacement response for that mode.

Note that the plots are drawn to different vertical scales.

Note that these maxima generally do not occur at the same times. Also note that the magnitudes of the modal response quantities is a function of the mode normalization technique used.

Example 2 (continued)

Modal Response Histories



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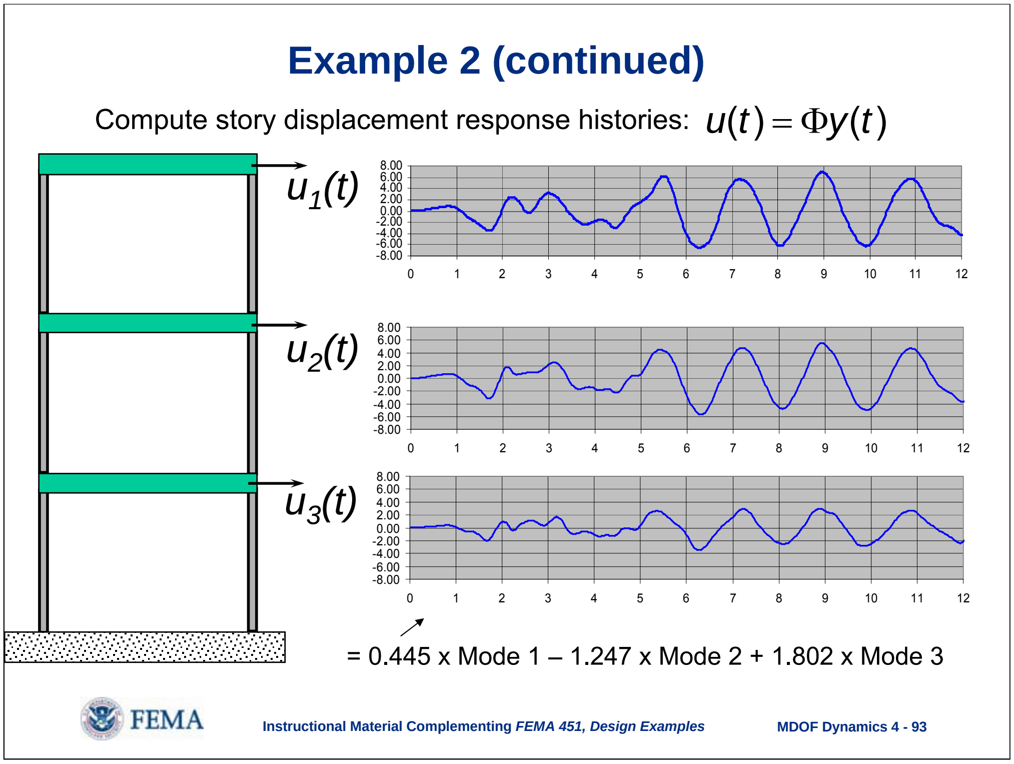
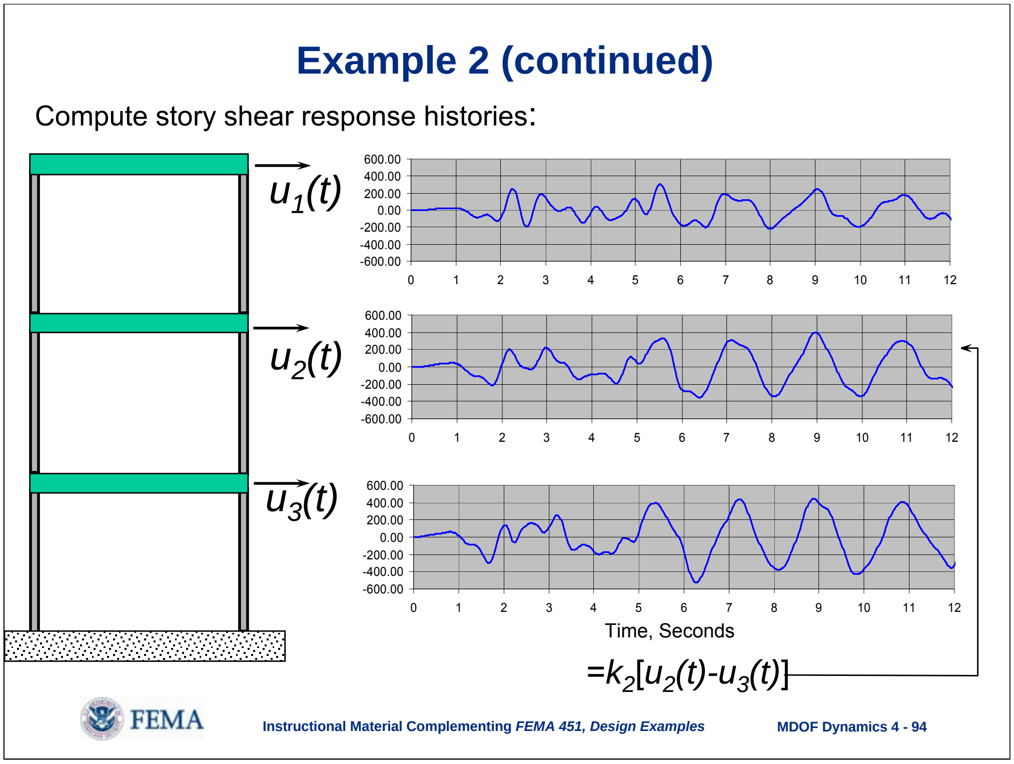
This is the same as the previous slide, but the modal displacements are drawn to the same scale. Note that the displacements in Mode 3 are very small when compared to Modes 1 and 2. This would seem to indicate that Mode 3 could be eliminated from the analysis with little introduction of error.

