1

MEET THE FINITE ELEMENT METHOD

1.1 WHAT IS THE FINITE ELEMENT METHOD?

The finite element method is a numerical analysis technique for obtaining approximate solutions to a wide variety of engineering problems. Although originally developed to study stresses in complex airframe structures, it has since been extended and applied to the broad field of continuum mechanics. Because of its diversity and flexibility as an analysis tool, it is receiving much attention in engineering schools and in industry.

Although this brief comment on the finite element method answers the question posed by the section heading, it does not give us the operational definition we need to apply the method to a particular problem. Such an operational definition—along with a description of the fundamentals of the method—requires considerably more than one paragraph to develop. Hence Part I of this book is devoted to basic concepts and fundamental theory. Before discussing more aspects of the finite element method, we should first consider some of the circumstances leading to its inception, and we should briefly contrast it with other numerical schemes.

In more and more engineering situations today, we find that it is necessary to obtain approximate numerical solutions to problems rather than exact closed-form solutions. For example, we may want to find the load capacity of a plate that has several stiffeners and odd-shaped holes, the concentration of pollutants during nonuniform atmospheric conditions, or the rate of fluid flow through a passage of arbitrary shape. Without too much effort, we can write down the governing equations and boundary conditions for these problems, but we see immediately that no simple analytical solution can be found. The difficulty in these three examples lies in the fact that either the geometry or some other feature of the problem is irregular or "arbitrary." Analytical solutions to problems of this type seldom exist; yet these are the kinds of problems that engineers are called upon to solve.

The resourcefulness of the analyst usually comes to the rescue and provides several alternatives to overcome this dilemma. One possibility is to make simplifying assumptions—to ignore the difficulties and reduce the problem to one that can be handled. Sometimes this procedure works; but, more often than not, it leads to serious inaccuracies or wrong answers. Now that computers are widely available, a more viable alternative is to retain the complexities of the problem and find an approximate numerical solution.

Several approximate numerical analysis methods have evolved over the years; a commonly used method is the finite difference [1][1] scheme. The familiar finite difference model of a problem gives a *pointwise* approximation to the governing equations. This model (formed by writing difference equations for an array of grid points) is improved as more points are used. With finite difference techniques we can treat some fairly difficult problems; but, for example, when we encounter irregular geometries or an unusual specification of boundary conditions, we find that finite difference techniques become hard to use.

Unlike the finite difference method, which envisions the solution region as an array of grid points, the finite element method envisions the solution region as built up of many small, interconnected subregions or elements. A finite element model of a problem gives a *piecewise* approximation to the governing equations. The basic premise of the finite element method is that a solution region can be analytically modeled or approximated by replacing it with an assemblage of discrete elements. Since these elements can be put together in a variety of ways, they can be used to represent exceedingly complex shapes.

As an example of how a finite difference model and a finite element model might be used to represent a complex geometrical shape, consider the turbine blade cross section in Figure 1.1. For this device we may want to find the distribution of displacements and stresses for a given force loading or the distribution of temperature for a given thermal loading. The interior coolant passage of the blade, along with its exterior shape, gives it a nonsimple geometry.

A uniform finite difference mesh would reasonably cover the blade (the solution region), but the boundaries must be approximated by a series of horizontal and vertical lines (or "stair steps"). On the other hand, the finite element

[1]Numbers in brackets denote references at the end of the chapter.

