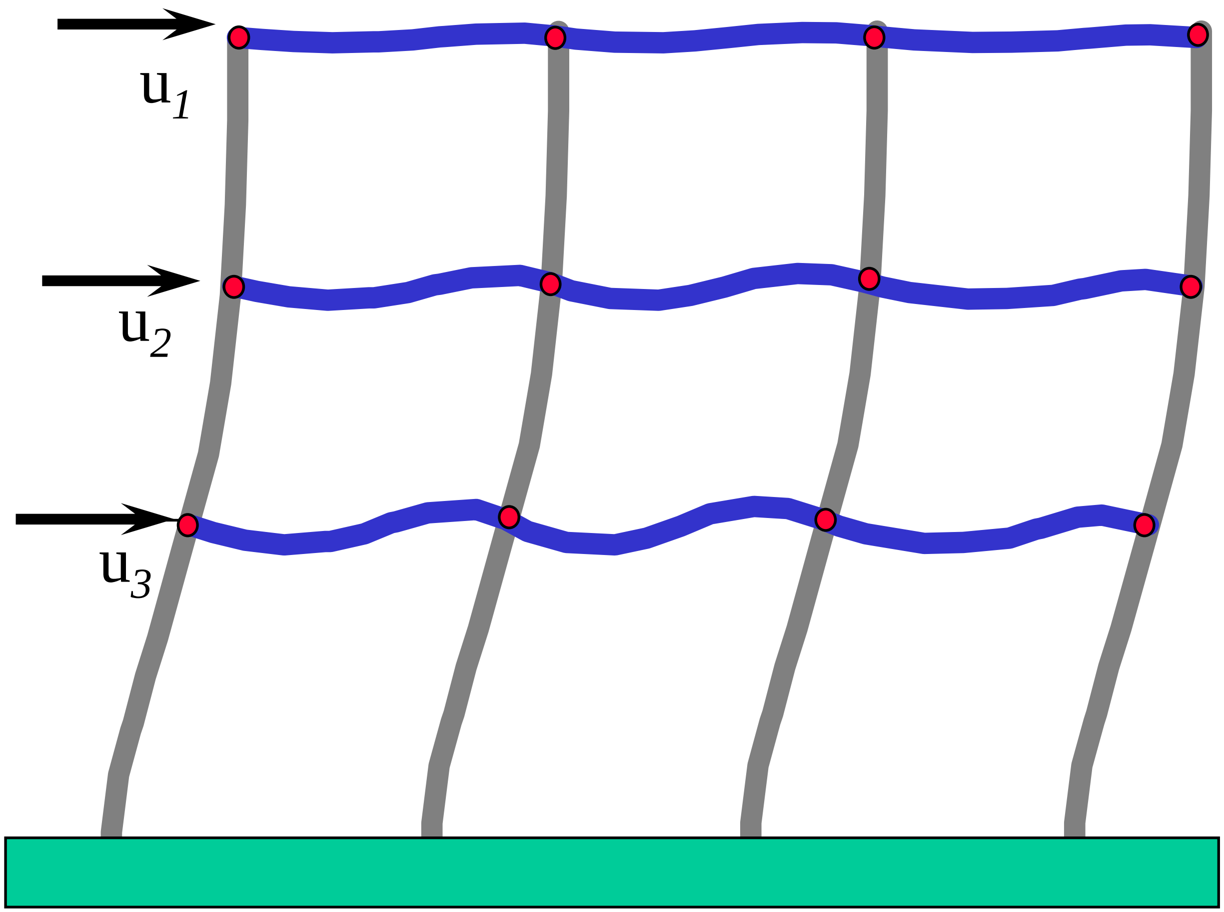
Structural Dynamics of Linear Elastic Multiple-Degrees-of-Freedom (MDOF) Systems



Structural Dynamics of Elastic MDOF Systems

- Equations of motion for MDOF systems
- Uncoupling of equations through use of natural mode shapes
- Solution of uncoupled equations
- Recombination of computed response
- Modal response history analysis
- Modal response spectrum analysis
- Equivalent lateral force procedure

Symbol Styles Used in this Topic

$\mathbf{M}$
$\mathbf{U}$

Matrix or vector (column matrix)

m
u

Element of matrix or vector or set

(often shown with subscripts)

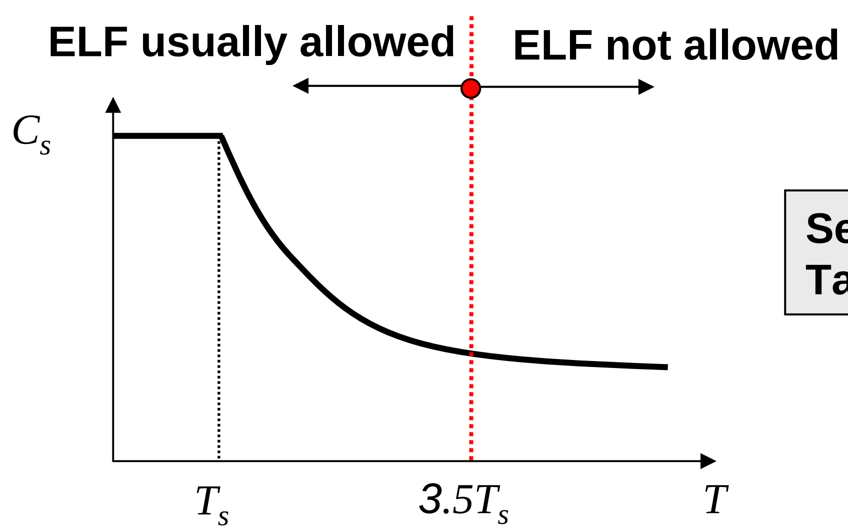
W
g

Scalars

Relevance to ASCE 7-05

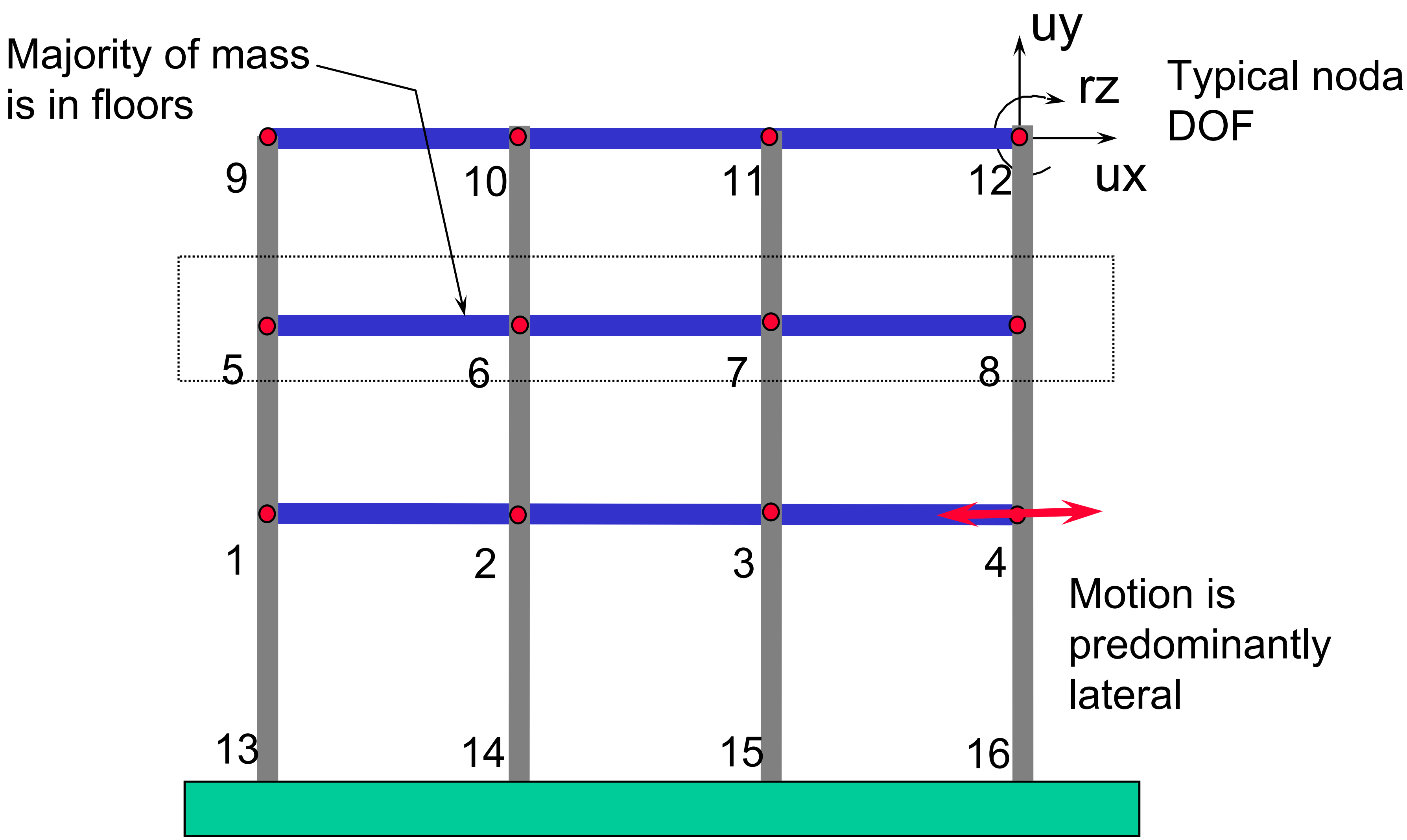
ASCE 7-05 provides guidance for three specific analysis procedures:

- Equivalent lateral force (ELF) analysis
- Modal superposition analysis (MSA)
- Response history analysis (RHA)

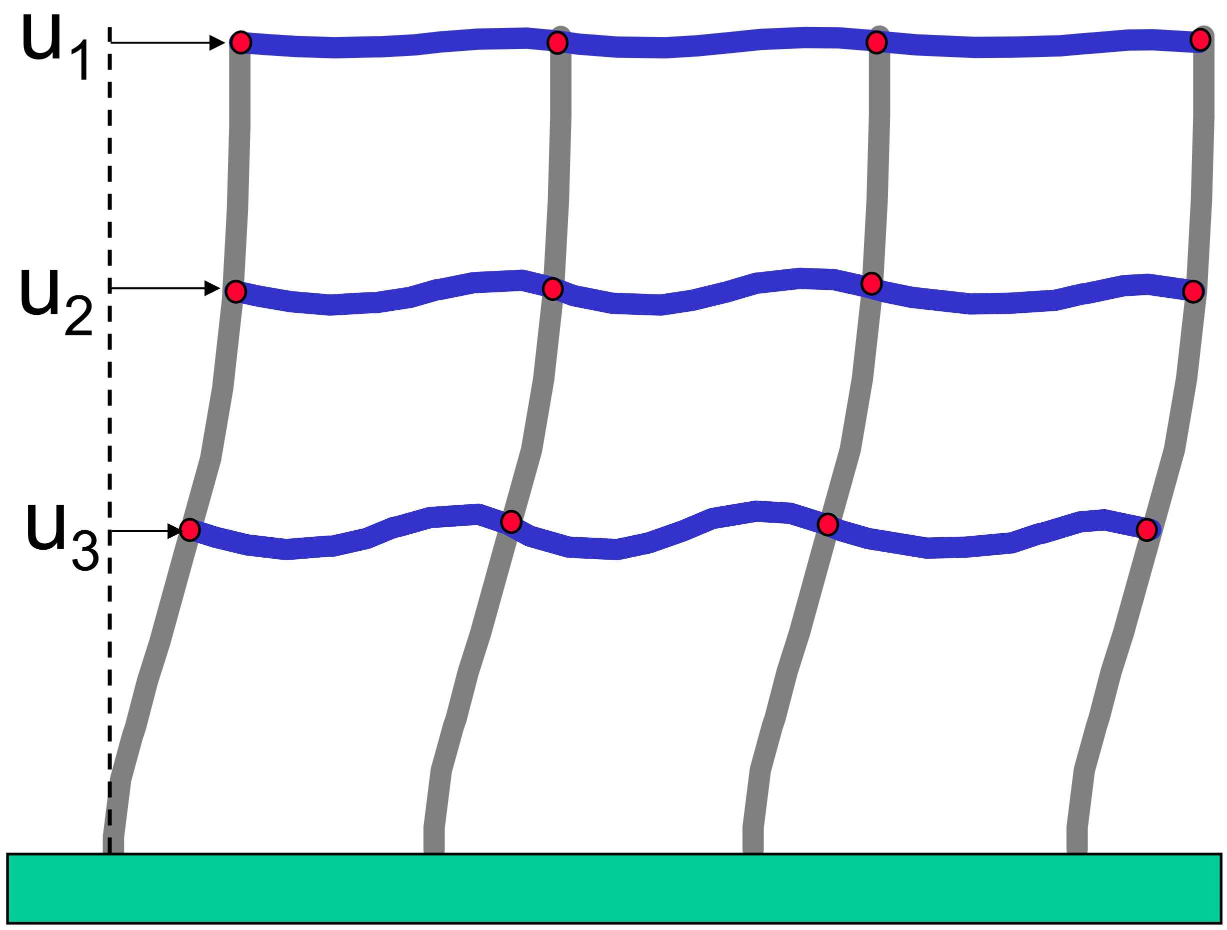


See ASCE 7-05 Table 12.6-1

Planar Frame with 36 Degrees of Freedom

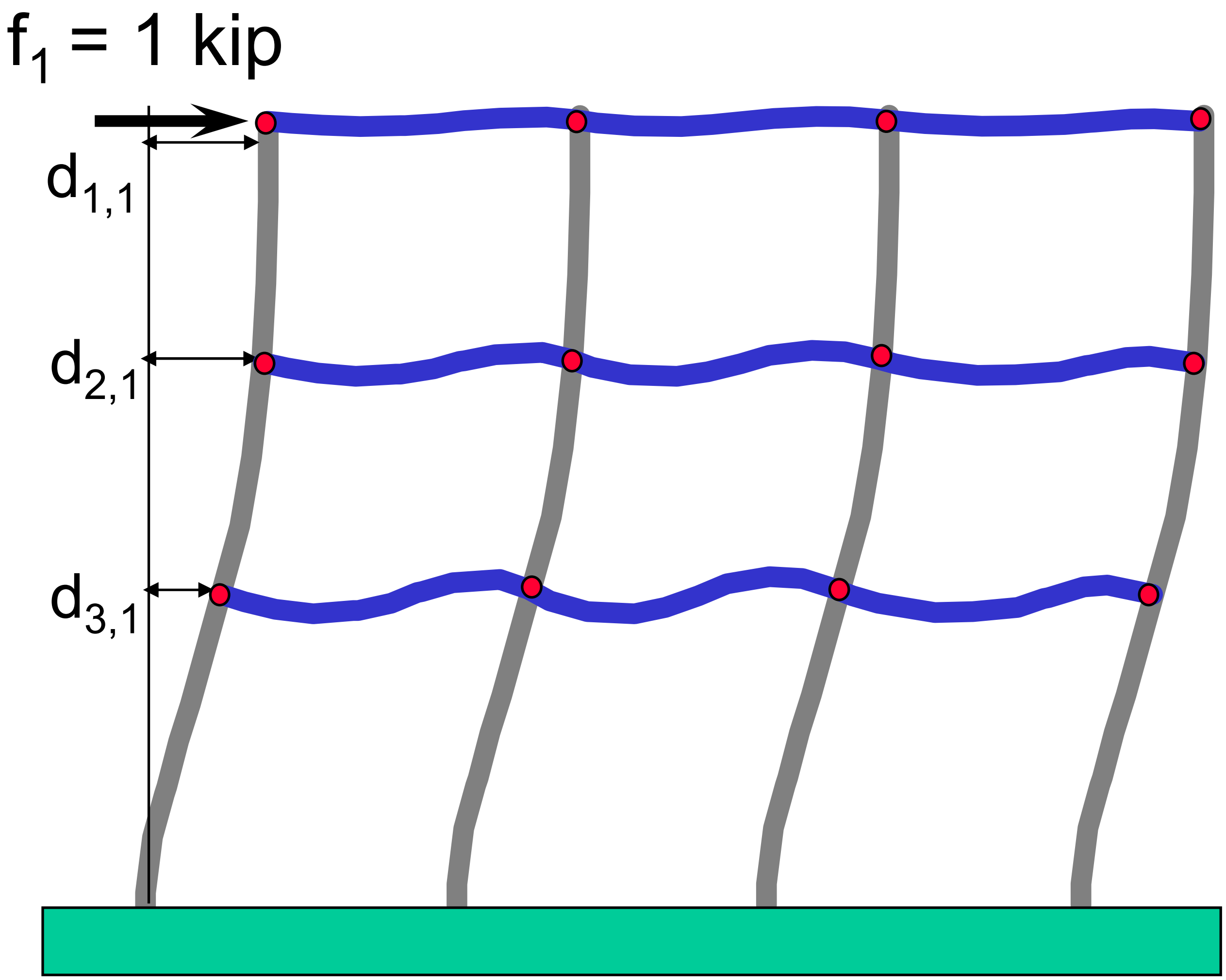


Planar Frame with 36 Static Degrees of Freedom But with Only THREE Dynamic DOF



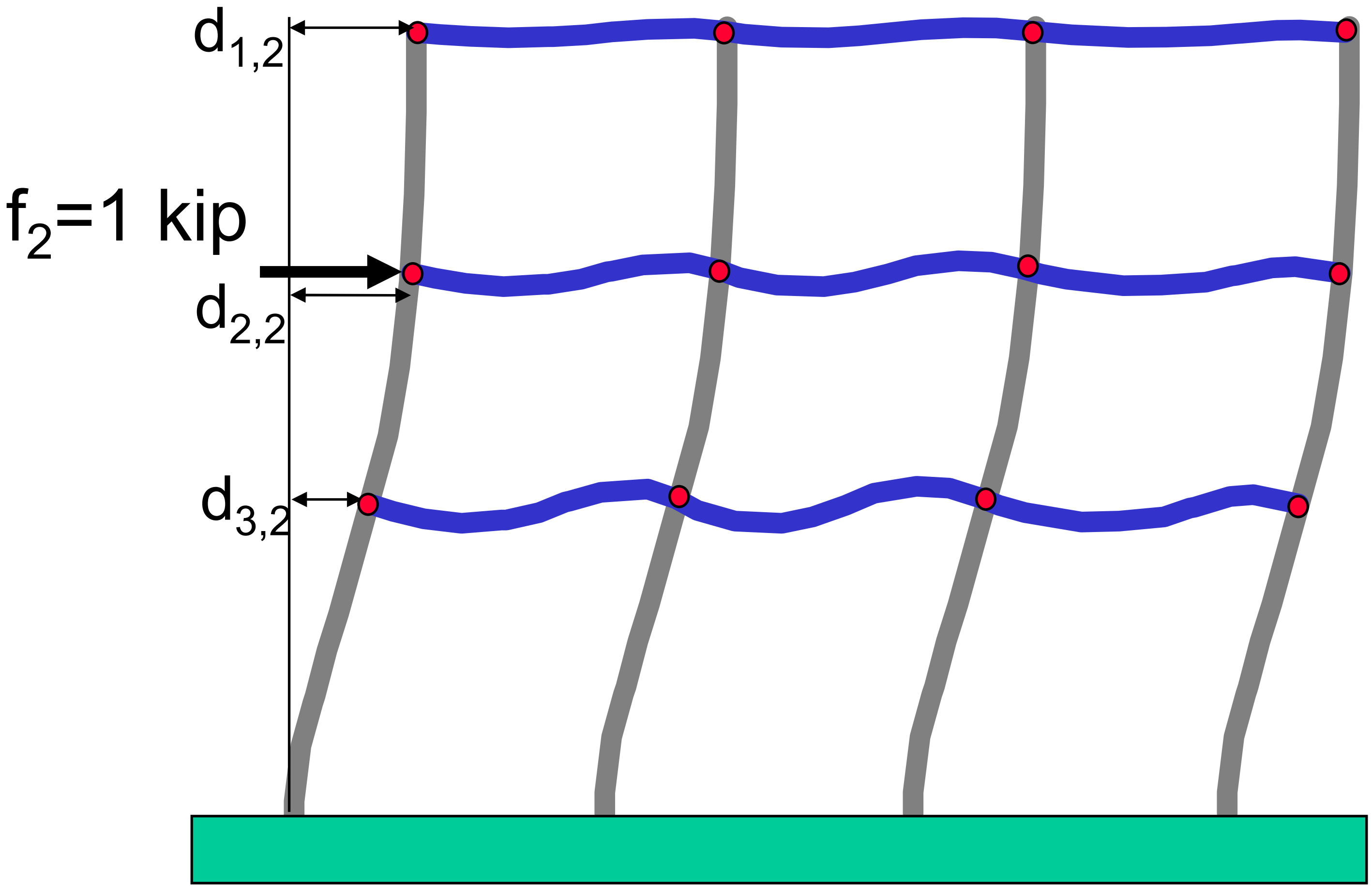
$$U = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Development of Flexibility Matrix



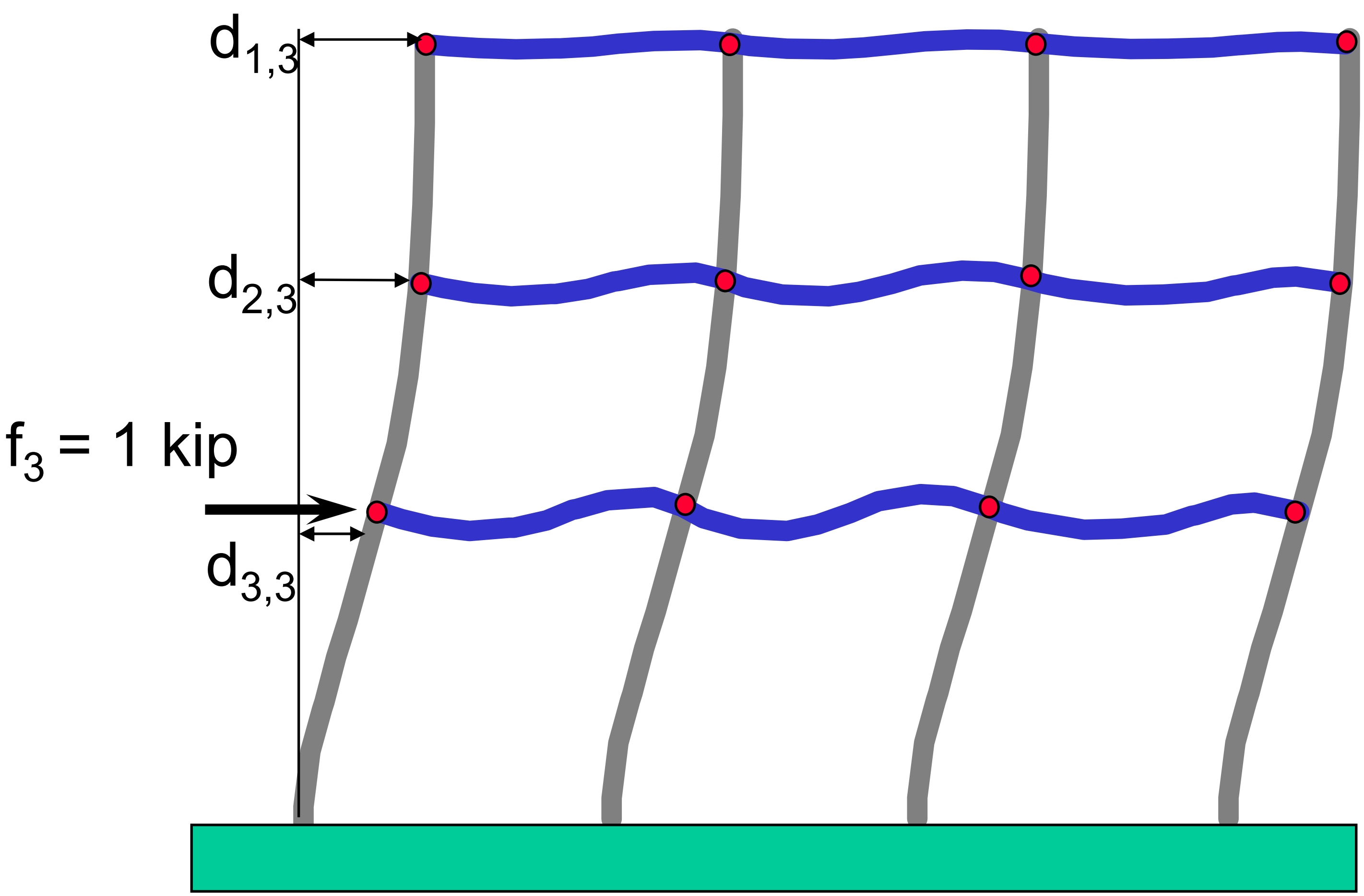
$$\begin{bmatrix} d_{1,1} & \bullet & \bullet \\ d_{2,1} & \bullet & \bullet \\ d_{3,1} & \bullet & \bullet \end{bmatrix}$$

Development of Flexibility Matrix
(continued)



$$\begin{bmatrix} d_{1,1} & d_{1,2} & \bullet \\ d_{2,1} & d_{2,2} & \bullet \\ d_{3,1} & d_{3,2} & \bullet \end{bmatrix}$$

Development of Flexibility Matrix
(continued)



$$\begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} \\ d_{2,1} & d_{2,2} & d_{2,3} \\ d_{3,1} & d_{3,2} & d_{3,3} \end{bmatrix}$$

Concept of Linear Combination of Shapes (Flexibility)

$$U = \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} \\ d_{2,1} & d_{2,2} & d_{2,3} \\ d_{3,1} & d_{3,2} & d_{3,3} \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}$$

$$U = \begin{Bmatrix} d_{1,1} \\ d_{2,1} \\ d_{3,1} \end{Bmatrix} f_1 + \begin{Bmatrix} d_{1,2} \\ d_{2,2} \\ d_{3,2} \end{Bmatrix} f_2 + \begin{Bmatrix} d_{1,3} \\ d_{2,3} \\ d_{3,3} \end{Bmatrix} f_3$$

$$DF = U \qquad K = D^{-1} \qquad KU = F$$

Static Condensation

$$\begin{bmatrix} K_{m,m} & K_{m,n} \\ K_{n,m} & K_{n,n} \end{bmatrix} \begin{Bmatrix} U_m \\ U_n \end{Bmatrix} = \begin{Bmatrix} F_m \\ \{0\} \end{Bmatrix}$$

DOF with mass

Massless DOF

1 $K_{m,m}U_m + K_{m,n}U_n = F_m$

2 $K_{n,m}U_m + K_{n,n}U_n = \{0\}$

Static Condensation
(continued)

Rearrange (2) $$U_n = -K_{n,n}^{-1} K_{n,m} U_m$$

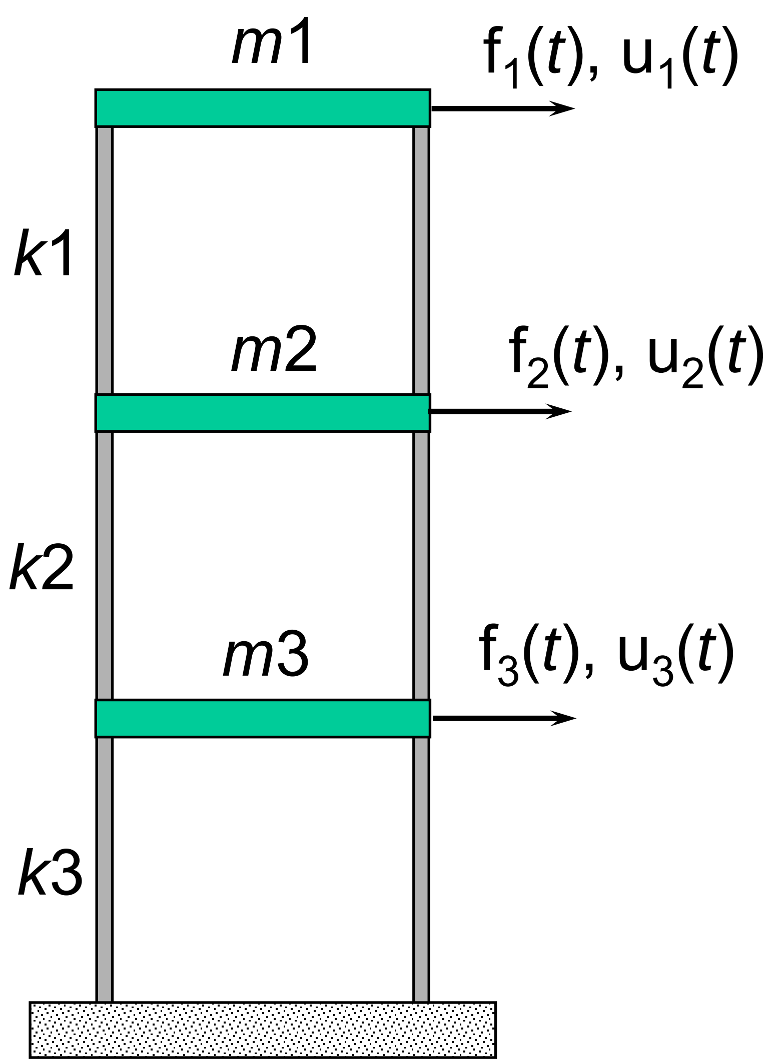
Plug into (1) $$K_{m,m} U_m - K_{m,n} K_{n,n}^{-1} K_{n,m} U_m = F_m$$

Simplify $$\left[K_{m,m} - K_{m,n} K_{n,n}^{-1} K_{n,m} \right] U_m = F_m$$

$$\boxed{\hat{K} = K_{m,m} - K_{m,n} K_{n,n}^{-1} K_{n,m}}$$

Condensed stiffness matrix

Idealized Structural Property Matrices



$$K = \begin{bmatrix} k1 & -k1 & 0 \\ -k1 & k1+k2 & -k2 \\ 0 & -k2 & k2+k3 \end{bmatrix}$$

$$M = \begin{bmatrix} m1 & 0 & 0 \\ 0 & m2 & 0 \\ 0 & 0 & m3 \end{bmatrix}$$

$$F(t) = \begin{Bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{Bmatrix} \qquad U(t) = \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix}$$

Note: Damping to be shown later

Coupled Equations of Motion for Undamped Forced Vibration

$$M\ddot{U}(t) + KU(t) = F(t)$$

$$\begin{bmatrix} m1 & 0 & 0 \\ 0 & m2 & 0 \\ 0 & 0 & m3 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \ddot{u}_3(t) \end{Bmatrix} + \begin{bmatrix} k1 & -k1 & 0 \\ -k1 & k1+k2 & -k2 \\ 0 & -k3 & k2+k3 \end{bmatrix} \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{Bmatrix}$$

DOF 1 $m1\ddot{u}_1(t) + k1u_1(t) - k1u_2(t) = f_1(t)$

DOF 2 $m2\ddot{u}_2(t) - k1u_1(t) + k1u_2(t) + k2u_2(t) - k2u_3(t) = f_2(t)$

DOF 3 $m3\ddot{u}_3(t) - k2u_2(t) + k2u_3(t) + k3u_3(t) = f_3(t)$

Developing a Way To Solve the Equations of Motion

- This will be done by a transformation of coordinates from *normal coordinates* (displacements at the nodes) To *modal coordinates* (amplitudes of the natural Mode shapes).
- Because of the *orthogonality property* of the natural mode shapes, the equations of motion become uncoupled, allowing them to be solved as SDOF equations.
- After solving, we can transform back to the normal coordinates.

