1

Fundamental Concepts

1.1 INTRODUCTION

Among the various numerical methods available, finite element method (FEM) is at present very widely used in every engineering analysis. Several engineering problems will be defined in terms of governing equations written in one, two and three dimensions. Usually these problems are expressed in the form of ordinary or partial differential equations. The problems of structural machanics such as deformation, trusses, stress analysis of automotive aircraft, building and bridge structures, magnetic flux, seepage etc. have been reduced to a system of linear simultaneous equations.

Solution of these equations gives us the approximate behaviour of the continuum. These facts suggest that we need to keep pace with the developments by understanding the basic theory, modeling techniques, and computational aspects of the finite element method. Applications range from problems relating to heat transfer, fluid flow, lubrication, soil machanics, electric and magnetic fields, structural engineering and discussions related to structural analysis problems.

1.2 A BRIEF HISTORY OF FEM

Although the name "finite element" is of recent origin, the basic concept has been used for centuries. The basic concept of finite elements originates from advances in aircraft structural analysis. One of the earlier problems dealt with this concept was by ancient mathematicians. In 1941, **Hrenikoff** found a solution of elasticity problems using the "framework method".

Courant introduced piecewise polynomial interpolation (or continuous trial functions) over triangular subregions to model tortion problems appeared in 1943.

In 1956, Turner et al. derived stiffness matrices for truss, beam, and other elements and presented their new findings.

Clough first introduced the term finite element, appeared in 1960. The two landmark papers to which the origin of the FEM is generally traced are those due to Turner, Clough, Martin and Topp and Argyris and Kelsey. The first book on finite elements by O.C. Zienkiewicz and Cheng was published in 1967. The finite element method was applied to the problems which are non-linear and large deformations in nature, appeared in the late 1960's and early 1970's. A book on nonlinear continua appeared in 1972. It is curious to note that the mathematicians continue to put the finite element method on sound theoretical ground whereas the engineers continue to find interesting extensions in various branches of engineering. In 1959, Greenstadt (279) utilised this technique to discretise cell. He defined the unknowns through a series of functions for each cell, proper variational principle for them and satisfied continuity requirements to tie together the cells giving fundamentalists of finite element technique. Thereafter in 1960s, Clough analysed plane elasticity problems further by this method. The method started getting its base when in 1963, it was recognised by Besseling and Melosh (174) as a form of Ritz's method of approximations. Zienkiewicz and Cheng (280), in 1965, discussed in a broad way and recognised that the method can handle all field problems which can be expressed in variational forms. After this, the technique was recognised by many and went on very fast.

Finite element method, today, is widely used in almost all fields of science and engineering, such as, aeroelasticity, aerodynamics, fluid flows, pipe and channel flows, thermodynamics, soil mechanics, foundation engineering, geotechnical engineering, structural engineering and structural dynamics, pile foundations, machine foundations, nuclear containment systems, lubrication, bearings, fluid and soil structure interactions, electrical technology, textile engineering, cable systems etc. Numerous software packages have been developed of which some of the most popular ones are SAP, ANSYS, ADINA, STAAD, STRUDL.

SAP and ANSYS are of the oldest softwares. SAP has got some of the best facilities for Dynamic analyses even till today. ANSYS is like an encyclopaedia of finite element packages. It has got almost all types of elements, with many facilities for many fields of machanics.

The accuracy of finite element method has always been a question to be answered. Based on Ritz's summation principle for approximating a solution with the help of parameters known at some selected points, the method always is of an approximate one. However, the accuracy can be controlled through

(i) Considering sufficient number of nodes and elements for a problem.
(ii) Choice of suitable approximating functions.
(iii) Choice of suitable numerical integration scheme.

