Giuseppe Pelosi
University of Florence
Via C. Lombroso 6/17
I-50134 Florence, Italy
Tel: 055-4796-759
Fax: 055-4796-767
E-mail: giuseppe.pelosi@unifi.it, g.pelosi@ieee.org

The Finite-Element Method, Part I: R. L. Courant

G. Pelosi

Department of Electronics and Telecommunications
University of Florence, Via C. Lombroso 6/17, 50134, Florence, Italy
E-mail: Giuseppe.pelosi@unifi.it

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Foreword

This time, the historical corner of the *IEEE Antennas and Propagation Magazine* is focused on finite elements. It is organized in two parts. The first part, published in this issue, contains a brief biography of Richard L. Courant, the father of finite elements. It also contains the appendix "Numerical Treatment of the Plane Torsion Problem for Multiply-Connected Domains," included in the R. L. Courant paper, "Variational Methods for the Solution of Problems of Equilibrium and Vibration" [1]. This appendix was the first official appearance of the Finite-Element technique. The second part, which will be published in the next *Magazine* issue, in June, contains a paper by Ron Ferrari (Cambridge University, England) entitled "P. P. Silvester (1935-1996) – An Innovator in Electromagnetic Numerical Modeling." This article originated from the special session organized at the 8th International Workshop on Finite Elements for Microwave Engineering, held in Stellenbosch, South Africa, in May 2006 (see also the report by D. B. Davidson [2]).

The Finite-Element Method, in its presently accepted forms, can be credited to no lesser a person than Richard L. Courant. When he prepared the published version of his 1942 address to the American Mathematical Society, he added a two-page appendix to show, by example, how the variational methods first described by Lord Rayleigh could be put to wider use in potential theory. Using piecewise-linear approximants on a set of triangles, which he called "elements," he dashed off a few two-dimensional examples – and the Finite-Element Method was born. It remained dormant for half a generation, probably waiting for computers to be invented. Courant's approach, amplified by the mathematical ideas of J. L. Synge [3], next appeared in the work of R. J. Duffin [4].

Finite-element activity in electrical engineering began in earnest another decade later, about 1968-1969. A paper giving the finite-element formulation of the classical hollow-waveguide problem was presented at the 1968 URSI symposium, and subsequently published in the Italian technical journal *Alta Frequenza* [5]. A concurrent but independent paper by P. L. Arlett, A. K. Bahrani, and O. C. Zienkiewicz [6] addressed waveguides and cavities, but unfortunately based the development on an incorrect formulation of the electromagnetic field problem. There followed a rapid succession of papers on magnetostatics, dielectrically loaded waveguides [7], and other well-known boundary-value problems of electromagnetics. The method was quickly applied to integral operators as well, both in microwave devices and wire antenna problems.

The approach to finite elements throughout the early period was variational. This made it relatively hard to follow for classically trained engineers, few of whom had any training or knowledge of variational methods. In spite of this difficulty, finite-element methods spread quickly in the eighties, and assumed a dominant role in several technical areas. One way of assessing the growth of finite-element applications is to examine the INSPEC and COMPENDEX bibliographic databases over the 1968-2005 period. In 1968, finite-element papers with electrical engineering content amounted to a small handful. By 2005, the total had reached about 1600, with some hundreds of articles being added each year. Figure 1 shows the growth of the finite-element litera-