Solutions for System in Undamped Free Vibration (Natural Mode Shapes and Frequencies)

$$M\ddot{U}(t) + KU(t) = \{0\}$$

Assume $\quad \mathrm{U}(t) = \phi \sin \omega t \quad \ddot{\mathrm{U}}(t) = -\omega^2 \phi \sin \omega t$

Then $K\phi - \omega^2 M\phi = \{0\}$ has three (n) solutions:

$$\phi_1 = \begin{Bmatrix} \phi_{1,1} \\ \phi_{2,1} \\ \phi_{3,1} \end{Bmatrix}, \quad \omega_1 \qquad \phi_2 = \begin{Bmatrix} \phi_{1,2} \\ \phi_{2,2} \\ \phi_{3,2} \end{Bmatrix}, \quad \omega_2 \qquad \phi_3 = \begin{Bmatrix} \phi_{1,3} \\ \phi_{2,3} \\ \phi_{3,3} \end{Bmatrix}, \quad \omega_3$$

Natural frequency

Natural mode shape

Solutions for System in Undamped Free Vibration (continued)

For a SINGLE Mode

$$K\Phi = M\Phi\Omega^2$$ For ALL Modes

Where: $\Phi = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \end{bmatrix}$

$$\Omega^2 = \begin{bmatrix} \omega_1^2 & & \\ & \omega_2^2 & \\ & & \omega_3^2 \end{bmatrix}$$

$$K\phi = \omega^2 M\phi$$

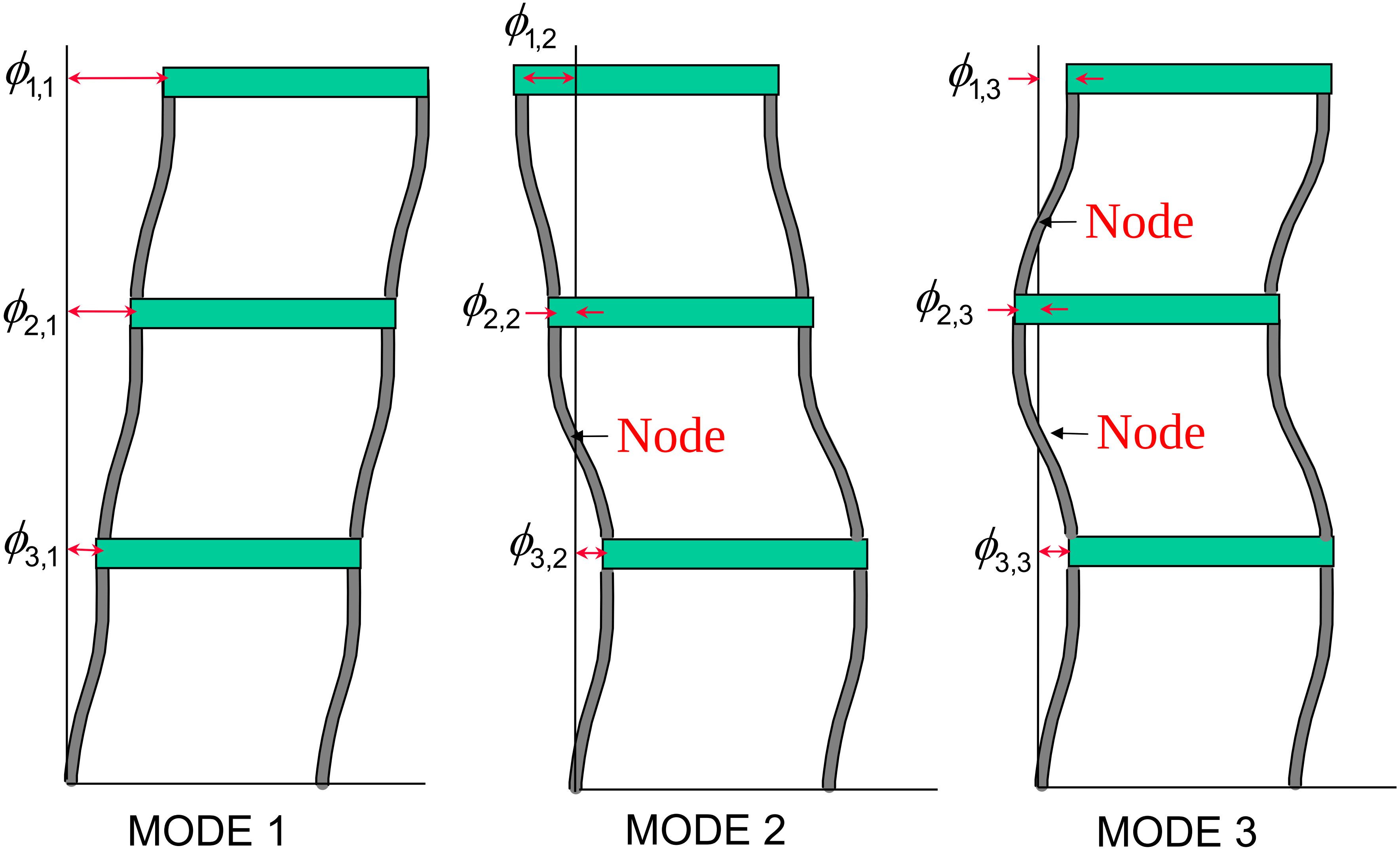
Note: Mode shape has arbitrary scale; usually

$$\phi_{1,i} = 1.0$$

or

$$\Phi^T M\Phi = I$$

Mode Shapes for Idealized 3-Story Frame



Concept of Linear Combination of Mode Shapes (Transformation of Coordinates)

$$U = \Phi Y$$

Mode shape

$$U = \begin{bmatrix} \phi_{1,1} & \phi_{1,2} & \phi_{1,3} \\ \phi_{2,1} & \phi_{2,2} & \phi_{2,3} \\ \phi_{3,1} & \phi_{3,2} & \phi_{3,3} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \qquad U = \begin{Bmatrix} \phi_{1,1} \\ \phi_{2,1} \\ \phi_{3,1} \end{Bmatrix} y_1 + \begin{Bmatrix} \phi_{1,2} \\ \phi_{2,2} \\ \phi_{3,2} \end{Bmatrix} y_2 + \begin{Bmatrix} \phi_{1,3} \\ \phi_{2,3} \\ \phi_{3,3} \end{Bmatrix} y_3$$

Modal coordinate = amplitude of mode shape

Orthogonality Conditions

$$\Phi = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \end{bmatrix}$$

Generalized mass

$$\Phi^T M \Phi = \begin{bmatrix} m_1^* & & \\ & m_2^* & \\ & & m_3^* \end{bmatrix}$$

Generalized stiffness

$$\Phi^T K \Phi = \begin{bmatrix} k_1^* & & \\ & k_2^* & \\ & & k_3^* \end{bmatrix}$$

Generalized damping

$$\Phi^T C \Phi = \begin{bmatrix} c_1^* & & \\ & c_2^* & \\ & & c_3^* \end{bmatrix}$$

Generalized force

$$\Phi^T \mathrm{F}(t) = \begin{Bmatrix} \mathrm{f}_1^*(t) \\ \mathrm{f}_2^*(t) \\ \mathrm{f}_3^*(t) \end{Bmatrix}$$

Development of Uncoupled Equations of Motion

MDOF equation of motion: $M\ddot{U} + C\dot{U} + KU = F(t)$

Transformation of coordinates: $U = \Phi Y$

Substitution: $M\Phi\ddot{Y} + C\Phi\dot{Y} + K\Phi Y = F(t)$

Premultiply by Φ^T: $\Phi^T M\Phi\ddot{Y} + \Phi^T C\Phi\dot{Y} + \Phi^T K\Phi Y = \Phi^T F(t)$

Using orthogonality conditions, uncoupled equations of motion are:

$$\begin{bmatrix} m_1^* & & \\ & m_2^* & \\ & & m_3^* \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{Bmatrix} + \begin{bmatrix} c_1^* & & \\ & c_2^* & \\ & & c_3^* \end{bmatrix} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{Bmatrix} + \begin{bmatrix} k_1^* & & \\ & k_2^* & \\ & & k_3^* \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} f_1^*(t) \\ f_2^*(t) \\ f_3^*(t) \end{Bmatrix}$$

Development of Uncoupled Equations of Motion (Explicit Form)

Mode 1 $$m_1^* \ddot{y}_1 + c_1^* \dot{y}_1 + k_1^* y_1 = f_1^*(t)$$

Mode 2 $$m_2^* \ddot{y}_2 + c_2^* \dot{y}_2 + k_2^* y_2 = f_2^*(t)$$

Mode 3 $$m_3^* \ddot{y}_3 + c_3^* \dot{y}_3 + k_3^* y_3 = f_3^*(t)$$

Development of Uncoupled Equations of Motion (Explicit Form)

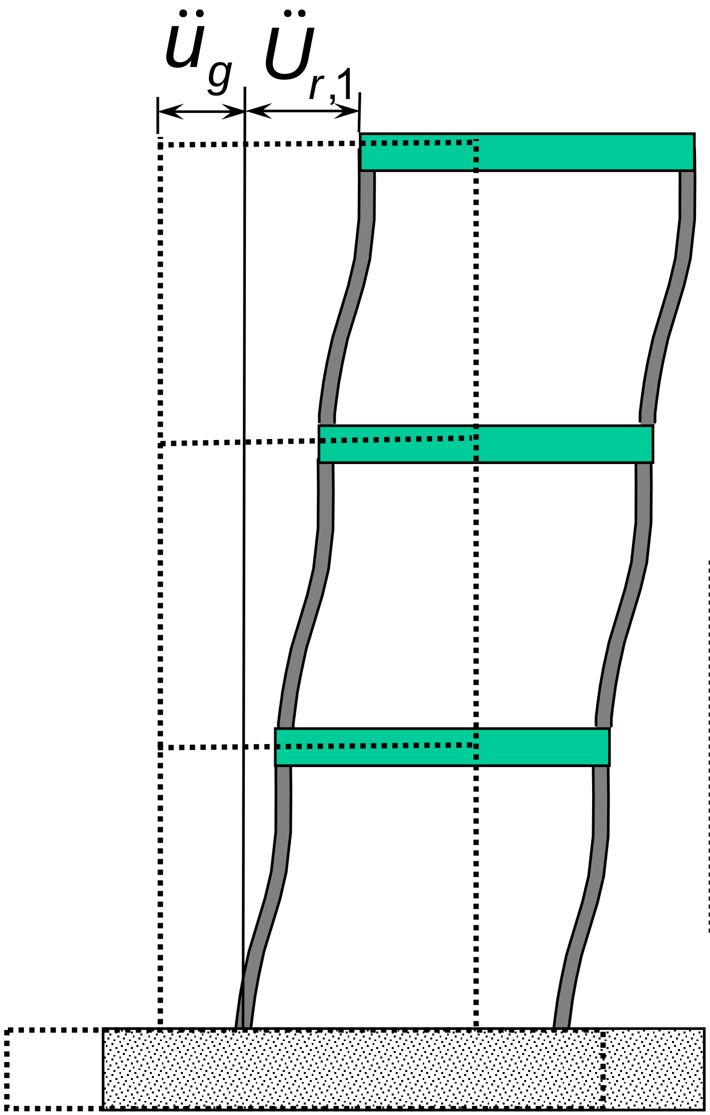
Simplify by dividing through by m* and defining $\xi_i = \dfrac{c_i^*}{2m_i^*\omega_i}$

Mode 1 $$\ddot{y}_1 + 2\xi_1\omega_1\dot{y}_1 + \omega_1^2 y_1 = f_1^*(t)/m_1^*$$

Mode 2 $$\ddot{y}_2 + 2\xi_2\omega_2\dot{y}_2 + \omega_2^2 y_2 = f_2^*(t)/m_2^*$$

Mode 3 $$\ddot{y}_3 + 2\xi_3\omega_3\dot{y}_3 + \omega_3^2 y_3 = f_3^*(t)/m_3^*$$

Earthquake “Loading” for MDOF System



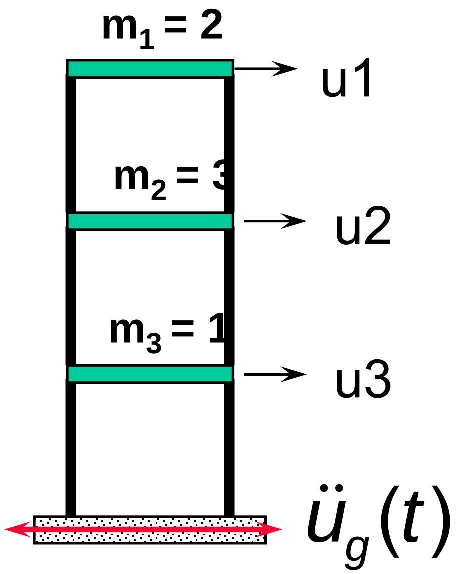
$$F_I(t) = M\begin{Bmatrix} \ddot{u}_g(t) + \ddot{u}_{r,1}(t) \\ \ddot{u}_g(t) + \ddot{u}_{r,2}(t) \\ \ddot{u}_g(t) + \ddot{u}_{r,3}(t) \end{Bmatrix} =$$

$$\boxed{M\begin{Bmatrix} 1.0 \\ 1.0 \\ 1.0 \end{Bmatrix}\ddot{u}_g(t)} + M\begin{Bmatrix} \ddot{u}_{r,1}(t) \\ \ddot{u}_{r,2}(t) \\ \ddot{u}_{r,3}(t) \end{Bmatrix}$$

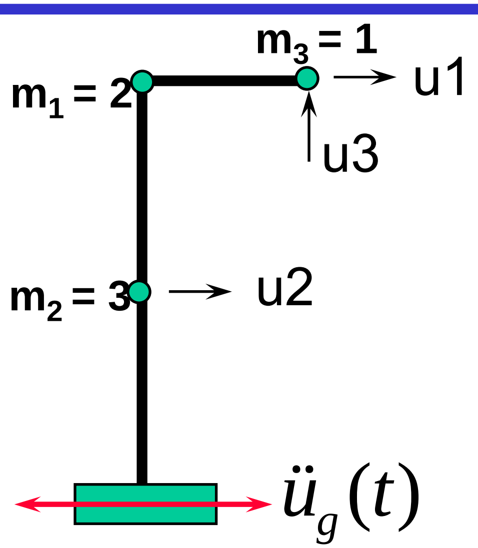
Move to RHS as $F_{EFF}(t) = -\,M\,R\ddot{u}_g(t)$

Modal Earthquake Loading

$$F^*(t) = -\Phi^T M R \ddot{u}_g(t)$$



$$M = \begin{bmatrix} 2 & & \\ & 3 & \\ & & 1 \end{bmatrix} \qquad R = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$



$$M = \begin{bmatrix} 2+1 & & \\ & 3 & \\ & & 1 \end{bmatrix} \qquad R = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

Definition of Modal Participation Factor

For earthquakes: $f_i^*(t) = -\phi_i^T M R \ddot{u}_g(t)$

Typical modal equation:

$$\ddot{y}_i + 2\xi_i\omega_i\dot{y}_i + \omega_i^2 y_i = \frac{f_i^*(t)}{m_i^*} = -\boxed{\frac{\phi_i^T MR}{m_i^*}}\ddot{u}_g(t)$$

Modal participation factor p_i

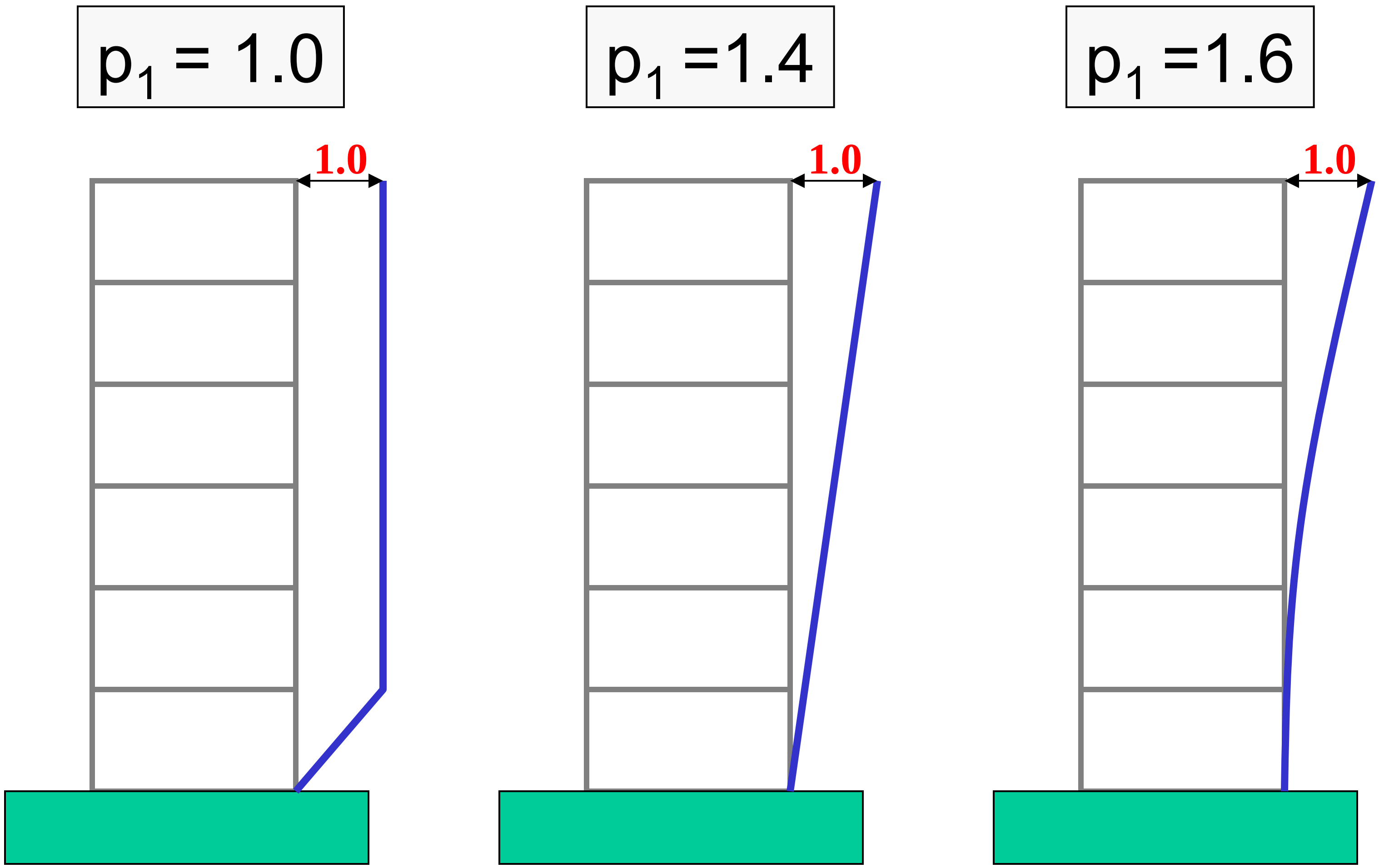
Caution Regarding Modal Participation Factor

$$p_i = \frac{\phi_i^T M R}{m_i^*}$$

$$\boxed{\phi_i^T M \phi_i}$$

Its value is dependent on the (arbitrary) method used to scale the mode shapes.

Variation of First Mode Participation Factor with First Mode Shape



Concept of Effective Modal Mass

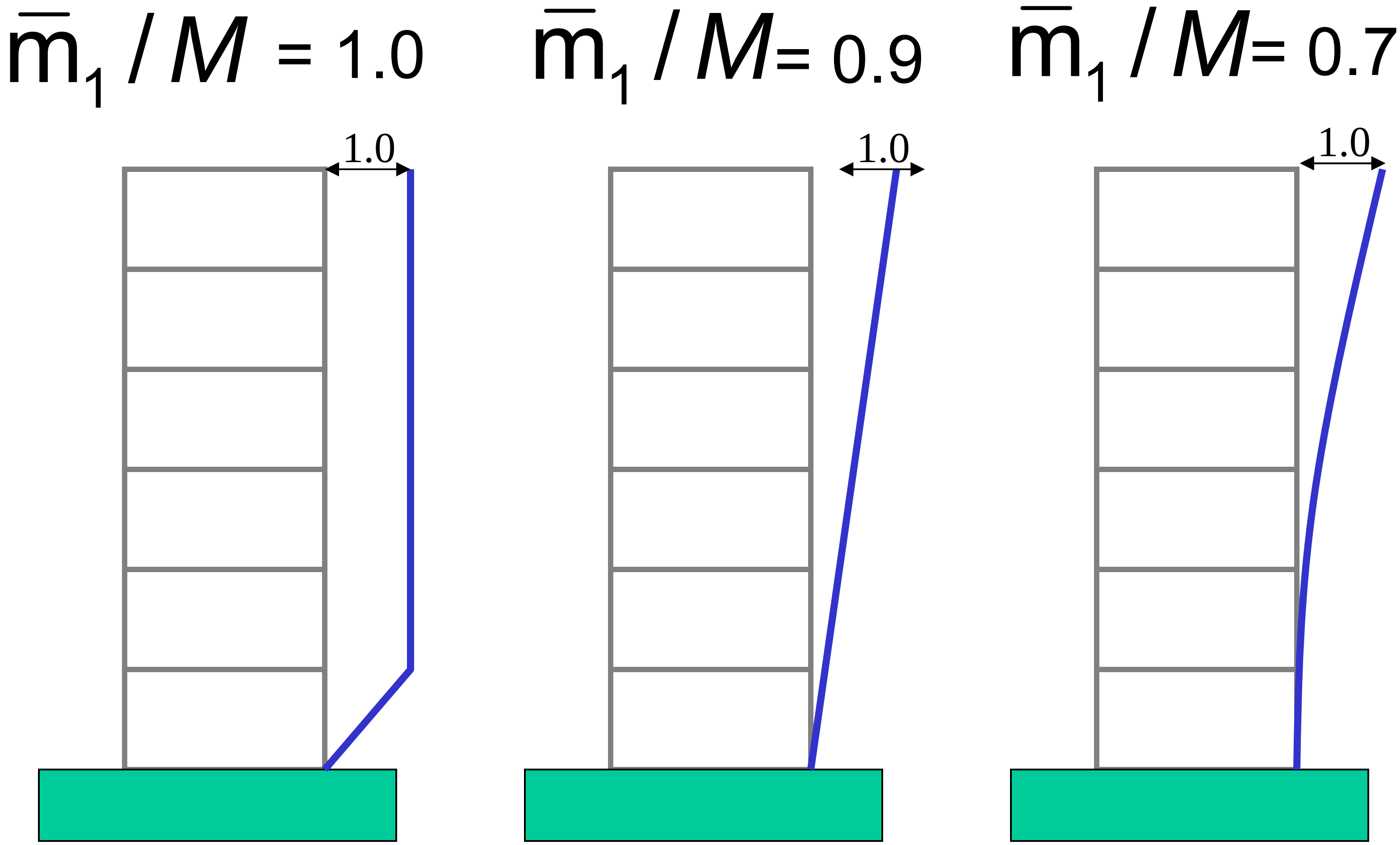
For each Mode *I*, $\bar{m}_i = p_i^2 m_i^*$

- The sum of the effective modal mass for all modes is
  equal to the total structural mass.

- The value of effective modal mass is *independent* of mode shape scaling.

- Use enough modes in the analysis to provide a total effective mass not less than 90% of the total structural mass.

Variation of First Mode Effective Mass with First Mode Shape



Derivation of Effective Modal Mass
(continued)

For each mode:

$$\ddot{y}_i + 2\xi_i\omega_i\dot{y}_i + \omega_i^2 y_i = -p_i\ddot{u}_g$$

SDOF system:

$$\ddot{q}_i + 2\xi_i\omega_i\dot{q}_i + \omega_i^2 q_i = -\ddot{u}_g$$

Modal response history, $q_i(t)$ is obtained by first solving the SDOF system.

Derivation of Effective Modal Mass

(continued)

From previous slide $$y_i(t) = p_i\, q_i(t)$$

Recall $$u_i(t) = \phi_i\, y_i(t)$$

Substitute $$u_i(t) = p_i\, \phi_i\, q_i(t)$$

Derivation of Effective Modal Mass
(continued)

Applied "static" forces required to produce $u_i(t)$:

$$V_i(t) = Ku_i(t) = P_i K\phi_i q_i(t)$$

Recall: $$K\phi_i = \omega_i^2 M\phi_i$$

Substitute:

$$V_i(t) = M\phi_i P_i \omega_i^2 q_i(t)$$

Derivation of Effective Modal Mass
(continued)

Total shear in mode: $\bar{V}_i = V_i^T R$

$$\bar{V}_i = (M\phi_i)^T RP_i\omega^2 q_i(t) = \phi_i^T MRP_i \boxed{\omega^2 q_i(t)}$$

"Acceleration" in mode

Define effective modal mass:

$$\bar{M}_i = \phi_i^T MRP_i$$

and

$$\bar{V}_i = \bar{M}_i \omega^2 q_i(t)$$

Derivation of Effective Modal Mass
(continued)

$$\bar{M}_i = \phi_i^T MRP_i = \left[\frac{\phi_i^T MR}{\phi_i^T M \phi_i}\right]_i \phi_i^T M \phi_i P_i$$

$$\bar{M}_i = P_i^2 m_i^*$$

Development of a Modal Damping Matrix

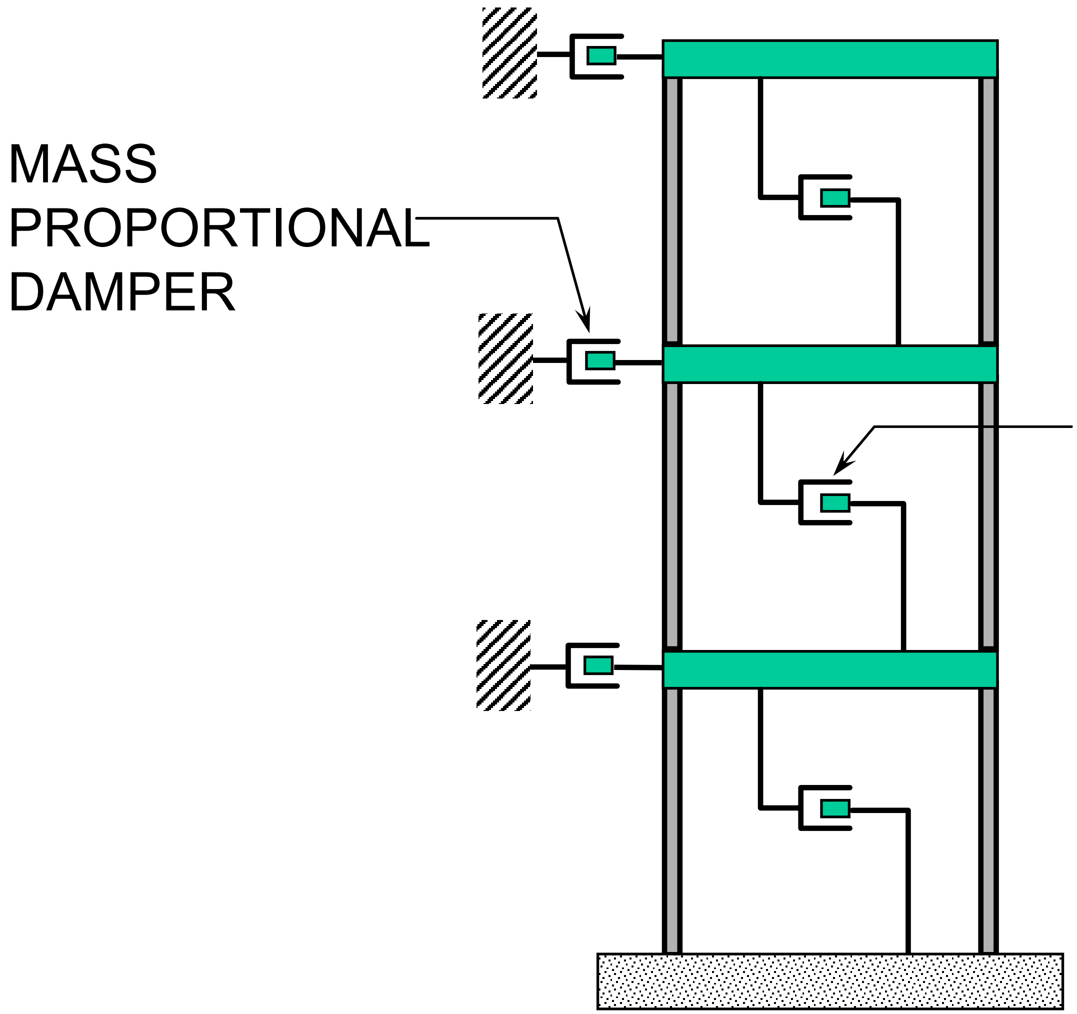
In previous development, we have assumed:

$$\Phi^T C \Phi = \begin{bmatrix} c_1^* & & \\ & c_2^* & \\ & & c_3^* \end{bmatrix}$$