Mathematicians and engineers have simultaneously been developing the approximation technique and some of the noted researchers in the field of FEM are listed below:

Researcher	*Period*
1. Rayleigh	1870
2. Ritz	1909
3. Richardson	1910
4. Galerkin	1915
5. Liebman	1918
6. Biezenokoch	1923
7. Southwell	1940
8. Hrenikoff	1941

Researcher	*Period*
9. Mchenry	1943
10. Courant	1943
11. Prager and Synge	1947
12. Newmark	1949
13. Morsh and Feshback	1953
14. Mchahon	1953
15. Argyris	1955
16. Turner and his group	1956

Researcher	Period	Researcher	Period
17. Clough, Martin and Topp	1956	20. Besseling	1963
18. Greenstadt	1959	21. Melosh	1963
19. Varga	1962	22. Zienkiewicz	1965

FEM Vs. Finite Difference Method (FDM)

1. In FEM, the final differential equation is surpassed, whereas it is the differential equation to start with in the finite difference method.
2. The approximation involved in FEM is physical in nature, as the actual continuum is replaced by finite elements. The element for mutation is mathematically exact whereas the finite difference technique involves the exact representation of the continuum in terms of the differential equation and on this actual physical system, the mathematical model is approximated.
3. FDM needs larger number of nodes to get good results while FEM needs fewer nodes.
4. FDM does not give the value at any point except at node points. It does not give any approximating function to evaluate the basic values (deflections, in case of solid mechanics) using the nodal values. FEM can give the values at the point. However, the values obtained at points other than nodes are by using suitable interpolation formulae.
5. FDM makes **pointwise approximation** to the governing equations i.e. it ensures continuity only at the node points. Continuity along the sides of grid lines are not ensured. FEM makes piecewise approximation i.e. it ensures the continuity at node points as well as along the sides of the element.
6. FEM caters irregular geometry whereas FDM makes stair type approximation to sloping and curved boundaries.

Finite Element Method Vs. Classical Methods

1. In FEM, we can obtain approximate solutions for the exact equations formed, whereas in classical methods exact equations are formed and exact solutions are obtained.
2. The FEM provides the solutions for all problems whereas classical method yields solutions for a few standard cases.
3. If structure consists of more than one materials, it is difficult to use classical method, but finite element can be used without any difficulty.
4. When material property is not isotropic, solutions for the problems become very difficult in classical method. Only few simple cases have been tried successfully by researchers. FEM can handle structures with anisotropic properties also without any difficulty.
5. FEM is superior to the classical methods in the sense that the problems which cannot be tackled by classical methods without making drastic assumptions.

1.3 NEED FOR STUDYING FEM

One may ask the question: "What is the need to study finite element method when there is a number of users-friendly packages available in the market?" This argument is not sound. The mathematicians continue to put the finite element method on sound theoretical ground whereas the engineers continue to find interesting extensions in various branches of engineering. Hence, the FE knowledge makes a good engineer better while just user without the knowledge of FE may produce more dangerous results. In order to use FEM packages properly, the user must know the following points clearly:

1. How to discretise the domain to get good results?
2. Identification of variables.
3. Which elements are to be used for solving the problems in hand?
4. Incorporation of boundary conditions.
5. Solution of simultaneous equations.
6. Choice of approximating functions.
7. Identification of variables.
8. How the element properties are developed and what are their limitations?

The FEM is a product of the computer age, and the application of the method to solve practical problems requires use of computer programs for analysis. Nowadays no such programs are developed as there are several commercial finite element packages available that can solve varieties of problems. Some of these packages are ANSYS, ALGOR, NISA, ABAQUS, NASTRAN etc. having pre- and postprocessors that give graphical pictures of the structure before and after loading.

Pre-processor helps in the generation of the finite element mesh and prepares the data for direct input into the analysis. This becomes essential as the modern structures have complicated shapes and geometry. As the geometry and loading become more and more complicated the discretization also becomes very cumbersome and nodal numbering is really a strenuous one.

Some of these packages also address questions like, "which type of analysis is the best for the problem?", "what is the best element type for the application?", "how can one combine different types of elements?'' etc. The preprocessor graphically displays the structure to be analyzed. If the discretisation is not perfect then it can be modified before the analysis. Briefly, in pre-processing we build the model with defining geometry; specifying element type; defining material properties; creating meshes and nodes with numbering.

The post-processor is again a program that presents the results of the finite element analysis usually graphically and also performs further calculations as desired by the user on the results. This is a very attractive feature of the commercial packages.

Example: If at a node we have six degrees of freedom (3 displacements and 3 rotations) and 500 nodes, then nodal displacements will be 3000, i.e. we will get a print out of 3000 displacements. To get any physical idea of these will once again be a formidable task. On the other hand, a graphical picture incorporating these values and showing the structure in its displaced position will give a physical grasp of the problems. Briefly, in postprocessing we extract results such as displacements, stresses etc.; time–history relation wherever applicable; and graphical representation of the results.