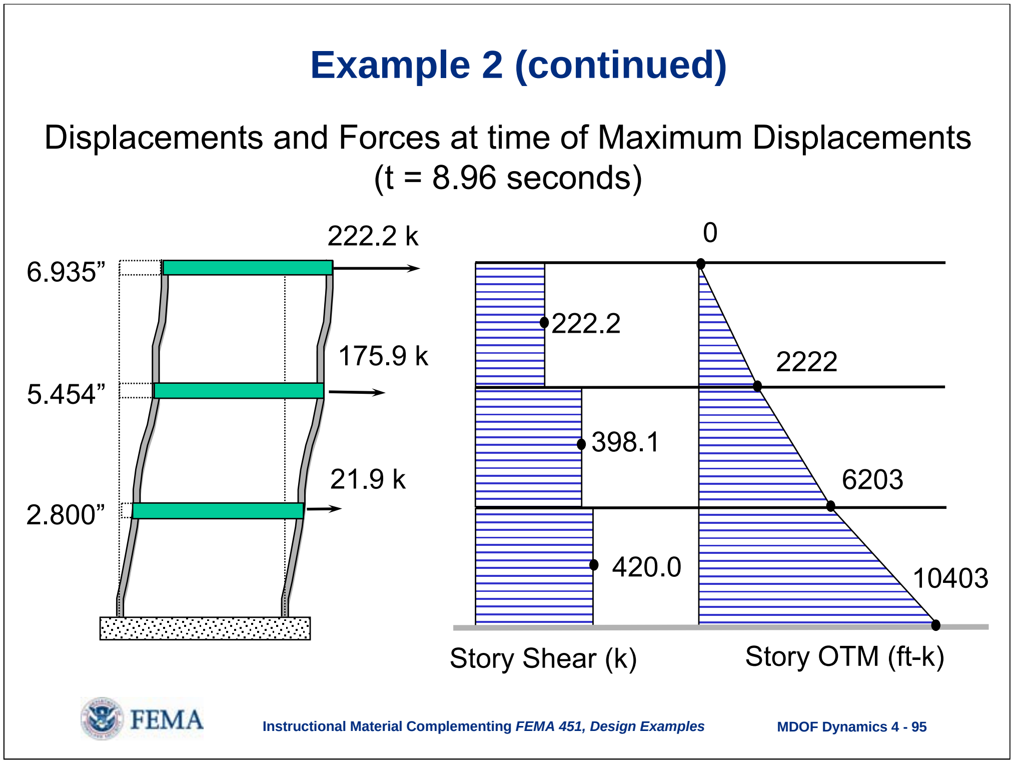
Note also the apparent difference in the frequency content of the modal responses.



By pre-multiplying the modal time histories by the mode shape matrix, the time histories of the displacements at each story are obtained. Essentially, we are transforming coordinates from modal space back to the original story DOF. The equation at the bottom of the slide shows how the time history of displacements at the first story was obtained. The factors in the equation are the amplitudes of the 1st, 2nd, and 3rd modes at DOF U_3



This slide shows how story shear response histories are obtained.

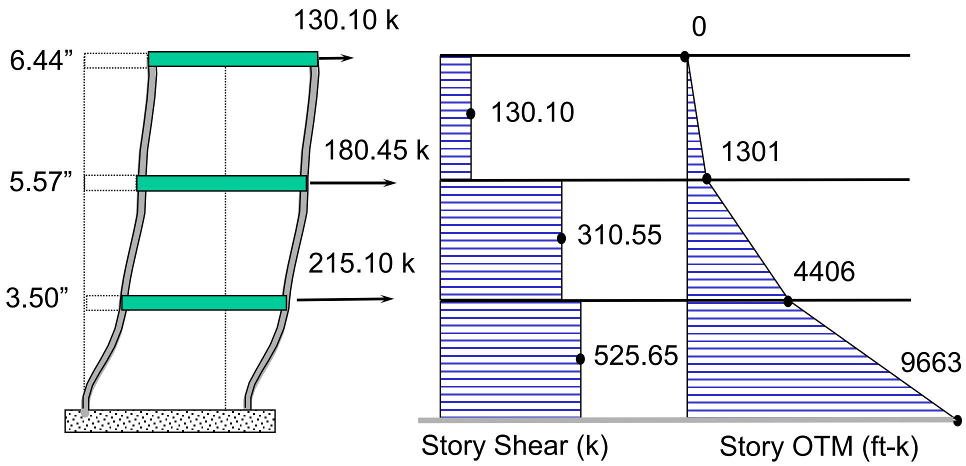


This is a summary “snap shot” of the response of the structure at the time of maximum displacement.

Note that the maximum displacement, shear, and overturning moments do not necessarily occur at the same point in time. (The following slide is needed to prove the point.)

Example 2 (continued)

Displacements and Forces at Time of Maximum Shear
(t = 6.26 sec)



This is a summary “snap shot” of the response of the structure at the time of maximum first-story shear.

Note that the maximum displacement, shear, and overturning moments do not necessarily occur at the same point in time. (The previous slide is needed to prove the point.)

Note that the inertial force pattern is not even close to “upper triangular.”

Modal Response Response Spectrum Method

- Instead of solving the time history problem for each mode, use a response spectrum to compute the **maximum** response in each mode.
- These maxima are generally **nonconcurrent**.
- Combine the maximum modal responses using some statistical technique, such as square root of the sum of the squares (SRSS) or complete quadratic combination (CQC).
- The technique is **approximate**.
- It is the basis for the equivalent lateral force (ELF) method.

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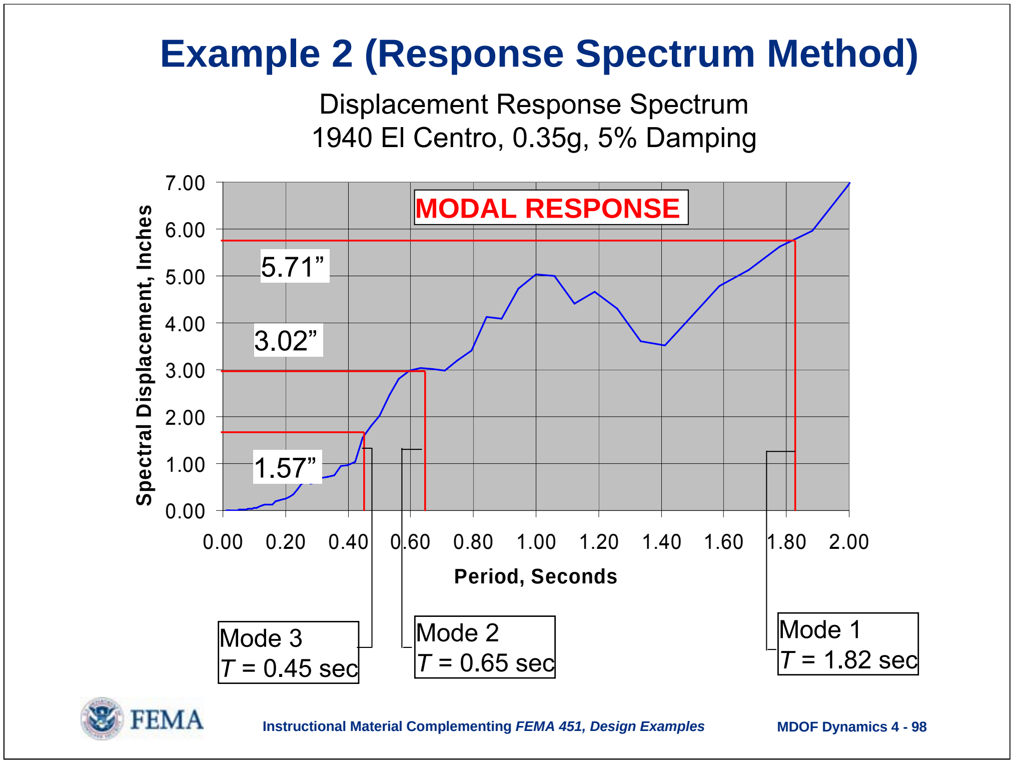
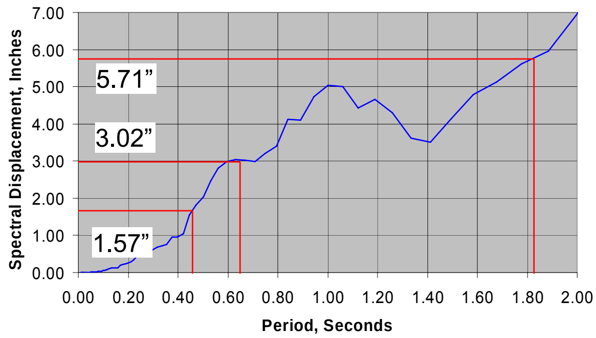
The next several slides introduce the modal response spectrum approach. Here, the individual modal displacements are obtained from a response spectrum and then the modal quantities are statistically combined. The method is approximate but generally accurate enough for design.



