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SHORT COMMUNICATIONS

EFFECTIVE QUADRATURE RULES FOR QUADRATIC SOLID ISOPARAMETRIC FINITE ELEMENTS

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In a recent note,[1] Irons demonstrated several integration formulae for use with solid isoparametric finite elements. The use of different formulae enabled variations in accuracy and running times to be achieved, and clearly one aims for the cheapest rule for a given degree of accuracy.

One of the most successful finite elements is the 20-node isoparametric solid element, and this has been used frequently in the CEGB. This note is mainly concerned with this particular element. The usual integrating rule used is the Gauss $3 \times 3 \times 3$ rule, with 27 points per element. In certain circumstances, namely shell-type structures subjected to bending modes, the use of a $2 \times 2 \times 2$ rule (8 points per element) has been shown to give good results with rapid convergence[2] in a manner very similar to the results of Zienkiewicz and co-workers[3] using a quadratic thick shell element.[4] However, this reduced rule does not always give satisfactory answers in membrane modes or in shells with solid attachments, and so, as with solid problems, alternative economies in integration techniques are desirable.

The two most economical rules giving the same order of accuracy as the Gauss $3 \times 3 \times 3$ rule appear to be the 14 point rule mentioned in Reference 1, and given originally by Hammer and Stroud[5] and the slightly cheaper, slightly less accurate, 13 point rule given originally by Stroud.[6] Details of the 14 point rule may be found in Reference 1 and are not repeated here. The 13 point rule is defined by the following co-ordinates (in the double unit cube) and weighting coefficients:

$$
\begin{array}{llll}
(0,\ 0,\ 0), & \text{coeff. A} & & \\
\pm(\lambda,\ \xi,\ \xi), & \pm(\xi,\ \lambda,\ \xi), & \pm(\xi,\ \xi,\ \lambda), & \text{coeff. B} \\
\pm(\mu,\ \mu,\ \lambda), & \pm(\mu,\ \gamma,\ \mu), & \pm(\gamma,\ \mu,\ \mu), & \text{coeff. C}
\end{array}
$$

where

$$
\begin{aligned}
\lambda &= 0{\cdot}88030430 \quad \xi = -0{\cdot}49584802 \\
\mu &= 0.79562143, \quad \gamma = 0{\cdot}025293237 \\
A &= 1{\cdot}68421056, \quad B = 0{\cdot}54498736, \quad C = 0{\cdot}507644216
\end{aligned}
$$

One observes that the locations of the integrating points are not completely symmetric in the cube.

In order to demonstrate the relative accuracies of these rules, 3 test problems were run using the $3 \times 3 \times 3$ Gauss rule, the 14 point rule and the 13 point rule. The comparisons also include results using the $2 \times 2 \times 2$, $4 \times 4 \times 4$ and $5 \times 5 \times 5$ Gauss rules.

Example 1

A cantilever, encastre at one end, and of dimensions $24 \times 8 \times 8$ units, was shear-loaded at the other end. Six 20-node brick elements were used, three along the length and two through the depth. The end deflection is shown in Table I together with the axial stress at the wall for the different integrating rules. The $4 \times 4 \times 4$ and $5 \times 5 \times 5$ Gauss rules are added to show the correctly integrated values, from which the percentage relative errors of the 13 and 14 point rules are derived. The $3 \times 3 \times 3$ rule is shown to integrate exactly, and the 14 point rule is more accurate than the 13 point rule. In this case, the $2 \times 2 \times 2$ rule also gives good results since the dominant mode is bending.

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Table I. Comparison of different integrating rules for the cantilever problem

Integrating rule	End deflection	Percentage error	Axial stress at wall	Percentage error
13	0·0933344	0·0220	1830·73	0·2689
14	0·0933570	0·0023	1825·55	0·0148
2 × 2 × 2	0·0924773	0·9401	1859·41	1·8397
3 × 3 × 3	0·0933549	0·0000	1825·82	0·0000
4 × 4 × 4	0·0933549		1825·82	
5 × 5 × 5	0·0933549		1825·82	