The commercial package ANSYS is used to show what has been explained so far. Hence, it is necessary that the users of FEM package should have sound knowledge of FEM.

1.4 BASIC CONCEPTS OF ELASTICITY

The concept of elasticity provides a more sophisticated tool of stress analysis than strength of materials approach. It can be observed that when a wet ball impinges on a hard floor a circular path is observed. This shows that the ball is compressed at the point of concussion. When a body is compressed it is slightly deformed, and almost immediately tends to recover its original shape. Internal forces are called into play to restore the body to its original shape.

But, when a glass ball is dropped on a marble floor it rebounds and attains a height almost equal to the original one ; but if dropped on a wooden floor the height regained is much smaller. Again if balls of different materials are dropped on the same floor the heights regained are different. The velocities of all these balls will be the same when they reach the floor, but the velocities of the rebound are different and so the heights reached are also different.

The property by virtue of which the differences in the velocities of the rebound are caused, and the bodies tend to revert to the original shapes is called **elasticity**.

All bodies are more or less elastic. They possess the property of elasticity in different degrees.

Stresses and Equilibrium

In continuum mechanics, two distinct types of forces are considered: body forces and surface forces. Forces which are distributed over the volume of the body are called body forces. That is, body forces are forces that act on every element of a material and hence on the entire volume of the material. These forces are expressed as 'force per unit mass of the element'. Self weight of a structure, inertia force, magnetic force are examples of such body forces.

Forces distributed over the surface of the body are called surface forces. That is, surface forces act on the surface of a material. This surface may be either a part or the whole of the boundary surface, if any, of the material or an imaginary surface visualized in the interior of the material. In other words, surface forces are external forces that act on the boundary surface of the material. These are usually expressed as 'force per unit surface area of the element'. Hydrostatic pressure, frictional force, wind forces, and forces exerted by a liquid on a solid immersed in it are examples of such surface forces. **Stresses** are system's internal bond forces acting at molecular level at all points and in all directions. **Strains** are defined as rate of change of displacements (deformations) with respect to the original dimensions (positions). In a deformed body they are present at all points and in all directions.

Preliminary Definitions and Results in Elasticity

Definition (1) (Stress): The intensity of internally distributed forces that tend to resist change in the shape of a body is defined as stress.

$$\text{Stress } (\sigma) = \frac{\text{Load}}{\text{Cross-sectional area}} = \frac{P}{A} \frac{(\text{N})}{(\text{mm}^2)}$$

Definition (2) (Axial strain or linear strain or primary strain): Change in length per unit length of linear dimension of a body is defined as linear strain.

$$\text{Strain } (e \text{ or } \varepsilon) = \frac{\text{Change in length}}{\text{Original length}} = \frac{(\delta l)\ \ (\text{mm})}{(l)\ \ (\text{mm})}$$

Definition (3) (Lateral strain or secondary strain): It is defined as the ratio of change in lateral dimension to original dimension.

$$\text{Lateral strain} = \frac{\text{Change in diameter}}{\text{Original diameter}} = \frac{\delta d}{d}$$

Definition (4) (Volumetric strain): It is defined as the ratio of change in volume to original volume.

$$\text{Volumetric strain} = \frac{\text{Change in volume}}{\text{Original volume}} = \frac{\delta V}{V}$$

Definition (5) (Poisson's ratio): It is the ratio of lateral strain to linear strain.

$$\text{Poisson's ratio} = \frac{\text{Lateral strain}}{\text{Linear strain}}$$

Definition (6) (Body): A body is a portion of matter occupying a finite portion of space (and so is bounded).

Definition (7) (Mass, volume, density): Mass is the quantity of matter in the body. The amount of space occupied by a body is called its volume. Mass per unit volume of the body is called its density.

Definition (8) (Particle): A body of indefinitely small dimensions is a particle. Or, a particle is regarded as a mathematical point endowed with a mass.

Note: Very often the dimensions of a body are not considered, and we treat the body as if its whole mass is concentrated at a single point, or for purposes of discussion we treat it as a particle.

Definition (9) (Displacement): The displacement of a moving particle is the change of its position. It has both magnitude and direction.

Definition (10) (Strain energy): When a member is gradually loaded, within its elastic limit, it deforms. On removal of the load it regains its original shape. During deformation it absorbs the energy and gives back the energy while regaining the shape. The energy stored is called "strain energy".