Two methods described herein:

- Rayleigh “proportional damping”
- Wilson “discrete modal damping”

Rayleigh Proportional Damping
(continued)



$$C = \alpha M + \beta K$$

Rayleigh Proportional Damping
(continued)

$$C = \alpha M + \beta K$$

For modal equations to be uncoupled:

$$2\omega_n \xi_n = \phi_n^T C \phi_n$$

Using orthogonality conditions:

$$2\omega_n \xi_n = \alpha + \beta \omega_n^2$$

$$\xi_n = \frac{1}{2\omega_n}\alpha + \frac{\omega_n}{2}\beta$$

Assumes

$$\Phi^T M \Phi = I$$

Rayleigh Proportional Damping
(continued)

Select damping value in two modes, ξ_m and ξ_n

Compute coefficients α and β:

$$\begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = 2\frac{\omega_m \omega_n}{\omega_n^2 - \omega_m^2} \begin{bmatrix} \omega_n & -\omega_m \\ -1/\omega_n & 1/\omega_m \end{bmatrix} \begin{Bmatrix} \xi_m \\ \xi_n \end{Bmatrix}$$

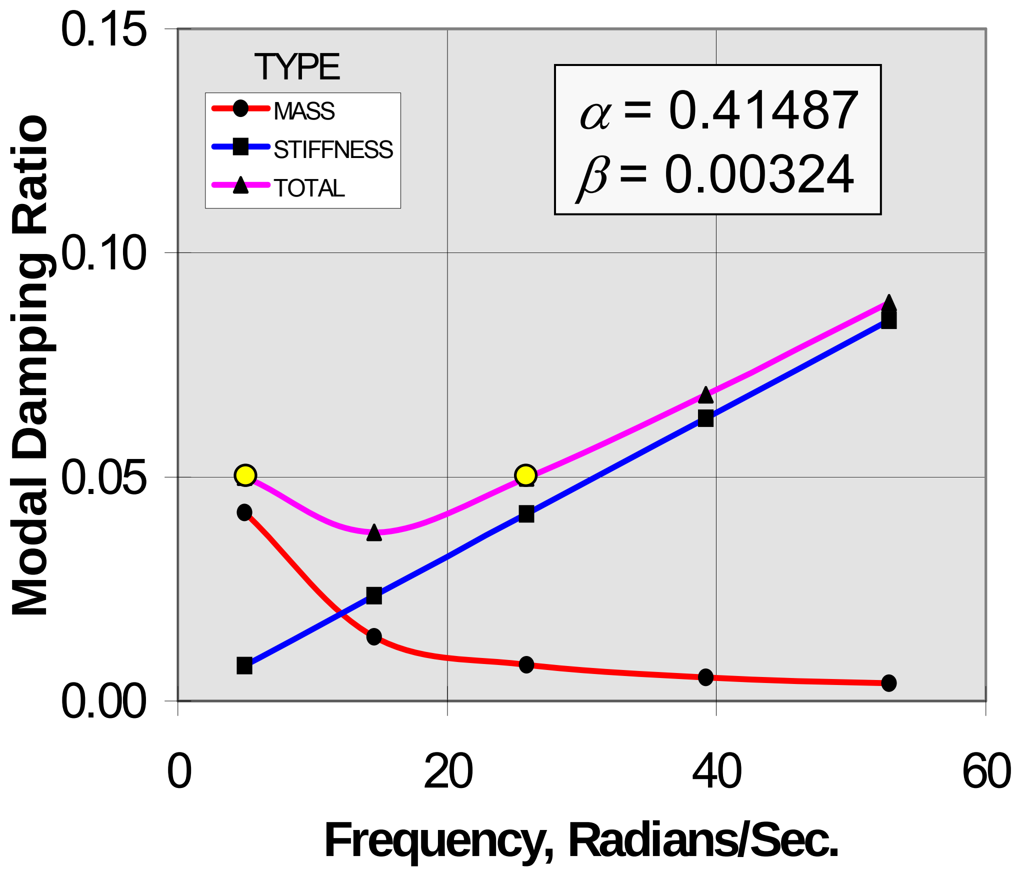
Form damping matrix $C = \alpha M + \beta K$

Rayleigh Proportional Damping (Example)

5% critical in Modes 1 and 3

Structural frequencies

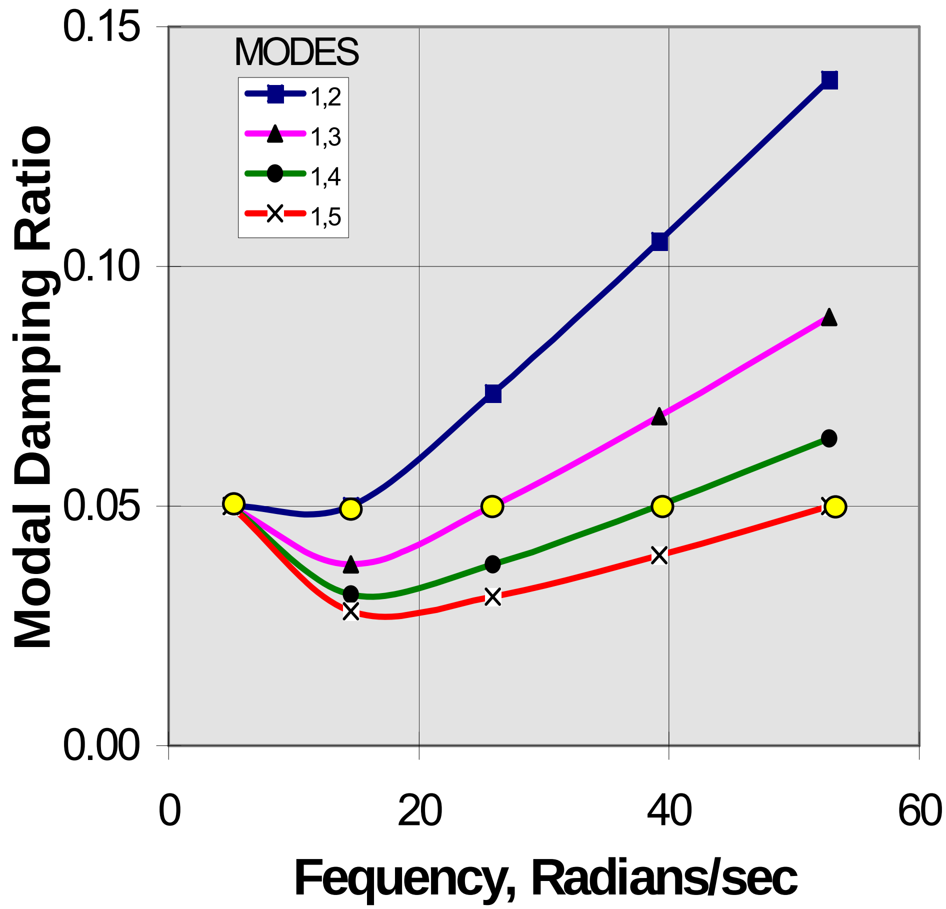
Mode	ω
1	**4.94**
2	14.6
3	**25.9**
4	39.2
5	52.8



Rayleigh Proportional Damping (Example)
5% Damping in Modes 1 & 2, 1 & 3, 1 & 4, or 1 & 5

Proportionality factors
(5% each indicated mode)

Modes	α	β
1 & 2	.36892	0.00513
1 & 3	.41487	0.00324
1 & 4	.43871	0.00227
1 & 5	.45174	0.00173



Wilson Damping

Directly specify modal damping values ξ_i^*

$$\Phi^T C \Phi = \begin{bmatrix} c_1^* & & \\ & c_2^* & \\ & & c_3^* \end{bmatrix} = \begin{bmatrix} 2m_1^*\omega_1\xi_1^* & & \\ & 2m_2^*\omega_2\boxed{\xi_2^*} & \\ & & 2m_3^*\omega_3\xi_3^* \end{bmatrix}$$

Formation of Explicit Damping Matrix From "Wilson" Modal Damping

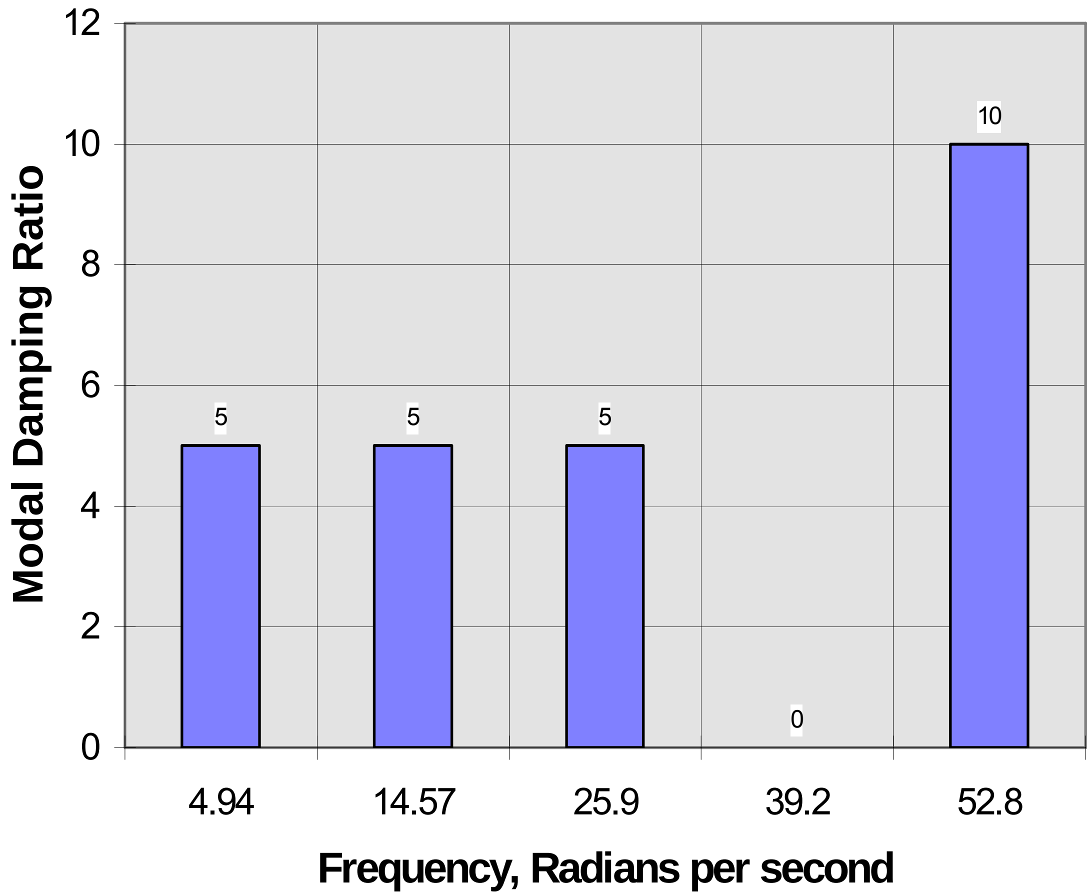
(NOT Usually Required)

$$\Phi^T C \Phi = \begin{bmatrix} 2\xi_1\omega_1 & & & & \\ & 2\xi_2\omega_2 & & & \\ & & \bullet\bullet\bullet & & \\ & & & 2\xi_{n-1}\omega_{n-1} & \\ & & & & 2\xi_n\omega_n \end{bmatrix} = c$$

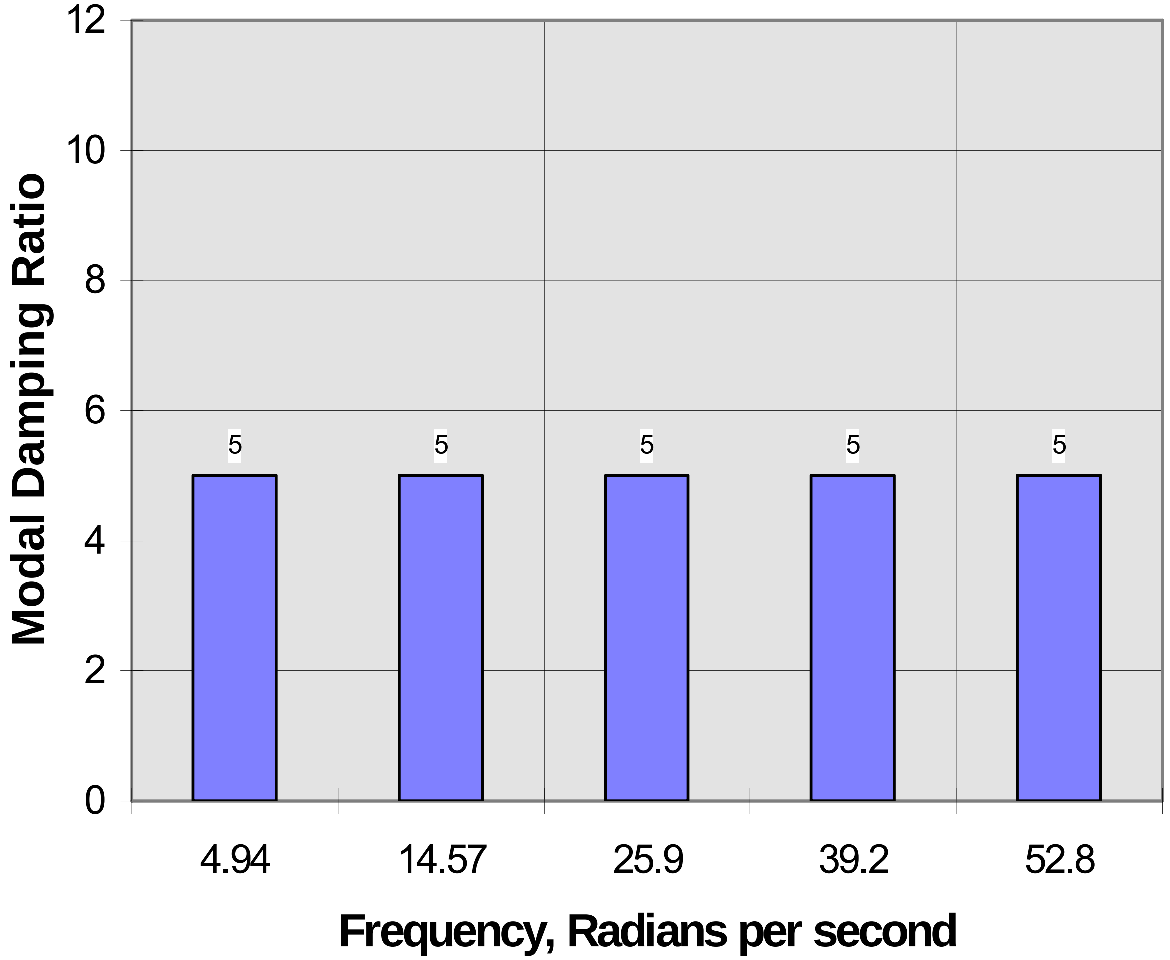
$$\left(\Phi^T\right)^{-1} c \Phi^{-1} = C$$

$$C = M\left[\sum_{i=1}^{n} 2\xi_i\omega_i\phi_i^T\phi_i\right]M$$

Wilson Damping (Example)
5% Damping in Modes 1 and 2, 3
10% in Mode 5, Zero in Mode 4



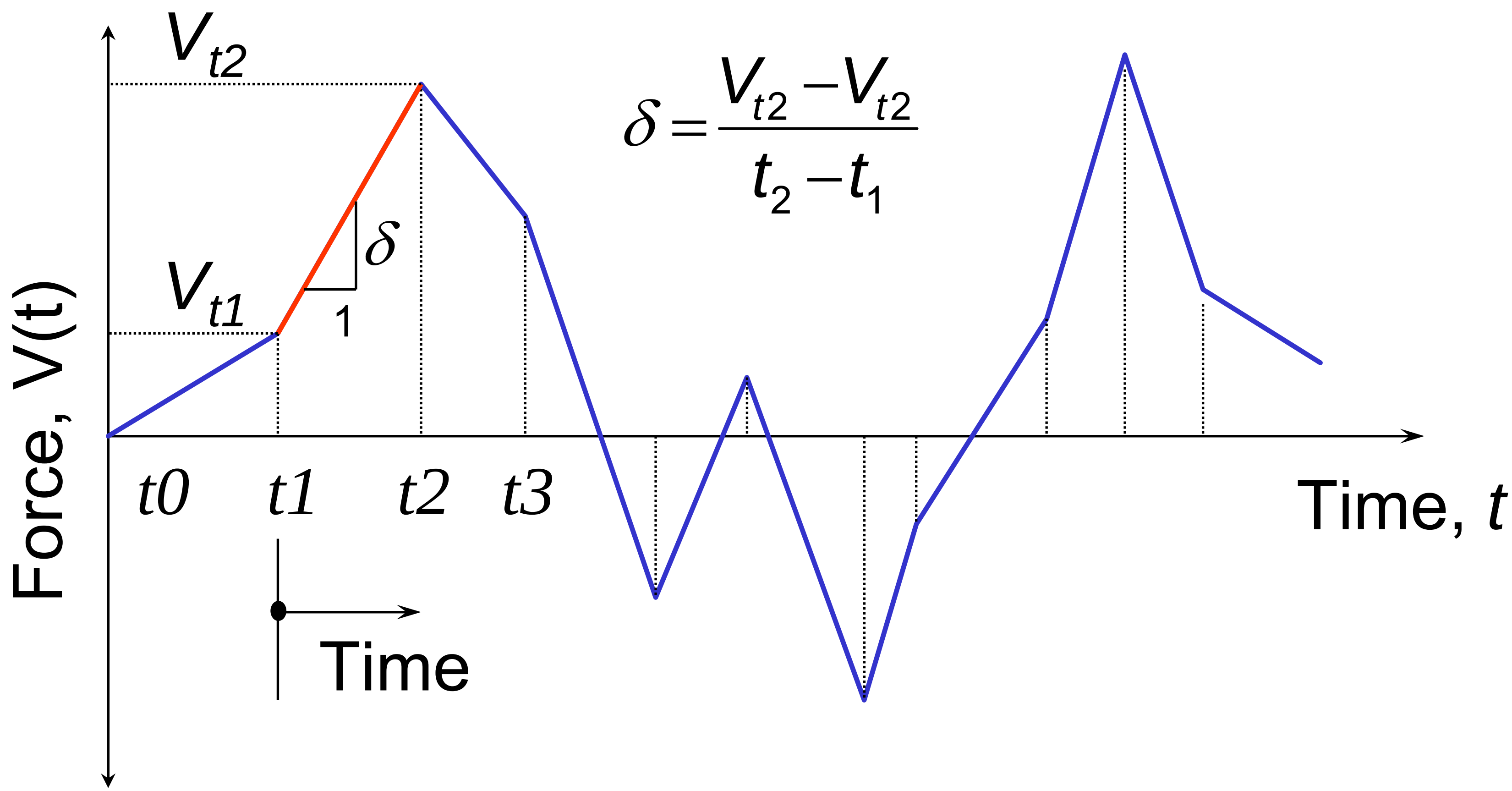
Wilson Damping (Example)
5% Damping in all Modes



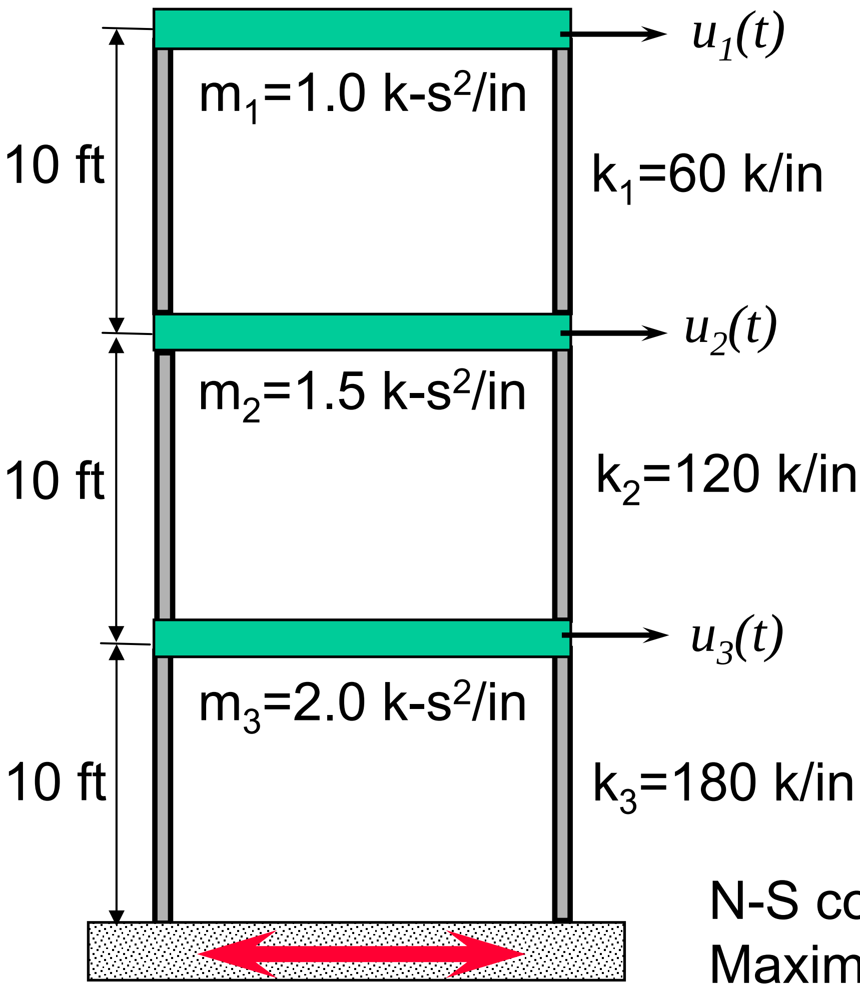
Solution of MDOF Equations of Motion

- Explicit (step by step) integration of *coupled* equations
- Explicit integration of FULL SET of *uncoupled* equations
- Explicit integration of PARTIAL SET of *uncoupled* Equations (approximate)
- Modal response spectrum analysis (approximate)

Computed Response for Piecewise Linear Loading



Example of MDOF Response of Structure Responding to 1940 El Centro Earthquake

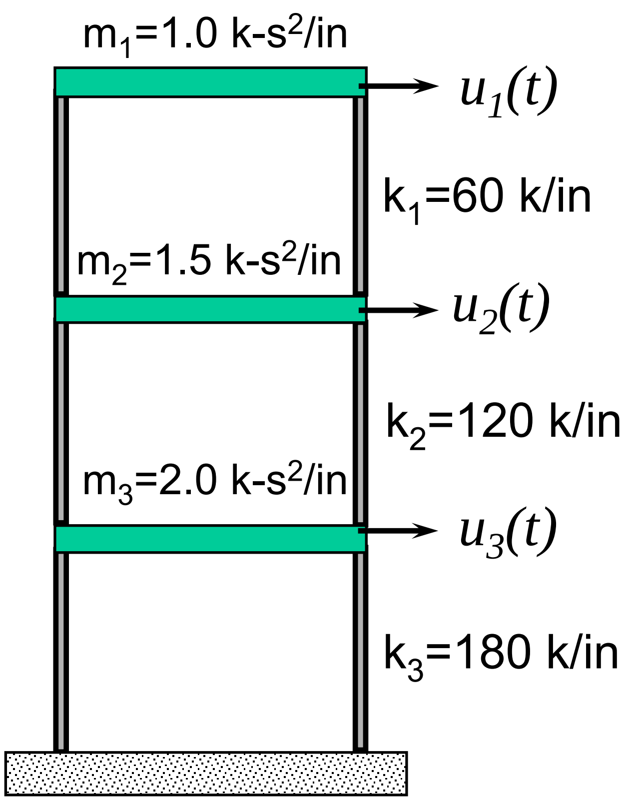


Example 1

Assume Wilson damping with 5% critical in each mode.

N-S component of 1940 El Centro earthquake
Maximum acceleration = 0.35 g

Example 1 (continued)

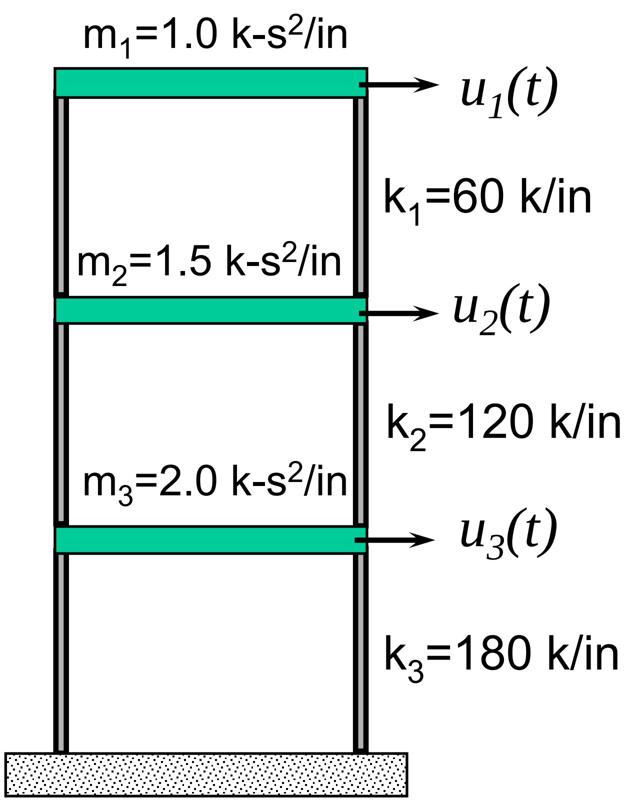


Form property matrices:

$$M = \begin{bmatrix} 1.0 & & \\ & 1.5 & \\ & & 2.0 \end{bmatrix} \text{kip} - \text{s}^2/\text{in}$$

$$K = \begin{bmatrix} 60 & -60 & 0 \\ -60 & 180 & -120 \\ 0 & -120 & 300 \end{bmatrix} \text{kip/in}$$

Example 1 (continued)



Solve eigenvalue problem:

$$K\Phi = M\Phi\Omega^2$$

$$\Omega^2 = \begin{bmatrix} 21.0 & & \\ & 96.6 & \\ & & 212.4 \end{bmatrix} \sec^{-2}$$

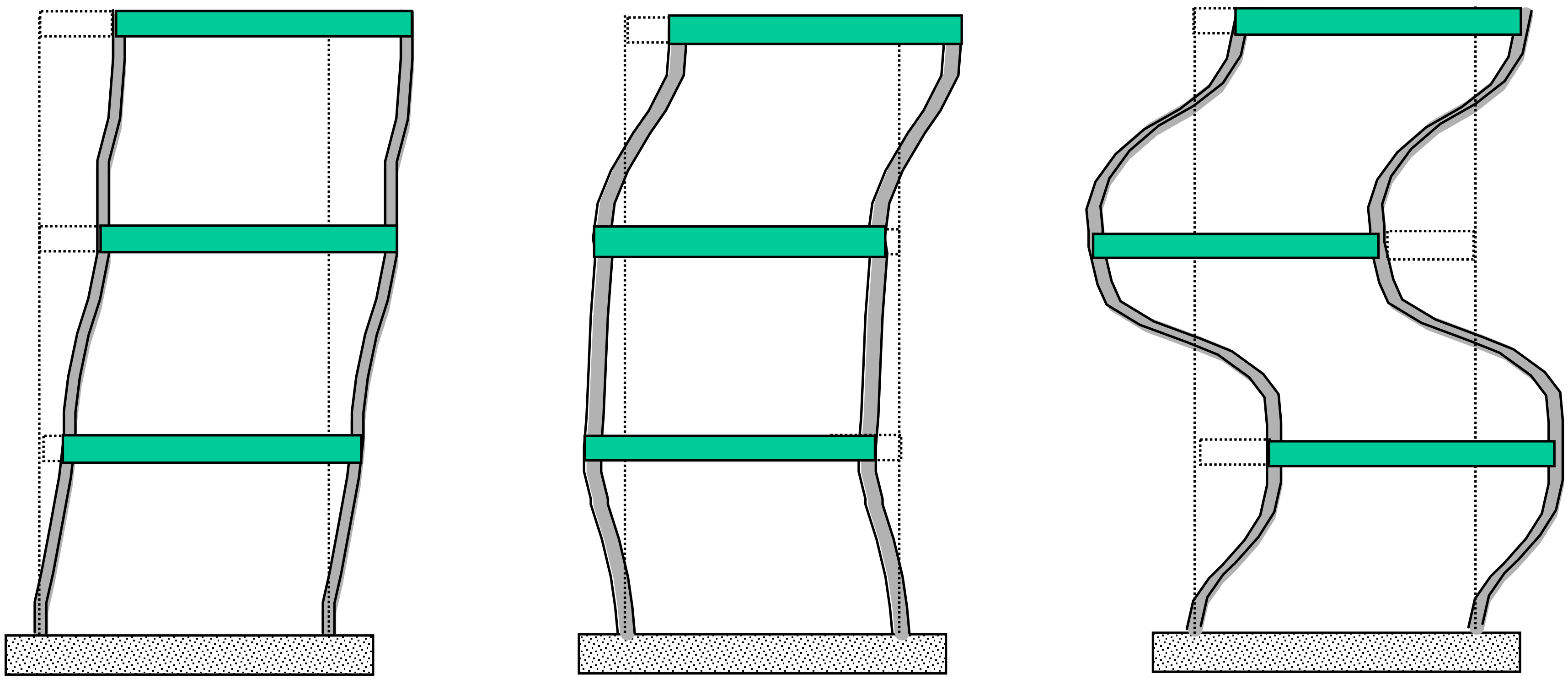
$$\Phi = \begin{bmatrix} 1.000 & 1.000 & 1.000 \\ 0.644 & -0.601 & -2.57 \\ 0.300 & -0.676 & 2.47 \end{bmatrix}$$

Normalization of Modes Using $\Phi^T M \Phi = I$

$$\Phi = \begin{bmatrix} 0.749 & 0.638 & 0.208 \\ 0.478 & -0.384 & -0.534 \\ 0.223 & -0.431 & 0.514 \end{bmatrix} \quad \text{vs} \quad \begin{bmatrix} 1.000 & 1.000 & 1.000 \\ 0.644 & -0.601 & -2.57 \\ 0.300 & -0.676 & 2.47 \end{bmatrix}$$