This slide shows the coordinates of the 5% damped El Centro displacement response spectrum at periods corresponding to Modes 1, 2 and 3 of the example structure.

In the response spectrum approach, these modal maxima are used in lieu of the full response histories obtained from direct integration of the uncoupled equations of motion.

Example 2 (continued)



Modal Equations of Motion

$$\ddot{y}_1 + 0.345\dot{y}_1 + 11.88y_1 = -1.22\ddot{v}_g(t)$$

$$\ddot{y}_2 + 0.966\dot{y}_2 + 93.29y_2 = 0.280\ddot{v}_g(t)$$

$$\ddot{y}_3 + 1.395\dot{y}_3 + 194.83y_3 = -0.060\ddot{v}_g(t)$$

Modal Maxima

$$\bar{y}_1 = 1.22 * 5.71 = 6.966"$$

$$\bar{y}_2 = 0.28 * 3.02 = 0.845"$$

$$\bar{y}_3 = 0.060 * 1.57 = 0.094"$$

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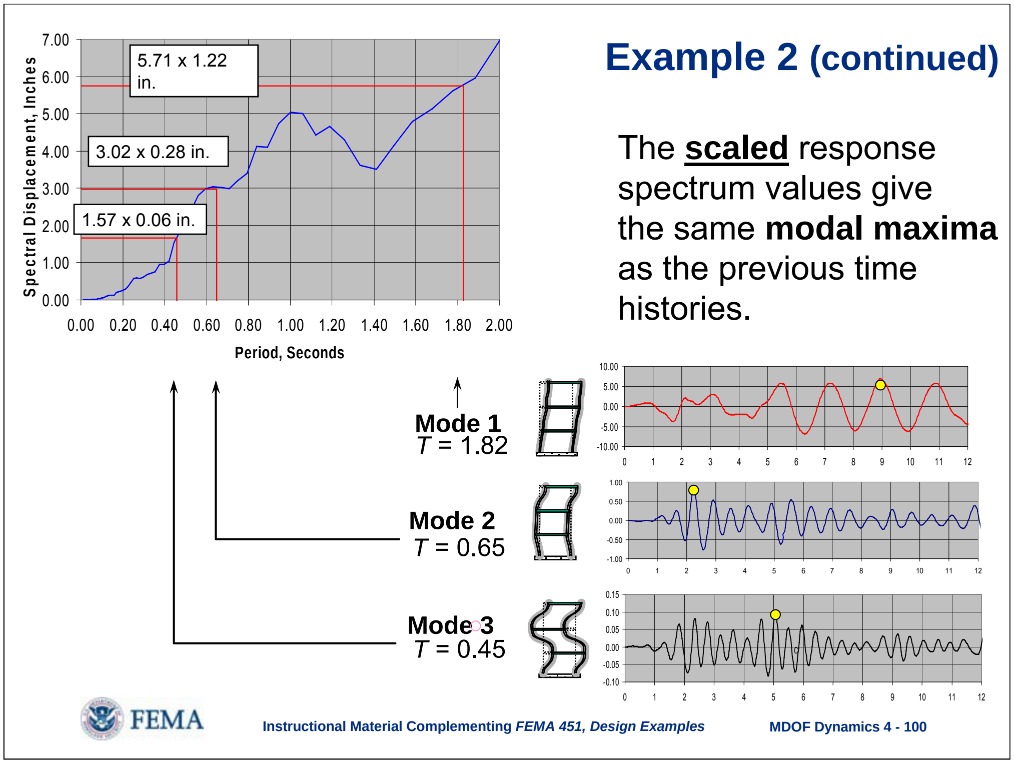
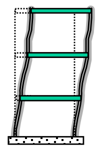
Before the response spectrum coordinates can be used, they need to be scaled by the appropriate factors shown on the right hand side of the individual modal equations of motion. Recall that these scale factors are the modal participation factors. There is no need to retain the signs because the signs will be lost when the modes are combined.



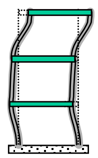
This slide shows that the scaled response spectrum coordinates are the same as the maxima obtained from the response history calculations. It also emphasizes that the modal maxima occur <u>at different points in time</u>.

Example 2 (continued)

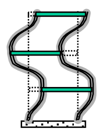
Computing **Nonconcurrent** Story Displacements

Mode 1 

$$\begin{Bmatrix} 1.000 \\ 0.802 \\ 0.445 \end{Bmatrix} 6.966 = \begin{Bmatrix} 6.966 \\ 5.586 \\ 3.100 \end{Bmatrix}$$

Mode 2 

$$\begin{Bmatrix} 1.000 \\ -0.555 \\ -1.247 \end{Bmatrix} 0.845 = \begin{Bmatrix} 0.845 \\ -0.469 \\ -1.053 \end{Bmatrix}$$

Mode 3 

$$\begin{Bmatrix} 1.000 \\ -2.247 \\ 1.802 \end{Bmatrix} 0.094 = \begin{Bmatrix} 0.094 \\ -0.211 \\ 0.169 \end{Bmatrix}$$

The modal displacements are obtained by multiplying the mode shapes by the values from the response spectrum.