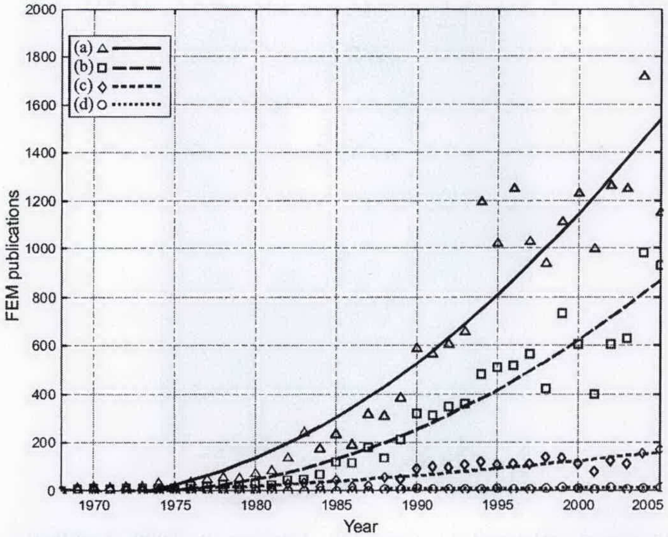

Figure 1. The annual production of finite-element papers in electrical engineering, 1969–2005: (a) INSPEC + COMPENDEX databases, all publications; (b) INSPEC + COMPENDEX databases, journal articles; (c) IEEE journal articles; (d) IEE journal articles.

ture of electrical engineering, as recorded in the INSPEC database. As will be evident from this graph, the rate of publication (i.e., the annual output of articles) has doubled about every four years since 1969, although it has slowed somewhat since the turn of the last decade.

R.L. Courant: "Numerical Treatment of the Plane Torsion Problem for Multiply-Connected Domains"

Figure 2 is the appendix, "Numerical Treatment of the Plane Torsion Problem for Multiply-Connected Domains," from [1], dealing with finite elements. §I, 2a (cited in the first line) defines S as the *"total stiffness"* of the column with respect to torsion, $S = -\iint u\,dxdy$, and u is the value of the unknown function, ϕ. This leads to the minimum value of the functional $D(\phi) = \iint \left(\phi_x^2 + \phi_y^2 + 2\phi\right) dxdy$ with prescribed values on cross-section contours. The function u gives the stress in the cross section of the column by differentiation.

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APPENDIX[8]

NUMERICAL TREATMENT OF THE PLANE TORSION PROBLEM FOR MULTIPLY-CONNECTED DOMAINS

The computation of the stiffness S defined in §I, 2a furnishes an example of independent interest which permits to compare the practical merits of some of the methods described in this address. Numerical calculations were carried out for the cross sections of the following diagrams, a square from which a smaller square is cut out; and a square, from which four squares are cut out. In the first case our quadratic frame was supposed to be bounded by the lines $x = \pm 1$, $y = \pm 1$ and $x = \pm 3/4$, $y = \pm 3/4$. To apply the Rayleigh-Ritz method for the domain as a whole would already be cumbersome because of the boundary conditions for admissible functions ϕ. However, this difficulty disappears if we exploit the symmetry of the domain and the resulting symmetry of the solution; thus we may confine ourselves to considering only one-eighth of the domain B^*, namely the quadrangle $ABCD$. For this polygon any function of the type

$$\phi = a(1 - x)[1 + (x - 3/4)P]$$

where $P(x, y)$ is a polynomial, is admissible, and its substitution in the integral leads to simple linear equations for the cofficients. Thus for the simplest attempt

$$\phi = a(1 - x)$$

which leaves only one constant a to be determined, we find with a

[8] Addition not contained in the original address.

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negligible amount of numerical labor $S = .339$ and $c = -.11$. A refined attempt with the function

$$\phi = a(1 - x)[1 + \alpha(x - 3/4)y]$$

yielded $S = .340$ and $c = -.109$ with little more labor.

These results were checked with those obtained by our generalized method of finite differences where arbitrary triangular nets are permitted. The diagrams are self-explanatory. Unknown are the

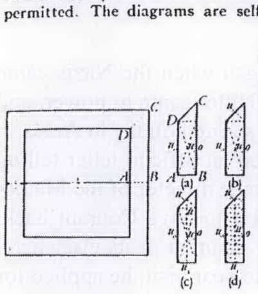

FIG. 2

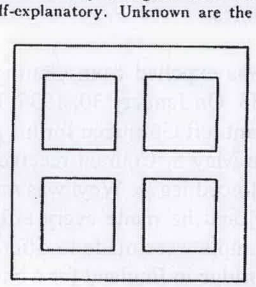

FIG. 3

net-point-values u_i, $(c = u_0)$. In the net-triangles our functions were chosen as linear, so that the variational problem results in linear equations for the u_i. The results, easily obtainable, were: case (a) with two unknowns: $S = .344$, $u_0 = -.11$; case (b) with three unknowns: $S = .352$, $u_0 = -.11$; case (c) with five unknowns $S = .353$, $u_0 = -.11$; case (d) with nine unknowns, corresponding to the ordinary difference method $S = .353$, $u_0 = -.11$.