Definition (11) (Work): A force acting on a particle is said to do work when the point of application of the force moves in the direction of the line of action of the force.

Definition (12) (Work done): Whenever a force S moves through a small distance , dx, then work is said to be done.

$$\begin{aligned}\text{Work done} &= \text{Force} \times \text{Distance moved}\\ &= Sdx\end{aligned}$$

(i) If S and dx are in the same direction, then that work is said to be positive.

(ii) If S and dx are in opposite directions (such as frictional force), then the work is said to be negative.

Example: For a linearly elastic rod shown in Fig. 1.1, the force varies linearly from O to P when the displacement increases from O to Δ.

$$\text{Small work done} = (+S)\,(+dx)$$

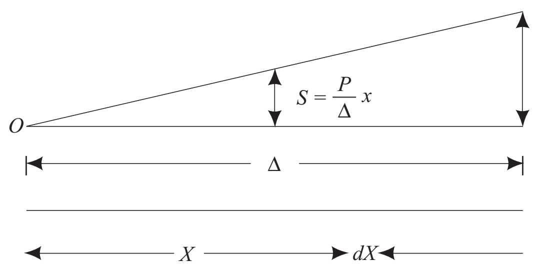


Fig. 1.1

We know $$S = \frac{P}{\Delta}x$$

Since in a linearly elastic material, force varies linearly with deformation

$$\text{Work done} = \int_O^\Delta S dx = \int_O^\Delta \frac{P}{\Delta} x dx$$

$$= \frac{1}{2} P\Delta \qquad \text{...[1]}$$

$$= (\text{average force} \times \text{distance moved})$$

Eq. (1) represents external work

$$\therefore \qquad \text{External work} = +\frac{1}{2} P\Delta$$

Definition (13) (Internal work): It is observed that at every section of Fig. 1.2, the material offers a resistance equal to the external force whose direction is opposite to that of the external force. But magnitude is same. This resistance undergoes the same displacement but with a difference. That is, in this case the force moves against the direction of displacement. Hence, work is negative.

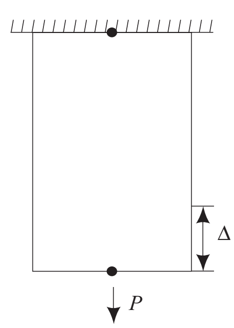


Fig. 1.2 Deformed configuration

$\therefore$ $$\text{Internal work} = -\frac{1}{2} P\Delta \quad ...[2]$$

Definition (14) (Strain energy due to shear stress): Strain energy is the work done by shear force. It is given by the relation

$$U = \left(\frac{q^2}{2c} \times V\right)$$

where q = shear stress
c = modulus of rigidity
V = volume of the given material

Definition (15) (Column, strut): Any member subjected to compressive stress is called strut. Strut may be vertical, horizontal, or inclined. The vertical strut is called as column.

Definition (16) (Isotropic material): Materials having the same properties in all directions.

Definition (17) (Anisotropic material): Anisotropic materials possess different machanical properties in different directions with reference to their crystallographic planes.

Definition (18) (Orthotropic): If the material has three orthogonal planes of symmetry it is said to be orthotropic.

Definition (19) (Structure): Structure is defined as an assemblage of finite elements, interconnected at finite number of joints called nodes as shown in Fig. 1.3.

The deformations anywhere in the structure can be defined in terms of nodal displacements. For the truss shown in Fig. 1.3 if the displacements at the four joints are known, deformations at any other point can be uniquely determined.

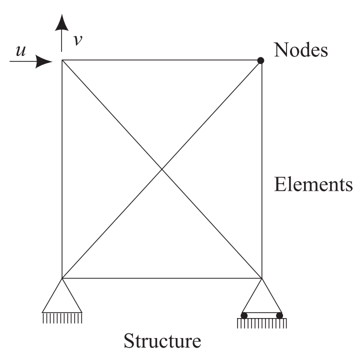


Fig. 1.1 Truss

Definition (20) (Degrees of freedom [Generalized coordinates]): These are the number of independent coordinates which must be specified to uniquely define all the displacements.