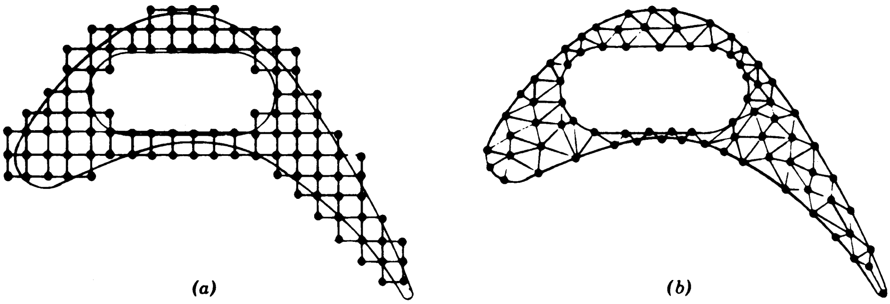

Figure 1.1 (*a*) Finite difference and (*b*) finite element discretizations of a turbine blade profile.

model (using the simplest two-dimensional element—the triangle) gives a better approximation to the region. Also, a better approximation to the boundary shape results because the curved boundary is represented by straight lines of any inclination. This example is not intended to suggest that finite element models are decidedly better than finite difference models for all problems. The only purpose of the example is to demonstrate that the finite element method is particularly well suited for problems with complex geometries.

Still another numerical analysis method is the boundary element method (boundary integral equation method) [2–4]. This method uses Green's theorem to reduce the dimensionality of the problem; a volume problem is reduced to a surface problem, a surface problem is reduced to a line problem. The turbine blade cross section example of Figure 1.1 would have no interior mesh, but rather a mesh of connected points along the exterior boundary and a mesh of connected points along the interior boundary. This method is computationally less efficient than finite elements and is not widely used in industry. It is popular for acoustic problems [5] and is sometimes used as a hybrid method in conjunction with finite elements.

1.2 HOW THE FINITE ELEMENT METHOD WORKS

We have been alluding to the essence of the finite element method, but now we shall discuss it in greater detail. In a continuum[2] problem of any dimension the field variable (whether it is pressure, temperature, displacement, stress, or some other quantity) possesses infinitely many values because it is a function of each generic point in the body or solution region. Consequently, the problem is one with an infinite number of unknowns. The finite element discretization procedures reduce the problem to one of a finite number of unknowns by dividing

[2] We define a continuum to be a body of matter (solid, liquid, or gas) or simply a region of space in which a particular phenomenon is occurring.

the solution region into elements and by expressing the unknown field variable in terms of assumed approximating functions within each element. The approximating functions (sometimes called *interpolation functions*) are defined in terms of the values of the field variables at specified points called *nodes* or *nodal points*. Nodes usually lie on the element boundaries where adjacent elements are connected. In addition to boundary nodes, an element may also have a few interior nodes. The nodal values of the field variable and the interpolation functions for the elements completely define the behavior of the field variable within the elements.

For the finite element representation of a problem the nodal values of the field variable become the unknowns. Once these unknowns are found, the interpolation functions define the field variable throughout the assemblage of elements.

Clearly, the nature of the solution and the degree of approximation depend not only on the size and number of the elements used but also on the interpolation functions selected. As one would expect, we cannot choose functions arbitrarily, because certain compatibility conditions should be satisfied. Often functions are chosen so that the field variable or its derivatives are continuous across adjoining element boundaries. The essential guidelines for choosing interpolation functions are discussed in Chapters 3 and 5. These are applied to the formulation of different kinds of elements.

Thus far we have briefly discussed the concept of modeling an arbitrarily shaped solution region with an assemblage of discrete elements, and we have pointed out that interpolation functions must be defined for each element. We have not yet mentioned, however, an important feature of the finite element method that sets it apart from other numerical methods. This feature is the ability to formulate solutions for individual elements before putting them together to represent the entire problem. This means, for example, that if we are treating a problem in stress analysis, we find the force–displacement or stiffness characteristics of each individual element and then assemble the elements to find the stiffness of the whole structure. In essence, a complex problem reduces to considering a series of greatly simplified problems.

Another advantage of the finite element method is the variety of ways in which one can formulate the properties of individual elements. There are basically three different approaches. The first approach to obtaining element properties is called the *direct approach* because its origin is traceable to the direct stiffness method of structural analysis. Although the direct approach can be used only for relatively simple problems, it is presented in Chapter 2 because it is the easiest to understand when meeting the finite element method for the first time. The direct approach suggests the need for matrix algebra (Appendix A) in dealing with the finite element equations.