Example 2 (continued)

Modal Combination Techniques (For Displacement)

Sum of absolute values:

$$\begin{Bmatrix} 6.966+0.845+0.108 \\ 5.586+0.469+0.211 \\ 3.100+1.053+0.169 \end{Bmatrix} = \begin{Bmatrix} 7.919 \\ 6.266 \\ 4.322 \end{Bmatrix}$$

At time of maximum displacement

"Exact"

$$\boxed{\begin{Bmatrix} 6.935 \\ 5.454 \\ 2.800 \end{Bmatrix}}$$

Square root of the sum of the squares

$$\begin{Bmatrix} \sqrt{6.966^2+0.845^2+0.108^2} \\ \sqrt{5.586^2+0.469^2+0.211^2} \\ \sqrt{3.100^2+1.053^2+0.169^2} \end{Bmatrix} = \begin{Bmatrix} 7.02 \\ 5.61 \\ 3.28 \end{Bmatrix}$$

Envelope of story displacement

$$\boxed{\begin{Bmatrix} 6.935 \\ 5.675 \\ 2.965 \end{Bmatrix}}$$

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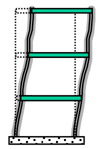
This slide shows two of the most common modal combination techniques, sum of absolute values and square root of the sum of the squares (SRSS). For very complicated structures with closely spaced modes, the complete quadratic combination (CQC) is preferred. The CQC method reduces to the SRSS method when the modes are not closely spaced. *SAP2000*, ETABS, RAMFRAME, and most commercial programs use the CQC approach.

Note the similarity between SRSS and the response history results. The comparison is particularly good when the response spectrum values are compared with the ENVELOPE values from the response history.

Example 2 (continued)

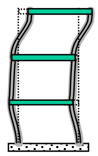
Computing Interstory Drifts

Mode 1



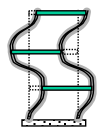
$$\begin{Bmatrix} 6.966-5.586 \\ 5.586-3.100 \\ 3.100-0 \end{Bmatrix} = \begin{Bmatrix} 1.380 \\ 2.486 \\ 3.100 \end{Bmatrix}$$

Mode 2



$$\begin{Bmatrix} 0.845-(-0.469) \\ -0.469-(-1.053) \\ -1.053-0 \end{Bmatrix} = \begin{Bmatrix} 1.314 \\ 0.584 \\ -1.053 \end{Bmatrix}$$

Mode 3



$$\begin{Bmatrix} 0.108-(-0.211) \\ -0.211-0.169 \\ 0.169-0 \end{Bmatrix} = \begin{Bmatrix} 0.319 \\ -0.380 \\ 0.169 \end{Bmatrix}$$

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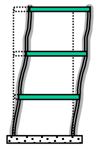
In this slide, modal interstory drifts are obtained from the modal displacements. These will be used to compute modal story shears.

Note that the interstory drifts are NOT obtained from the SRSSed displacements shown in the previous slide.

Example 2 (continued)

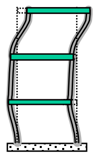
Computing Interstory Shears (Using Drift)

Mode 1



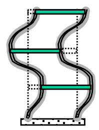
$$\begin{Bmatrix} 1.380(150) \\ 2.486(150) \\ 3.100(150) \end{Bmatrix} = \begin{Bmatrix} 207.0 \\ 372.9 \\ 465.0 \end{Bmatrix}$$

Mode 2



$$\begin{Bmatrix} 1.314(150) \\ 0.584(150) \\ -1.053(150) \end{Bmatrix} = \begin{Bmatrix} 197.1 \\ 87.6 \\ -157.9 \end{Bmatrix}$$

Mode 3



$$\begin{Bmatrix} 0.319(150) \\ -0.380(150) \\ 0.169(150) \end{Bmatrix} = \begin{Bmatrix} 47.9 \\ -57.0 \\ 25.4 \end{Bmatrix}$$

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Computation of modal story shears from modal interstory drift. To obtain the shears the drifts remultiplied by story stiffness.

Example 2 (continued)

Computing Interstory Shears: SRSS Combination

$$\begin{Bmatrix} \sqrt{207^2 + 197.1^2 + 47.9^2} \\ \sqrt{372.9^2 + 87.6^2 + 57^2} \\ \sqrt{465^2 + 157.9^2 + 25.4^2} \end{Bmatrix} = \begin{Bmatrix} 289.81 \\ 387.27 \\ 491.73 \end{Bmatrix}$$

“Exact”

$$\begin{Bmatrix} 130.1 \\ 310.5 \\ 525.7 \end{Bmatrix}$$

At time of max. shear

“Exact”

$$\begin{Bmatrix} 222.2 \\ 398.1 \\ 420.0 \end{Bmatrix}$$

At time of max. displacement

“Exact”

$$\begin{Bmatrix} 304.0 \\ 398.5 \\ 525.7 \end{Bmatrix}$$

Envelope = maximum per story

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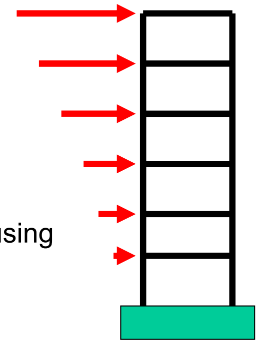
Calculation of total story shear using SRSS. Note that the story shears were not obtained from the SRSS response of the modal displacements.

Note the remarkable similarity in results from the exact envelope values, and from the response spectrum approach. This degree of correlation is somewhat unusual.

ASCE 7 Allows an Approximate Modal Analysis Technique Called the Equivalent Lateral Force Procedure

- Empirical period of vibration
- Smoothed response spectrum
- Compute total base shear, **V**, as if SDOF
- Distribute **V** along height assuming “regular” geometry
- Compute displacements and member forces using standard procedures



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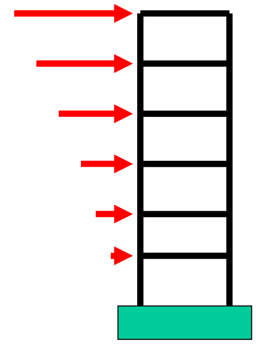
In the next several slides the equivalent lateral force method of analysis will be derived and demonstrated. The basis of the method is presented on this slide.

