My Fifty Years with Finite Elements

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My Fifty Years with Finite Elements

Outline: Presentation summarizes:

● Historical overview
  – Early FEM developments by R.W. Clough
  – Berkeley-Swansea connection.

● Some of the work of people who have influenced FEM and me!
  – Near incompressibility treatment
  – Time integration algorithms
  – Computational mechanics at UC
  – Finite deformation

● Some challenges I see today.
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Early History
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Berkeley Campus 1940
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UC Engineering Buildings ~1960
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Karl Pister's 1962-63 Graduates!

1962

1995
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- **FEM: The engineering beginning**

- Clough spent summers of 1952-53 and 1953-54 at Boeing.


- Credits Turner with idea of *elements* to determine frequencies of Delta wing aircraft.

- Names FEM in 1960

- My first class in 1957!

R.W. Clough (1956)
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1956: First Berkeley and Engineering FEM paper.

JOURNAL OF THE
AERONAUTICAL SCIENCES

VOLUME 23  SEPTEMBER, 1956  NUMBER 9

Stiffness and Deflection Analysis of Complex Structures

M. J. TURNER,* R. W. CLOUGH,† H. C. MARTIN,‡ AND L. J. TOPP**
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- FEM: The beginning
  - Direct physical construction: Plane stress elasticity

(a) Triangle and geometry  
(b) Uniform stress state

- Linear displacements: Constant Strain
  Nodal forces by equilibrium.
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- FEM: Results from first paper

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**Fig. 12.** Clamped rectangular plate subjected to uniform tensile loading.

**Fig. 13.** Nodes and supports for clamped rectangular plate.
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• Earlier contribution from mathematics.

• 1941 presentation to American Math Society.


• Solved Laplace equation problem (torsion).

• Not known to engineers in mid-1950’s.

R. Courant (1941)
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FEM Software Development:

- In addition to theoretical studies, FEM programs written.
- 1956: First campus machine IBM 701 – in Cory Hall.
- Early program in assembly code (before FORmula TRANslator developed by Backus, et al. at IBM – 1954-57).
- Ed Wilson (Clough’s student) prepared first UC program in FORTRAN II (1958) using IBM 704.
- Wilson’s programs formed basis of all our early efforts.
- All early programs used 3-node triangle as basic element.
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Early Computing Environment

Keypunch room

IBM Key Punch
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- See Wilson’s web page for more on early FEM research by Clough:
  
  http://www.edwilson.org

EARLY FINITE ELEMENT RESEARCH AT BERKELEY

by

Ray W. Clough
Nishkian Professor of Structural Engineering, Emeritus
University of California, Berkeley

and

Edward L. Wilson
T. Y. Lin Professor of Structural Engineering, Emeritus
University of California, Berkeley
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Nearly Incompressible Analyses
1960-68: Solid propellant rocket analyses

Propellant materials had properties making nearly incompressible ($\nu \approx 0.5$) and time dependent

Displacement method gave poor results for $\nu > 0.4$.

Improved using mixed methods – displacement/pressure

Based on paper by: Herrmann & Toms: J. Appl. Mech, 1964

Used 4-triangle quadrilateral with constant pressure.

Developed 2-d programs for elastic and thermoviscoelastic materials.
Constitutive form: (L.R. Herrmann)

\[ \sigma_{ij} = 2\mu \left[ \epsilon_{ij} + (\nu H - e_T) \delta_{ij} \right] \]

\[ H = \frac{3\sigma_{kk}}{2\mu(1 + \nu)} \]

\[ \epsilon_{kk} - e_T = (1 - 2\nu)H \]

Elements (composite)

Interpolations: \( u_i \) linear, \( H \) constant. (Before BB papers!)
THERMOMECHANICAL ANALYSIS OF VISCOELASTIC SOLIDS

ROBERT L. TAYLOR
Associate Professor of Civil Engineering

KARL S. PISTER
Professor of Engineering Science

and

GERALD L. GOUDREAU
Graduate Student in Civil Engineering
University of California, Berkeley
Thermoviscoelasticity:

