HISTORICAL COMMENTS ON FINITE ELEMENTS

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1. INTRODUCTION

Finite elements; perhaps no other family of approximation methods has had a greater impact on the theory and practice of numerical methods during the twentieth century. Finite element methods have now been used in virtually every conceivable area of engineering that can make use of models of nature characterized by partial differential equations.

Why have finite element methods been so popular in both the engineering and mathematical community? I feel that a principal reason for the success and popularity of these methods is that they are based on the weak, variational, formulation of boundary and initial value problems. This is a critial property, not only because it provides a proper setting for the existence of very irregular solutions to differential equations (e.g. distributions), but also because the solution appears in the integral of a quantity over a domain. The simple fact that the integral of a measurable function over an arbitrary domain can be broken up into the sum of integrals over an arbitrary collection of almost disjoint subdomains whose union is the original domain, is a vital property. Because of it, the analysis of a problem can literally be made locally, over a typical subdomain, and by making the subdomain sufficiently small one can argue that polynomial functions of various degrees are adequate for representing the local behavior of the solution. This summability of integrals is exploited in every finite element program. It allows the analysts to focus their attention on a typical finite element domain and to develop an approximation independent of the ultimate location of that element in the final mesh.

The simple integral property also has important implications in physics and in most problems in continuum mechanics. Indeed, the classical balance laws of mechanics are global, in the sense that they are integral laws applying to a given mass of material, a fluid or solid. From the onset, only regularity of the primitive variables sufficient for these global conservation laws to make sense is needed. Moreover, since these laws are supposed to be fundamental axioms of physics, they must hold over every finite portion of the material: every finite element of the continuum. Thus once again, one is encouraged to think of approximate methods defined by integral formulations over typical pieces of a continuum to be studied.

2. THE ORIGIN OF FINITE ELEMENTS

When did finite elements begin? It is difficult to trace the origins of finite element methods because of a basic problem in defining precisely what constitutes a "finite element method". To most mathematicians, it is a method of piecewise polynomial approximation and, therefore, its origins are frequently traced to the appendix of a paper by COURANT [1943] in which piecewise linear approximations of the Dirichlet problem over a network of triangles is discussed. Also, the "interpretation of finite differences" by POLYA [1952] is regarded as embodying piecewiese-polynomial approximation aspects of finite elements.

On the other hand, the approximation of variational problems on a mesh of triangles goes back much further: 92 years. In 1851, SCHELLBACH [1851] proposed a finite-element-like solution to Plateau's problem of determining the surface S of minimum area enclosed by a given closed curve. SCHELLBACH used an approximation S_h of S by a mesh of triangles over which the surface was represented by piecewise linear functions, and he then obtained an approximation of the solution to Plateau's problem by minimizing S_h with respect to the coordinates of hexagons formed by six elements (see WILLIAMSON [1980]). Not quite the conventional finite element approach, but certainly as much a finite element technique as that of COURANT.

Some say that there is even an earlier work that uses some of the ideas underlying finite element methods: LEIBNIZ himself employed a piecewise linear approximation of the Brachistochrone problem proposed by BERNOULLI in 1696 (see the historical volume, LEIBNIZ [1962]). With the help of his newly developed calculus tools, LEIBNIZ derived the governing differential equation for the problem, the solution of which is a cycloid. However, most would agree that to credit this work as a finite element approximation is somewhat stretching the point. LEIBNIZ had no intention of approximating a differential equaiton; rather, his purpose was to derive one. Two and a half centuries later it was realized that useful approximations of differential equations could be determined by not necessarily taking infinitesimal elements as in the calculus, but by keeping the elements finite in size. This idea is, in fact, the basis of the term "finite elements".

There is also some difference in the process of laying a mesh of triangles over a domain on the one hand and generating the domain of approximation by piecing together triangles on the other. While these processes may look the same in some cases, they may differ dramatically in how the boundary conditions are imposed. Thus, neither SCHELLBACH nor COURANT, nor for that matter SYNGE who used triangular meshes many years later, were particularly careful as to how boundary conditions were to be imposed or as to how the boundary of the domain was to be modeled by elements, issues that are now recognized as an important feature of finite element methodologies. If a finite element method is one in which a global approximation of a partial differential equation is built up from a sequence of local approximations over subdomains, then credit must go back to the early papers of HRENNIKOFF [1941], and perhaps beyond, who chose to solve plane elasticity problems by breaking up the domain of the displacements into little finite pieces, over which the stiffnesses were approximated using bars, beams, and spring elements. A similar "lattice analogy" was used by McHENRY [1943]. While these works are draped in the most primitive physical terms, it is nevertheless clear that the methods involve some sort of crude piecewise linear or piecewise cubic approximation over rectangular cells. Miraculously, the methods also seem to be convergent.