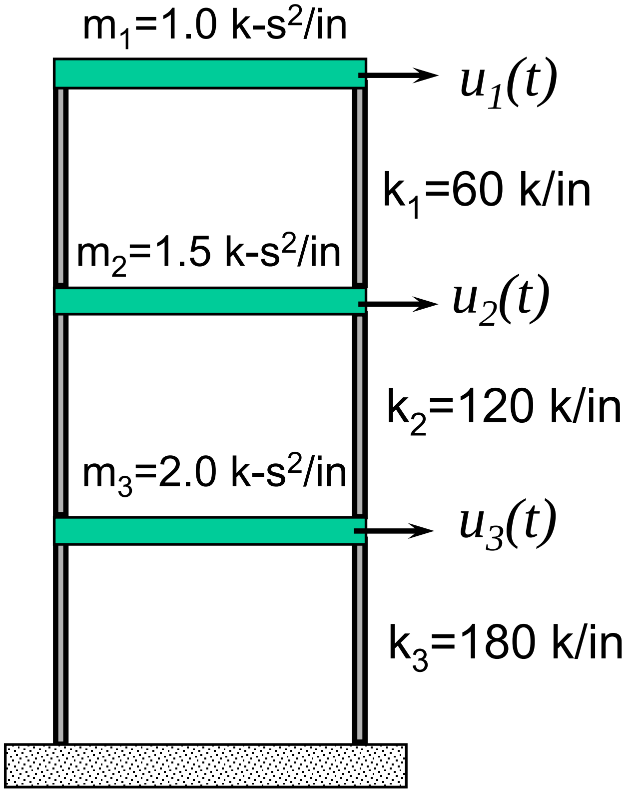
Example 1 (continued)

Mode Shapes and Periods of Vibration



MODE 1	MODE 2	MODE 3
ω = 4.58 rad/sec	ω = 9.83 rad/sec	ω = 14.57 rad/sec
T = 1.37 sec	T = 0.639 sec	T = 0.431 sec

Example 1 (continued)

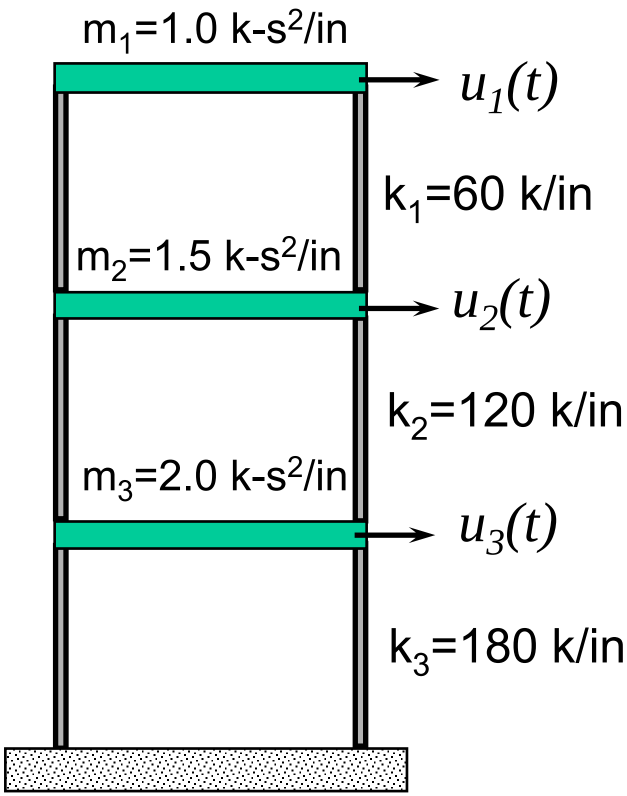


$$\omega_n = \begin{Bmatrix} 4.58 \\ 9.83 \\ 14.57 \end{Bmatrix} \text{rad/sec} \qquad T_n = \begin{Bmatrix} 1.37 \\ 0.639 \\ 0.431 \end{Bmatrix} \text{sec}$$

Compute Generalized Mass:

$$M^* = \Phi^T M \Phi = \begin{bmatrix} 1.801 & & \\ & 2.455 & \\ & & 23.10 \end{bmatrix} \text{kip} - \text{sec}^2/\text{in}$$

Example 1 (continued)



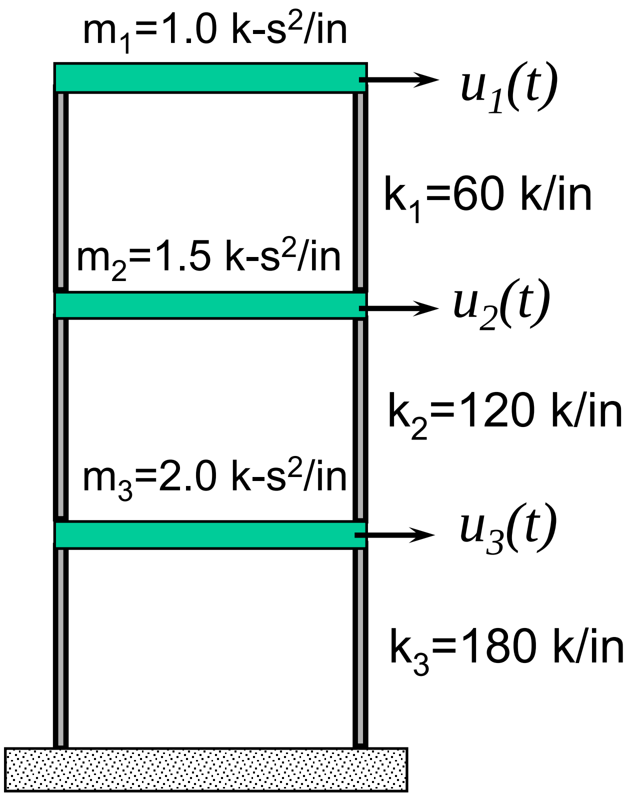
Compute generalized loading:

$$V^{*}(t) = -\Phi^{T} M R \ddot{v}_g(t)$$

$$V_n^{*} = -\begin{Bmatrix} 2.566 \\ -1.254 \\ 2.080 \end{Bmatrix} \ddot{v}_g(t)$$

Example 1 (continued)

Write uncoupled (modal) equations of motion:



$$\ddot{y}_1 + 2\xi_1\omega_1\dot{y}_1 + \omega_1^2 y_1 = V_1^*(t)/m_1^*$$

$$\ddot{y}_2 + 2\xi_2\omega_2\dot{y}_2 + \omega_2^2 y_2 = V_2^*(t)/m_2^*$$

$$\ddot{y}_3 + 2\xi_3\omega_3\dot{y}_3 + \omega_3^2 y_3 = V_3^*(t)/m_3^*$$

$$\ddot{y}_1 + 0.458\dot{y}_1 + 21.0 y_1 = -1.425\ddot{v}_g(t)$$

$$\ddot{y}_2 + 0.983\dot{y}_2 + 96.6 y_2 = 0.511\ddot{v}_g(t)$$

$$\ddot{y}_3 + 1.457\dot{y}_3 + 212.4 y_3 = -0.090\ddot{v}_g(t)$$

Modal Participation Factors

Mode	1	1.425	1.911
Mode	2	–0.511	–0.799
Mode	3	0.090	0.435

Modal scaling $\phi_{i,1} = 1.0$ $\phi_i^T M \phi_i = 1.0$

Modal Participation Factors (continued)

$$1.425\begin{Bmatrix}1.000\\0.644\\0.300\end{Bmatrix}=1.911\begin{Bmatrix}0.744\\0.480\\0.223\end{Bmatrix}$$

using $\phi_{1,1}=1$ using $\phi_1{}^T M\phi_1=1$

Effective Modal Mass

$$\bar{M}_n = P_n^{\,2} m_n^*$$

	$\bar{M}_n$	%	Accum%
Mode 1	3.66	81	81
Mode 2	0.64	14	95
Mode 3	0.20	5	100%
	4.50	100%	

Example 1 (continued)

Solving modal equation via NONLIN:

For Mode 1:

$$\ddot{y}_1 + 2\xi_1\omega_1\dot{y}_1 + \omega_1^2 y_1 = V_1^*(t)/m_1^*$$

$$1.00\ddot{y}_1 + 0.458\dot{y}_1 + 21.0y_1 = -1.425\ddot{v}_g(t)$$

M = 1.00 kip-sec^2/in

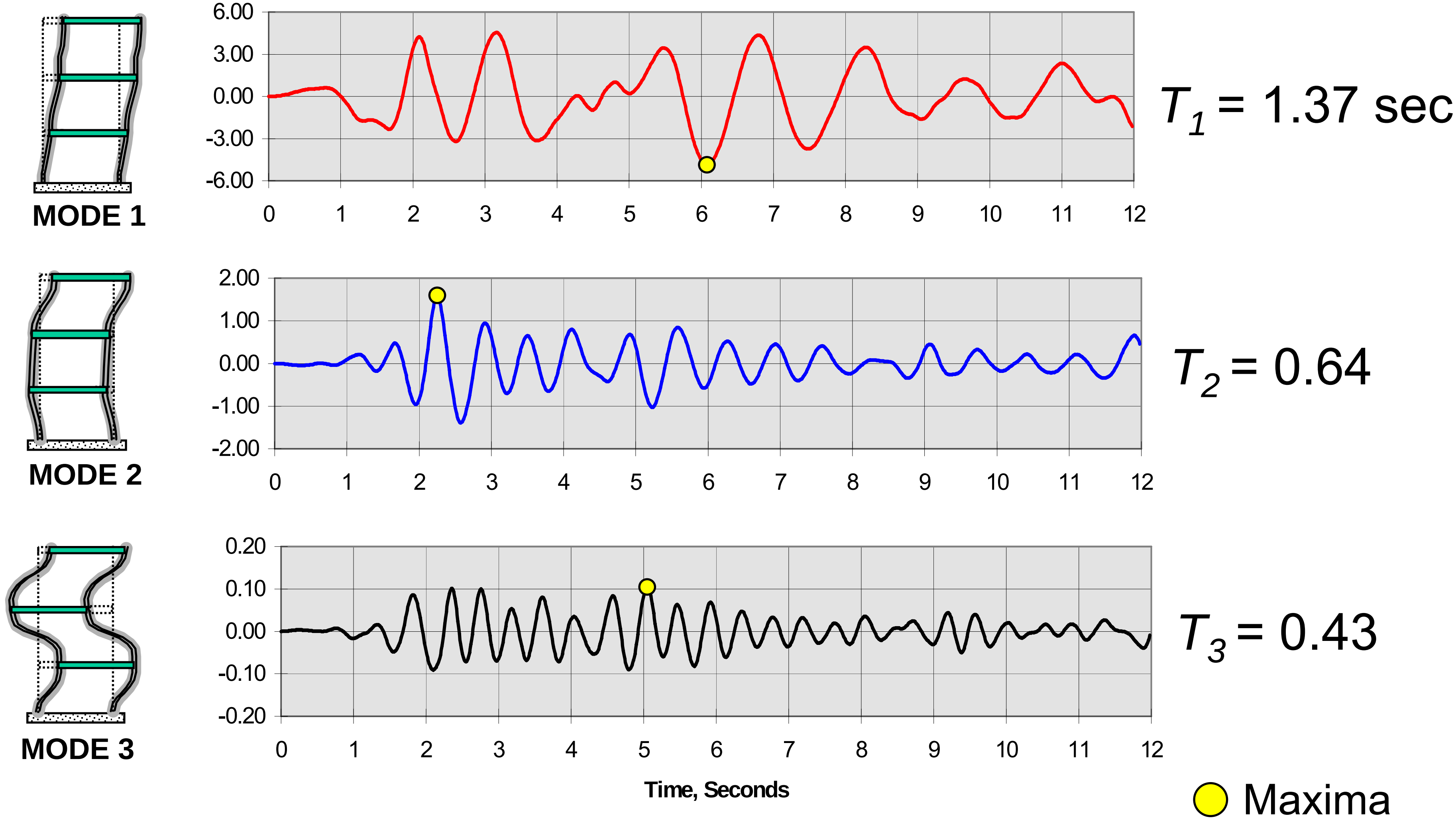
C = 0.458 kip-sec/in

K_1 = 21.0 kips/inch

Scale ground acceleration by factor 1.425

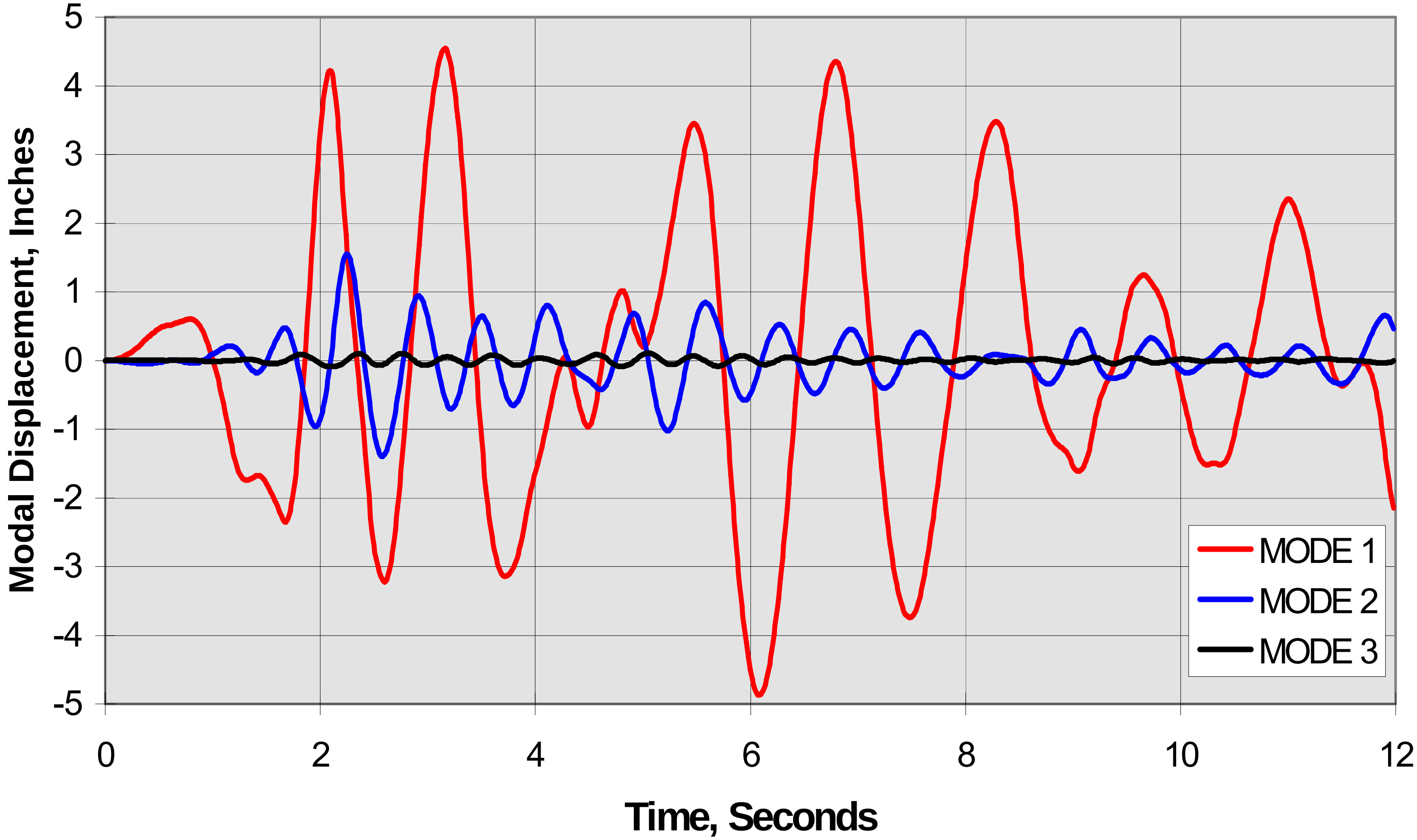
Example 1 (continued)

Modal Displacement Response Histories (from NONLIN)



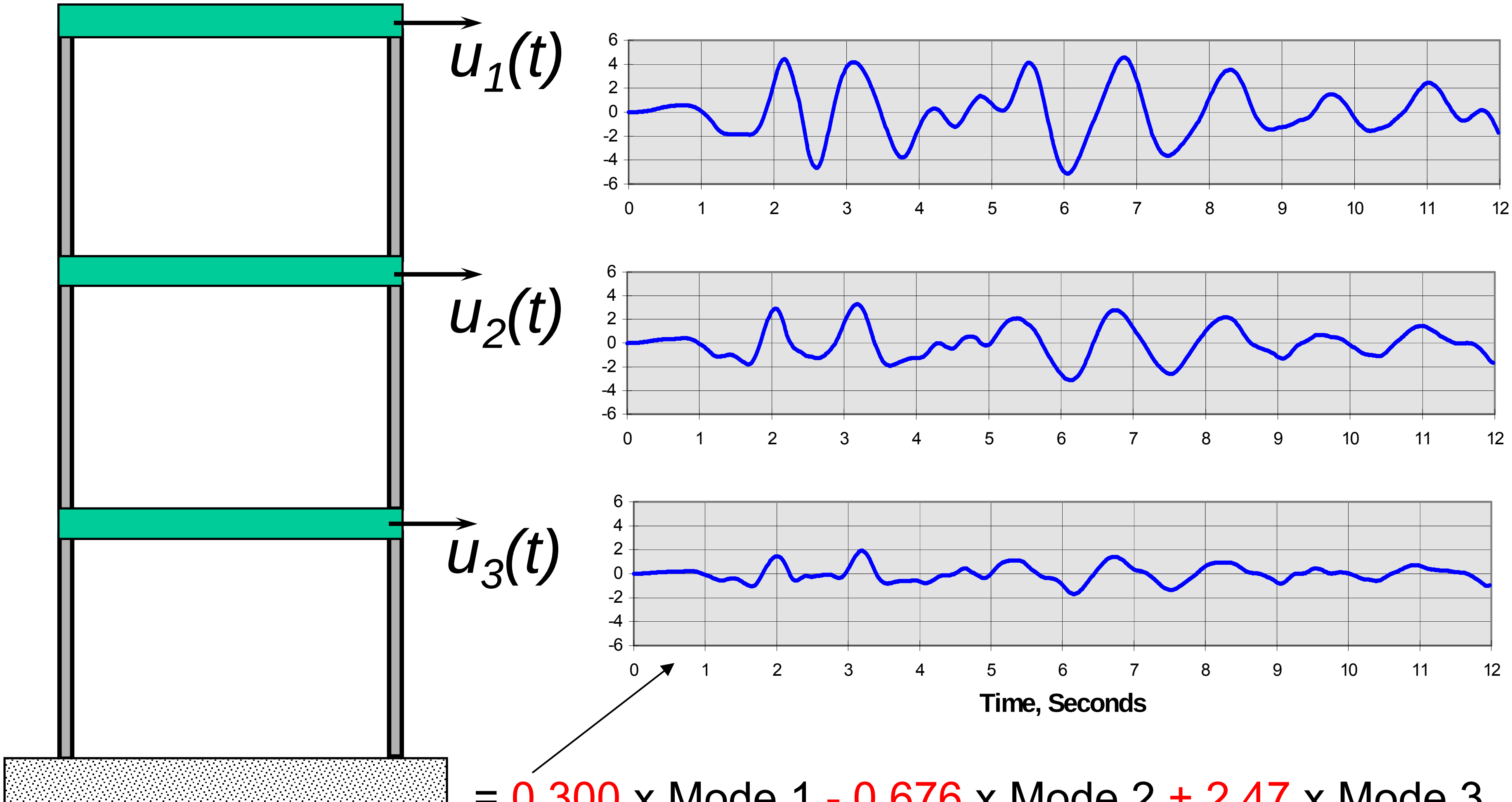
Example 1 (continued)

Modal Response Histories:



Example 1 (continued)

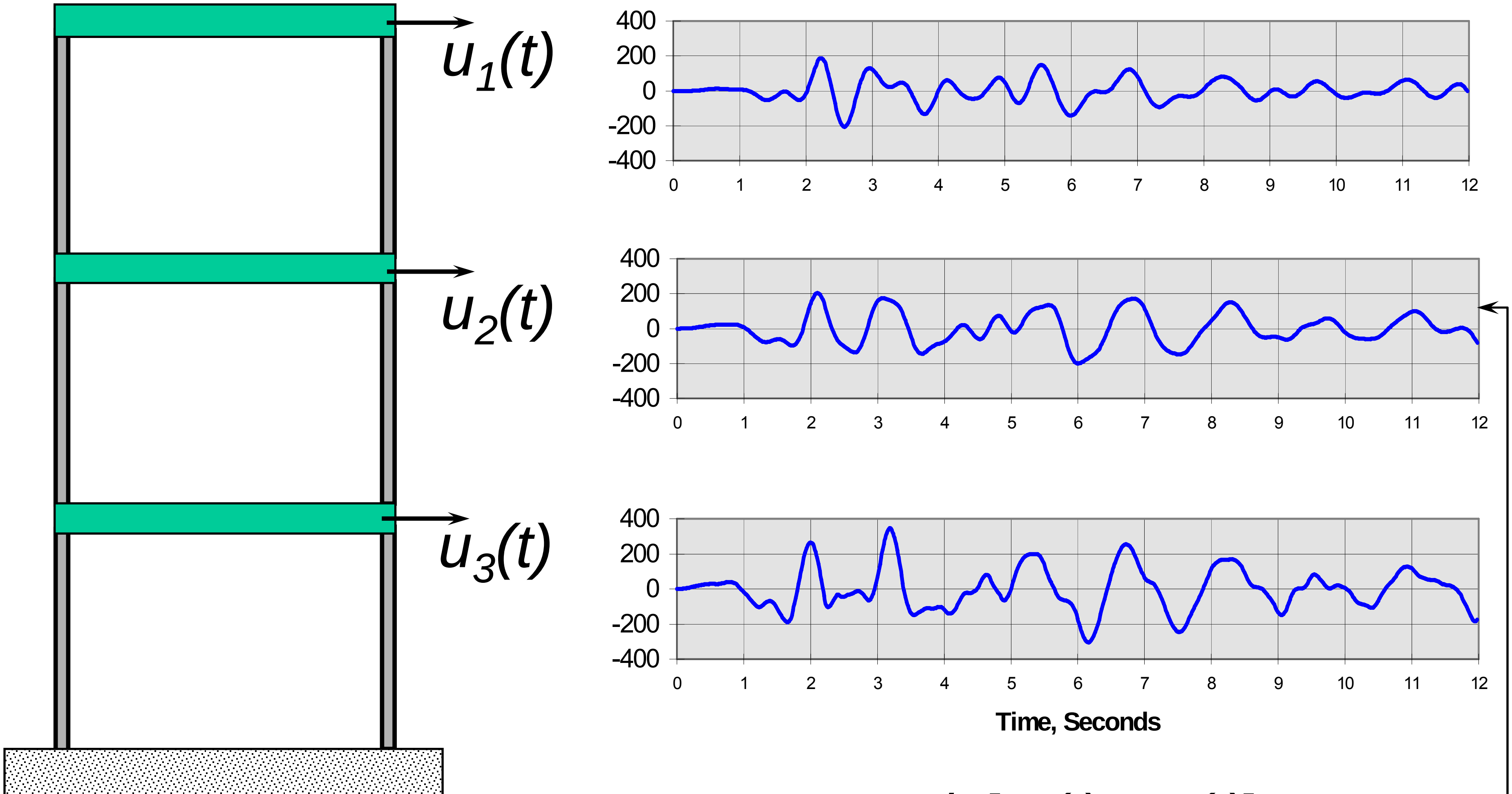
Compute story displacement response histories: $u(t) = \Phi y(t)$



= 0.300 x Mode 1 - 0.676 x Mode 2 + 2.47 x Mode 3

Example 1 (continued)

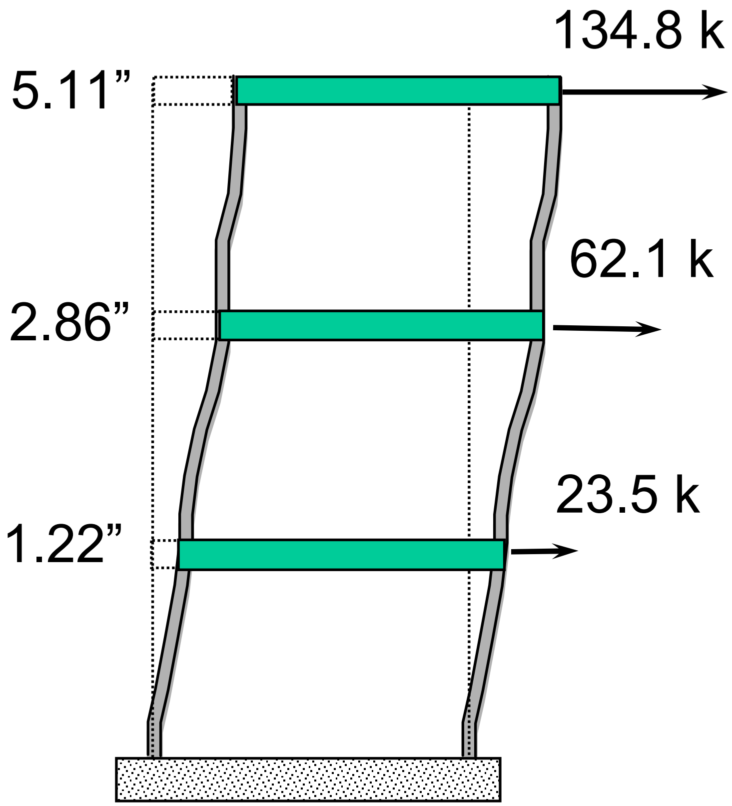
Compute story shear response histories:

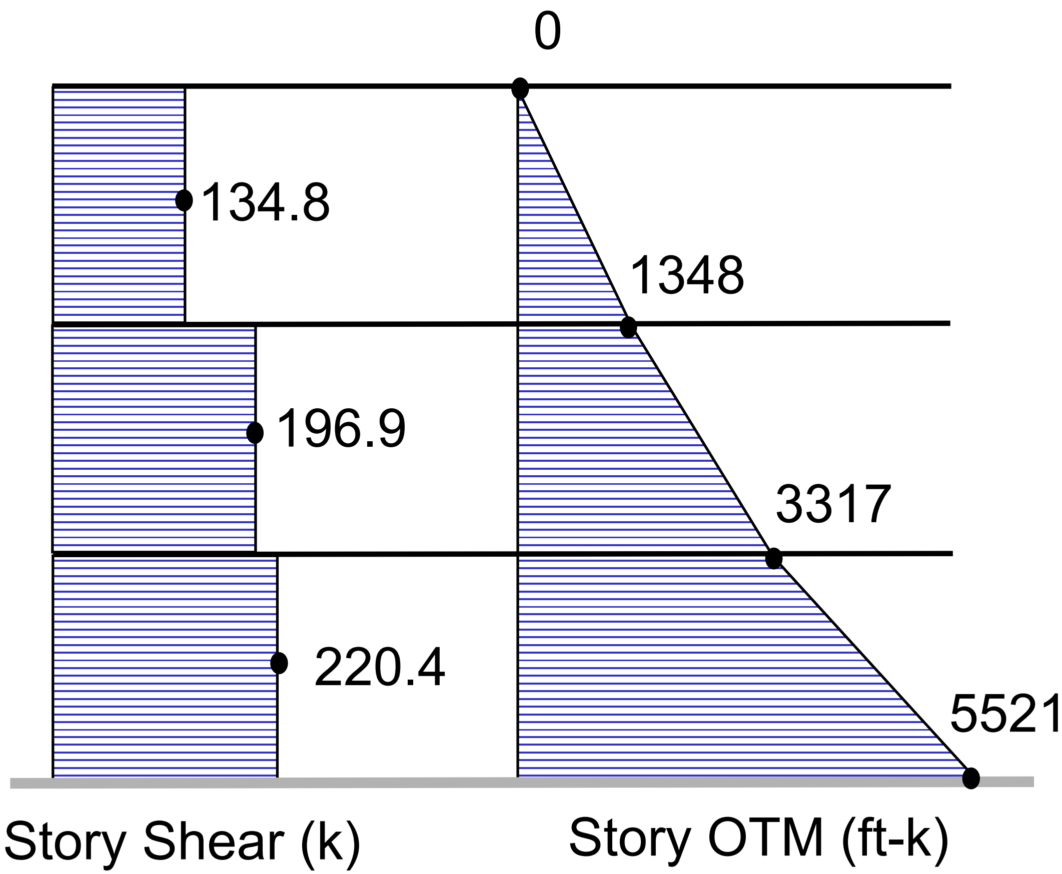


$$= k_2[u_2(t) - u_3(t)]$$

Example 1 (continued)