Element properties obtained by the direct approach can also be determined by the *variational approach*. The variational approach relies on the calculus of variations (Appendix B) and involves extremizing a *functional*. For problems in solid mechanics the functional turns out to be the potential energy, the complementary energy, or some variant of these, such as the Reissner variational principle. Knowledge of the variational approach (Chapter 3) is necessary to work beyond the introductory level and to extend the finite element method to a wide variety of engineering problems. Whereas the direct approach can be used to formulate element properties for only the simplest element shapes, the variational approach can be employed for both simple and sophisticated element shapes.

A third and even more versatile approach to deriving element properties has its basis in mathematics and is known as the *weighted residuals approach* (Chapter 4). The weighted residuals approach begins with the governing equations of the problem and proceeds without relying on a variational statement. This approach is advantageous because it thereby becomes possible to extend the finite element method to problems where no functional is available. The method of weighted residuals is widely used to derive element properties for nonstructural applications such as heat transfer and fluid mechanics.

Regardless of the approach used to find the element properties, the solution of a continuum problem by the finite element method always follows an orderly step-by-step process. To summarize in general terms how the finite element method works we will succinctly list these steps now; they will be developed in detail later.

1. Discretize the Continuum. The first step is to divide the continuum or solution region into elements. In the example of Figure 1.1 the turbine blade has been divided into triangular elements that might be used to find the temperature distribution or stress distribution in the blade. A variety of element shapes (such as those cataloged in Chapter 5) may be used, and different element shapes may be employed in the same solution region. Indeed, when analyzing an elastic structure that has different types of components such as plates and beams, it is not only desirable but also necessary to use different elements in the same solution. Although the number and the type of elements in a given problem are matters of engineering judgment, the analyst can rely on the experience of others for guidelines. The discussion of applications in Chapters 6–9 reveals many of these useful guidelines.

2. Select Interpolation Functions. The next step is to assign nodes to each element and then choose the interpolation function to represent the variation of the field variable over the element. The field variable may be a scalar, a vector, or a higher-order tensor. Often, polynomials are selected as interpolation functions for the field variable because they are easy to integrate and differentiate. The degree of the polynomial chosen depends on the number of nodes assigned to the element, the nature and number of unknowns at each node, and certain continuity requirements imposed at the nodes and along the element boundaries. The magnitude of the field variable as well as the magnitude of its derivatives may be the unknowns at the nodes.

3. Find the Element Properties. Once the finite element model has been established (that is, once the elements and their interpolation functions have been selected), we are ready to determine the matrix equations expressing the properties of the individual elements. For this task we may use one of the three approaches just mentioned: the direct approach, the variational approach, or the weighted residuals approach.

4. Assemble the Element Properties to Obtain the System Equations. To find the properties of the overall system modeled by the network of elements we must "assemble" all the element properties. In other words, we combine the matrix equations expressing the behavior of the elements and form the matrix equations expressing the behavior of the entire system. The matrix equations for the system have the same form as the equations for an individual element except that they contain many more terms because they include all nodes.

The basis for the assembly procedure stems from the fact that at a node, where elements are interconnected, the value of the field variable is the same for each element sharing that node. A unique feature of the finite element method is that the system equations are generated by assembly of the individual *element* equations. In contrast, in the finite difference method the system equations are generated by writing nodal equations. In Chapter 2 we demonstrate how the assembly process leads to the system equations.

5. Impose the Boundary Conditions. Before the system equations are ready for solution they must be modified to account for the boundary conditions of the problem. At this stage we impose known nodal values of the dependent variables or nodal loads. In Chapter 2 we will see examples of how nodal boundary conditions are introduced.