To the average practitioner who uses them, finite elements are much more than a method of piecewise polynomial approximation. The whole process of partitioning of domain, assembling elements, applying loads and boundary conditions, and, of course, along with it, local polynomial approximation, are all components of the finite element method.

If this is so, then one must acknowledge the early papers of GABRIEL KRON who developed his "tensor analysis of networks" in 1939 and applied his "method of tearing" and "network analysis" to the generation of global systems from large numbers of individual components in the 1940's and 1950's (KRON [1939]; see also KRON [1953], [1955]). Of course, KRON never necessarily regarded his method as one of approximating partial differential equations; rather, the properties of each component were regarded as exactly specified, and the issue was an algebraic one of connecting them all appropriately together.

In the early 1950's, ARGYRIS [1954] began to put these ideas together into what some call a primitive finite element method: he extended and generalized the combinatoric method of KRON and other ideas that were being developed in the literature on system theory at the time, and added to it variational methods of approximation, a fundamental step toward true finite element methodology.

Around the same time, SYNGE [1956] described his "method of the hypercircle" in which he also spoke of piecewise linear approximations on triangular meshes, but not in a rich variational setting and not in a way in which approximations were built by either partitioning a domain into triangles or assembling triangles to approximate a domain (indeed Synge's treatment of boundary conditons was clearly not in the spirit of finite elements, even though he was keenly aware of the importance of convergence criteria and of the "angle condition" for triangles, later studied in some depth by others).

It must be noted that during the mid-1950's there was a number of independent studies underway which made use of "matrix methods" for the analysis of aircraft structures. A principal contributor to this methodology was LEVY [1953] who introduced the "direct stiffness method" wherein he approximated the structural behavior of aircraft wings using assemblies of box beams, torsion boxes, rods and shear panels. These assuredly represent some sort of crude local polynomial approximation in the same spirit as the HRENNIKOFF and McHENRY approaches. The direct stiffness method of LEVY had a great impact on the structural analysis of aircraft, and aircraft companies throughout the United States began to adopt and apply some variant of this method or of the methods of ARGYRIS to complex aircraft structural analyses. During this same period, similar structural analysis methods were being developed and used in Europe, particularly in England, and one must mention in this regard the work of TAIG [1961] in which shear lag in aircraft wing panels was approximated using basically a bilinear finite element method of approximation. Similar element-like approximations were used in many aircraft industries as components in various matrix-methods of structural analyses. Thus the precedent was established for piecewise approximations of some kind by the mid-1950's.

To a large segment of the engineering community, the work representing the beginning of finite elements was that contained in the pioneering paper of TURNER, CLOUGH, MARTIN, and TOPP [1956] in which a genuine attempt was made at both a local approximation (of the partial differential equations of linear elasticity) and the use of assembly strategies essential to finite element methodology. It is interesting that in this paper local element properties were derived without the use of variational principles. It was not until 1960 that CLOUGH [1960] actually dubbed these techniques as "finite element methods" in a landmark paper on the analysis of linear plane elasticity problems.

The 1960's were the formative years of finite element methods. Once it was perceived by the engineering community that useful finite element methods could be derived from variational principles, variationally based methods significantly dominated all the literature for almost a decade. If an operator was unsymmetric, it was thought that the solution of the associated problem was beyond the scope of finite elements, since it did not lend itself to a traditional extremum variational approximation in the spirit of RAYLEIGH and RITZ.