Example 2

A pressure-loaded cylinder was considered with a radius/thickness ratio of 50 to 1. Here, four 20-node brick elements were used around the half-circumference (a half used with symmetric cuts) and two axially, making eight elements in all. The thinness of the cylinder gave a very distorted set of elements to test the integrating rules, and generally the results were excellent. Table II shows the various results for the hoop stress in the midsurface at the centre of the half-cylinder, the radial deflection at a typical point on the inner surface, and the radial stress at a typical point on the inner surface, these values being sensibly constant over the respective surface. The 4 × 4 × 4 and 5 × 5 × 5 point rules are again added to show the correctly integrated values, and the percentage relative error of the 13, 14 and 3 × 3 × 3 point rules are derived from these (the 5 × 5 × 5 point rule specifically, since minor variations between the 4 × 4 × 4 and 5 × 5 × 5 point rules exist). The results are again very good, although this time the 13 point rule is better than the 14 point rule. The errors with the radial stress are larger than usual because that particular component is of small order compared with other components. The results for the 2 × 2 × 2 rule are very inaccurate, particularly for the stresses, since no bending exists.

Table II. Comparison of different integrating rules for moderately thin cylinders

Integrating rule	Hoop stress	Percentage error	Radial deflection	Percentage error	Radial stress	Percentage error
13	$0{\cdot}495400 \times 10^5$	0·0164	2·48089	0·0016	$-0{\cdot}100050 \times 10^4$	1·9435
14	$0{\cdot}495513 \times 10^5$	0·0392	2·48085	0·0032	$-0{\cdot}976273 \times 10^3$	4·3179
2 × 2 × 2	$-0{\cdot}344747 \times 10^5$	168·6010	2.78646	12·3151	$-0{\cdot}169223 \times 10^6$	
3 × 3 × 3	$0{\cdot}495517 \times 10^5$	0·0400	2.48085	0·0032	$-0{\cdot}974910 \times 10^3$	4·4515
4 × 4 × 4	$0{\cdot}495315 \times 10^5$		2·48093		$-0{\cdot}102125 \times 10^4$	
5 × 5 × 5	$0{\cdot}495319 \times 10^5$		2·48093		$-0{\cdot}102033 \times 10^4$	

Example 3

The problem of Example 2 was repeated with the radius/thickness ratio increased to 1,000 : 1. This problem was used to test the integrating rules in extremely distorted elements. The same reference values as in Example 2 were used (Table III). This time the relative errors are expressed in terms of the simple results for hoop stress and deflection, with pressure of 1,000 units. The radial stress should equal − 1,000 units, and is seen to be poor, particularly in the 3 × 3 × 3 case, again because of the relatively small order of this component and roundoff effects. The 13 and 14 point rules give better

Table III. Comparison of different integrating rules for extremely thin cylinders

Integrating rule	Hoop stress	Percentage error	Radial deflection	Percentage error	Radial stress	Percentage error
13	$0{\cdot}100055 \times 10^7$	0·055	$0{\cdot}998858 \times 10^3$	0·1142	$-0{\cdot}669000 \times 10^3$	33.1
14	$0{\cdot}100071 \times 10^7$	0·071	$0{\cdot}998833 \times 10^3$	0·1167	$-0{\cdot}737000 \times 10^3$	26·3
2 × 2 × 2	$-0{\cdot}568722 \times 10^9$	—	$0{\cdot}399235 \times 10^5$	—	$-0{\cdot}112392 \times 10^{10}$	—
3 × 3 × 3	$0{\cdot}100216 \times 10^7$	0·216	$0{\cdot}998777 \times 10^3$	0·1223	$0{\cdot}116084 \times 10^5$	1,060·84

accuracy in this problem than the $3 \times 3 \times 3$ rule. The results for the $2 \times 2 \times 2$ rule are extremely inaccurate.

The beam problem of Example 1 has been repeated with two 15-node quadratic triangle prism elements in place of each 20-node brick element, and again comparing the results from the $3 \times 3 \times 3$ rule with the 13 and 14 point rules. The accuracy of the different rules were of the same orders as in Example 1. These elements are used frequently with the 20-node elements to accommodate mesh size changes without incurring unduly distorted elements.