The appeal of this method is that the structural analysis requires only an application of static lateral forces to the structure, similar to the application of wind loads.

Equivalent Lateral Force Procedure

- Method is based on **first mode** response.
- Higher modes can be included empirically.
- Has been calibrated to provide a reasonable estimate of the envelope
  of story shear, NOT to provide accurate estimates of story force.
- May result in overestimate of overturning moment.



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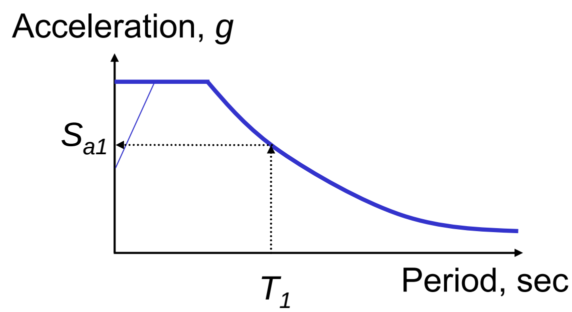
Additional features of the ELF method are described.

The overestimate of overturning moment resulted in “overturning moment reductions” in previous versions of the code. ASCE 7-05 still allows an overturning moment reduction of 25% at the foundation but no longer allows reductions in the above-grade structure. See Section 12.13.4 of ASCE 7-05.

Equivalent Lateral Force Procedure

- Assume first mode effective mass = total Mass = $M = W/g$
- Use response spectrum to obtain total acceleration @ T_1



$$V_B = (S_{a1}g)M = (S_{a1}g)\frac{W}{g} = S_{a1}W$$

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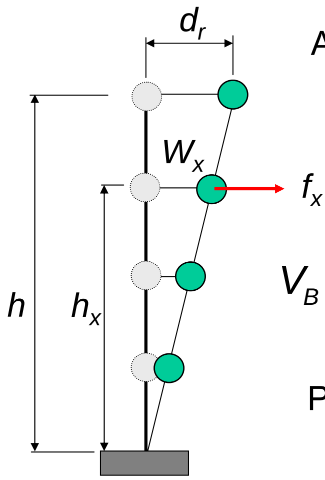
The first step in the ELF analysis is the computation of the base shear. The base shear is simply the spectral acceleration (expressed as a fraction of gravity) times the weight of the system. The acceleration, S_{a1}, is determined from the 5% damped elastic spectrum (modified for inelastic effects, as explained in a separate topic). T_1 is the first mode period of vibration of the structure.

Note that a modal participation factor for the first mode is assumed to be 1.0, and that 100% of the weight is assigned to the first mode.

Equivalent Lateral Force Procedure



Assume linear first mode response

$$f_x(t) = \frac{h_x}{h} d_r(t)\omega_1^2 \frac{W_x}{g}$$

$$V_B(t) = \sum_{i=1}^{nstories} f_i(t) = \frac{d_r(t)\omega_1^2}{hg} \sum_{i=1}^{nstories} h_i W_i$$

Portion of base shear applied to story *i*

$$\boxed{\frac{f_x(t)}{V_B(t)} = \frac{h_x W_x}{\sum_{i=1}^{nstories} h_i W_i}}$$

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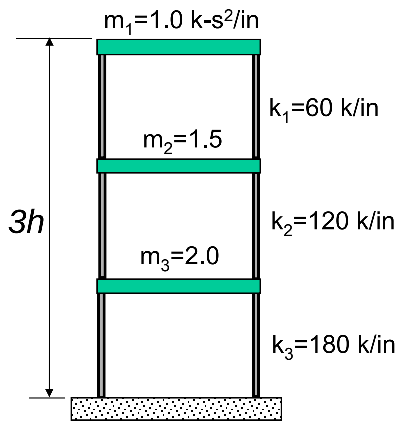
Once the base shear is determined, the next step is to distribute the shear along the height of the structure. This is done by assuming that the first mode shape is a straight line and that total accelerations can be approximated as the displacements times the frequency squared.

Given the displacement at any height, the acceleration is determined and the inertia force is computed as the weight at that height times the acceleration. The inertial force are summed to produce the total base shear. The ratio of the story force to the base shear is given by the final expression.

Note that the linear shape assumption is only valid for structures with T <= 0.5 sec. For longer periods the exponent k adjusts for higher modes.

ELF Procedure Example



Recall
T_1 = 1.37 sec

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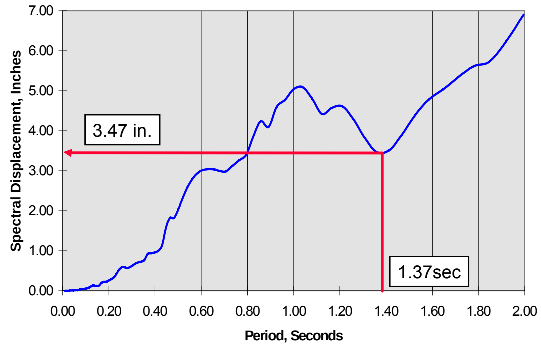
The example that was analyzed using response history analysis and response spectrum analysis will not be analyzed using ELF.

Note that this example has significant vertical irregularities in both mass and stiffness, and theoretically, can not be analyzed by ELF according to the code. See Table 12.6.1 of ASCE 7-05. For the purpose of this course, the analysis will proceed even though the code provision is violated.

ELF Procedure Example

Total weight = M x g = (1.0 + 1.5 + 2.0) 386.4 = 1738 kips

Spectral acceleration = $w^2 S_D$ = $(2p/1.37)^2$ x 3.47 = 72.7 in/sec^2 = 0.188g



Base shear = $S_a W$ = 0.188 x 1738 = 327 kips

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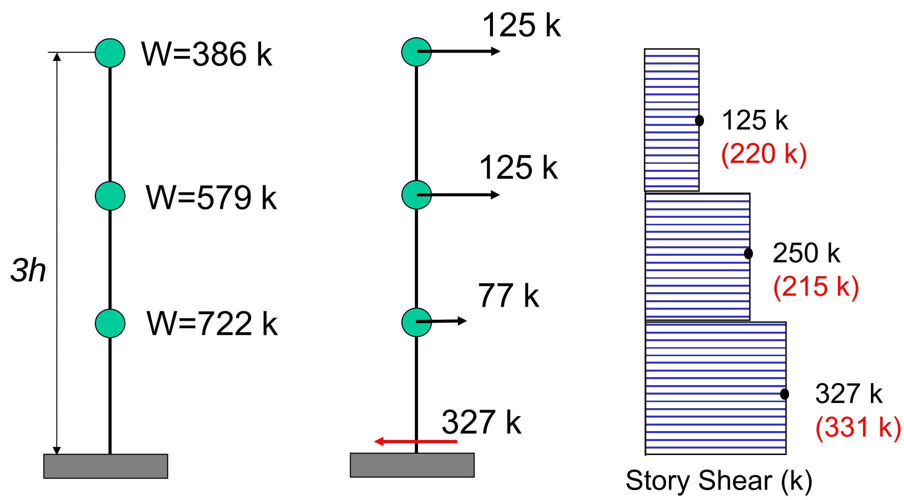
Instead of using the code spectrum, the example uses the actual 5% damped El Centro spectrum. This is done to provide better comparison with the previous analysis methods.

