QUADRATURE RULES FOR BRICK BASED FINITE ELEMENTS

BRUCE M. IRONS
University of Wales, Swansea

Already isoparametric hexahedral (brick) finite elements with 20 or 32 nodes[1] are highly competitive in practice,[2] despite the observation that 50 per cent of the total computation is often absorbed in numerically integrating the coefficients of the equations.[2] This cost is approximately halved[3] by a method based, essentially, on using a 9×9 $[\mathbb{D}]$ matrix which operates on $\partial u/\partial x, \partial u/\partial y, \partial u/\partial z, \partial v/\partial x, \ldots \partial w/\partial z$—a technique which, moreover, is more general than the classical $\sum \mathbb{B}^{\mathrm{T}}\,\mathbb{D}\mathbb{B}\times$constant algorithm.[4]

The purpose of this note is to demonstrate how one may further halve the cost by using simpler integration formulae having the same order of truncation error. We compare certain Gaussian-type rules, some of them new and all of them designed to integrate complete polynomials, with the corresponding product-Gauss rules which are normally used.[5] The former integrate correctly $\sum \mathbb{C}_{ijk}\, x^i y^j z^k$, $i+j+k \leqslant n$, while the latter integrate correctly a much larger number of terms, those with $i,j,k \leqslant n$. All these rules have been checked by computer. They are now presented in the form:

$$
\begin{aligned}
\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = {} & \mathbb{A}_1 f(0,0,0)\;(1\text{ term})\\
& + \mathbb{B}_6\{f(-b,0,0)+f(b,0,0)+f(0,-b,0)+\ldots 6\text{ terms}\}\\
& + \mathbb{C}_8\{f(-c,-c,-c)+f(c,-c,-c)+\ldots 8\text{ terms}\}\\
& + \mathbb{D}_{12}\{f(-d,-d,0)+\ldots+f(d,0,-d)+\ldots 12\text{ terms}\}
\end{aligned}
$$

The rules are listed below, labelled according to the number of points they use.

Rule 6 (i.e. a 6-point rule). $\mathbb{B}_6 = 8/6$, $b = 1$. That is, we take the mean of the six mid-face values. This rule is accurate to the complete cubic in x, y, z, i.e. 20 terms. A multiplying constant and x, y, z are freely chosen at 6 points, i.e. 24 constants are chosen. The efficiency is defined as 20/24 so that the rule is nearly Gaussian.

This is an excellent rule. Since the mid-face values are so representative, we should evidently calculate stresses at mid-face in brick elements, rather than at corners—which are the worst possible positions!

Rule 8G (i.e. the $2\times 2\times 2$ product—Gauss rule). Included for comparison with Rule 6.

Rule 14 with $\mathbb{B}_6 = 0{\cdot}886426593$, $b = 0{\cdot}795822426$, $\mathbb{C}_8 = 0{\cdot}335180055$ and $c = 0{\cdot}758786911$. Accurate to the complete quintic, like Rule 27G below. Another excellent rule, with precisely Gaussian efficiency (56/56), small multiplying constants, and moderately small sextic errors. (The good efficiency is surprising when the term $\mathbb{A}_1$ is absent.)

Rule 15a with $\mathbb{A}_1 = 1{\cdot}564444444$, $\mathbb{B}_6 = 0{\cdot}355555556$, $b = 1$,
$\mathbb{C}_8 = 0{\cdot}537777778$ and $c = 0{\cdot}674199862$.

A slightly less effective rule, whose surplus constant is chosen to make it modular with Rules 1 and 6 above, allowing a flexible strategy.

Received 16 January 1970

Rule 15b with $A_1 = 0{\cdot}712137436$, $B_6 = 0{\cdot}686227234$, $b = 0{\cdot}848418011$, $C_8 = 0{\cdot}396312395$ and $c = 0{\cdot}727662441$. Another rule with surplus constant, now modular with Rule 27a below.

Rule 19 with $A_1 = 2{\cdot}074074074$, $B_6 = -0{\cdot}24691358$, $D_{12} = 0{\cdot}617283951$ and

$$b = d = 0{\cdot}774596669.$$

A rule previously recommended[6] but apparently much less efficient (56/76).

Rule 27a with $A_1 = 0{\cdot}788073483$, $B_6 = 0{\cdot}499369002$, $b = 0{\cdot}848418011$, $C_8 = 0{\cdot}478508449$, $c = 0{\cdot}652816472$, $D_{12} = 0{\cdot}032303742$ and $d = 1{\cdot}106412899$. A super-efficient rule (120/108) correct to complete heptic like Rule 64G below. (This is unusual among Gaussian rules in that it has sampling points outside the domain.)

Rule 27G (i.e. the $3 \times 3 \times 3$ product—Gauss rule). Included for comparison with Rule 14 etc.

Rule 64G (i.e. the $4 \times 4 \times 4$ product—Gauss rule). Included for comparison with Rule 27a.

Table I. Errors of rules

Rule No.	Quartic terms		Sixth degree terms			Eighth degree terms			
	x^4	$x^2 y^2$	x^6	$x^4 y^2$	$x^2 y^2 z^2$	x^8	$x^6 y^2$	$x^4 y^4$	$x^4 y^2 z^2$
6	1·1	−0·89							
8G	−0·71	0	−0·85	−0·24	0				
14	0	0	−0·18	−0·02	0·22				
15a	0	0	−0·03	−0·13	0·11				
15b	0	0	−0·16	−0·06	0·17				
19	0	0	−0·18	0	−0·30	−0·31	−0·06	0	−0·18
27a	0	0	0	0	0	0·09	0·04	0·10	−0·05
27G	0	0	−0·18	0	0	−0·31	−0·06	0	0
64G	0	0	0	0	0	−0·05	0	0	0

REFERENCES

1. B. M. Irons, 'Engineering applications of numerical integration in stiffness methods', *AIAA Jnl*, 4, 2035–2037 (1966).
2. O. C. Zienkiewicz, B. M. Irons and J. Ergatoudis, 'Three-dimensional analysis of arch dams and their foundations', *Symp. Arch Dams*, Paper 4, *Proc. Instn civ. Engrs*, London (1968).
3. B. M. Irons, 'Economical computer techniques for numerically integrated finite elements', *Int. J. num. Meth. Engng*, 1, 201 (1969).
4. O. C. Zienkiewicz, *The Finite Element Method*, 2nd edn., McGraw-Hill, New York, 1970, Chap. 19.
5. A. H. Stroud and Don Secrest, *Gaussian Quadrature Formulas*, Prentice-Hall, Englewood Cliffs, 1966, p. 40.
6. J. C. P. Miller, 'Numerical quadrature in two or more dimensions', *Maths Comput.* 14, 130–138 (1960).