For the truss shown in Fig. 1.3 the degree of freedom is 5. This is also called kinematic degree of indeterminacy. The structure (finite element) has a fixed number of nodes. Each node is capable of deformation. The deformation includes displacements, rotations and/or strains. These are collectively known as Nodal degrees of freedom.

Boundary conditions

A structure cannot resist any external load or its own weight without proper boundary conditions. Because of all these the problems in structural machanics fall into boundary value problems essentially.

Work, Energy

Consider a structure in equilibrium under the action of forces $\{Q\}$ with a deformed configuration represented by generalized displacement coordinates $\{q\}$. Consider a conservative system and the loads are applied gradually. If W denotes the work done by the forces $\{Q\}$ on displacement $\{q\}$, then the external work W done by forces $\{Q\}$ on displacements $\{q\}$ (Fig. 1.4) is equal to the strain energy U stored in the structure (energy is conserved).

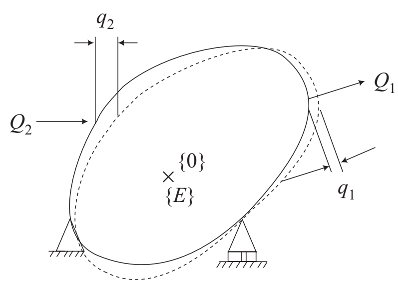


Fig. 1.4 Structure in equilibrium under the action of generalized forces

The structure shown in Fig. 1.5 in state A is the initial configuration. $\{\sigma\}$ are the internal stresses which are in equilibrium with the external loads $\{Q\}$ and the strains {E} are compatible with the displacement $\{q\}$.

Now, we shall study the response of the structure for small variation in displacements $\{\delta q\}$ and forces $\{\delta Q\}$ from initial position.

Due to small variation of displacements and forces there will be changes in work, complementary work, strain energy and complementary energy.

The small variation in displacements Δq takes the structure from equilibrium state A to state B. The incremental changes in work ΔW and strain energy ΔU are given by vertical strips of shaded area as shown in Fig. 1.6(a) (i) and (ii). Magnifying the Fig. 1.6(a) (i) and (ii) as in Figs. 1.6(b) (i) and (ii)

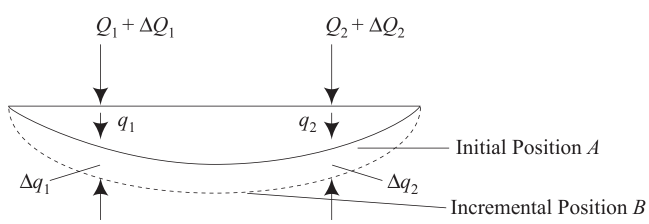


Fig. 1.5 Initial and deformed configuration of the structures

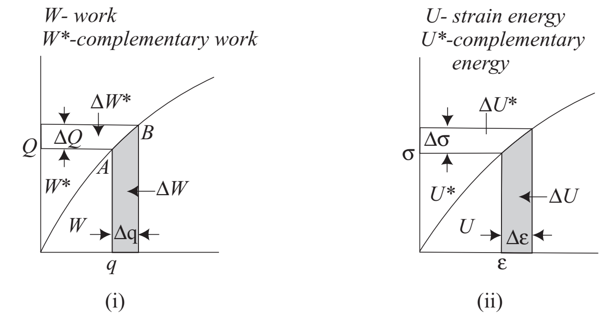


Fig. 1.6(a) Work and energy

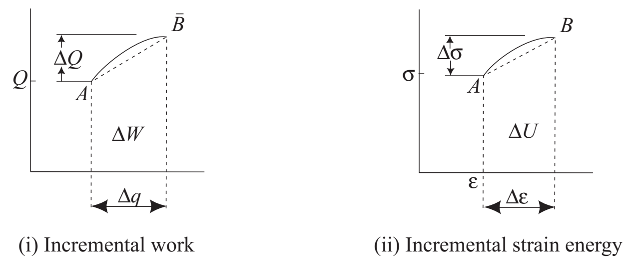


Fig. 1.6(b)

$$\Delta W = \Sigma Q_j \, \Delta V_j + \frac{1}{2} \, \Sigma \, \Delta \, Q_j \, \Delta V_j + \text{higher order terms}$$

In matrix form

$$\Delta W = \{\Delta q\}^T \{Q\} + \frac{1}{2} \{\Delta q\}^T \{\Delta Q\} \qquad \text{...[1]}$$