Note that unlike the response spectrum method, the modal participation factor is not used in the ELF method. Effectively, the participation factor is taken as 1.0 (and it is assumed that the first mode shape is normalized to have a value of 1.0 at the top of the structure).

ELF Procedure Example (Story Forces)

$$f_3 = \frac{386(3h)}{386(3h) + 579(2h) + 722(h)} = 0.381V_B = 0.375(327) = 125\ kips$$



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The story forces and shears are computed as shown. The values in red (in parenthesis) are from the response spectrum analysis using SRSS. The forces in the lower two levels compare pretty well, but the ELF method has done a very poor job predicting the shear in the upper level.

ELF Procedure Example (Story Displacements)

Units = inches

Time History (Envelope)	Modal Response Spectrum	**ELF**
$\begin{Bmatrix} 5.15 \\ 3.18 \\ 1.93 \end{Bmatrix}$	$\begin{Bmatrix} 5.18 \\ 3.33 \\ 1.84 \end{Bmatrix}$	$\begin{Bmatrix} 5.98 \\ 3.89 \\ 1.82 \end{Bmatrix}$

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The displacement computed from the various methods are compared here. The ELF method has done a reasonably good job predicting the displacements.

ELF Procedure Example (Summary)

- ELF procedure gives **good correlation** with base shear (327 kips ELF vs 331 kips modal response spectrum).
- ELF story force distribution is **not** as **good**. ELF underestimates shears in upper stories.
- ELF gives reasonable correlation with displacements.

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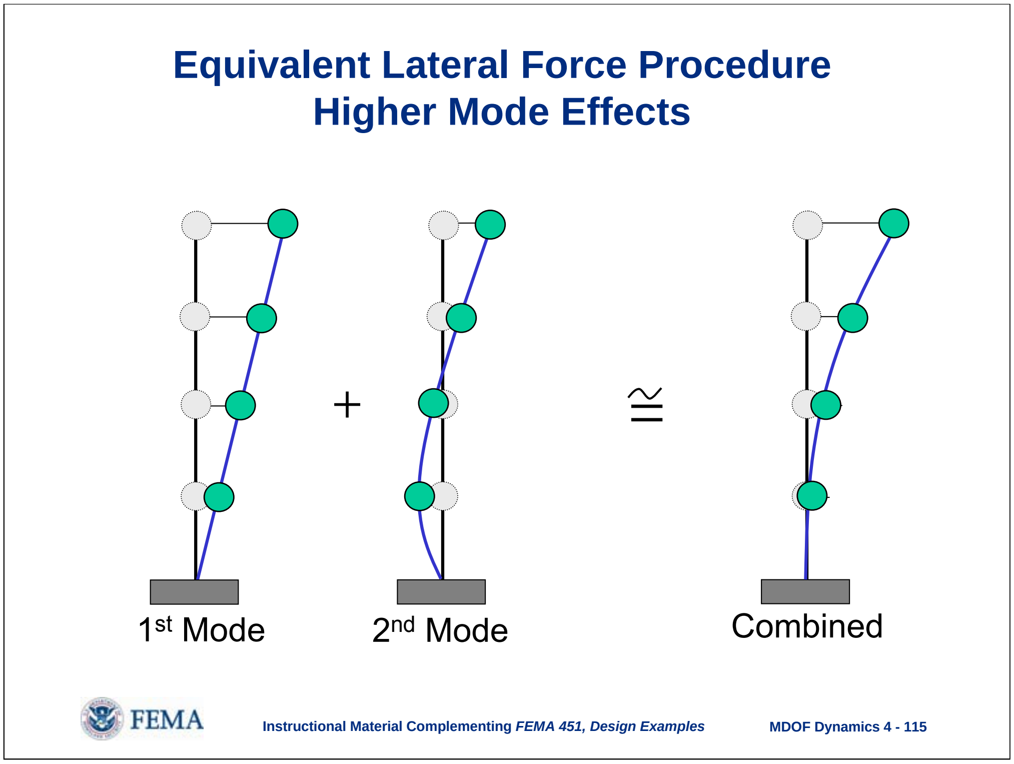
A basic summary of the ELF method is provided (with respect to the Response Spectrum approach). The fundamental question is whether the method is good enough for design. The answer is probably yes with the exception of the very poor prediction of the shear in the upper level of the structure.



The main reason for the poor performance for the example is that the ELF method as developed so far is based on first mode behavior only. If higher modes are included, accelerations and inertial forces will be higher at the top. As explained later, the code adjusts the ELF method by assuming a "combined" first and second mode as shown.

ASCE 7-05 ELF Approach

- Uses empirical period of vibration
- Uses smoothed response spectrum
- Has correction for higher modes
- Has correction for overturning moment
- Has limitations on use

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The ELF method as presented in ASCE-7 is very similar to that presented earlier. The following slides explain the differences and the limitations on use.

Approximate Periods of Vibration

$$T_a = C_t h_n^x$$

C_t = 0.028, x = 0.8 for steel moment frames
C_t = 0.016, x = 0.9 for concrete moment frames
C_t = 0.030, x = 0.75 for eccentrically braced frames
C_t = 0.020, x = 0.75 for all other systems
Note: For building structures **only**!

$$T_a = 0.1N$$

For moment frames < 12 stories in height, minimum story height of 10 feet. N = number of stories.

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In the code-based approach, the period is computed using the empirical formulas, and then adjusted for local seismicity (next slide).

Adjustment Factor on Approximate Period

$$\boxed{T = T_a C_u \leq T_{computed}}$$

S_{D1}	C_u
> 0.40g	1.4
0.30g	1.4
0.20g	1.5
0.15g	1.6
< 0.1g	1.7

Applicable **only** if $T_{computed}$ comes from a "properly substantiated analysis."