Many workers in the field feel that the famous Dayton conferences on finite elements (at the Air Force Flight Dynamics Laboratory in Dayton, Ohio, U.S.A.) represented landmarks in the development of the field (see PRZEMINIECKI et al. [1966]). Held in 1965, 1968, 1970, these meetings brought specialists from all over the world to discuss their latest triumphs and failures, and the pages of the proceedings, particularly the earlier volumes, were filled with remarkable and innovative accomplishments from a technical community just beginning to learn the richness and power of this new collection of ideas. In these volumes one can find many of the premier papers of now well-known methods. In the first volume alone one can find mixed finite element methods (HERRMANN [1966], Hermite approximations (PESTEL [1966]), C^1-cubic approximations (BOGNER, FOX, and SCHMIT [1966]) hybrid methods (PIAN [1966]) and other contributions. In later volumes, further assaults on nonlinear problems and special element formulations can be found.

Near the end of the sixties and earlv seventies there finally emerged the realization that the method could be applied to unsymmetric operators without difficulty and thus problems in fluid mechanics were brought within the realm of application of finite element methods; in particular, finite element models of the full Navier-Stokes equations were first presented during this period (ODEN [1969], ODEN and SOMOGYI [1969], ODEN [1970]).

The early textbook by ZIENKIEWICZ and CHANG [1967] did much to popularize the method with the practicing engineering community. However, the most important factor leading to the rise in popularity during the late 1960's and early 1970's was not purely the publication of special formulations and algorithms, but the fact that the method was being very successfully used to solve difficult engineering problems. Much of the techology used during this period was due to BRUCE IRONS, who with his colleagues and students developed a multitude of techniques for the successful implementation of finite elements. These included frontal solution technique (IRONS [1970]), the patch test (IRONS and RAZZAQUE [1972]), isoparametric elements

(ERGATOUDIS, IRONS and ZIENKIEWICZ [1966]), and numerical integration schemes (IRONS [1966]) and many more. The scope of finite element applications in the 1970's would have been significantly diminished without these contributions.

3. THE MATHEMATICAL THEORY

The mathematical theory of finite elements was slow to emerge from this caldron of activity. Many of the works on "variational finite difference methods" which appeared in the mid-to-late 1960's actually captured the essence of convergence requirements of finite element methods. Thus, the 1965 work of FENG KANG [1965] on such methods, published in Chinese and unknown to the western world for over a decade, is regarded by many as containing the first proof of convergence of finite-element methods. The mathematical theory of finite elements, which addressed mathematical issues connected with purely finite element schemes, began around 1968 and several papers were published that year on the subject. One of the first papers in this period to address the problem of convergence of the finite method in a rigorous way and in which a-priori error estimates for bilinear approximations of a problem in a plane elasticity are obtained, is the often overlooked paper of JOHNSON and McCLAY [1968], which appeared in the *Journal of Applied Mechanics*. This paper correctly developed error estimates in energy norms, and even atempted to characterize the deterioration of convergence rates due to corner singularities.

Also in 1968 there appeared the important mathematical paper of ZLÂMAL [1968] in which a detailed analysis of interpolation properties of a class of triangular elements and their application to second-order and fourth-order linear elliptic boundary-value problems is discussed. This paper attracted the interest of a large segment of the numerical analysis community and several very good mathematicians began to work on finite element methodologies. In the same year, CIARLET [1968] published a rigorous proof of convergence of a finite element approximation of a class of linear two-point boundary-value problems in which piecewise linear shape functions were used. By using a discrete maximum principle he was able to prove L^∞ estimates. We also mention the work of OLIVEIRA [1968] on convergence of finite element methods which established corrected rates-of-convergence of certain problems in appropriate energy norms.

By 1972, finite element methods had emerged as an important new area of numerical analysis in applied mathematics. Mathematical conferences were held on the subject on a regular basis, and there began to emerge a rich volume of literature on mathematical aspects of the method applied to elliptic problems, eigenvalue problems, and parabolic problems. A conference of special significance in this period was held at the University of Maryland in 1972 and featured a penetrating series of lectures by IVO BABUŠKA (see BABUŠKA and AZIZ [1973]) and several important mathematical papers by leading specialists in the mathematics of finite elements, all collected in the volume edited by AZIZ [1972].

One unfamiliar with aspects of the history of finite elements may be led to the erroneous conclusion that the method of finite elements emerged from the growing wealth of information on partial differential equations, weak solutions of boundary-value problems, Sobolev spaces, and the associated approximation theory for elliptic variational boundary-value problems. This is a natural mistake, because the seeds for the modern theory of partial differential equations were sown about the same time as those for the development of modern finite element methods, but in an entirely different garden.