6. Solve the System Equations. The assembly process gives a set of simultaneous equations that we solve to obtain the unknown nodal values of the problem. If the problem describes steady or equilibrium behavior, then we must solve a set of linear or nonlinear algebraic equations. In Chapter 10 we briefly discuss standard solution techniques for solving these equations. If the problem is unsteady, the nodal unknowns are a function of time, and we must solve a set of linear or nonlinear ordinary differential equations. We describe techniques for solving time-dependent equations in Part II of the book in Chapters 6–9.

7. Make Additional Computations If Desired. Many times we use the solution of the system equations to calculate other important parameters. For example, in a structural problem the nodal unknowns are displacement components. From these displacements we calculate element strains and stresses. Similarly, in a heat-conduction problem the nodal unknowns are temperatures, and from these we calculate element heat fluxes.

1.3 A BRIEF HISTORY OF THE METHOD

Although the label *finite element method* first appeared in 1960, when it was used by Clough [6] in a paper on plane elasticity problems, the ideas of finite element analysis date back much further. In fact, the questions Who originated the finite element method? and When did it begin? have three different answers depending on whether one asks an applied mathematician, a physicist, or an engineer. All of these specialists have some justification for claiming the finite element method as their own, because each developed the essential ideas independently at different times and for different reasons. The applied mathematicians were concerned with boundary value problems of continuum mechanics; in particular, they wanted to find approximate upper and lower bounds for eigenvalues. The physicists were also interested in solving continuum problems, but they sought means to obtain piecewise approximate functions to represent their continuous functions. Faced with increasingly complex problems in aerospace structures, engineers were searching for a way in which to find the stiffness influence coefficients of shell-type structures reinforced by ribs and spars. The efforts of these three groups resulted in three sets of papers with distinctly different viewpoints.

The first efforts to use piecewise continuous functions defined over triangular domains appear in the applied mathematics literature with the work of Courant [7] in 1943. Courant used an assemblage of triangular elements and the principle of minimum potential energy to study the St. Venant torsion problem.

In 1959 Greenstadt [8], motivated by a discussion in the book by Morse and Feshback [9], outlined a discretization approach involving "cells" instead of points; that is, he imagined the solution domain to be divided into a set of contiguous subdomains. In his theory he describes a procedure for representing the unknown function by a series of functions, each associated with one cell. After assigning approximating functions and evaluating the appropriate variational principle to each cell, he uses continuity requirements to tie together the equations for all the cells. By this means he reduces a continuous problem to a discrete one. Greenstadt's theory allows for irregularly shaped cell meshes and contains many of the essential and fundamental ideas that serve as the mathematical basis for the finite element method as we know it today.

As the popularity of the finite element method began to grow in the engineering and physics communities, more applied mathematicians became interested in giving the method for a firm mathematical foundation. As a result, a number of studies were aimed at estimating discretization error, rates of convergence, and stability for different types of finite element approximations. These studies most often focused on the special case of linear elliptic boundary value problems. Since the late 1960s the mathematical literature on the finite element method has grown more than in any previous period. In this book we shall not study the rigorous mathematical basis of the finite element method, because such knowledge is unnecessary for most practical applications. Instead we shall call upon pertinent results when they are needed.

While the mathematicians were developing and using finite element concepts, the physicists were also busy with similar ideas. The work of Prager and Synge [10] leading to the development of the hypercircle method is a key example. As a concept in function space, the hypercircle method was originally developed in connection with classical elasticity theory to give its minimum principles a geometric interpretation. Outgrowths of the hypercircle method (such as the one suggested by Synge [11]) can be applied to the solution of continuum problems in much the same way as finite element techniques can be applied.

Physical intuition first brought finite element concepts to the engineering community. In the 1930s when a structural engineer encountered a truss problem such as the one shown in Figure 1.2*a*, he immediately knew how to solve for component stresses and deflections as well as the overall strength of the unit. First, he would recognize that the truss was simply an assembly of rods whose force–deflection characteristics he knew well. Then he would combine these individual characteristics according to the laws of equilibrium and solve the resulting system of equations for the unknown forces and deflections for the overall system.