- Constitution: Spherical/Deviatoric split

\[
\sigma_{ij} = p \delta_{ij} + s_{ij} \quad ; \quad \epsilon_{kl} = \theta \delta_{kl} + e_{kl}
\]

\[
p = 3 K (\theta - \alpha \Delta T) \quad ; \quad s_{ij} = 2 G \int_{-\infty}^{t} G(\xi(t) - \xi(\tau), T_0) \frac{\partial e_{ij}}{\partial \tau} \, d\tau
\]

- Thermorheologically simple

\[
\xi(t) = \int_{0}^{t} \phi(T(t')) \, dt'
\]

- Relaxation function: Prony series representation

\[
G(\xi) = G \left[ \mu_0 + \sum_{m=1}^{M} \mu_m \exp(\xi/\lambda_m) \right] \quad ; \quad \mu_0 + \sum_{m=1}^{M} \mu_m = 1
\]
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- Integration of Prony series form of constitution

\[
s = 2G \left[ \mu_0 e + \sum_{m=1}^{M} \mu_m q^m \right]
\]

- Recursion for increment \( t_n \) to \( t_{n+1} \)

\[
q^m = \int_{-\infty}^{t} \exp[-(\xi(t) - \xi(\tau))/\lambda_m] \frac{\partial e}{\partial \tau} \, d\tau
\]

\[
 \approx \exp[-\delta \xi_{n+1}/\lambda_m] \left[ e_{n+1} - e_n \right] + \Delta q_{n+1}^m \left[ e_{n+1} - e_n \right]
\]

where \( \Delta \xi_{n+1} = \xi(t_{n+1}) - \xi(t_n) \approx \frac{1}{2} \left( \phi(T_n) + \phi(T_{n+1}) \right) \Delta t \):

\[
\Delta q_{n+1}^m = \frac{\lambda_m}{\Delta \xi_{n+1}} \left[ 1 - \exp(-\Delta \xi_{n+1}/\lambda_m) \right]
\]

\[
= 1 - \frac{1}{2} \left( \frac{\Delta \xi_{n+1}}{\lambda_m} \right) + \frac{1}{3!} \left( \frac{\Delta \xi_{n+1}}{\lambda_m} \right)^2 - \frac{1}{4!} \left( \frac{\Delta \xi_{n+1}}{\lambda_m} \right)^3 + \ldots
\]
Example: Thin-walled cylinder

- Normalized temperature

\[ \bar{\theta}(x, \rho) = (1 - x)[1 - \exp(-2\rho)] \]

where \( \rho \) is normalized time, is \( x \) normalized thickness distance

- Properties

\[ K = 2.5 \times 10^{10} \]
\[ G = 8.3 \times 10^9 \]
\[ \mu_0 = 0.001 \]
\[ \mu_1 = 0.999 \]

- Shift function for polymethylmethacrylate

\[ \phi(\bar{\theta}) = 3981.1 \exp[-6.2172(1 - \bar{\theta})(1.3333 + \bar{\theta} + 1.095 \bar{\theta}^2)] \]
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Connections to Swansea
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• 1965-70: FEM Major Advancements
  • Thin/thick plates & shells (Zienkiewicz, et al. – 1965ff)

• Swansea clearly an active FEM research location!

• 1968: Introduced to Zienkiewicz by Clough on airplane to 2nd Wright-Patterson conference.