In the late 1940's, LAURENT SCHWARTZ was putting together his theory of distributions around a decade after the notion of generalized functions and their use in partial differential equations appeared in the pioneering work of SOBOLEV. A long list of other names could be added to the list of contributors to the modern theory of partial differential equations, but that is not our purpose here. Rather, we must only note that the rich mathematical theory of partial differential equations which began in the 1940's and 50's, blossomed in the 1960's, and is now an integral part of the foundations of not only partial differential equations but also approximation theory, did not lead naturally to the variational methods of approximation such as finite elements, but grew independently and parallel to that development for almost two decades. It was a happy accident, or perhaps an unavoidable occurrence, that in the late 1960's these two independent subjects, finite element methodology and the theory of approximation of partial differential equations via functional analysis methods, united in an inseparable way, so much so that it is difficult to appreciate the fact that they were ever separate.

The 1970's must mark the decade of the mathematics of finite elements. During this period, great strides were made in determining a-priori error estimates for a variety of finite element methods, for linear elliptic boundary-value problems, for eigenvalue problems, and certain classes of linear and nonlinear parabolic problems; also, some preliminary work on finite element applications to hyperbolic equations was done. It is both inappropriate and perhaps impossible to provide an adequate survey of this large volume of literature, but it is possible to present an albeit biased reference to some of the major works along the way.

An important component in the theory of finite elements is an interpolation theory: how well can a given finite element method approximate functions of a given class locally over a typical finite element? A great deal was known about this subject from the literature on approximation theory and spline analysis, but its particularization to finite elements involves technical difficulties. One can find results on finite element interpolation in a number of early papers, including those of ZLÂMAL [1968], BRAMBLE and ZLAMAL [1970], BABUŠKA [1970, 1971], and BABUŠKA and AZIZ [1972]. But the elegant work on Lagrange and Hermite interpolations of finite elements by CIARLET and RAVIART [1972a] must stand as a very important contribution to this vital aspect of finite element theory. A landmark work on the mathematics of finite elements appeared in 1972 in the remarkably comprehensive and penetrating memoir of BABUŠKA and AZIZ [1972] on the mathematical foundations of finite element methods. Here one can find interwoven with the theory of Sobolev spaces and elliptic problems, general results on approximation theory that have direct bearing on finite element methods. The fundamental work of NITSCHE [1975] on L^∞ estimates for general classes of linear elliptic problems must stand out as one of the most important contributions of the seventies. STRANG [1972], in an important communication, pointed out "variational crimes", inherent in many finite element methods, such as improper numerical quadrature, the use of nonconforming elements, improper satisfaction of boundary conditions, etc., all common practices in applications, but all frequently leading to exceptable numerical schemes. In the same year, CIARLET and RAVIART [1972b,c] also contributed penetrating studies of these issues. Many of the advances of the 1970's drew upon earlier results on variational methods of approximation based on the Ritz method and finite differences; for example the fundamental Aubin-Nitsche method for lifting the order of convergence to lower Sobolev norms (see AUBIN [1967] and NITSCHE [1968]) used such results. In 1974, the important paper of BREZZI [1974] used such earlier results on saddle-point problems and laid the groundwork for a multitude of papers on problems with constraints and on the stability of various finite element procedures. While convergence of special types of finite element strategies such as mixed methods and hybrid methods had been attempted in the early 1970's (e.g. ODEN [1973]), the BREZZI results, and the methods of BABUŠKA for constrained problems, provided a general framework for sudying virtually all mixed and hybrid finite elements (e.g. RAVIART [1975], RAVIART and THOMAS [1977], BABUSKA, ODEN and LEE [1977]).

The penetrating work of SCHATZ and WHALBIN [1976] on interior estimates and problems represented notable contributions to the growing mathematical theory of finite elements. The important work of DOUGLAS and DUPONT (e.g. [1970], [1973]; DUPONT [1973]) on finite element methods for parabolic problems and hyperbolic problems must be mentioned along with the idea of elliptic projections of WHEELER [1973] which provided a useful technique for deriving error bounds for time-dependent problems.