This procedure worked well whenever the structure in question had a finite number of interconnection points, but then the following question arose: What can we do when we encounter an elastic continuum structure such as a plate that has an infinite number of interconnection points? For example, in Figure 1.2*b*, if a plate replaces the truss, the problem becomes considerably more difficult. Intuitively, Hrenikoff [12] reasoned that this difficulty could be overcome by assuming the continuum structure to be divided into elements or structural sections (beams) interconnected at only a finite number of node points. Under this assumption the problem reduces to that of a conventional structure, which could be handled by the old methods. Attempts to apply Hrenikoff's "framework method" were successful, and thus the seed to finite element techniques began to germinate in the engineering community.

Shortly after Hrenikoff, McHenry [13] and Newmark [14] offered further development of these discretization ideas, while Kron [15, 16] studied topological properties of discrete systems. There followed a ten-year spell of inactivity,

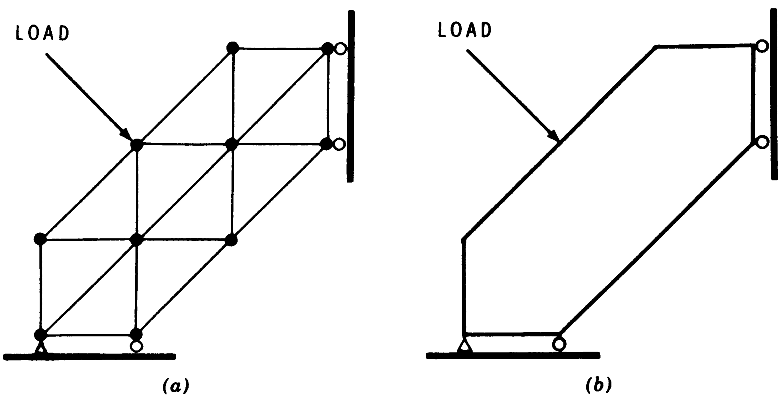

Figure 1.2 Example of (*a*) a truss and (*b*) a similarly shaped plate supporting the same load.

which was broken in 1954 when Argyris and his collaborators [17–21] began to publish a series of papers extensively covering linear structural analysis and efficient solution techniques well suited to automatic digital computation.

The actual solution of plane stress problems by means of triangular elements whose properties were determined from the equations of elasticity theory was first given in the now classical 1956 paper of Turner, Clough, Martin, and Topp [22]. These investigators were the first to introduce what is now known as the direct stiffness method for determining finite element properties. Their studies, along with the advent of the digital computer at that time, opened the way to the solution of complex plane elasticity problems. After further treatment of the plane elasticity problem by Clough [6] in 1960, engineers began to recognize the efficacy of the finite element method. In a 1980 paper Clough [23] gives his personal account of the origins of the method, describing the sequence of events from the original efforts at Boeing that produced reference 18 to the paper [6] in which he introduced the label of *the finite element method*.

In 1965 the finite element method received an even broader interpretation when Zienkiewicz and Cheung [24] reported that it is applicable to all field problems that can be cast into variational form. During the late 1960s and early 1970s (while mathematicians were working on establishing errors, bounds, and convergence criteria for finite element approximations) engineers and other practitioners of the finite element method were also studying similar concepts for various problems in the area of solid mechanics.

In the years since 1960 the finite element method has received widespread acceptance in engineering. Thousands of papers, hundreds of conferences, and many books appeared on the subject. The number of books published over this period illustrates the exponential growth. The first edition of this book in 1974 lists fewer than ten finite element books. In the second edition in 1982 we list fewer than 40 finite element books. When we wrote the third edition in the early 1990s, finite element books were so numerous that we were no longer able to list them. A 1991 bibliography [25] lists nearly 400 finite element books in English and other languages. The bibliography also identifies over 200 international finite element symposia, conferences, and short courses that took place between 1964 and 1991. A recent web search on the phrase "finite element*" using the AltaVista search engine yielded almost 200,000 *pages* of results. Clearly, these trends show the amazingly rapid worldwide growth of the method. Table 1.1 shows a time line of developments in computer hardware and software compared with developments in finite elements.