These results show in themselves and by comparison that the generalized method of triangular nets seems to have advantages. It was applied with similar success to the case of a square with four holes, and it is obviously adaptable to any type of domain, much more so than the Rayleigh-Ritz procedure in which the construction of admissible functions would usually offer decisive obstacles.

In a separate publication it will be shown how the method can be extended also to problems of plates and to other problems involving higher derivatives.

Of course, one must not expect good local results from a method

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using so few elements. However, it might be expected that a smooth interpolation of the net functions obtained will yield functions which themselves with their derivatives are fairly good approximations to the actual quantities.

Figure 2. The appendix, "Numerical Treatment of the Plane Torsion Problem for Multiply-Connected Domains," from [1].

Richard L. Courant

Richard (Figure 3) was the eldest child of Siegmund and Martha Courant. When Richard was nine, his family moved to Breslau. Richard attended school there, entering the König-Wilhelm Gymnasium. With little previous education, Richard struggled at school at first, even being "less than satisfactory" at arithmetic. When Richard was fourteen years old, he began to tutor to make enough money to support himself. Soon after this, a tragedy struck the family when Jakob, Richard's uncle, in severe business difficulties, committed suicide; shortly after, Siegmund was declared bankrupt. The following two years must have been times of extreme difficulty for Richard, attending school and still supporting himself tutoring.

Courant had studied at Breslau with two fellow students, Otto Toeplitz and Ernst Hellinger. These two, several years older than Courant and further on with their education, were by this stage studying at Göttingen. They wrote to Courant, telling him how exciting it was there, particularly because of Hilbert. In the spring of 1907, Courant left Breslau, spent a semester at Zurich, and then began his studies at Göttingen on November 1, 1907.

R. L. Courant received his doctorate from the University of Göttingen in 1910 under David Hilbert. For the next four years, he taught mathematics at Göttingen. Then at age 26, his teaching and research were interrupted by the onset of World War I. A few years after the war, Courant returned to Göttingen, where he founded and directed (from 1922 to 1933) the University's Mathematics Institute. The Institute was just a name and a group of people until 1927, when a building was constructed. While at the institute, he wrote a two-volume treatise with D. Hilbert, *Methoden der mathematischen Physik* (1924), later translated into English as *Methods of Mathematical Physics*, a work that furthered the evolution of quantum mechanics.

Courant was expelled from Göttingen when the Nazis came to power in 1933. On January 30, 1933, Hitler came to power, and in March, Courant left Göttingen for his spring holiday in Arosa in Switzerland. On May 5, Courant received an official letter telling him he was on forced leave. Weyl was made director of the Mathematics Institute, and he made every effort to have Courant back. Meanwhile, attempts were made to offer Courant posts elsewhere. Invited to Cambridge in England for a one-year visit, he applied for leave (slightly strange that he had to do so, since he was already on forced leave). His forced leave was changed to ordinary leave, and Courant left for England, going to New York University the following year.

The first few months in New York were difficult. He was poorly paid, there were no mathematicians of quality on the staff, and the students he taught he found very badly prepared. In 1936, he was offered a professorship at New York University, and given the task of building a graduate center. Again, he had a mostly successful collaboration with his students. In 1940-41, he worked on a new book with Herbert Robbins, a young topologist from Harvard. This book was *What is Mathematics?* and it records Courant's views of mathematics.

Figure 3. Richard L. Courant, January 8, 1888, Lublinitz, Prussia, Germany (now Lubliniec, Poland) – January 27, 1972, New Rochelle, New York, USA.

Courant suffered a stroke on November 19, 1971, and was taken to a hospital in New Rochelle, where he died two months later.

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