A comparison of the 13 point rule with the $3 \times 3 \times 3$ rule has been conducted on a very large three-dimensional structure of complex shape, containing 159 20-node and 15-node elements and 3,912 degrees of freedom. Inspection of the displacements and stresses of largest magnitude showed that the relative errors between the two rules varied by up to 4 per cent. The forward solution execution time was reduced by 75 per cent.

It is concluded from these results that the more economical 13 and 14 point rules are of similar accuracy to the standard $3 \times 3 \times 3$ rule for both very distorted and regular shaped elements with 20 and 15 nodes, and that the quality of accuracy is maintained in very large problems as well as in small test problems. The $2 \times 2 \times 2$ rule is very accurate for certain types of structure (generally shells) subjected to bending modes, but its success in more general applications and loading systems is not guaranteed, and so the rule must be used with care.

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BOOK REVIEWS

APPLIED NUMERICAL METHODS, B. Carnahan, H. A. Luther and J. O. Wilkes, Wiley, New York, 1969. No. of pages: 604. Price: £6·60.

The book is written for those who are engaged in day to day digital computer programming in FORTRAN language. The emphasis throughout the book is on the unified but independent treatment of program logic derived from numerical analysis and calculations done by the computers. The authors have successfully presented various numerical methods together with appropriate programming.

The book is divided into eight chapters covering interpolation and approximation, numerical integration, solution of equations, matrices and related topics, systems of equations, approximation to the solution of ordinary differential equations, approximation to the solution of partial differential equations, statistical methods and numerical methods, with forty well presented computer examples. Use of comment cards in program listings, data and output and flow diagrams make the presentation of computer examples self explanatory and useful.

The book is written for undergraduate students of mathematics, engineering and sciences to bring out the inter-relationship between numerical methods and use of digital computers for calculations with the emphasis on the application of certain numerical procedures. Examples and exercises deal with numerical models corresponding to realistic physical situations mainly taken from the realm of chemical engineering.

The emphasis on usefulness to the beginner means that rigorous proofs of procedures and certain more advanced processes had to be excluded. It appears that sections dealing with non-linear equations and situations suffer most from these criteria.

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SIMULATION MODELING, Forest Paul Wyman, Wiley (Series in Management and Administration), London, 1970. No. of pages: 264. Price: £3·75.

There exists no really useful teaching text on simulation, but this is probably because the subject is difficult to teach except by way of examples and projects. The author clearly recognizes this, and the book makes no attempt to lay down general ideas but teaches through a series of examples. In this way it forms a very good short postgraduate course. Most of the text is easy to read and understand, and there is no doubt that a student who followed the examples of simulation in detail would have mastered the basic ideas to the point where he would be a useful man to employ.

The book does, however, assume the use of Simscript throughout, and it would not, therefore, be useful, except to someone proposing to use that language. The examples given in the text have been run on a Control Data 6400 Computer and the language is also available for users of I.B.M. machines; but there may be many people in the country who normally use computers for which Simscript is not available or who are happily using another language such as C.S.L.

The book is not, in itself, a manual although much of it is self-explanatory. It would be desirable to use a manual in conjunction with the book.

Although most of the examples are realistic, ranging from machine shop loading to the spread of a plague, they are not very complex, and it would be interesting to see a further example in which the language is used to simulate a modest factory in which stores and processes form a network, i.e. when several function in parallel and when several function in sequence.

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SPARSE MATRICES AND THEIR APPLICATIONS, Edited by Donald J. Rose and Ralph A. Willoughby, Plenum Press, New York, 1972. No. of pages: 215. Price: $17·50.

Efficient methods for handling sparse matrix operations are critical to the feasibility of numerical methods in practical application. These operations are replicated in many diverse engineering fields, and it is clear that an important service is performed by furnishing a single forum for individuals responsible for the development and use of sparse matrix algorithms. One such forum was the conference held at the IBM Thomas J. Watson Research Center at Yorktown Heights, New York, in September 1972, the papers of which are collected in this volume. Fifteen papers, grouped under four topic headings, are represented. The areas of Computational Circuit Design, Linear Programming, Partial Differential Equations and Special Topics comprise the four headings. Especially useful is the introductory section of twenty-two pages in which the editors review and place in context each of the proceedings papers. The editing has been done with more care than is usual for proceedings of technical conferences, as demonstrated by the construction of a unified bibliography to which all citations in the respective papers are made.

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