1.4 RANGE OF APPLICATIONS

Applications of the finite element method divide into three categories, depending on the nature of the problem to be solved. In the first category are the problems known as *equilibrium problems* or time-independent problems. The majority of applications of the finite element method fall into this category.

TABLE 1.1 Time Line of Important Developments

Computer Technology	Year	Analysis Technology
	1941	Framework Method.
	1942	
	1943	Courant studies St. Venant torsion.
	1944	Equivalent Circuits of Electric Field.
	1945	
ENIAC 1 at University of Illinois.	1946	
Transistor invented.	1947	Hypercircle Method.
	1948	
	1949	
	1950	
	1951	
	1952	
IBM's first general purpose computer.	1953	
FORTRAN created.	1954	Argyris publishes efficient solution techniques.
	1955	
	1956	Classic paper solving plane stress.
	1957	
	1958	
	1959	Greenstadt's discretization approach.
	1960	Phrase *finite element* is coined.
First commercial integrated circuit.	1961	
	1962	
Mouse is patented.	1963	
CDC introduces CDC6600 with 60-bit words.	1964	First commercial offering of finite element software.
BASIC created. First graphics tablet.		

For the solution of equilibrium problems in the solid mechanics area, we need to find the displacement distribution and the stress distribution for a given mechanical or thermal loading. Similarly, for the solution of equilibrium problems in fluid mechanics, we need to find pressure, velocity, temperature, and density distributions under steady-state conditions.

In the second category are the so-called *eigenvalue problems* of solid and fluid mechanics. These are steady-state problems whose solution often requires the determination of natural frequencies and modes of vibration of solids and fluids. Examples of eigenvalue problems involving both solid and fluid mechanics appear in civil engineering when the interaction of lakes and dams is considered and in aerospace engineerng when the sloshing of liquid fuels in flexible tanks is involved. Another class of eigenvalue problems includes the stability of structures and the stability of laminar flows.

In the third category is the multitude of time-dependent or *propagation problems* of continuum mechanics. This category is composed of the problems that result when the time dimension is added to the problems of the first two categories.

Just about every branch of engineering is a potential user of the finite element method. But the mere fact that this method can be used to solve a particular problem does not mean that it is the most practical solution technique. Often several attractive techniques are available to solve a given problem. Each technique has its relative merits, and no technique enjoys the lofty distinction of being "the best" for all problems.

The range of possible applications of the finite element method extends to all engineering disciplines, but civil, mechanical, and aerospace engineers are the most frequent users of the method. In addition to structural analysis other areas of applications include heat transfer, fluid mechanics, electromagnetism, biomechanics, geomechanics, and acoustics. The method finds acceptance in multidisciplinary problems where there is a coupling between two or more of the disciplines. Examples include thermal structures where there is a natural coupling between heat transfer and displacements, as well as aeroelasticity where there is a strong coupling between external flow and the distortion of the wing.

1.5 COMMERCIAL FINITE ELEMENT SOFTWARE

The first commercial finite element software made its appearance in 1964. The Control Data Corporation sold it in a time-sharing environment. No preprocessors (mesh generators) were available, so engineers had to prepare data element by element and node by node. A keypunched IBM (Hollerith) card represented each element and each node. Batch-mode line plots were used to check geometry and to postprocess results. Turnaround occurred in days for simple problems. Only linear problems could be addressed. Nevertheless it represented a breakthrough in the complexity of the problem that could be handled in a practical time frame. Later, finite element software could be purchased or leased to run on corporate computers. Typically the corporate computer had been purchased to process financial data, so that computer availability to the engineer was restricted, perhaps to nights and weekends. The introduction of workstations circa 1980 brought several breakthrough advantages. Interactive graphics were practical and availability of computer power to solve problems on a dedicated basis was achieved. Finally, the introduction of personal computers (PCs) powerful enough to run finite element software provides extremely cost effective problem solving.