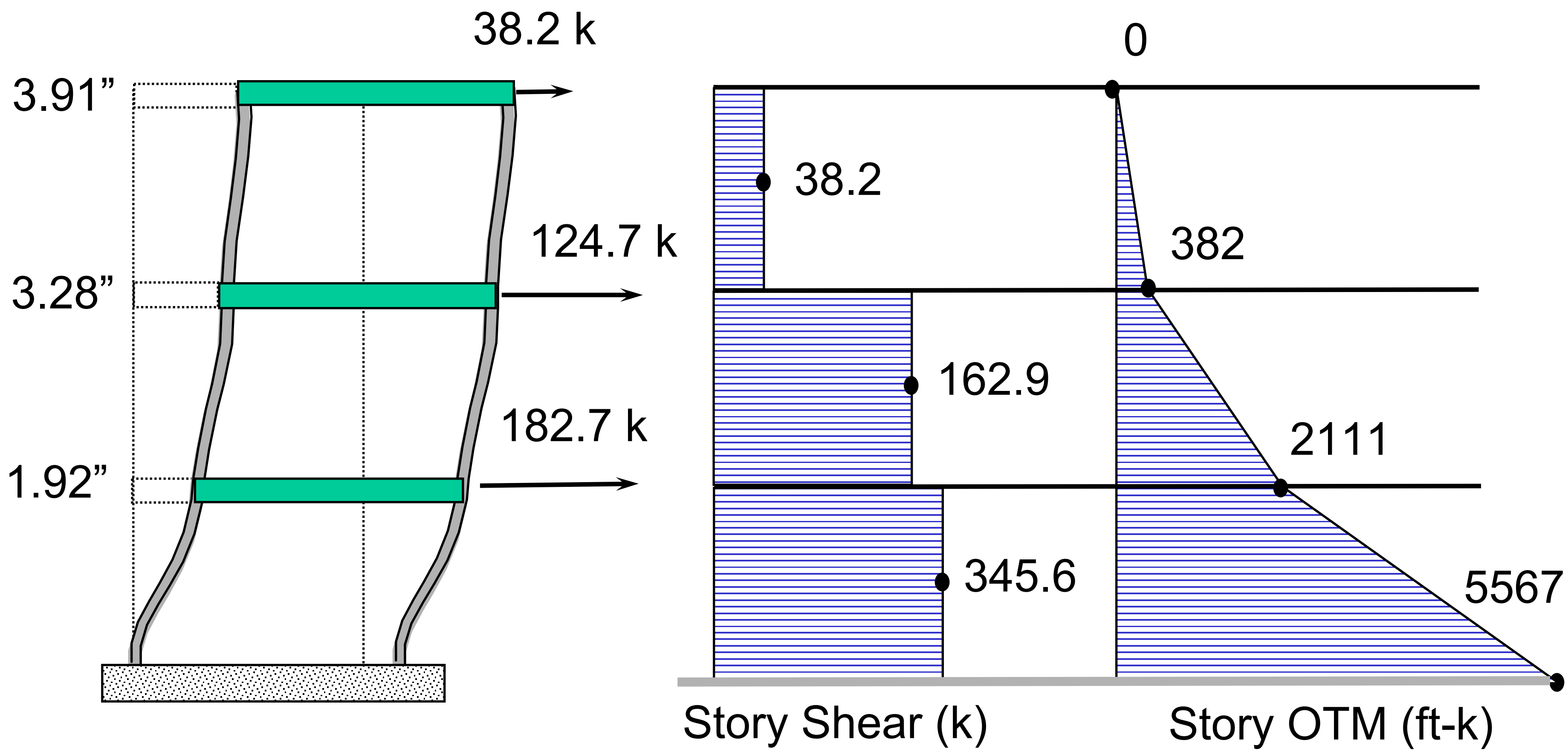
Displacements and forces at time of maximum displacements
(t = 6.04 sec)





Example 1 (continued)

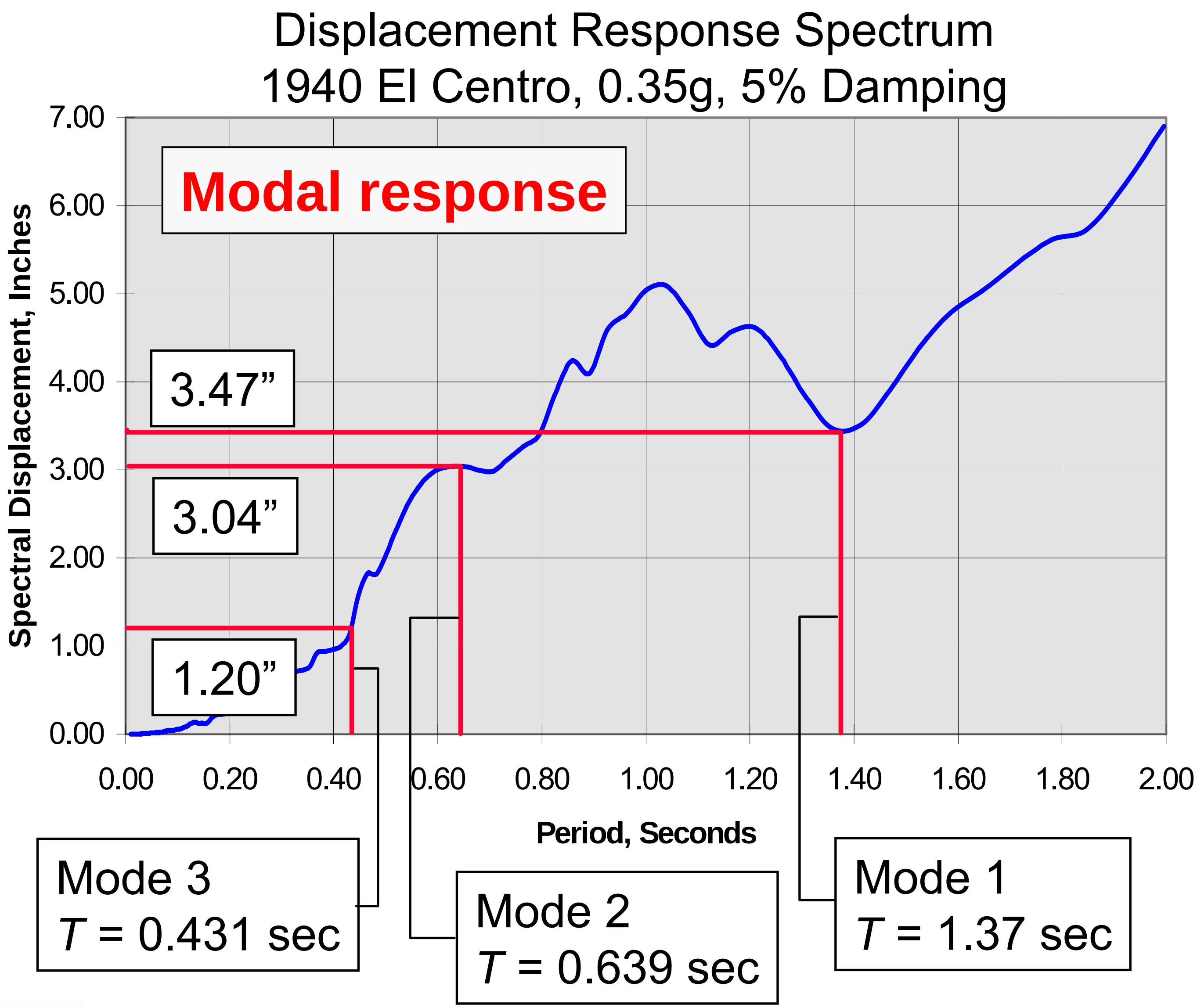
Displacements and forces at time of maximum shear
(t = 3.18 sec)



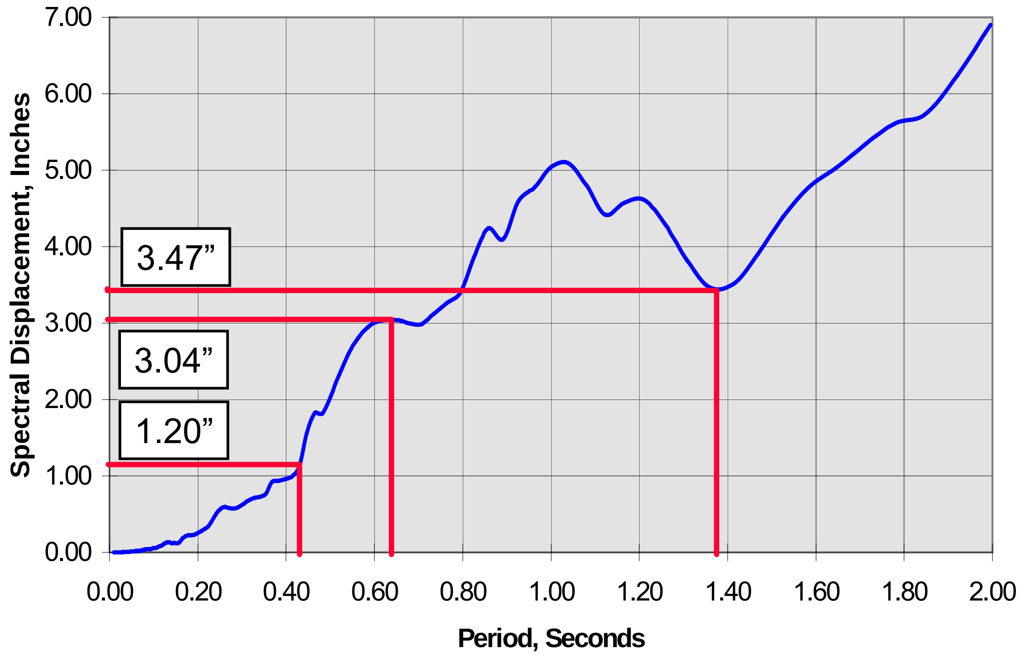
Modal Response Response Spectrum Method

- Instead of solving the time history problem for each mode, use a response spectrum to compute the **maximum** response in each mode.
- These maxima are generally **nonconcurrent**.
- Combine the maximum modal responses using some statistical technique, such as square root of the sum of the squares (SRSS) or complete quadratic combination (CQC).
- The technique is **approximate**.
- It is the basis for the equivalent lateral force (ELF) method.

Example 1 (Response Spectrum Method)



Example 1 (continued)



Modal Equations of Motion

$$\ddot{y}_1 + 0.458\dot{y}_1 + 21.0y_1 = -1.425\ddot{v}_g(t)$$

$$\ddot{y}_2 + 0.983\dot{y}_2 + 96.6y_2 = 0.511\ddot{v}_g(t)$$

$$\ddot{y}_3 + 1.457\dot{y}_3 + 212.4y_3 = -0.090\ddot{v}_g(t)$$

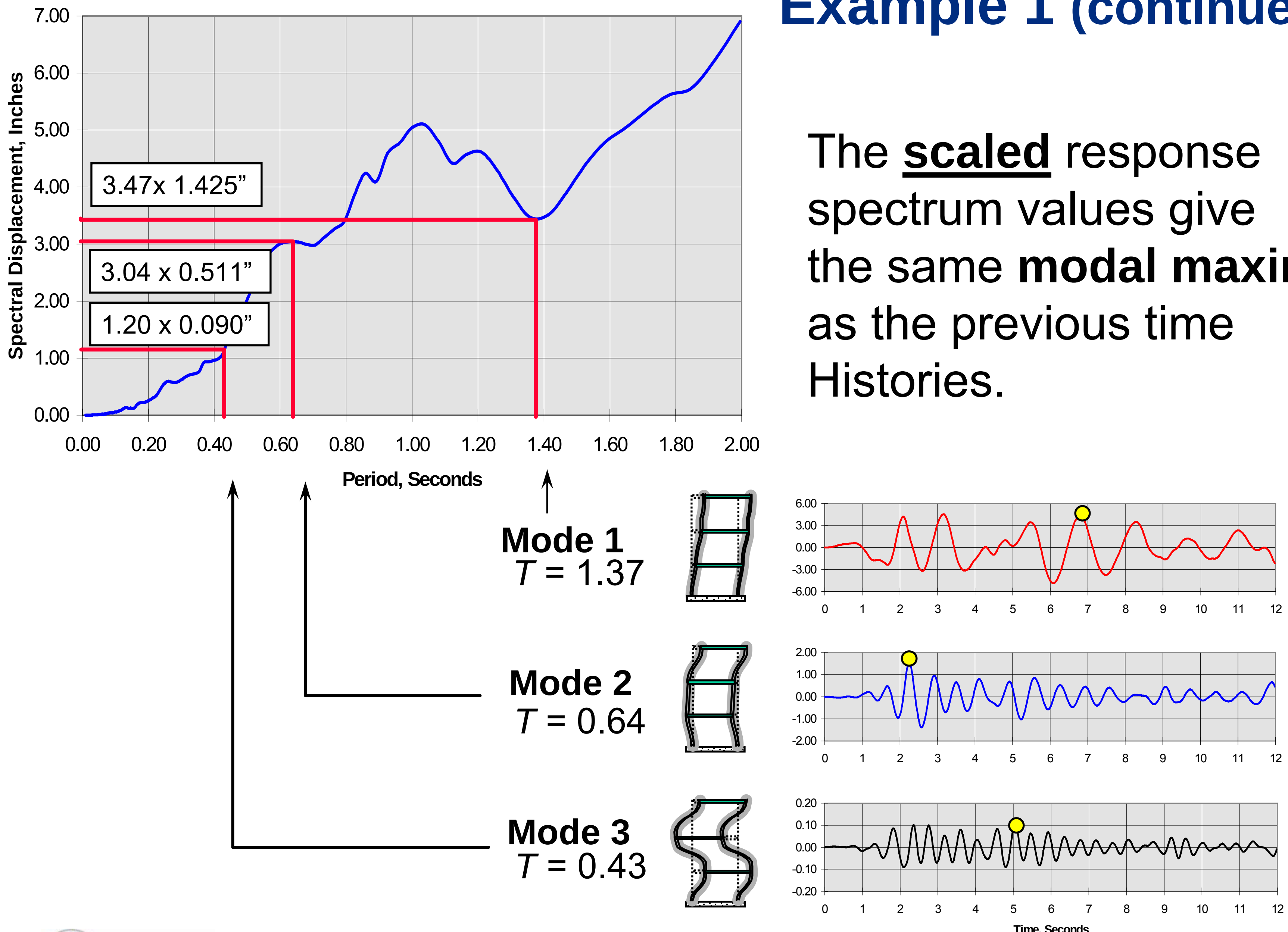
Modal Maxima

$$\bar{y}_1 = 1.425 * 3.47 = 4.94''$$

$$\bar{y}_2 = 0.511 * 3.04 = 1.55''$$

$$\bar{y}_3 = 0.090 * 1.20 = 0.108''$$

Example 1 (continued)

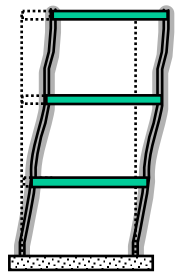


The **scaled** response spectrum values give the same **modal maxima** as the previous time Histories.

Example 1 (continued)

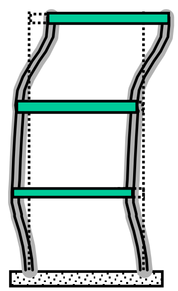
Computing **Nonconcurrent** Story Displacements

Mode 1



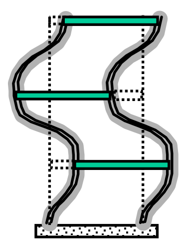
$$\begin{Bmatrix} 1.000 \\ 0.644 \\ 0.300 \end{Bmatrix} 4.940 = \begin{Bmatrix} 4.940 \\ 3.181 \\ 1.482 \end{Bmatrix}$$

Mode 2



$$\begin{Bmatrix} 1.000 \\ -0.601 \\ -0.676 \end{Bmatrix} 1.550 = \begin{Bmatrix} 1.550 \\ -0.931 \\ -1.048 \end{Bmatrix}$$

Mode 3



$$\begin{Bmatrix} 1.000 \\ -2.570 \\ 2.470 \end{Bmatrix} 0.108 = \begin{Bmatrix} 0.108 \\ -0.278 \\ 0.267 \end{Bmatrix}$$

Example 1 (continued)

Modal Combination Techniques (for Displacement)

Sum of Absolute Values:

$$\begin{Bmatrix} 4.940 + 1.550 + 0.108 \\ 3.181 + 0.931 + 0.278 \\ 1.482 + 1.048 + 0.267 \end{Bmatrix} = \begin{Bmatrix} 6.60 \\ 4.39 \\ 2.80 \end{Bmatrix}$$

At time of maximum displacement

"Exact"

$$\begin{Bmatrix} 5.15 \\ 2.86 \\ 1.22 \end{Bmatrix}$$

Square Root of the Sum of the Squares:

$$\begin{Bmatrix} \sqrt{4.940^2 + 1.550^2 + 0.108^2} \\ \sqrt{3.181^2 + 0.931^2 + 0.278^2} \\ \sqrt{1.482^2 + 1.048^2 + 0.267^2} \end{Bmatrix} = \begin{Bmatrix} 5.18 \\ 3.33 \\ 1.84 \end{Bmatrix}$$

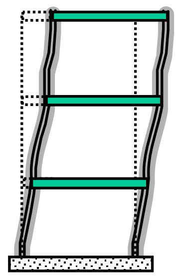
Envelope of story displacement

$$\begin{Bmatrix} 5.15 \\ 3.18 \\ 1.93 \end{Bmatrix}$$

Example 1 (continued)

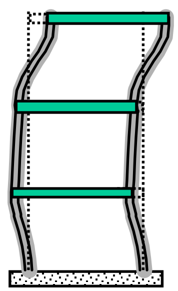
Computing Interstory Drifts

Mode 1



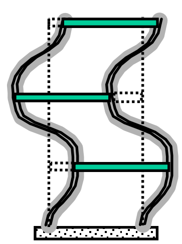
$$\begin{Bmatrix} 4.940-3.181 \\ 3.181-1.482 \\ 1.482-0 \end{Bmatrix} = \begin{Bmatrix} 1.759 \\ 1.699 \\ 1.482 \end{Bmatrix}$$

Mode 2



$$\begin{Bmatrix} 1.550-(-0.931) \\ -0.931-(-1.048) \\ -1.048-0 \end{Bmatrix} = \begin{Bmatrix} 2.481 \\ 0.117 \\ -1.048 \end{Bmatrix}$$

Mode 3

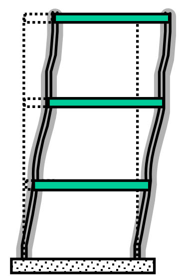


$$\begin{Bmatrix} 0.108-(-0.278) \\ -0.278-0.267 \\ 0.267-0 \end{Bmatrix} = \begin{Bmatrix} 0.386 \\ -0.545 \\ 0.267 \end{Bmatrix}$$

Example 1 (continued)

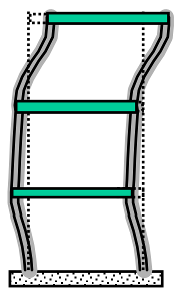
Computing Interstory Shears (Using Drift)

Mode 1



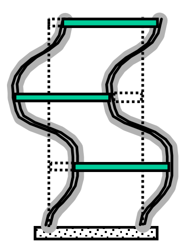
$$\begin{Bmatrix} 1.759(60) \\ 1.699(120) \\ 1.482(180) \end{Bmatrix} = \begin{Bmatrix} 105.5 \\ 203.9 \\ 266.8 \end{Bmatrix}$$

Mode 2



$$\begin{Bmatrix} 2.481(60) \\ 0.117(120) \\ -1.048(180) \end{Bmatrix} = \begin{Bmatrix} 148.9 \\ 14.0 \\ -188.6 \end{Bmatrix}$$

Mode 3



$$\begin{Bmatrix} 0.386(60) \\ -0.545(120) \\ 0.267(180) \end{Bmatrix} = \begin{Bmatrix} 23.2 \\ -65.4 \\ 48.1 \end{Bmatrix}$$

Example 1 (continued)

Computing Interstory Shears: SRSS Combination

$$\begin{Bmatrix} \sqrt{106^2 + 149^2 + 23.2^2} \\ \sqrt{204^2 + 14^2 + 65.4^2} \\ \sqrt{267^2 + 189^2 + 48.1^2} \end{Bmatrix} = \begin{Bmatrix} 220 \\ 215 \\ 331 \end{Bmatrix}$$

"Exact"	"Exact"	"Exact"
$\begin{Bmatrix} 38.2 \\ 163 \\ 346 \end{Bmatrix}$	$\begin{Bmatrix} 135 \\ 197 \\ 220 \end{Bmatrix}$	$\begin{Bmatrix} 207 \\ 203 \\ 346 \end{Bmatrix}$
At time of max. shear	At time of max. displacement	Envelope = maximum per story

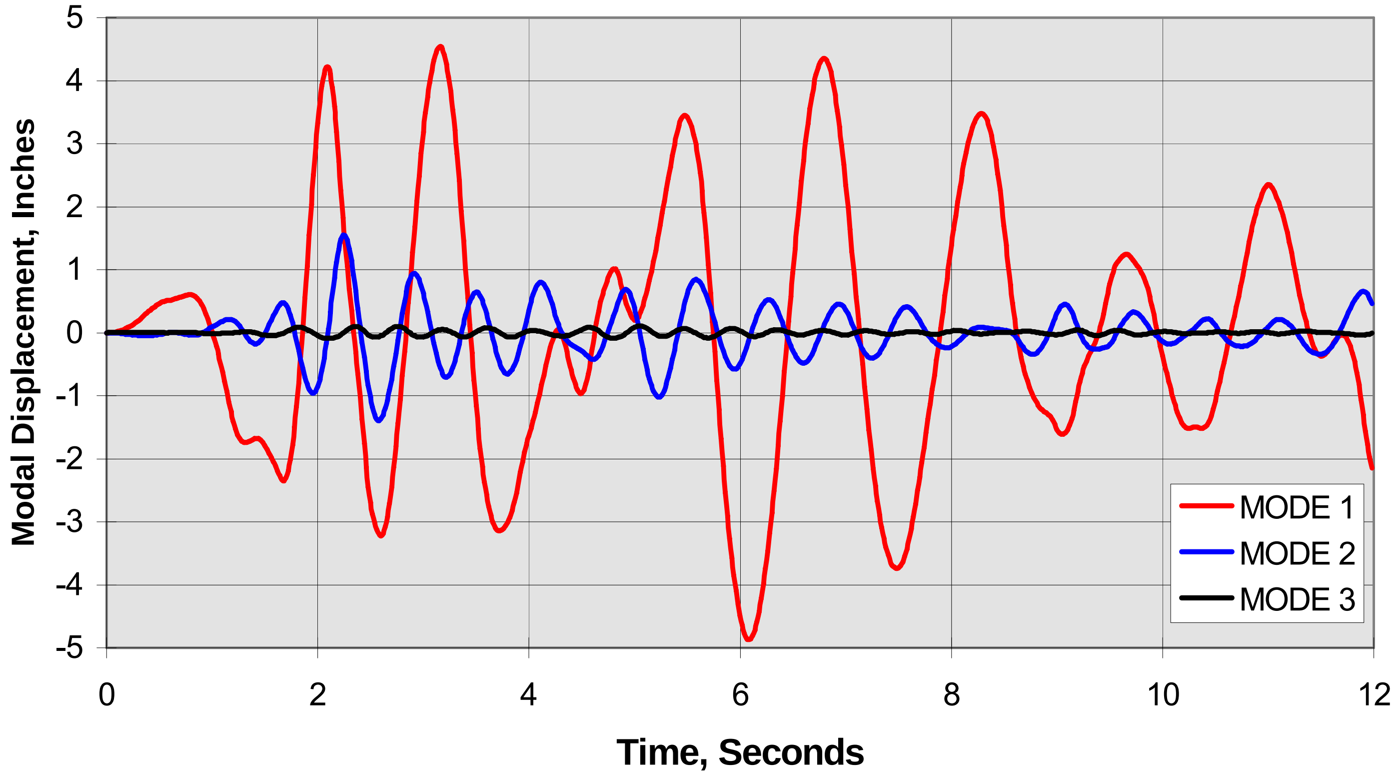
Caution:

Do NOT compute story shears from the story drifts derived from the SRSS of the story displacements.

Calculate the story shears in each mode (using modal drifts) and then SRSS the results.

Using Less than Full (Possible) Number of Natural Modes

Modal Response Histories:



Using Less than Full Number of Natural Modes

Time-History for **Mode** 1

$$y(t) = \begin{bmatrix} \boxed{y_1(t1) & y_1(t2) & y_1(t3) & y_1(t4) & y_1(t5) & y_1(t6) & y_1(t7) & y_1(t8) & .... & y_1(tn)} \\ y_2(t1) & y_2(t2) & y_2(t3) & y_2(t4) & y_2(t5) & y_2(t6) & y_2(t7) & y_2(t8) & .... & y_2(tn) \\ y_3(t1) & y_3(t2) & y_3(t3) & y_3(t4) & y_3(t5) & y_3(t6) & y_3(t7) & y_3(t8) & .... & y_3(tn) \end{bmatrix}$$

Transformation:

$$\boxed{\underset{3 \text{ x nt}}{u(t)} = \underbrace{\underset{3 \text{ x } 3}{\begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \end{bmatrix}} \underset{3 \text{ x nt}}{y(t)}}_{3 \text{ x nt}}}$$

Time History for **DOF** 1

$$u(t) = \begin{bmatrix} \boxed{u_1(t1) & u_1(t2) & u_1(t3) & u_1(t4) & u_1(t5) & u_1(t6) & u_1(t7) & u_1(t8) & .... & u_1(tn)} \\ u_2(t1) & u_2(t2) & u_2(t3) & u_2(t4) & u_2(t5) & u_2(t6) & u_2(t7) & u_2(t8) & .... & u_2(tn) \\ u_3(t1) & u_3(t2) & u_3(t3) & u_3(t4) & u_3(t5) & u_3(t6) & u_3(t7) & u_3(t8) & .... & u_3(tn) \end{bmatrix}$$

Using Less than Full Number of Natural Modes

Time History for **Mode** 1

$$y(t) = \begin{bmatrix} y_1(t1) & y_1(t2) & y_1(t3) & y_1(t4) & y_1(t5) & y_1(t6) & y_1(t7) & y_1(t8) & .... & y_1(tn) \\ y_2(t1) & y_2(t2) & y_2(t3) & y_2(t4) & y_2(t5) & y_2(t6) & y_2(t7) & y_2(t8) & .... & y_2(tn) \end{bmatrix}$$

NOTE: Mode 3 **NOT** Analyzed

Transformation:

$$\underbrace{\underset{3 \times nt}{u(t)} = \underset{3 \times 2}{\begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix}} \underset{2 \times nt}{y(t)}}_{3 \times nt}$$

Time history for **DOF** 1

$$u(t) = \begin{bmatrix} u_1(t1) & u_1(t2) & u_1(t3) & u_1(t4) & u_1(t5) & u_1(t6) & u_1(t7) & u_1(t8) & .... & u_1(tn) \\ u_2(t1) & u_2(t2) & u_2(t3) & u_2(t4) & u_2(t5) & u_2(t6) & u_2(t7) & u_2(t8) & .... & u_2(tn) \\ u_3(t1) & u_3(t2) & u_3(t3) & u_3(t4) & u_3(t5) & u_3(t6) & u_3(t7) & u_3(t8) & .... & u_3(tn) \end{bmatrix}$$

Using Less than Full Number of Natural Modes

(Modal Response Spectrum Technique)

Sum of absolute values:

$$\begin{Bmatrix} 4.940+1.550+0.108 \\ 3.181+0.931+0.278 \\ 1.482+1.048+0.267 \end{Bmatrix} = \begin{Bmatrix} 6.60 \\ 4.39 \\ 2.80 \end{Bmatrix} \qquad \begin{Bmatrix} 6.49 \\ 4.112 \\ 2.53 \end{Bmatrix}$$

At time of maximum displacement

Square root of the sum of the squares:

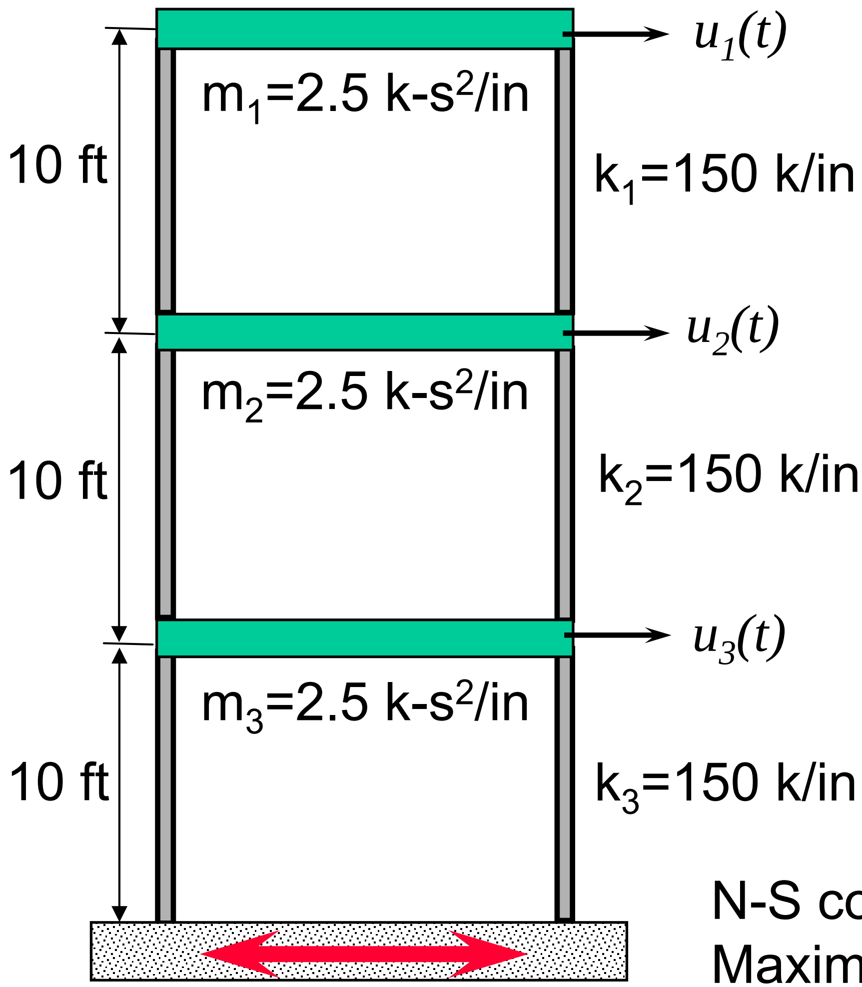
$$\begin{Bmatrix} \sqrt{4.940^2+1.550^2+0.108^2} \\ \sqrt{3.181^2+0.931^2+0.278^2} \\ \sqrt{1.482^2+1.048^2+0.267^2} \end{Bmatrix} = \begin{Bmatrix} 5.18 \\ 3.33 \\ 1.84 \end{Bmatrix} \qquad \begin{Bmatrix} 5.18 \\ 3.31 \\ 1.82 \end{Bmatrix}$$

3 modes 2 modes

“Exact”:

$$\begin{Bmatrix} 5.15 \\ 2.86 \\ 1.22 \end{Bmatrix}$$

Example of MDOF Response of Structure Responding to 1940 El Centro Earthquake

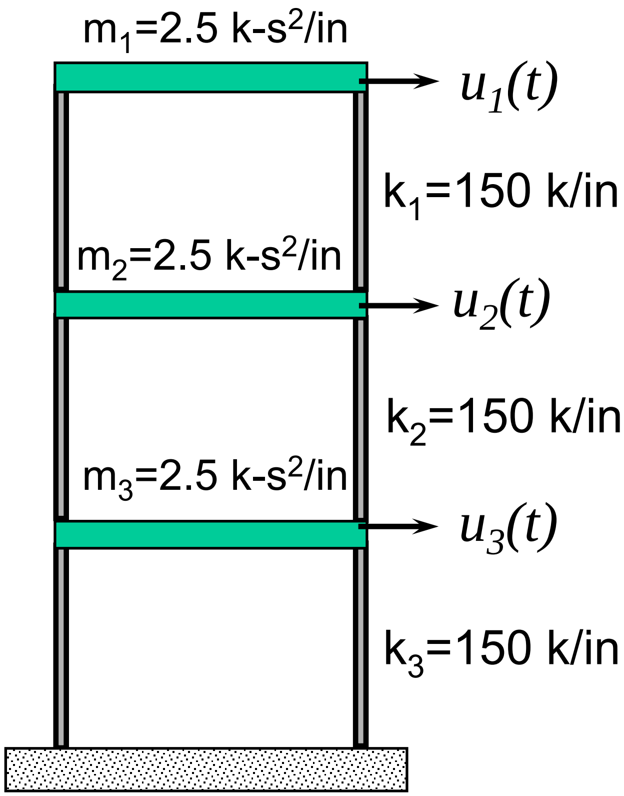


Example 2

Assume Wilson damping with 5% critical in each mode.