The 1970's also represented a decade in which the generality of finite element methods began to be appreciated over a large portion of the mathematics and scientific community and it was during this period that significant applications to highly nonlinear problems were made. The fact that very general nonlinear phenomena in continuum mechanics, including problems of finite deformation of solids and of flow of viscous fluids could be modeled by finite elements and solved on existing computers was demonstrated in the early seventies (e.g. ODEN [1972]), and, by the end of that decade, several "general purpose" finite element programs were in use by engineers to treat broad classes of nonlinar problems in solid mechanics and heat transfer. The mathematical theory for nonlinear problems also was advanced in this period, and the important work of FALK [1974] on finite element approximations of variational inequalities should be mentioned.

It is not too inaccurate to say that by 1980, a solid foundation for the mathematical theory of finite elements for linear problems had been established and that significant advances in both theory and application into nonlinear problems existed. The open questions that remain are difficult ones and their solution will require a good understanding of the mathematical properties of the method.

4. PERSONAL REFLECTIONS AND ACKNOWLEDGEMENTS

I remember very well my own introduction to finite elements. I had read thoroughly the work of AGYRIS and others on "matrix methods in structural mechanics" and had developed notes on the subject while teaching graduate courses in solid mechanics in the early 1960's, but none of the literature of the day had much impact on my university research at the time, if the research of anyone in the university community. The aircraft industry was actively developing the subject during this period and was far ahead of universities in studying and implementing these methods.

Then, in 1963, I had the good fortune to enter the aerospace industry for a brief period of time and to meet and begin joint work with GILBERT BEST, who had been charged with the responsibility of developing a large general-purpose finite element code for use in aircraft structural analysis. Only the two of us worked on the project, but by fall 1963 we had produced some quite general results and one of the early working codes on finite elements. This code had features in it that were not fully duplicated for more than a decade. I still have copies of our elaborate report on that work (BEST and ODEN [1963]).

It was BEST who demonstrated to me the strength and versatility of the method. In our work, noted above, we developed mixed methods, assumed stress methods, hybrid methods, we explored algorithms for optimization problems, nonlinear problems, bifurcation and vibration problems, and did detailed tests on stability and convergence of various methods by numerical experimentation. We developed finite elements for beams, plates, shells, for composite materials, for three-dimensional problems in elasticity, for thermal analysis, and linear dynamic analysis. Some of our methods were failures; most were effective and useful. Since convergence properties and criteria were not to come on the scene for another decade, our only way to test many of the more complex algorithms was to code them and compute solutions for test problems.

I went on to return to academia in 1964 and among my first chores was to develop a graduate course on finite element methods. At the same time, I taught mathematics and continuum mechanics, and it became clear to me that finite elements and electronic computing offered hope of transforming nonlinear continuum mechanics from a qualitative and academic subject into something useful in modern scientific computing and engineering. Toward this end, I began work with graduate students in 1965 that led to successful numerical analyses of problems in finite-strain elasticity (1965, 1966), elastoplasticity (1967), thermoelasticity (1967), thermoviscoelasticity (1969), and incompressible and compressible viscous fluid flow (1968, 1969). These works, many summarized in ODEN [1972], include early (perhaps the first) uses of Discrete-Kirchhoff elements, incremental elasto-plastic algorithms, conjugate-gradient methods for nonlinear finite element systems, continuation methods, dynamic relaxation schemes, Taylor-Galerkin algorithms (then called "finite-element based Lax-Wendroff schemes"), primitive-variable formulations in incompresible flow, curvilinear elements, and penalty formulations; all these subjects have been resurected in more recent times and have been studied in far more detail and better style and depth than was possible in the 1960's.

While my later work, work in the '70's and '80's, was influenced by the competent mathematicians (and friends) who developed the subject during the period (BABUŠKA, CIARLET, STRANG, DOUGLAS, NITSCHE, and many others), the work and guidance of G. BEST was basic to may interest in this subject, and I dedicate this note to him.

I should also add that portions of this paper are excerpts from an article to appear in the Handbook of Numerical Analysis, edited by J.L. LIONS and P.G. CIARLET, North Holland Publishing Co., Amsterdam. I am grateful to North Holland for granting permission to use this material in the present volume.

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