Today we have hundreds of commercial software packages to choose from. A small number of these dominate the market. It is difficult to make compar-

TABLE 1.2 Leading Commercial Finite Element Software Companies

Company Name	Product Name	Web Site
Hibbitt, Karlsson & Sorensen	ABAQUS	www.hks.com
Ansys, Incorporated	ANSYS	www.ansys.com
Structural Data Research Corp.	SDRC-Ideas	www.sdrc.com
Parametric Technology, Inc.	RASNA	www.ptc.com
MSC Software Corp.	MSC/NASTRAN	www.mscsoftware.com

isons purely on a finite element basis, because the software houses are often diversified. Data from Daratech suggest that the companies listed in Table 1.2 are dominant providers of general-purpose finite element software. Choice among these, or other providers, involves a complex set of criteria, usually including: analysis versatility, ease of use, efficiency, cost, technical support, training, and even the labor pool locally available to use particular software.

In contrast to the early days, we can now use computer-aided design (CAD) software or solid modelers to generate complex geometries, at either the component or assembly level. We can (with some restrictions) automatically generate elements and nodes, by merely indicating the desired nodal density. Software is available that works in conjunction with finite elements to generate structures of optimum topology, shape, or size. Nonlinear analyses including contact, large deflection, and nonlinear material behavior are routinely addressed.

1.6 THE FUTURE OF THE FINITE ELEMENT METHOD

Our brief look at the history of the finite element method shows us that its early development was sporadic. The applied mathematicians, physicists, and engineers all dabbled with finite element concepts, but they did not recognize at first the diversity and the multitude of potential applications. After 1960 this situation changed and the tempo of development increased markedly. By 1972 the finite element method had become the most active field of interest in the numerical solution of continuum problems. It remains the dominant method today. Part of its strength is that it can be used in conjunction with other methods. Software components such as solvers can be used in a modular fashion, so that improvements in diverse areas can be rapidly assimilated. Having said that, we can still remark that major innovations in technology, for example, p-adaptive finite elements (Chapter 10), often take as long as a decade to move from academia to commercial practice. Therefore, academic publications are the best leading indicator of what's to come in commercial finite elements. Certainly, improved iterative solvers, meshless formulations, better error indicators, and special-purpose elements are on the list of things to come.

Although the finite element method can be used to solve a very large number of complex problems, there are still some practical engineering problems that are difficult to address because we lack an adequate theory of failure, or because we lack appropriate material data. This is not a finite element problem per se, but is of serious concern to any engineer who wants to supplant testing with analysis. (The use of analysis usually permits faster design turnaround, the exploration of widely varying environments, and the use of optimization tools. Furthermore, analysis is usually significantly cheaper than building prototypes and testing them.) The mechanical and thermal properties of many nonmetallic materials are difficult to acquire, especially over a range of temperatures. Fatigue data is often lacking. Fatigue failure theory often lags our ability to calculate changing complex stress states. Data on friction is often difficult to obtain. Calculations based on the assumption of Coulomb friction are often unrealistic. There is a general paucity of thermal data, especially regarding absorptivity and emissivity needed for radiation calculations. The World Wide Web should offer a means of placing material properties into accessible databases.

From a practitioner's viewpoint, the finite element method, like any other numerical analysis technique, can always be made more efficient and easier to use. As the method is applied to larger and more complex problems, it becomes increasingly important that the solution process remain economical. The rapid growth in engineering usage of computer technology will undoubtedly continue to have a significant effect on the advancement of the finite element method. Improved efficiency achieved by computer technology advancements such as parallel processing will surely occur. Since the mid 1970s interactive finite element programs on small but powerful personal computers and workstations have played a major role in the remarkable growth of computer-aided design. With continuing economic pressures to improve engineering productivity, this decade will see an accelerated role of the finite element method in the design process. This methodology is still exciting and an important part of an engineer's tool kit.

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