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Berkeley–Swansea Connection

• 1967: First FEM book

• 1977: 3rd edition – our first joint effort.
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Swansea in 1969
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Swansea – Shells work

REDUCED INTEGRATION TECHNIQUE IN GENERAL ANALYSIS OF PLATES AND SHELLS

O. C. ZIENKIEWICZ*
University of Wales, Swansea

R. L. TAYLOR†
University of California, Berkeley, California

AND

J. M. TOO‡
University of Wales, Swansea
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Swansea: Reduced Integration

Figure 2. (a) Parabolic isoparametric hexahedron; (b) degeneration to a shell element

Figure 3. A simply supported square plate under uniform load \( q_p \); (a) Plot of central deflection for element of Reference 1; (b) with reduced transverse shear integration; (c) displacement at centre versus thickness parameter
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The first Swansea year

- First FEAP
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Time Integration Developments
EVALUATION OF NUMERICAL INTEGRATION METHODS IN ELASTODYNAMICS

G.L. GOUUDREAU
Engineer, Lawrence Livermore Laboratory

and

R.L. TAYLOR
Associate Professor, Department of Civil Engineering
University of California, Berkeley
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Error and stability analysis of discrete elastic problem

- Modal equation (elastic and undamped) \( \ddot{d} + \omega^2 d = f \)
- Newmark
  \[ d_{n+1} = d_n + \Delta t v_n + \left( \frac{1}{2} - \beta \right) \Delta t^2 a_n + \beta \Delta t^2 a_{n+1} \]
  \[ v_{n+1} = v_n + (1 - \gamma) \Delta t a_n + \gamma \Delta t a_{n+1} \]
- Let \( \theta = \omega \Delta t; \delta = \gamma - \frac{1}{2} \) and \( \alpha^2 = \theta^2/(1 + \beta \theta^2) \)
- Eliminate velocity and acceleration and assume \( d_k = \lambda^k \) gives
  \[ \lambda^2 - (2 - \alpha^2 - \delta \alpha^2) \lambda + (1 - \delta \alpha^2) = 0 \]
  for homogeneous equation.
- Solution: \( \lambda = (1 - \alpha^2 \delta)^{1/2} \exp(\pm ia) \) where
  \[ a = \tan^{-1} \left[ \frac{\alpha \sqrt{1 - \frac{1}{4} \alpha^2 (1 + \delta)^2}}{1 - \frac{1}{2} \alpha^2 (1 + \delta)} \right] \]
Results:

- If $\gamma < \frac{1}{2}$ (or $\delta < 0$): Negative damping, **unstable**.
- If $\gamma > \frac{1}{2}$ (or $\delta > 0$): Positive damping
- For oscillatory response $1 + \frac{1}{4}\alpha^2(1 + \delta)^2 \geq 0$
- Gives stability limit on $\Delta t$: $\theta \leq \left[\frac{1}{4}(1 + \delta)^2 - \beta\right]^{-1/2}$
- **Unconditional stability** requires: $\beta \geq \frac{1}{4}(1 + \delta)^2$
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EARTHQUAKE ENGINEERING & STRUCTURAL DYNAMICS, 5, (1976) 283-292

IMPROVED NUMERICAL DISSIPATION FOR TIME INTEGRATION ALGORITHMS IN STRUCTURAL DYNAMICS

H.M. HILBER, T.J.R. HUGHES and R.L. TAYLOR
University of California, Berkeley
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- **HHT - Algorithm**

\[
\begin{align*}
M_{n+1}a_{n+1} + C_{n+1}v_{n+1} + (1 - \alpha)Kd_{n+1} &= F_{n+1} + \alpha Kd_n \\
d_{n+1} &= d_n + \Delta t v_n + \left(\frac{1}{2} - \beta\right) \Delta t^2 a_n \\
&\quad + \beta \Delta t^2 a_{n+1} \\
v_{n+1} &= v_n + (1 - \gamma) \Delta t a_n + \gamma \Delta t a_{n+1}
\end{align*}
\]

- **Parameters:**

\[
\begin{align*}
\beta &= \frac{1}{4} (1 - \alpha)^2 \\
\gamma &= \frac{1}{2} - \alpha \\
-\frac{1}{3} &\leq \alpha \leq 0
\end{align*}
\]
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Inelastic and Finite Deformation Problems
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- Inelastic, contact & finite deformation developments. (with T.J.R. Hughes, W. Kanok-nukulchia, & A. Curnier)