N-S component of 1940 El Centro earthquake
Maximum acceleration = 0.35 g

Example 2 (continued)

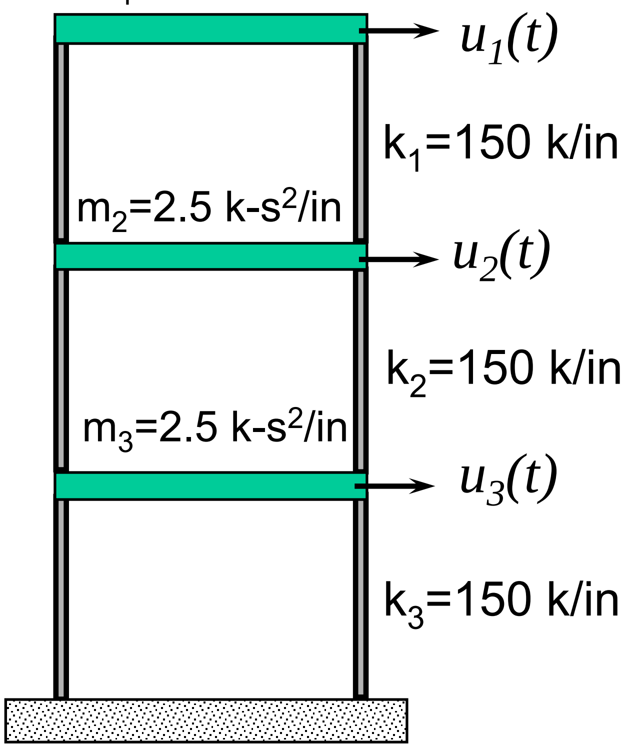


Form property matrices:

$$M = \begin{bmatrix} 2.5 & & \\ & 2.5 & \\ & & 2.5 \end{bmatrix} \text{kip} - \text{s}^2/\text{in}$$

$$K = \begin{bmatrix} 150 & -150 & 0 \\ -150 & 300 & -150 \\ 0 & -150 & 300 \end{bmatrix} \text{kip/in}$$

Example 2 (continued)



Solve = eigenvalue problem:

$$K\Phi = M\Phi\Omega^2$$

$$\Omega^2 = \begin{bmatrix} 11.9 & & \\ & 93.3 & \\ & & 194.8 \end{bmatrix} \sec^{-2}$$

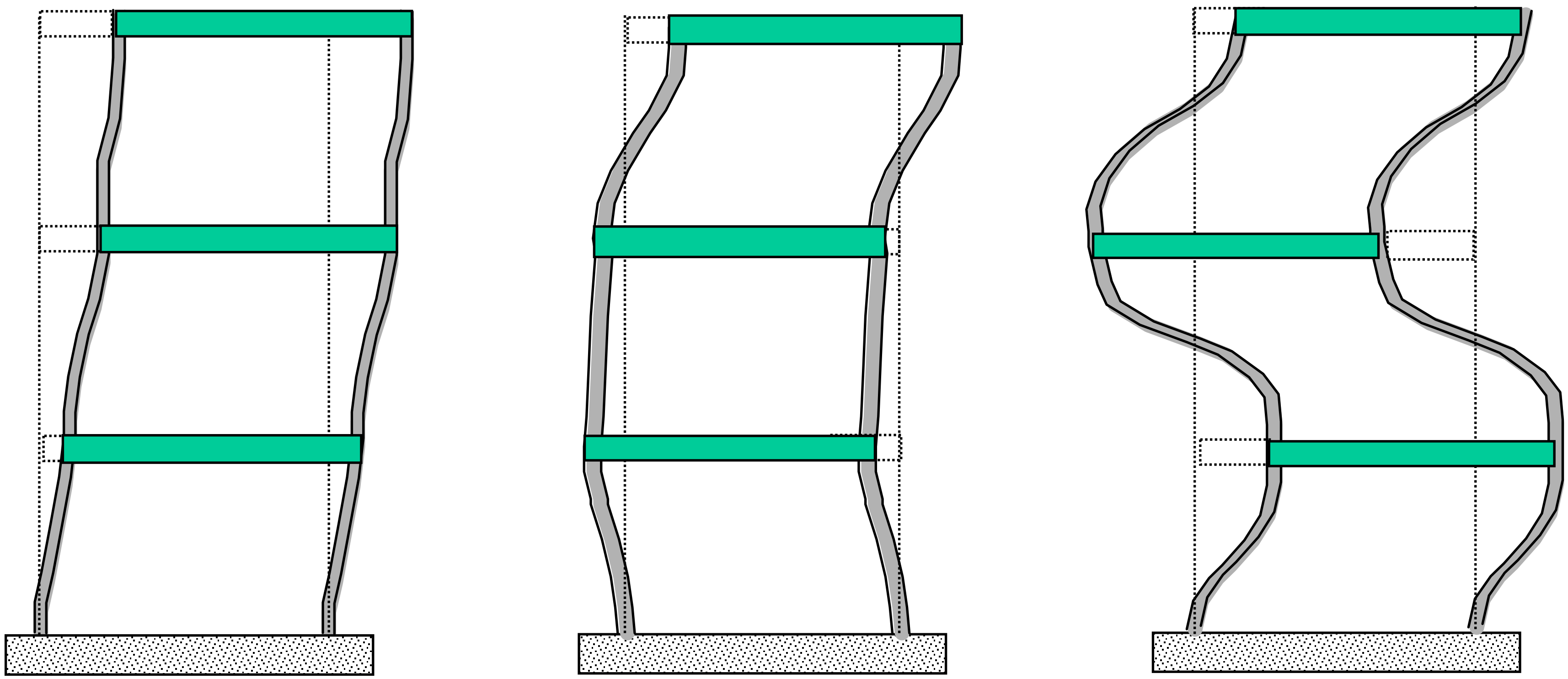
$$\Phi = \begin{bmatrix} 1.000 & 1.000 & 1.000 \\ 0.802 & -0.555 & -2.247 \\ 0.445 & -1.247 & 1.802 \end{bmatrix}$$

Normalization of Modes Using $\Phi^T M \Phi = I$

$$\Phi = \begin{bmatrix} 0.466 & 0.373 & 0.207 \\ 0.373 & -0.207 & -0.465 \\ 0.207 & -0.465 & 0.373 \end{bmatrix} \text{ vs } \begin{bmatrix} 1.000 & 1.000 & 1.000 \\ 0.802 & -0.555 & -2.247 \\ 0.445 & -1.247 & 1.802 \end{bmatrix}$$

Example 2 (continued)

Mode Shapes and Periods of Vibration



Mode 1

ω = 3.44 rad/sec

T = 1.82 sec

Mode 2

ω = 9.66 rad/sec

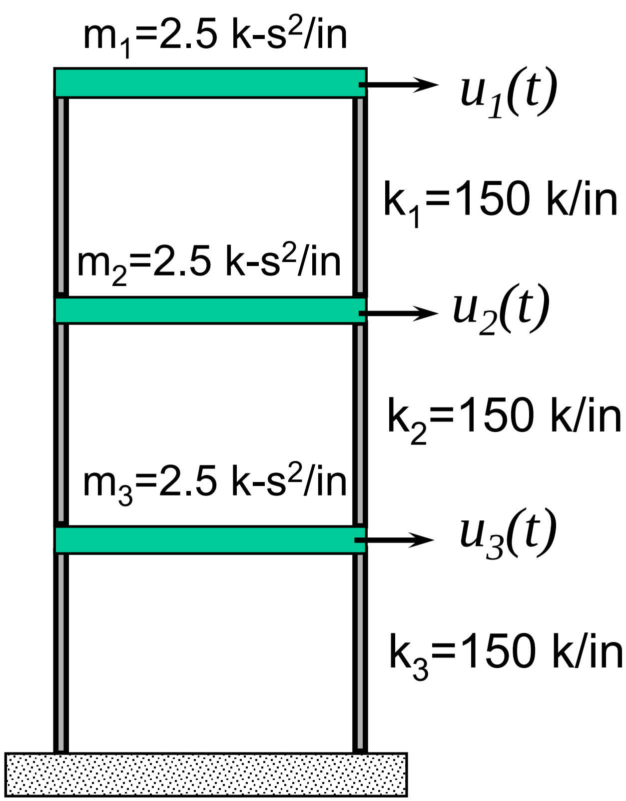
T = 0.65 sec

Mode 3

ω = 13.96 rad/sec

T = 0.45 sec

Example 2 (continued)

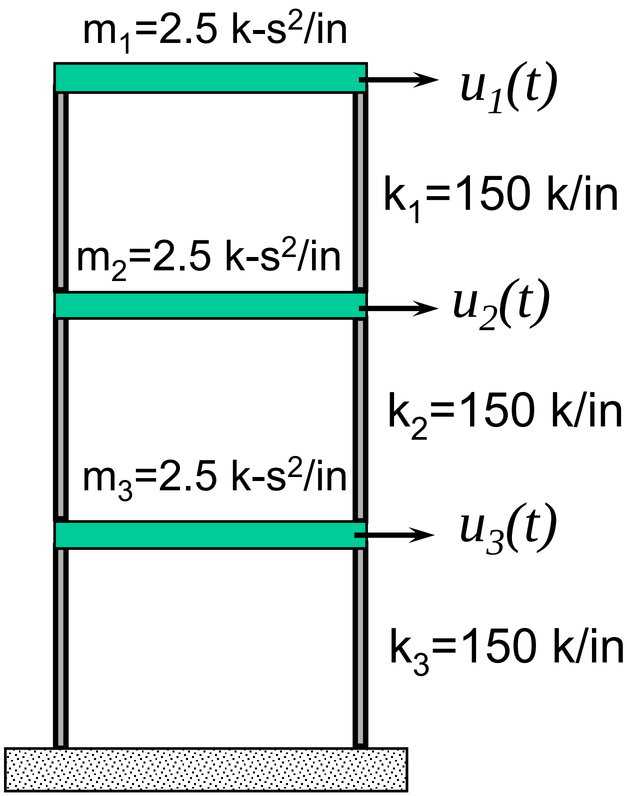


$$\omega_n = \begin{Bmatrix} 3.44 \\ 9.66 \\ 13.96 \end{Bmatrix} \text{rad/sec} \qquad T_n = \begin{Bmatrix} 1.82 \\ 0.65 \\ 0.45 \end{Bmatrix} \text{sec}$$

Compute generalized mass:

$$M^* = \Phi^T M \Phi = \begin{bmatrix} 4.603 & & \\ & 7.158 & \\ & & 23.241 \end{bmatrix} \text{kip} - \text{sec}^2/\text{in}$$

Example 2 (continued)



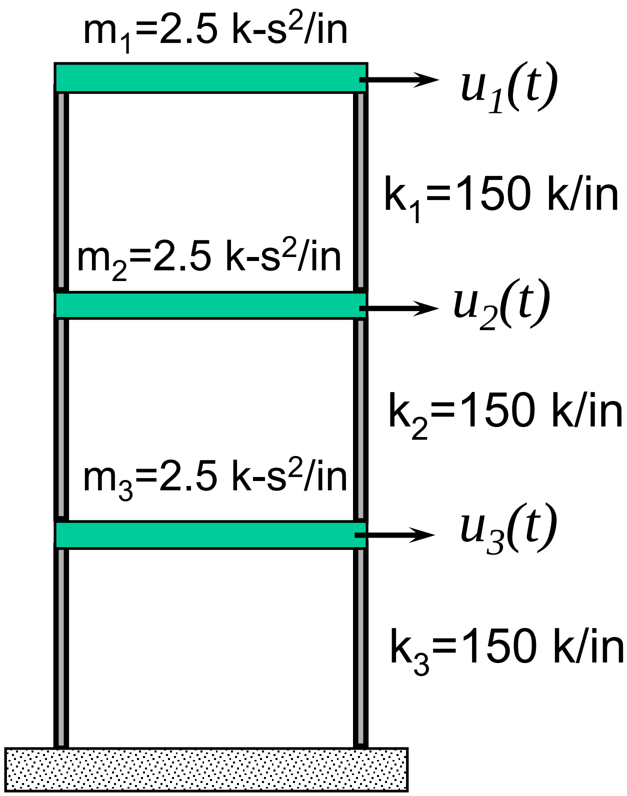
Compute generalized loading:

$$V^*(t) = -\Phi^T M R \ddot{v}_g(t)$$

$$V_n^* = \begin{Bmatrix} -5.617 \\ 2.005 \\ -1.388 \end{Bmatrix} \ddot{v}_g(t)$$

Example 2 (continued)

Write uncoupled (modal) equations of motion:



$$\ddot{y}_1 + 2\xi_1\omega_1\dot{y}_1 + \omega_1^2 y_1 = V_1^*(t)/m_1^*$$

$$\ddot{y}_2 + 2\xi_2\omega_2\dot{y}_2 + \omega_2^2 y_2 = V_2^*(t)/m_2^*$$

$$\ddot{y}_3 + 2\xi_3\omega_3\dot{y}_3 + \omega_3^2 y_3 = V_3^*(t)/m_3^*$$

$$\ddot{y}_1 + 0.345\dot{y}_1 + 11.88 y_1 = -1.22\ddot{v}_g(t)$$

$$\ddot{y}_2 + 0.966\dot{y}_2 + 93.29 y_2 = 0.280\ddot{v}_g(t)$$

$$\ddot{y}_3 + 1.395\dot{y}_3 + 194.83 y_3 = -0.06\ddot{v}_g(t)$$

Modal Participation Factors

$$\begin{array}{ll} \textit{Mode} & 1 \\ \textit{Mode} & 2 \\ \textit{Mode} & 3 \end{array} \boxed{\begin{array}{c} -1.22 \\ 0.28 \\ -0.060 \end{array}} \quad \boxed{\begin{array}{c} -2.615 \\ 0.748 \\ -0.287 \end{array}}$$

Modal scaling $\quad \phi_{i,1} = 1.0 \qquad \phi_i^{\,T} M \phi_i = 1.0$

Effective Modal Mass

$$\overline{M_n} = P_n{}^2 m_n$$

	$\overline{M}_n$	%	Accum%
Mode 1	6.856	91.40	91.40
Mode 2	0.562	7.50	98.90
Mode 3	0.083	1.10	100.0
	7.50	100%	

Example 2 (continued)

Solving modal equation via NONLIN:

For Mode 1:

$$\ddot{y}_1 + 2\xi_1\omega_1\dot{y}_1 + \omega_1^2 y_1 = V_1^*(t)/m_1^*$$

$$1.00\ddot{y}_1 + 0.345\dot{y}_1 + 11.88y_1 = -1.22\ddot{v}_g(t)$$

M = 1.00 kip-sec²/in

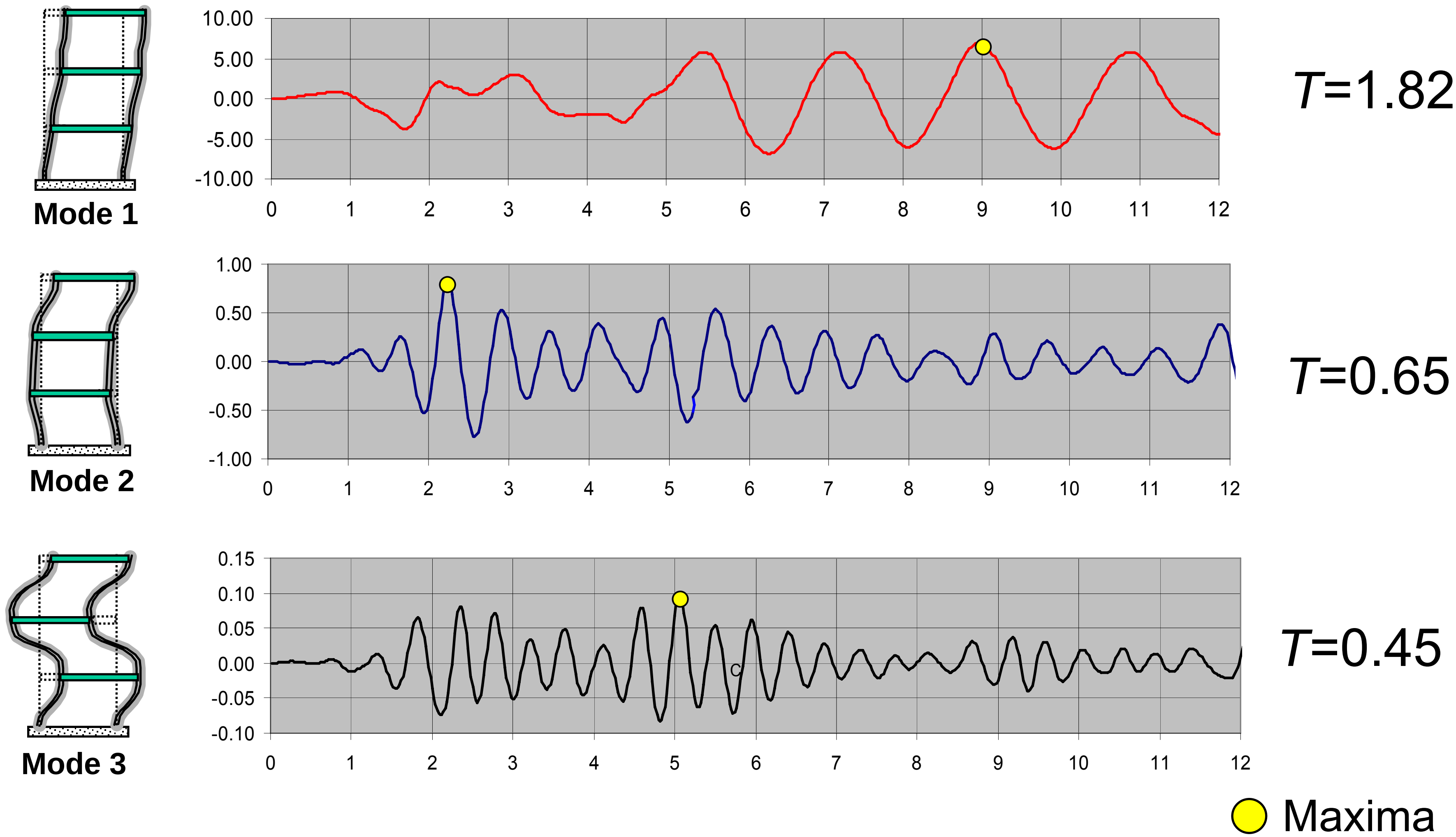
C = 0.345 kip-sec/in

K_1 = 11.88 kips/inch

Scale ground acceleration by factor 1.22

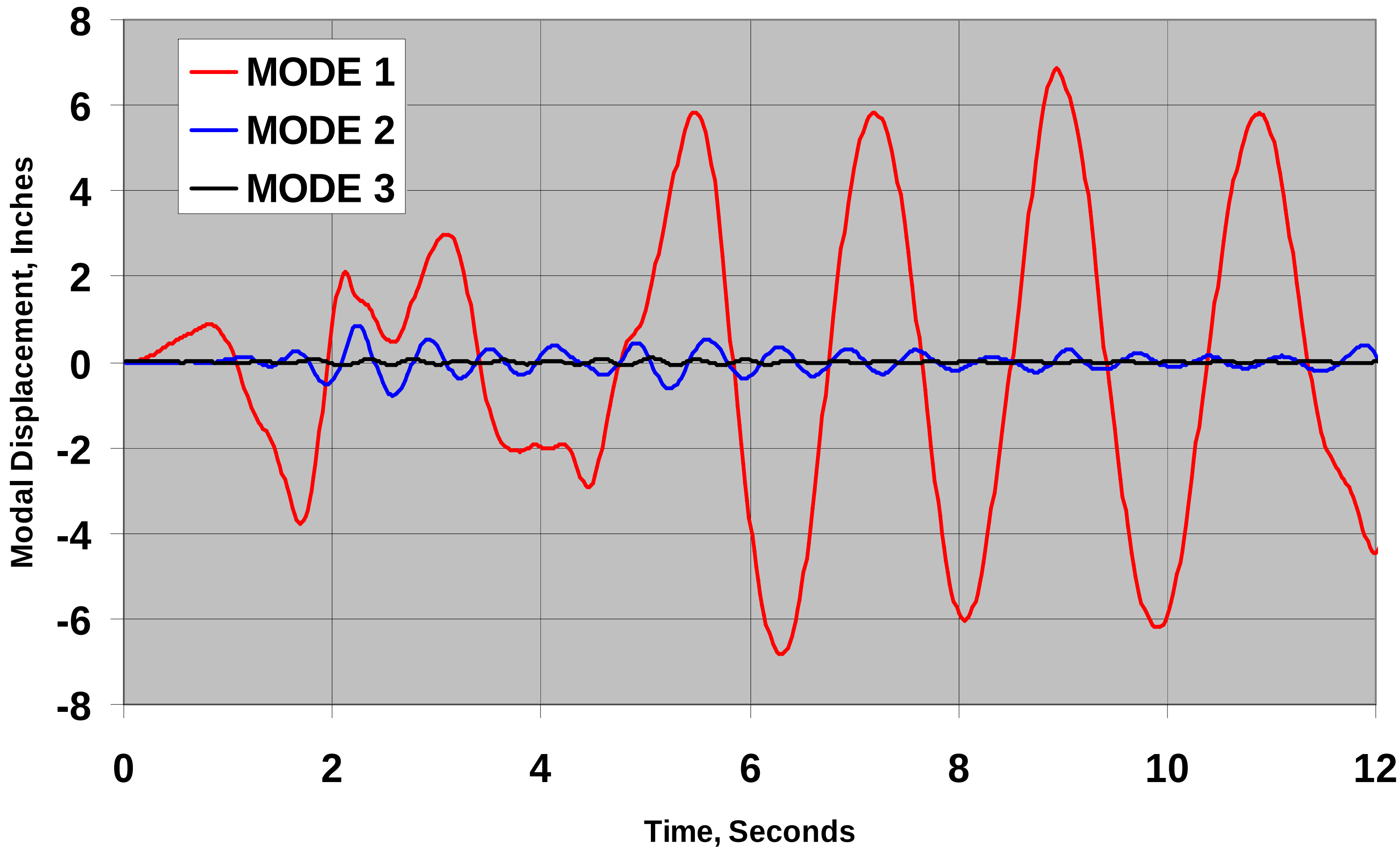
Example 2 (continued)

Modal Displacement Response Histories (from NONLIN)



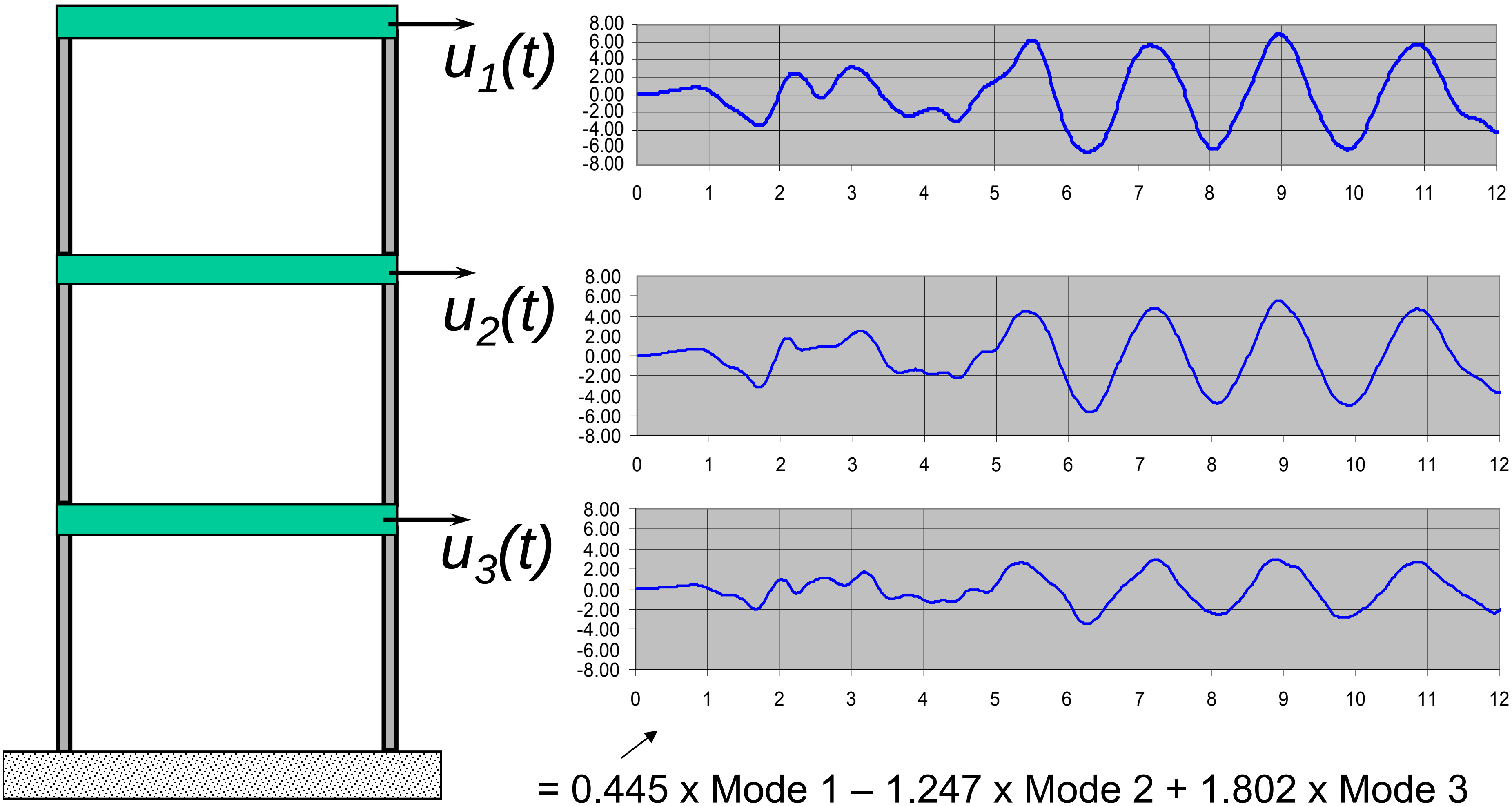
Example 2 (continued)

Modal Response Histories



Example 2 (continued)

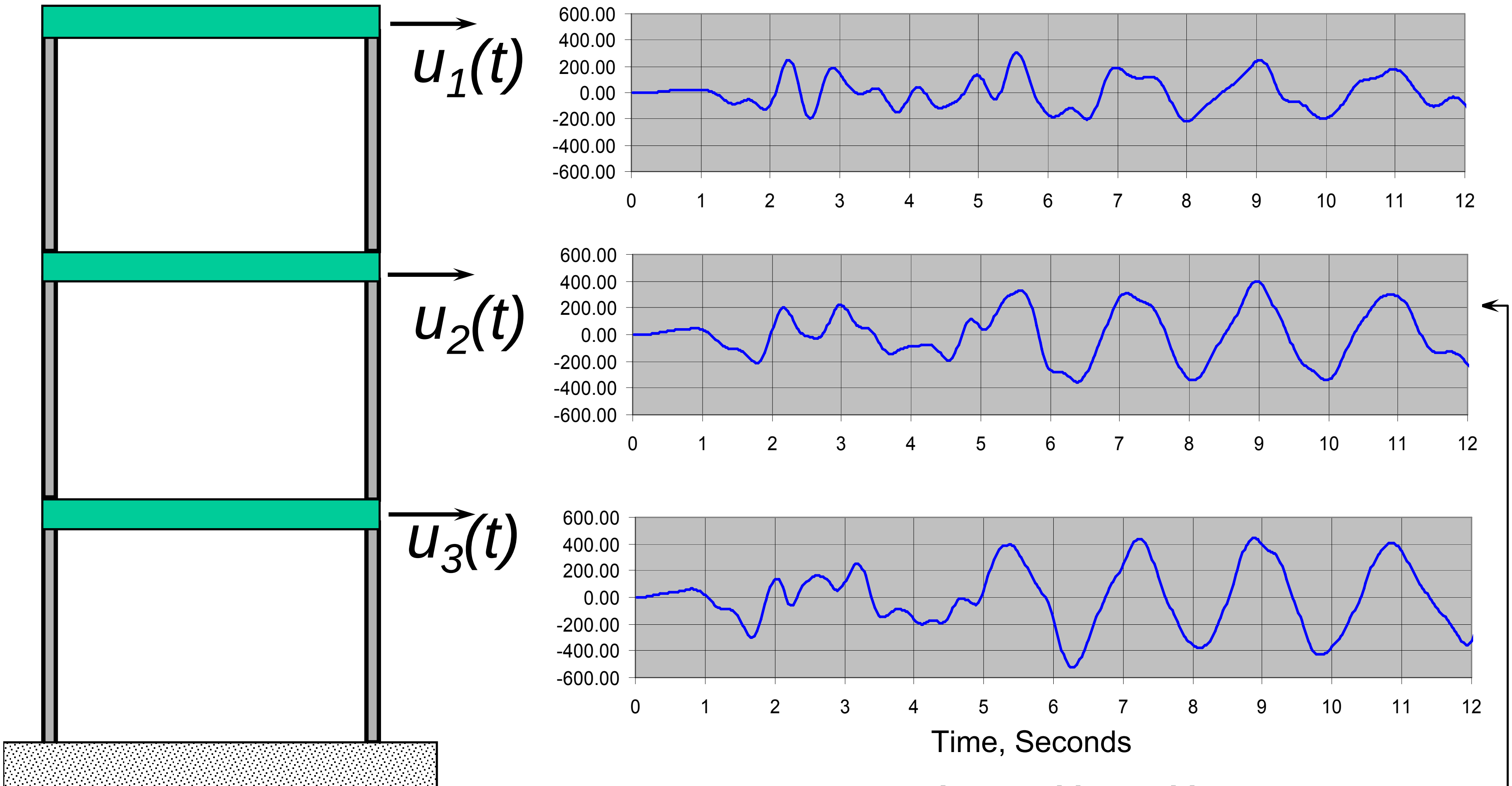
Compute story displacement response histories: $u(t) = \Phi y(t)$