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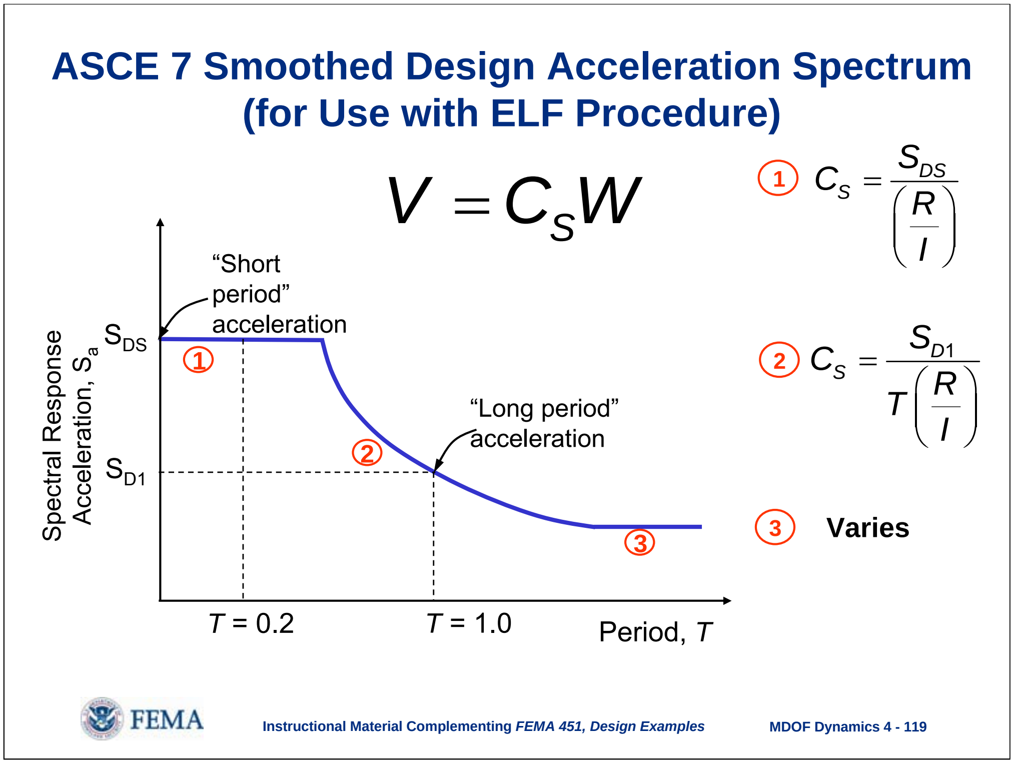
The period shown on this slide is the MAXIMUM period you are allowed to use, and this is only allowed if a detailed structural analysis has been used to provide a computed period.



The ASCE-7 spectrum has three basic branches. Various formulas are provided for the third part and, hence, the equation is not shown on the slide.

If the analyst used a true response spectrum, there are limitation on the base shear that is computed, and this limitation is tied to the ELF shear. More details are provided in the topic on seismic load analysis.

R is the *response modification factor*, a function of system inelastic behavior. This is covered in the topic on inelastic behavior. For now, use R = 1, which implies linear elastic behavior.

I is the *importance factor* which depends on the Seismic Use Group. I = 1.5 for essential facilities, 1.25 for important high occupancy structures, and 1.0 for normal structures. For now, use I = 1.

A preview of the terms R and I is provided. More detail is given in later topics.

Distribution of Forces Along Height

$$F_x = C_{vx} V$$

$$C_{vx} = \frac{w_x h_x^k}{\sum_{i=1}^{n} w_i h_i^k}$$

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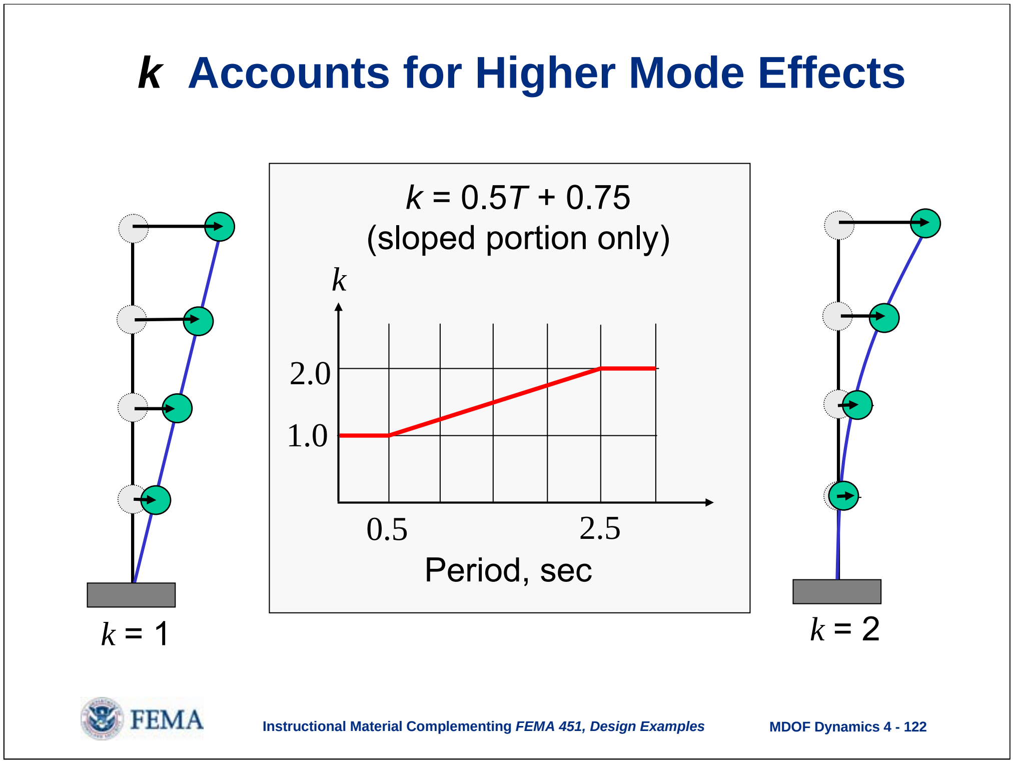
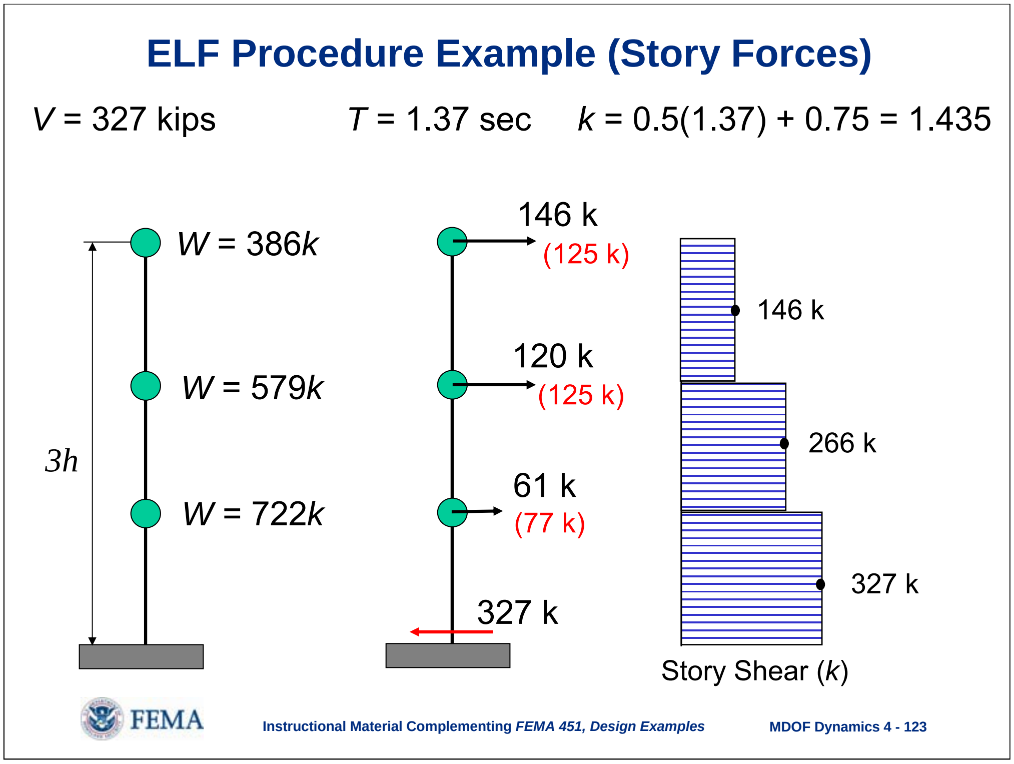
ASCE 7 provides an empirical "fix" for the case where higher modes will influence the accuracy of the ELF method. Computationally, the correction is provided by the use of the exponent k on the height term of the expression that is used to compute the relative value of base shear applied at each level of the structure.