- First UC Computational Mechanics Course (Spring 1975) (Taught by: K.S. Pister & T.J.R. Hughes)

- Established much of notation and methods we use today
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The Juan Simo Years
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- J.C. Simo (1952-1994)
- 1981-94: Interactions with many!
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• 1981-94: Interactions with J.C. Simo

  • Developed method of solution for:
    • Integration of plasticity (plane strain & plane stress);
    • Enhanced strain elements;
    • Elasticity & viscoelasticity constitution in principal stretches;
    • Flexible-rigid body solutions;
    • Energy-momentum conserving integration methods;
  
  • Contributed to development of FEAP for finite deformation
Material Modeling: Elasto-Plastic

- Developed algorithm for $J_2$ (Mises) plasticity
- Included isotropic and kinematic hardening
- Linearized return map algorithm
- Unaware Hibbitt had done perfect plasticity in Abaqus
- Later did plane stress case also.
Material Modeling: $J_2$ Elasto-Plastic Model

- Graphically, return map for $J_2$ form is
### Table 5
Energy norm values for step 4

<table>
<thead>
<tr>
<th>Iteration</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<tr>
<td>Continuum</td>
<td>0.14e + 2</td>
<td>0.80e - 2</td>
<td>0.61e - 3</td>
<td>0.18e - 3</td>
<td>0.89e - 4</td>
<td>0.47e - 7</td>
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<td>Consistent</td>
<td>0.14e + 2</td>
<td>0.11e - 1</td>
<td>0.77e - 4</td>
<td>0.10e - 9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Iteration</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>Continuum</td>
<td>0.27e - 4</td>
<td>0.16e - 4</td>
<td>0.97e - 5</td>
<td>0.59e - 5</td>
<td>0.36e - 5</td>
<td>0.22e - 5</td>
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<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Iteration</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
</tr>
<tr>
<td>Continuum</td>
<td>0.13e - 5</td>
<td>0.85e - 6</td>
<td>0.52e - 6</td>
<td>0.32e - 6</td>
<td>0.20e - 6</td>
<td>0.12e - 6</td>
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<tr>
<td>Consistent</td>
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<td></td>
<td></td>
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<tr>
<td>Iteration</td>
<td>19</td>
<td>20</td>
<td>21</td>
<td>22</td>
<td>23</td>
<td></td>
</tr>
<tr>
<td>Continuum</td>
<td>0.77e - 7</td>
<td>0.47e - 7</td>
<td>0.29e - 7</td>
<td>0.18e - 7</td>
<td>0.11e - 7</td>
<td></td>
</tr>
<tr>
<td>Consistent</td>
<td></td>
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</tr>
</tbody>
</table>
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Material Modeling: Finite Elasticity
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Today
By mid 1990’s FEM for solids was fairly well established

- Solve finite deformation solids, rods & shells
  - Treat near incompressibility; integrate inelastic constitutive models, etc.
- Sparse solvers and eigen-problem methods available.
- Research and commercial software available.
- Personal computer/workstation costs reasonable.

- Thermal, Fluids, Electro-magnetics solvers also available.
Some Challenges for Today (and Tomorrow!)

- Multi-physics: Coupling of solids to multiple inputs.
  - Thermal; Electro-magnetics; Chemistry; Fluids, etc

- Multi-scale
  - Coupling between continuum-scale; meso-scale; etc.

- Automated analysis
  - From solid models to results with minimal user intervention
    - Challenges for element technology (tetrahedra)
    - Robust solvers (iterative & non-linear)
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Summary:

- Described some of what I have witnessed in the last 50 years.

- Key contributions always occurred in collaboration with others!

- *FEAP* remains my "hobby" – but still FEM is a lot of fun!

- I thank all (mentioned and unmentioned) I have had an opportunity to know during the last 50 years.

- Much accomplished, but much to do.

- I look forward to many more years of learning!
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Thank you for your attention!