= 0.445 x Mode 1 – 1.247 x Mode 2 + 1.802 x Mode 3

Example 2 (continued)

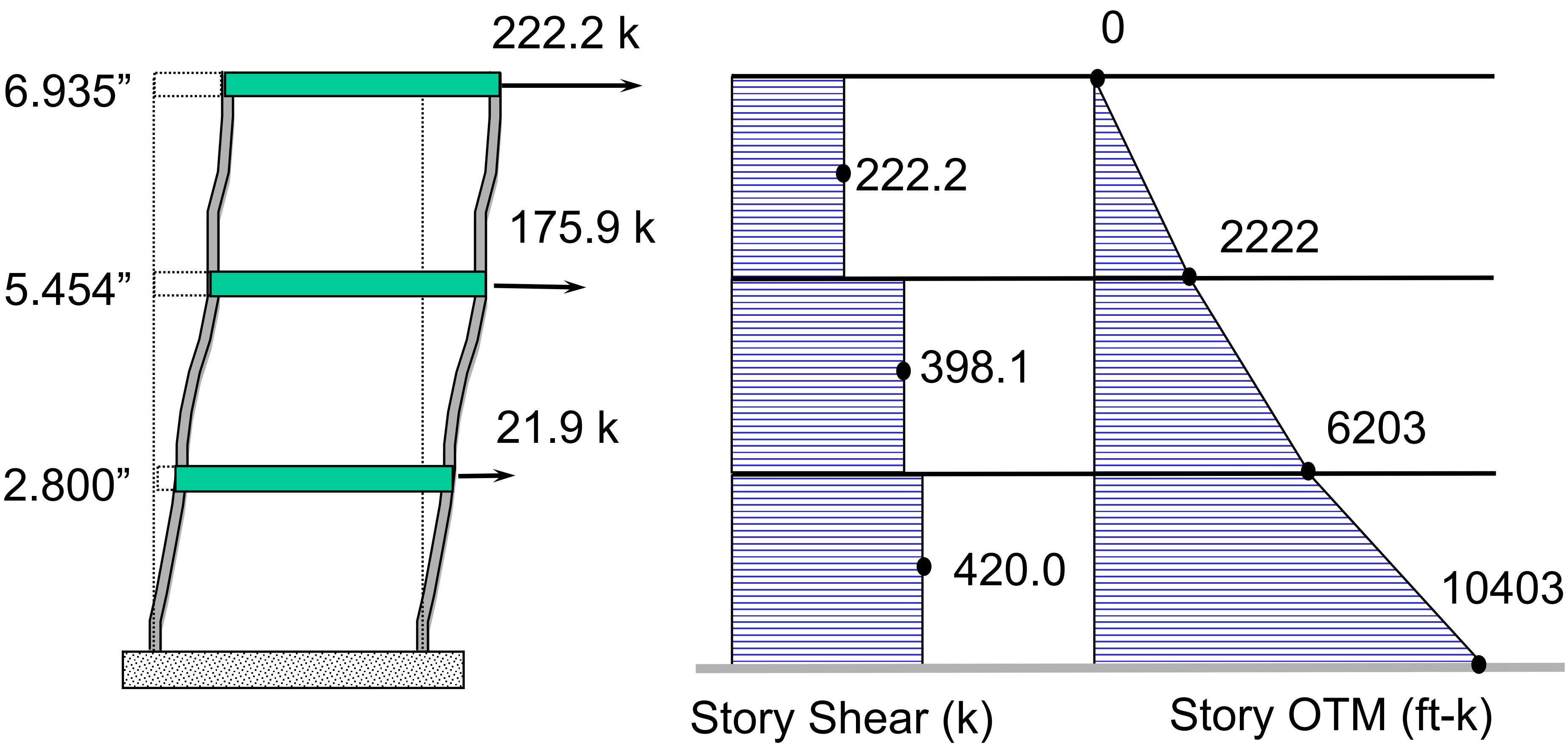
Compute story shear response histories:



$$=k_2[u_2(t)-u_3(t)]$$

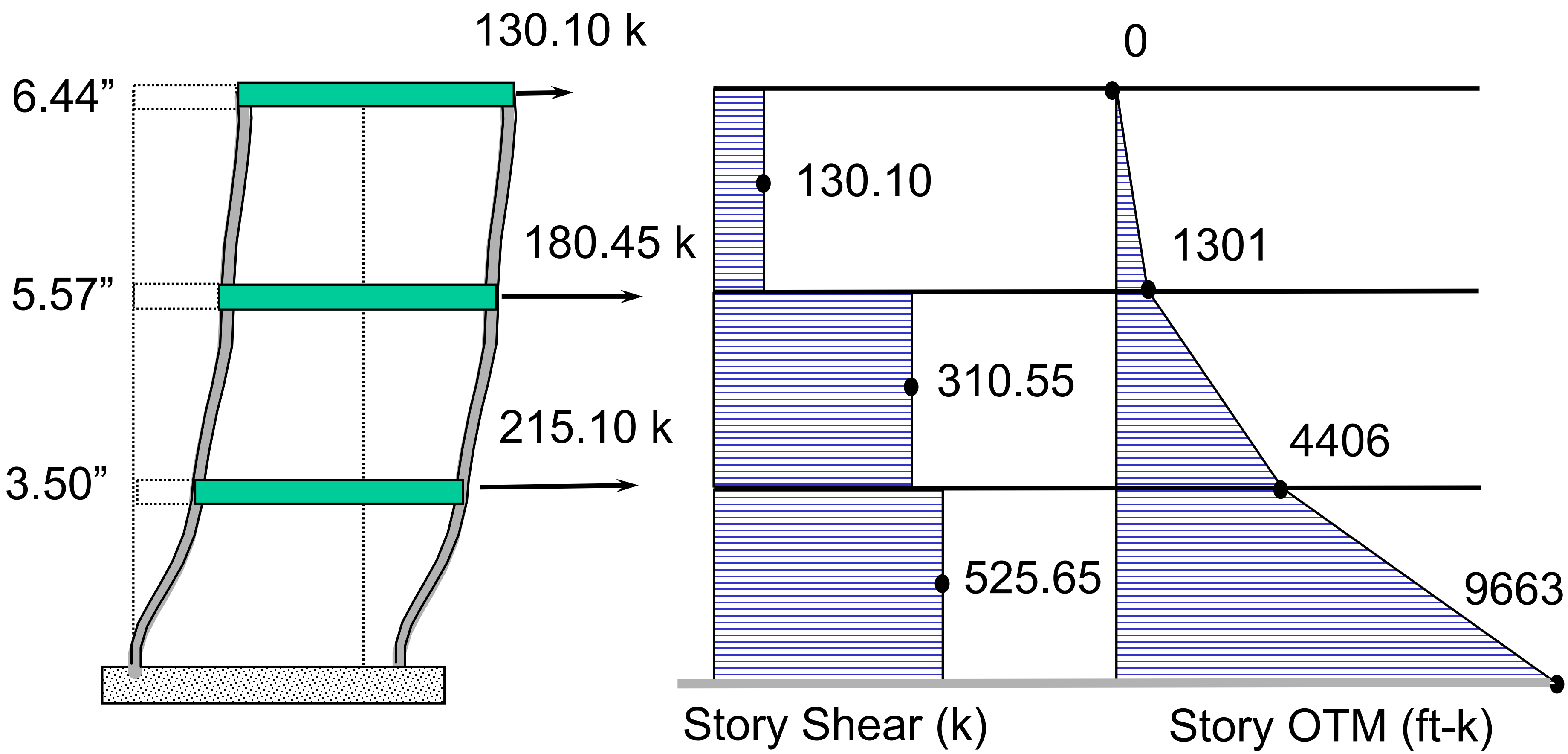
Example 2 (continued)

Displacements and Forces at time of Maximum Displacements
(t = 8.96 seconds)



Example 2 (continued)

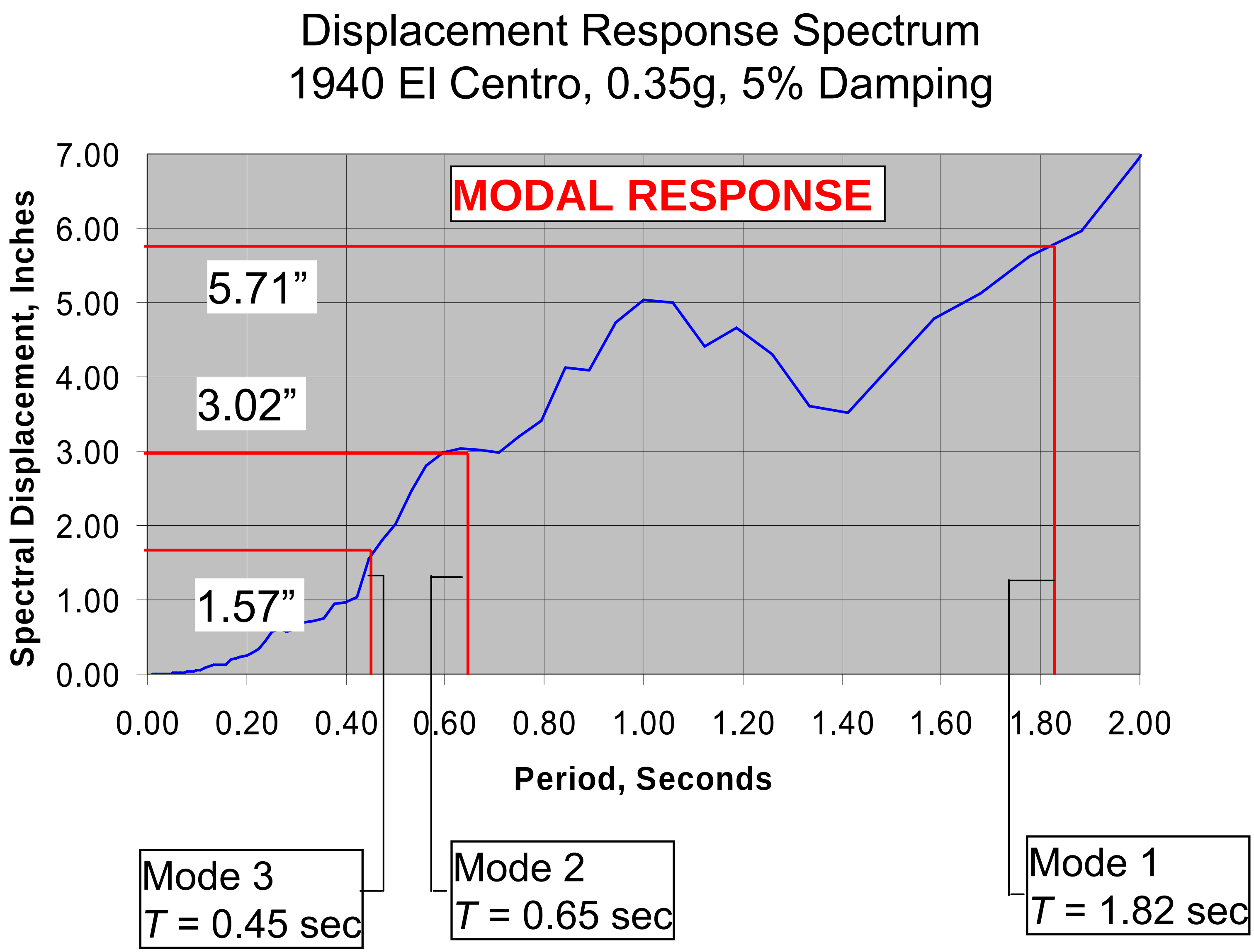
Displacements and Forces at Time of Maximum Shear
(t = 6.26 sec)



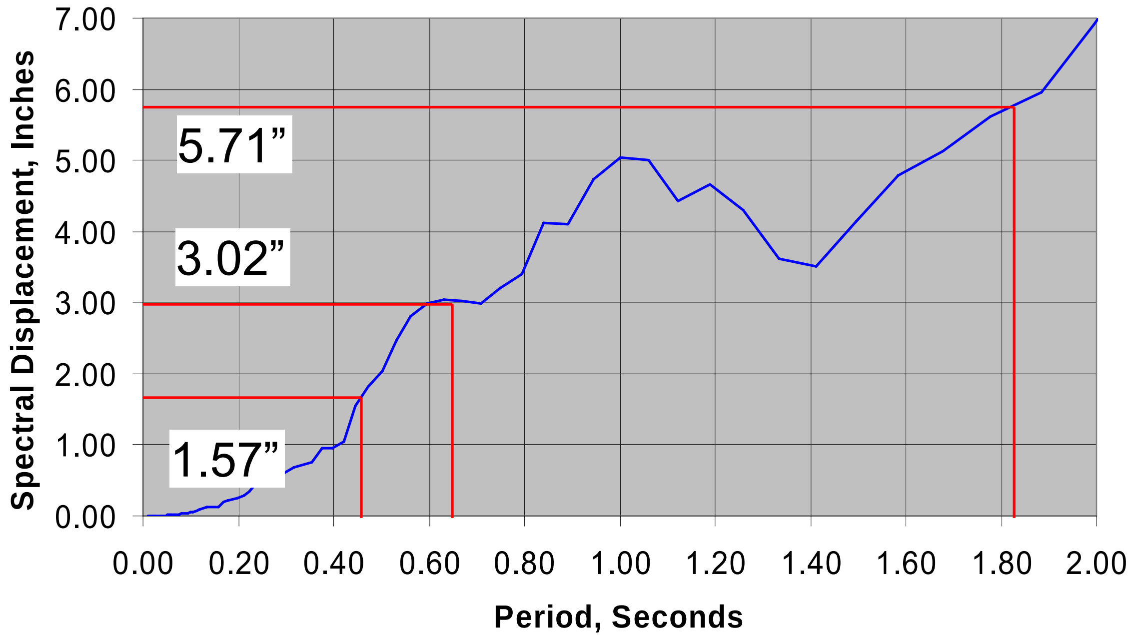
Modal Response Response Spectrum Method

- Instead of solving the time history problem for each mode, use a response spectrum to compute the **maximum** response in each mode.
- These maxima are generally **nonconcurrent**.
- Combine the maximum modal responses using some statistical technique, such as square root of the sum of the squares (SRSS) or complete quadratic combination (CQC).
- The technique is **approximate**.
- It is the basis for the equivalent lateral force (ELF) method.

Example 2 (Response Spectrum Method)



Example 2 (continued)



Modal Equations of Motion

$$\ddot{y}_1 + 0.345\dot{y}_1 + 11.88y_1 = -1.22\ddot{v}_g(t)$$

$$\ddot{y}_2 + 0.966\dot{y}_2 + 93.29y_2 = 0.280\ddot{v}_g(t)$$

$$\ddot{y}_3 + 1.395\dot{y}_3 + 194.83y_3 = -0.060\ddot{v}_g(t)$$

Modal Maxima

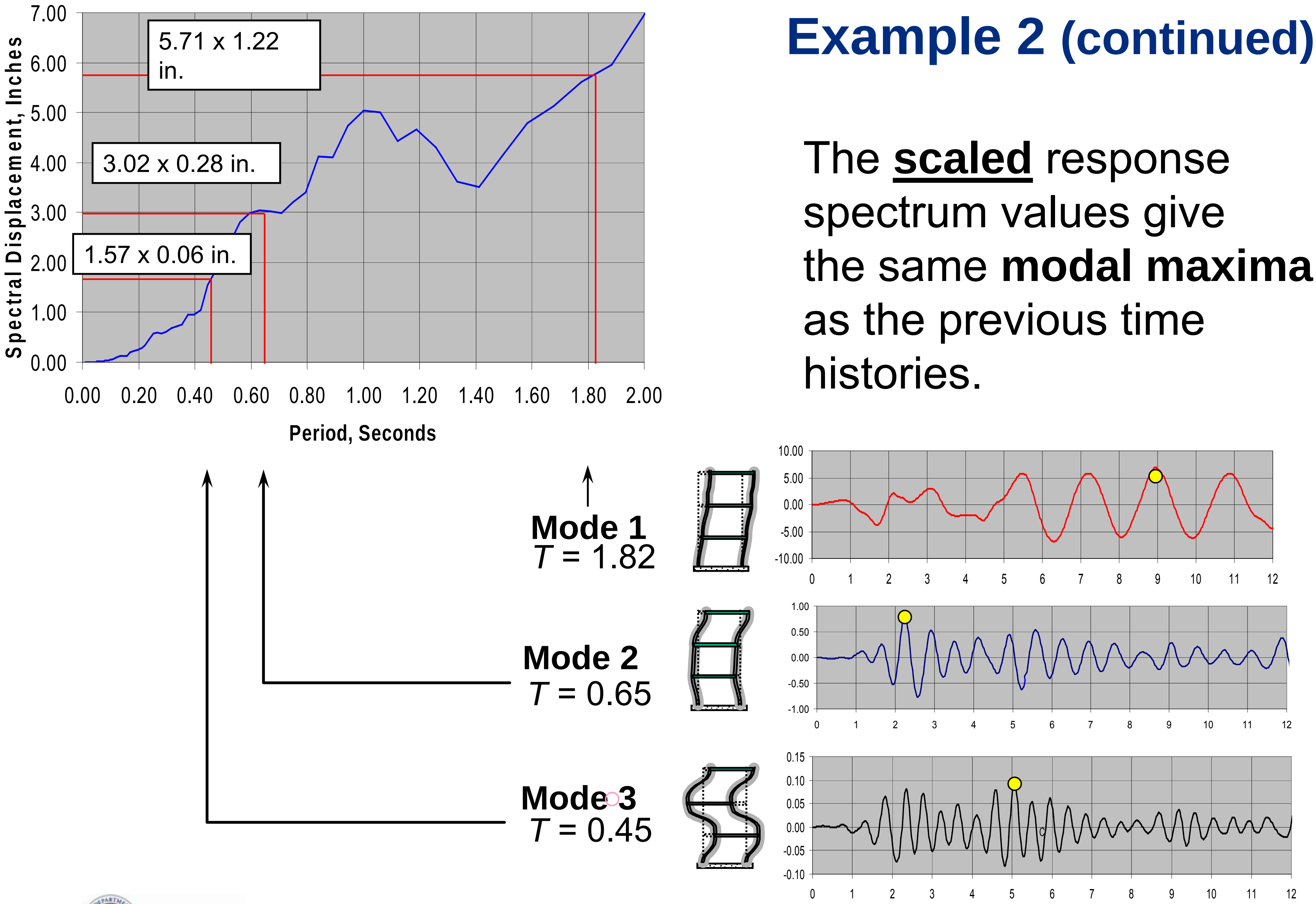
$$\bar{y}_1 = 1.22 * 5.71 = 6.966"$$

$$\bar{y}_2 = 0.28 * 3.02 = 0.845"$$

$$\bar{y}_3 = 0.060 * 1.57 = 0.094"$$

Example 2 (continued)

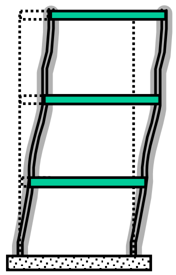
The **scaled** response spectrum values give the same **modal maxima** as the previous time histories.



Example 2 (continued)

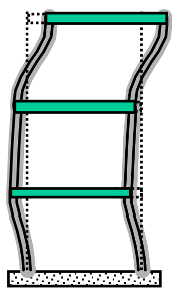
Computing **Nonconcurrent** Story Displacements

Mode 1



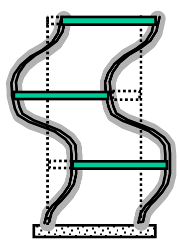
$$\begin{Bmatrix} 1.000 \\ 0.802 \\ 0.445 \end{Bmatrix} 6.966 = \begin{Bmatrix} 6.966 \\ 5.586 \\ 3.100 \end{Bmatrix}$$

Mode 2



$$\begin{Bmatrix} 1.000 \\ -0.555 \\ -1.247 \end{Bmatrix} 0.845 = \begin{Bmatrix} 0.845 \\ -0.469 \\ -1.053 \end{Bmatrix}$$

Mode 3



$$\begin{Bmatrix} 1.000 \\ -2.247 \\ 1.802 \end{Bmatrix} 0.094 = \begin{Bmatrix} 0.094 \\ -0.211 \\ 0.169 \end{Bmatrix}$$

Example 2 (continued)

Modal Combination Techniques (For Displacement)

Sum of absolute values:

$$\begin{Bmatrix} 6.966+0.845+0.108 \\ 5.586+0.469+0.211 \\ 3.100+1.053+0.169 \end{Bmatrix} = \begin{Bmatrix} 7.919 \\ 6.266 \\ 4.322 \end{Bmatrix}$$

At time of maximum displacement

$$\boxed{\begin{matrix} \text{"Exact"} \\ \begin{Bmatrix} 6.935 \\ 5.454 \\ 2.800 \end{Bmatrix} \end{matrix}}$$

Square root of the sum of the squares

$$\begin{Bmatrix} \sqrt{6.966^2+0.845^2+0.108^2} \\ \sqrt{5.586^2+0.469^2+0.211^2} \\ \sqrt{3.100^2+1.053^2+0.169^2} \end{Bmatrix} = \begin{Bmatrix} 7.02 \\ 5.61 \\ 3.28 \end{Bmatrix}$$

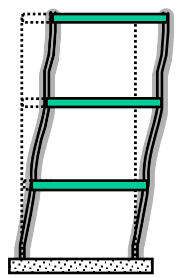
Envelope of story displacement

$$\boxed{\begin{Bmatrix} 6.935 \\ 5.675 \\ 2.965 \end{Bmatrix}}$$

Example 2 (continued)

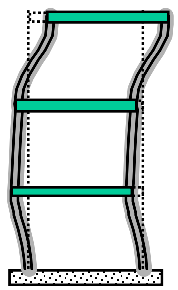
Computing Interstory Drifts

Mode 1



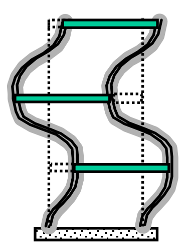
$$\begin{Bmatrix} 6.966-5.586 \\ 5.586-3.100 \\ 3.100-0 \end{Bmatrix} = \begin{Bmatrix} 1.380 \\ 2.486 \\ 3.100 \end{Bmatrix}$$

Mode 2



$$\begin{Bmatrix} 0.845-(-0.469) \\ -0.469-(-1.053) \\ -1.053-0 \end{Bmatrix} = \begin{Bmatrix} 1.314 \\ 0.584 \\ -1.053 \end{Bmatrix}$$

Mode 3

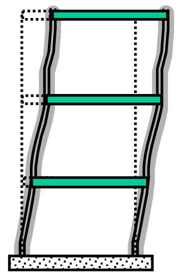


$$\begin{Bmatrix} 0.108-(-0.211) \\ -0.211-0.169 \\ 0.169-0 \end{Bmatrix} = \begin{Bmatrix} 0.319 \\ -0.380 \\ 0.169 \end{Bmatrix}$$

Example 2 (continued)

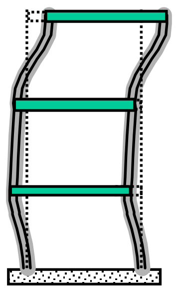
Computing Interstory Shears (Using Drift)

Mode 1



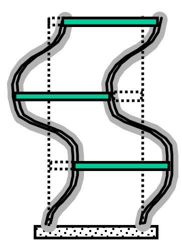
$$\begin{Bmatrix} 1.380(150) \\ 2.486(150) \\ 3.100(150) \end{Bmatrix} = \begin{Bmatrix} 207.0 \\ 372.9 \\ 465.0 \end{Bmatrix}$$

Mode 2



$$\begin{Bmatrix} 1.314(150) \\ 0.584(150) \\ -1.053(150) \end{Bmatrix} = \begin{Bmatrix} 197.1 \\ 87.6 \\ -157.9 \end{Bmatrix}$$

Mode 3



$$\begin{Bmatrix} 0.319(150) \\ -0.380(150) \\ 0.169(150) \end{Bmatrix} = \begin{Bmatrix} 47.9 \\ -57.0 \\ 25.4 \end{Bmatrix}$$

Example 2 (continued)

Computing Interstory Shears: SRSS Combination

$$\begin{Bmatrix} \sqrt{207^2 + 197.1^2 + 47.9^2} \\ \sqrt{372.9^2 + 87.6^2 + 57^2} \\ \sqrt{465^2 + 157.9^2 + 25.4^2} \end{Bmatrix} = \begin{Bmatrix} 289.81 \\ 387.27 \\ 491.73 \end{Bmatrix}$$

"Exact"

$$\begin{Bmatrix} 130.1 \\ 310.5 \\ 525.7 \end{Bmatrix}$$

At time of max. shear

"Exact"

$$\begin{Bmatrix} 222.2 \\ 398.1 \\ 420.0 \end{Bmatrix}$$

At time of max. displacement

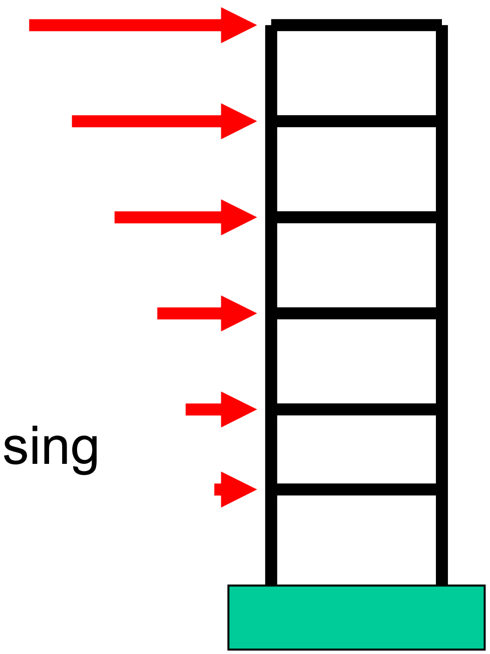
"Exact"

$$\begin{Bmatrix} 304.0 \\ 398.5 \\ 525.7 \end{Bmatrix}$$

Envelope = maximum per story

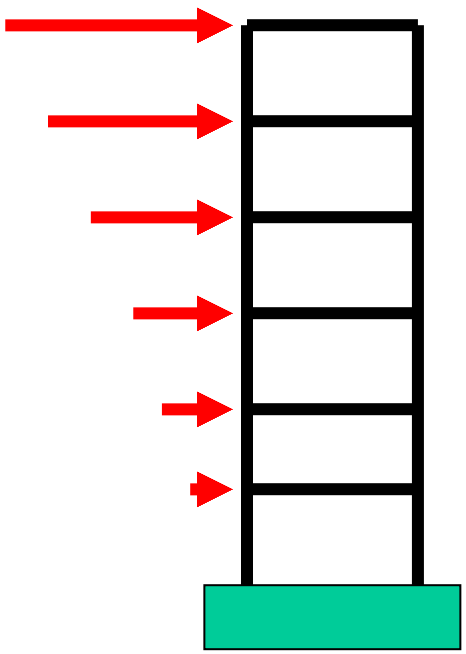
ASCE 7 Allows an Approximate Modal Analysis Technique Called the Equivalent Lateral Force Procedure

- Empirical period of vibration
- Smoothed response spectrum
- Compute total base shear, **V**, as if SDOF
- Distribute **V** along height assuming “regular” geometry
- Compute displacements and member forces using standard procedures



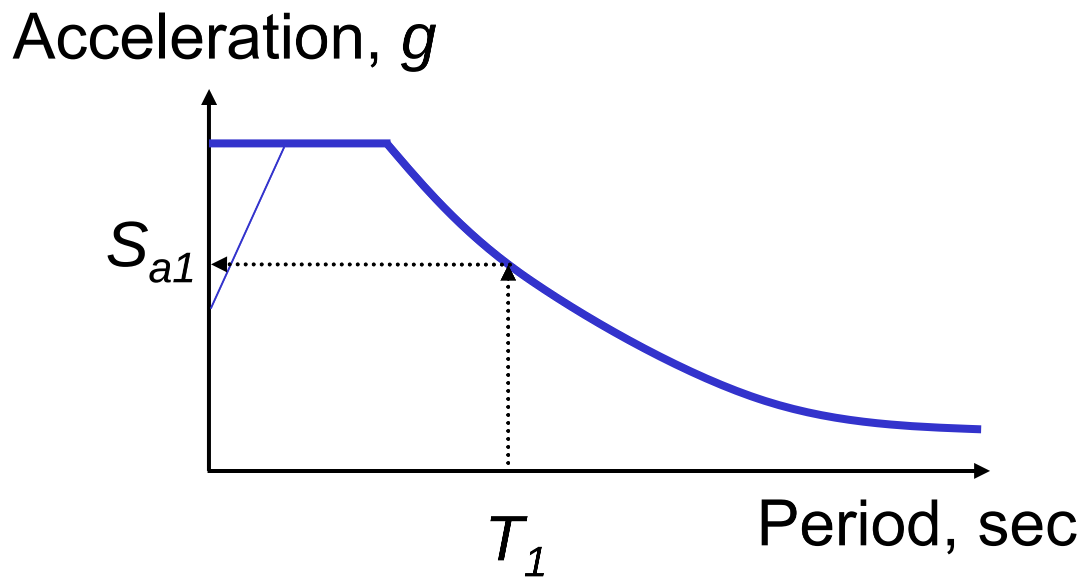
Equivalent Lateral Force Procedure

- Method is based on **first mode** response.
- Higher modes can be included empirically.
- Has been calibrated to provide a reasonable estimate of the envelope
  of story shear, NOT to provide accurate estimates of story force.
- May result in overestimate of overturning moment.