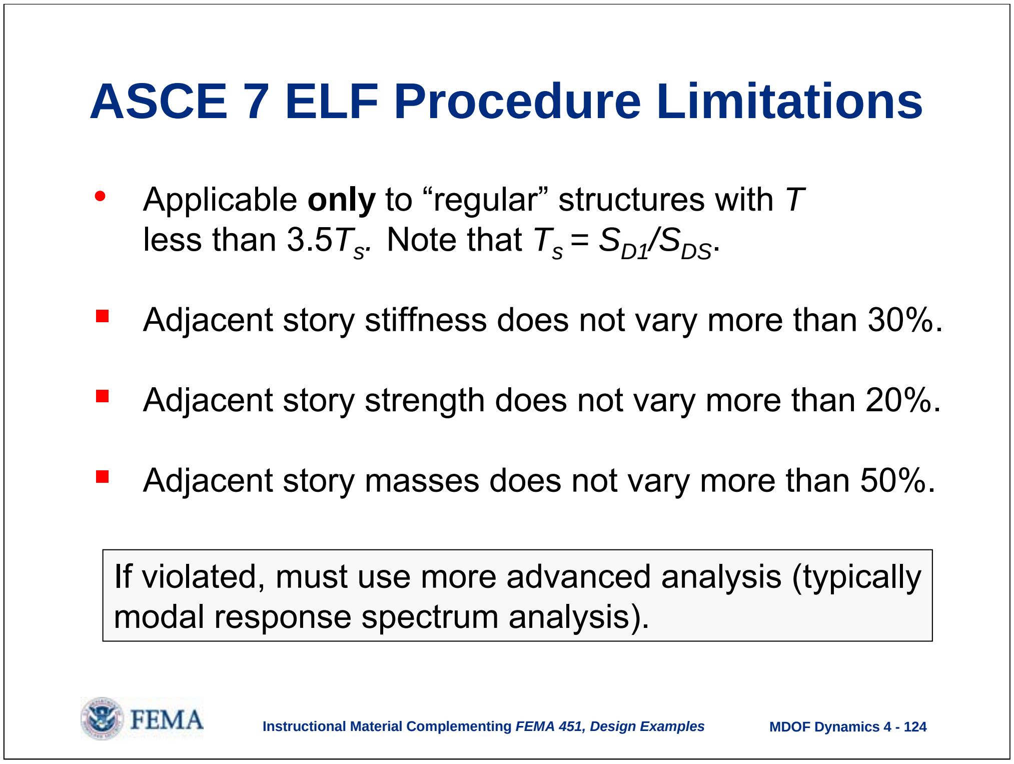
The higher the fundamental period, the more important the higher mode effect. Hence, ASCE ties the exponent k to the period. For short period structures (T < 0.5 sec), k=1 and the first mode shape is linear. For higher period (presumably taller) structures (T > 2), the mode shape is parabolic with k = 2.



The example has been reworked to include the "higher mode effect." Here the values shown in red (parenthesis) are for the case where k=1. There is a slight improvement as there is now higher shear towards the top. The shear is still less than that given be the response spectrum method, however.

A fundamental issue still not overcome in this example is that the structure has nonuniform mass and stiffness and, hence, is not particularly suitable for ELF analysis. It is for this reason that ASCE-7 has restrictions on the use of ELF.

Note that the structure with uniform mass and stiffness (Example 2) with exponent k=1.435 gives a much better match with ELF.



This slide shows the limitation on use of the ELF procedure as given in Table 12.3-2 of ASCE-7.

ASCE 7 ELF Procedure
Other Considerations Affecting Loading

- Orthogonal loading effects
- Redundancy
- Accidental torsion
- Torsional amplification
- P-delta effects
- Importance factor
- Ductility and overstrength

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Aside for distributing forces along the height of the structure, the standard has additional requirements. Many of these requirements are also applicable to designs based on response history and response spectrum analysis.

These items are covered in more detail in later topics.