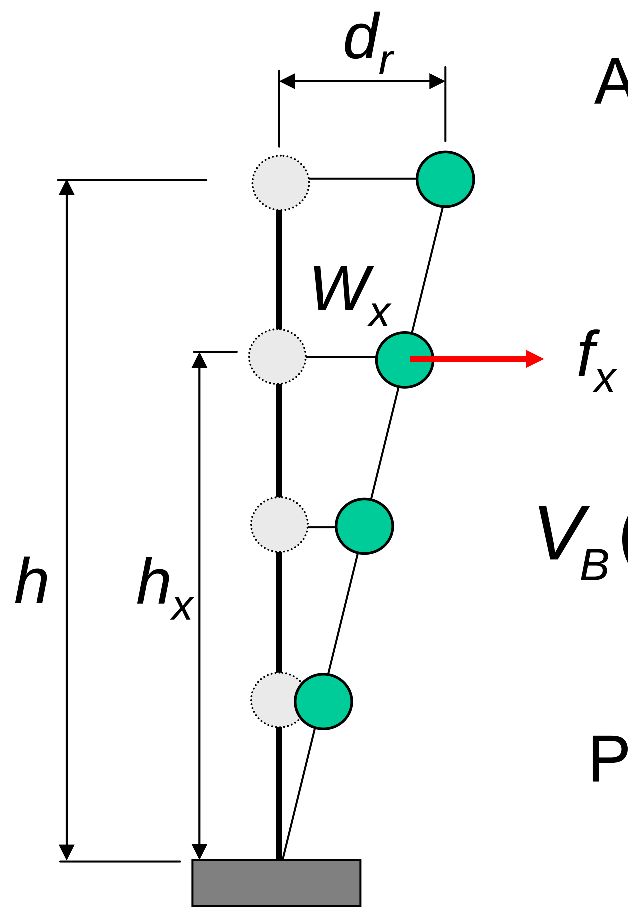
Equivalent Lateral Force Procedure

- Assume first mode effective mass = total Mass = $M = W/g$
- Use response spectrum to obtain total acceleration @ T_1



$$V_B = (S_{a1} g) M = (S_{a1} g) \frac{W}{g} = S_{a1} W$$

Equivalent Lateral Force Procedure



Assume linear first mode response

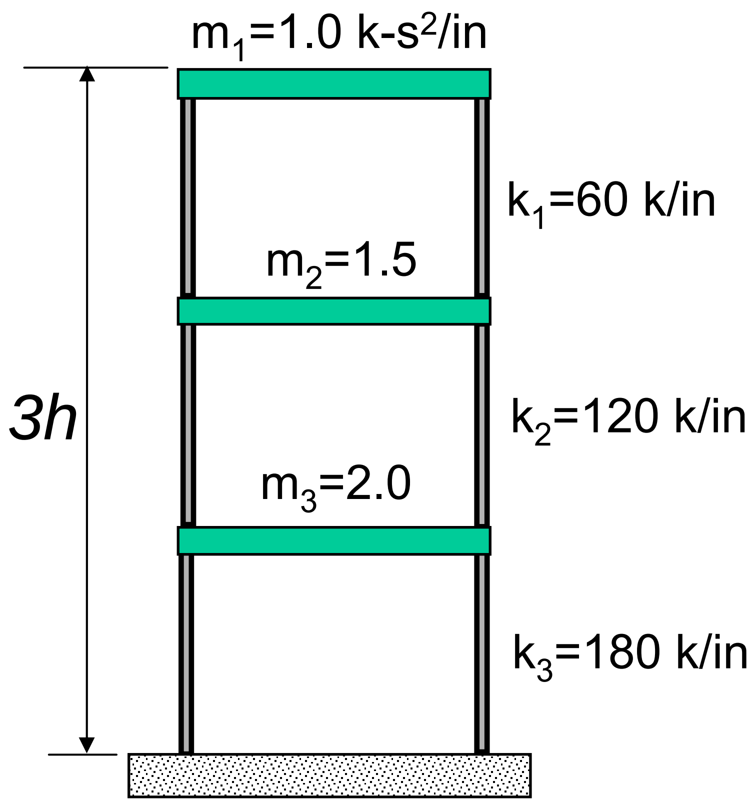
$$f_x(t) = \frac{h_x}{h} d_r(t) \omega_1^2 \frac{W_x}{g}$$

$$V_B(t) = \sum_{i=1}^{nstories} f_i(t) = \frac{d_r(t)\omega_1^2}{hg} \sum_{i=1}^{nstories} h_i W_i$$

Portion of base shear applied to story i

$$\boxed{\frac{f_x(t)}{V_B(t)} = \frac{h_x W_x}{\sum_{i=1}^{nstories} h_i W_i}}$$

ELF Procedure Example



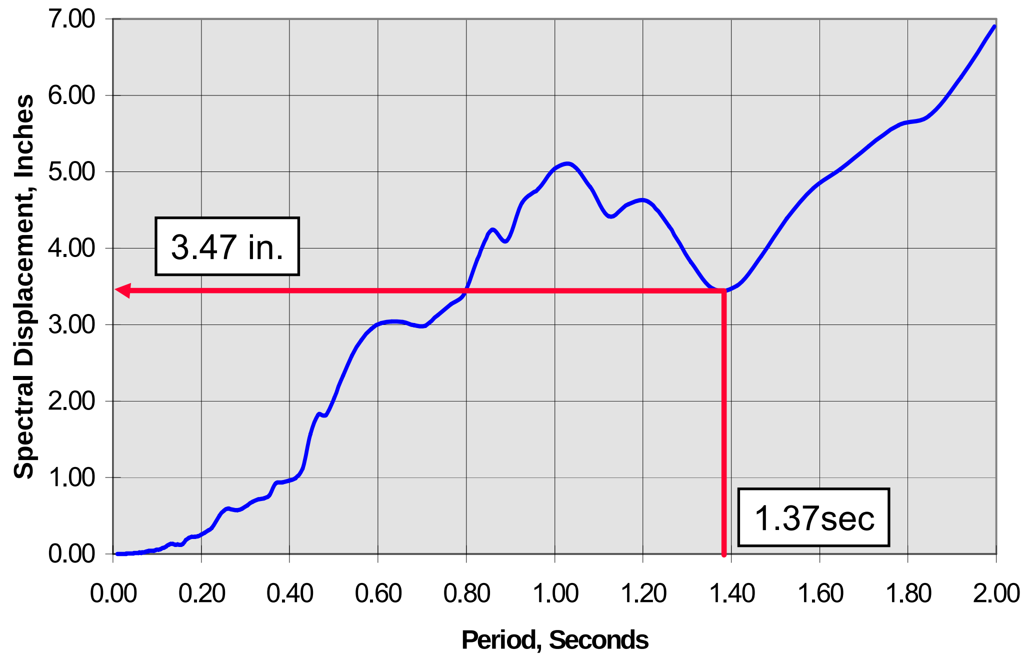
Recall

T_1 = 1.37 sec

ELF Procedure Example

Total weight = M x g = (1.0 + 1.5 + 2.0) 386.4 = 1738 kips

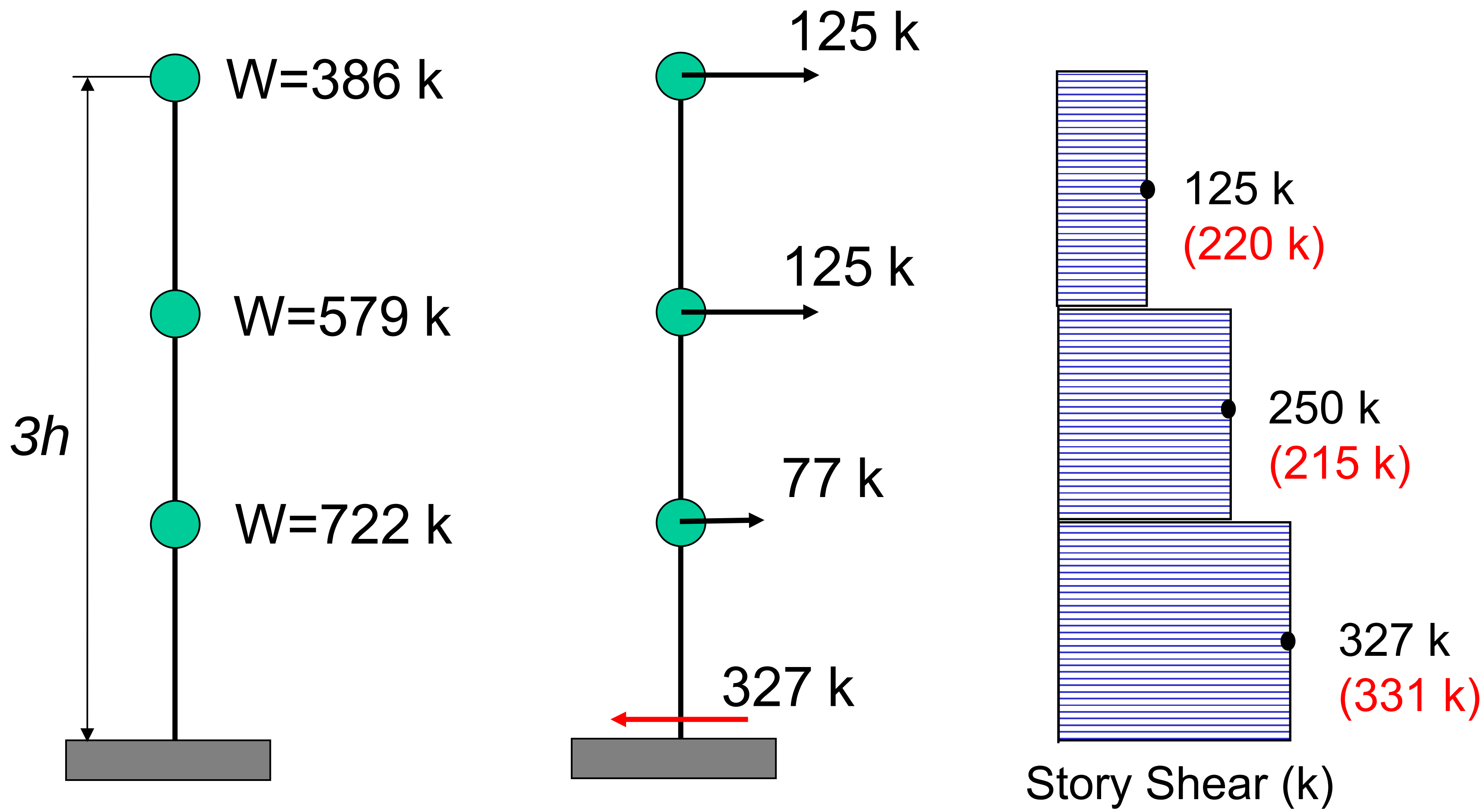
Spectral acceleration = $w^2 S_D$ = (2p/1.37)^2 x 3.47 = 72.7 in/sec^2 = 0.188g



Base shear = $S_a W$ = 0.188 x 1738 = 327 kips

ELF Procedure Example (Story Forces)

$$f_3 = \frac{386(3h)}{386(3h) + 579(2h) + 722(h)} = 0.381V_B = 0.375(327) = 125\ kips$$



ELF Procedure Example (Story Displacements)

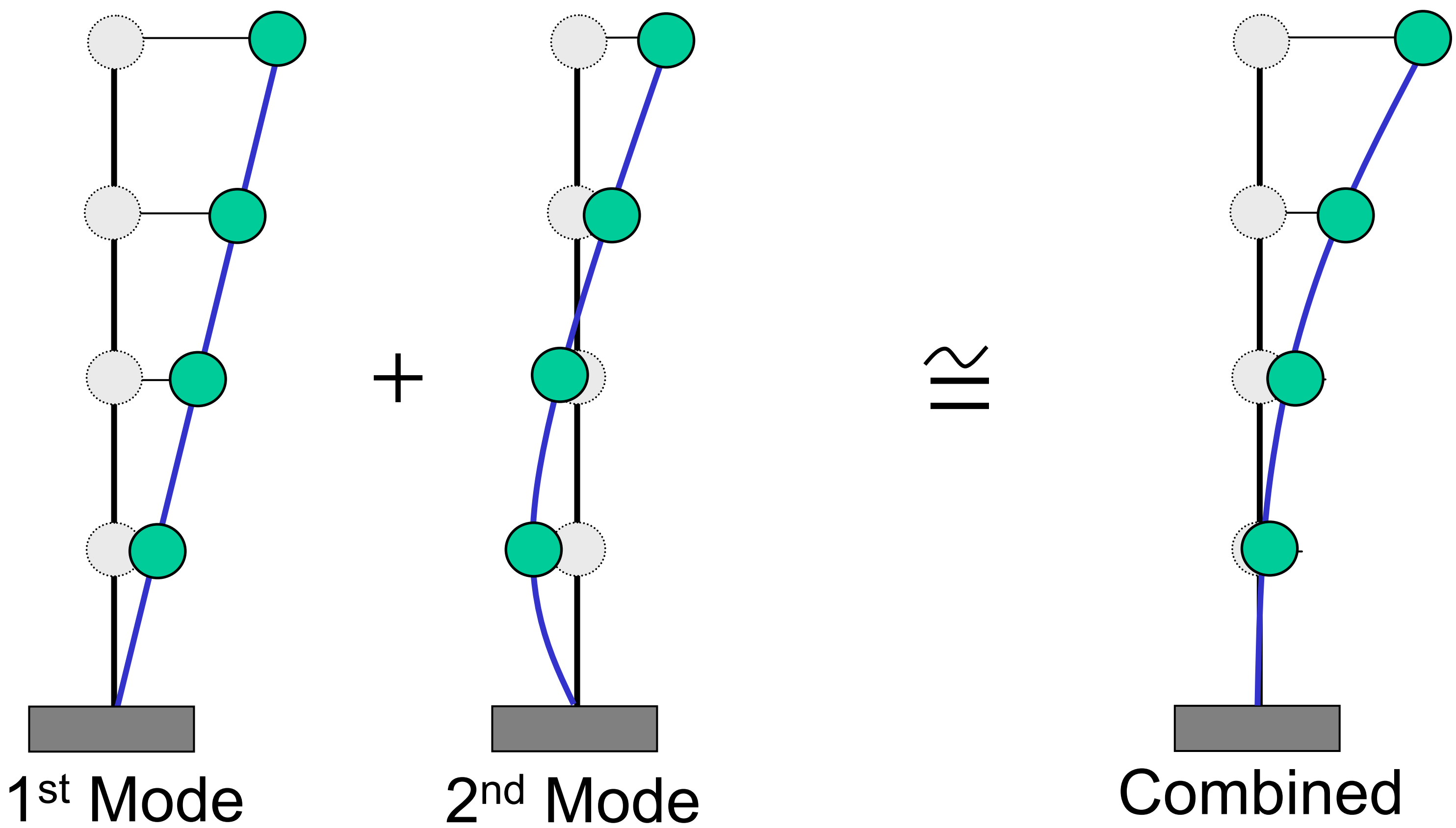
Units = inches

Time History (Envelope)	Modal Response Spectrum	**ELF**
$\begin{Bmatrix} 5.15 \\ 3.18 \\ 1.93 \end{Bmatrix}$	$\begin{Bmatrix} 5.18 \\ 3.33 \\ 1.84 \end{Bmatrix}$	$\begin{Bmatrix} 5.98 \\ 3.89 \\ 1.82 \end{Bmatrix}$

ELF Procedure Example (Summary)

- ELF procedure gives **good correlation** with base shear (327 kips ELF vs 331 kips modal response spectrum).
- ELF story force distribution is **not** as **good**. ELF underestimates shears in upper stories.
- ELF gives reasonable correlation with displacements.

Equivalent Lateral Force Procedure
Higher Mode Effects



ASCE 7-05 ELF Approach

- Uses empirical period of vibration
- Uses smoothed response spectrum
- Has correction for higher modes
- Has correction for overturning moment
- Has limitations on use

Approximate Periods of Vibration

$$\boxed{T_a = C_t h_n^x}$$

C_t = 0.028, x = 0.8 for steel moment frames
C_t = 0.016, x = 0.9 for concrete moment frames
C_t = 0.030, x = 0.75 for eccentrically braced frames
C_t = 0.020, x = 0.75 for all other systems
Note: For building structures **only**!

$$\boxed{T_a = 0.1N}$$

For moment frames < 12 stories in height, minimum story height of 10 feet. N = number of stories.

Adjustment Factor on Approximate Period

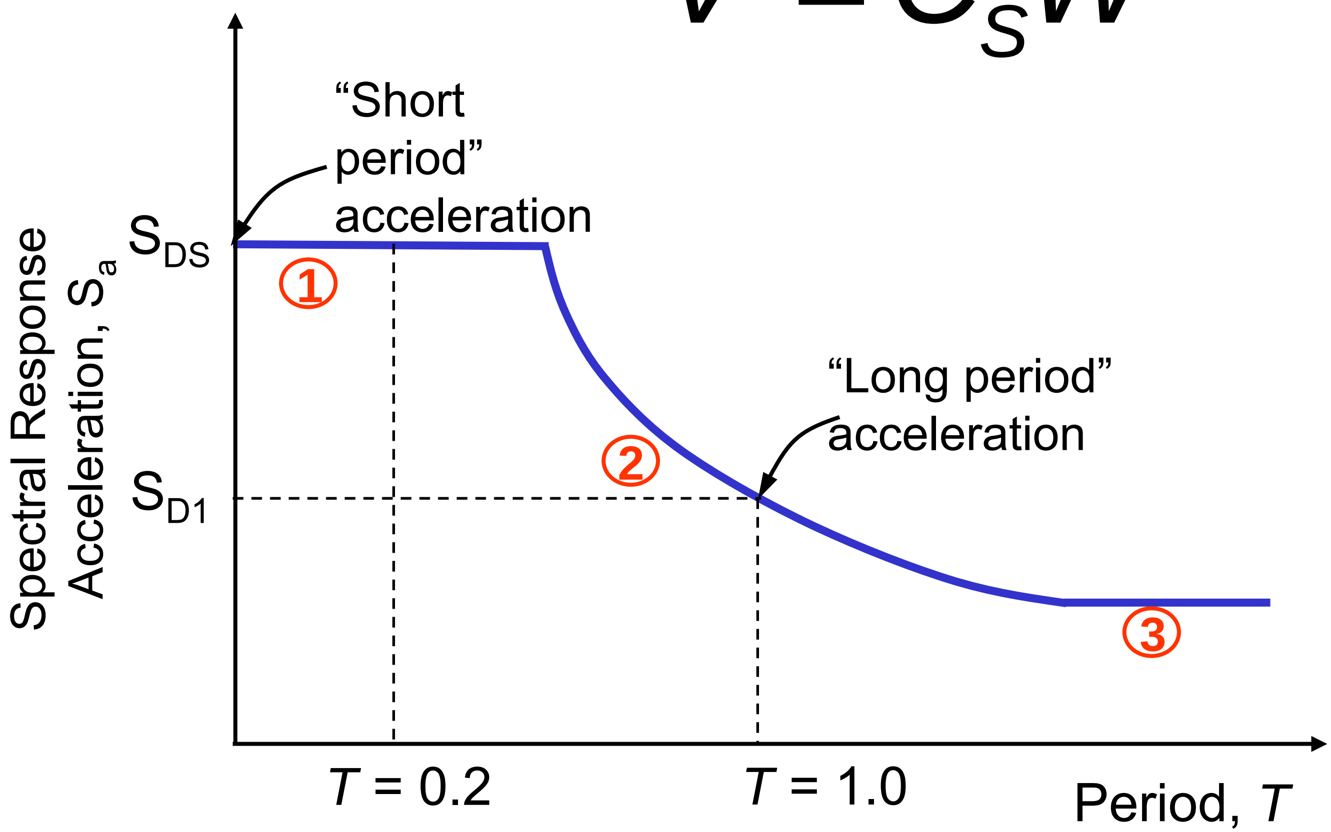
$$T = T_a C_u \leq T_{computed}$$

S_{D1}	C_u
> 0.40g	1.4
0.30g	1.4
0.20g	1.5
0.15g	1.6
< 0.1g	1.7

Applicable **only** if $T_{computed}$ comes from a "properly substantiated analysis."

ASCE 7 Smoothed Design Acceleration Spectrum (for Use with ELF Procedure)

$$V = C_S W$$



1 $C_S = \dfrac{S_{DS}}{\left(\dfrac{R}{I}\right)}$

2 $C_S = \dfrac{S_{D1}}{T\left(\dfrac{R}{I}\right)}$

3 **Varies**

R is the *response modification factor*, a function of system inelastic behavior. This is covered in the topic on inelastic behavior. For now, use R = 1, which implies linear elastic behavior.

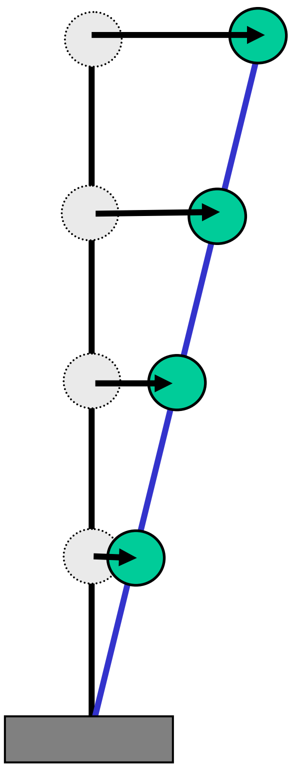
I is the *importance factor* which depends on the Seismic Use Group. I = 1.5 for essential facilities, 1.25 for important high occupancy structures, and 1.0 for normal structures. For now, use I = 1.

Distribution of Forces Along Height

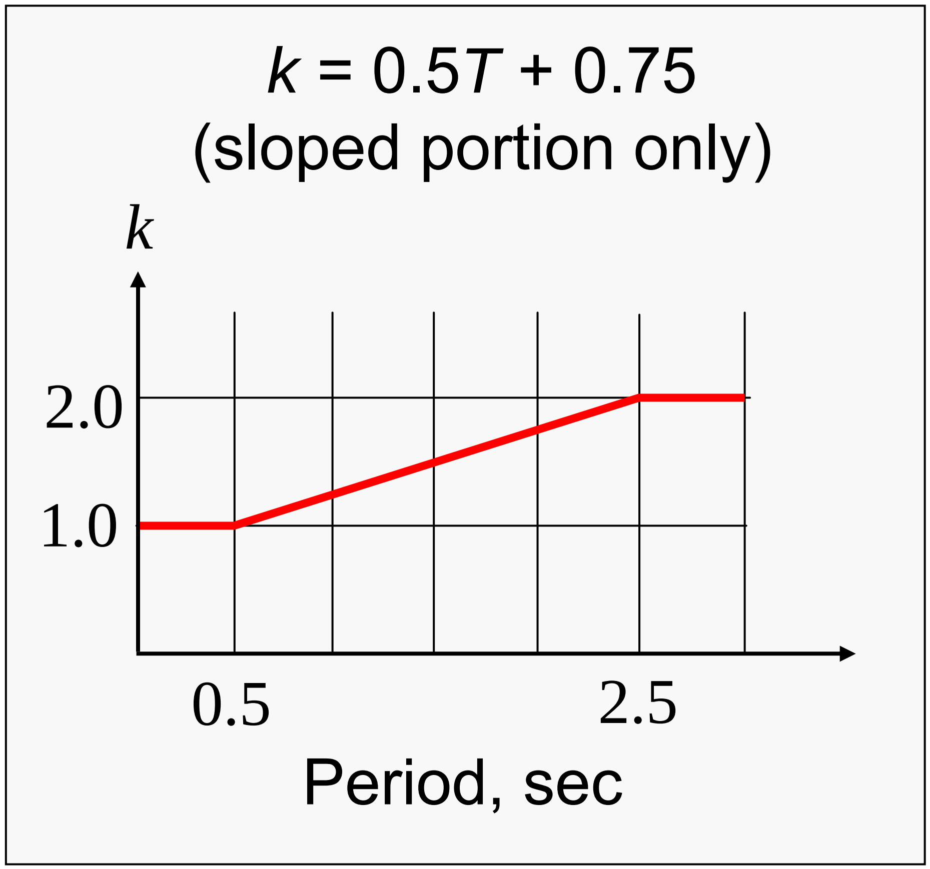
$$F_x = C_{vx} V$$

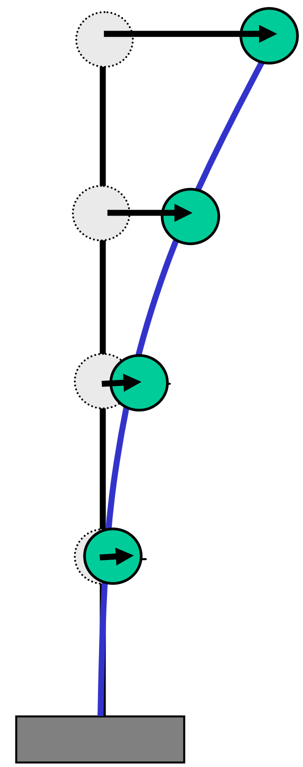
$$C_{vx} = \frac{w_x h_x^k}{\sum_{i=1}^{n} w_i h_i^k}$$

k Accounts for Higher Mode Effects



$k = 1$

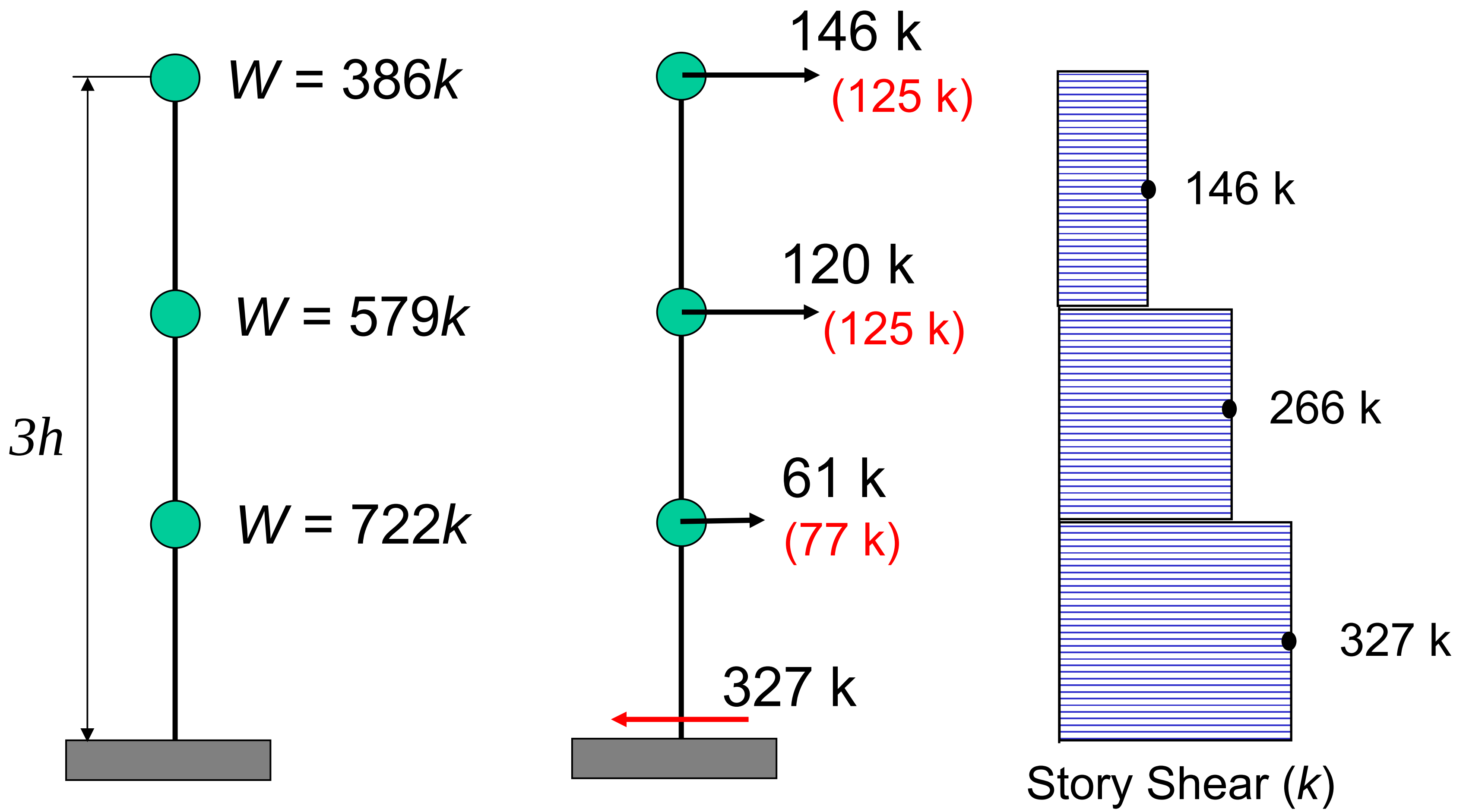




$k = 2$

ELF Procedure Example (Story Forces)

V = 327 kips T = 1.37 sec k = 0.5(1.37) + 0.75 = 1.435



ASCE 7 ELF Procedure Limitations

- Applicable **only** to "regular" structures with T less than $3.5T_s$. Note that $T_s = S_{D1}/S_{DS}$.
- Adjacent story stiffness does not vary more than 30%.
- Adjacent story strength does not vary more than 20%.
- Adjacent story masses does not vary more than 50%.

If violated, must use more advanced analysis (typically modal response spectrum analysis).

ASCE 7 ELF Procedure
Other Considerations Affecting Loading

- Orthogonal loading effects
- Redundancy
- Accidental torsion
- Torsional amplification
- P-delta effects
- Importance factor
- Ductility and overstrength