Similarly, ΔU can be written as

$$\Delta U = \int_v \{\Delta \, \varepsilon\}^T \; \{\sigma\} \, dv + \frac{1}{2} \int_v \{\Delta \, \varepsilon\}^T \; \{\Delta \sigma\} \, dv \qquad \text{...[2]}$$

Recalling that the product of stress and strain represents energy/unit volume, if we assume that $\{Q\}$ and $\{\sigma\}$ remain unchanged while considering the variation of W and U which is equivalent as saying

$$\{\Delta\sigma\} = 0 \qquad \text{...[3]}$$

$$\{\Delta Q\} = 0 \qquad \text{...[4]}$$

In the study of continuous media, we are concerned with the manner in which forces are transmitted through a medium. If V is the volume occupied by a three-dimensional body bounded by a surface S, then any point in the body is represented by x, y, z coordinates.

The external loads acting on the body include point forces, distributed force per unit area T, also called traction. Under the loads the body deforms. The deformation of a point x (= $[xyz]^T$) is given by the three components of the displacement vector

$$U = [u, v, w]^T \qquad \text{...[1]}$$

The external load (force vector) F consists of the following three types of loads that a body may normally be subjected to the vector (body force vector) f, given by

$$f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = [f_x f_y f_z]^T \qquad \text{...[2]}$$

where f_x, f_y, f_z are the components in the x, y, z directions, respectively. The work done by these forces can be expressed as

$$\text{Work done} = \int_v u^T f\, dv \qquad \text{...[3]}$$

The surface traction T can be expressed as

$$T = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = [T_x T_y T_z]^T \qquad \text{...[4]}$$

where T_x, T_y, T_z are the traction components in the x, y, z directions respectively. The work done by the forces will be

$$\text{Work done} = \int_A u^T T\, ds \qquad \text{...[5]}$$

An external load P acting at a point i is represented by its three components (Fig. 1.7a)

$$P_i = [P_x, P_y, P_z]_i^T \qquad \text{...[6]}$$

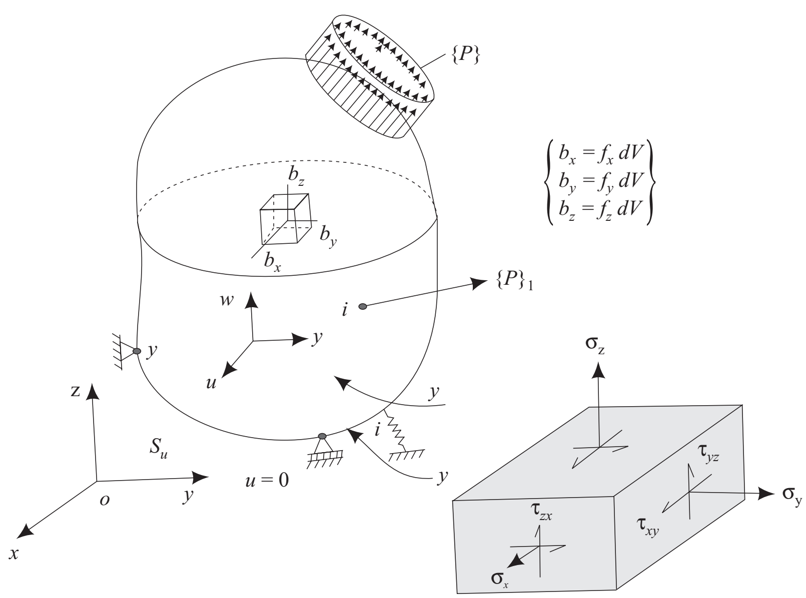


Fig. 1.7(a)

The state of stress in an elemental volume of a loaded body is defined in terms of six components of stress, expressed in a vector form as:

$$\sigma = \{\sigma\} = [\sigma_x\ \sigma_y\ \sigma_z\ \tau_{xy}\ \tau_{yz}\ \tau_{zx}] \qquad \text{...[7]}$$

where σ_x, σ_y, σ_z are normal stresses and τ_{yz}, τ_{xz}, τ_{xy} are shear stresses. The stresses must satisfy the equilibrium condition at each and every point within the body, considering the equilibrium of the elemental volume: $dV = dx\ dy\ dz$, (Fig. 1.7b).

Writing: $\Sigma F_x = 0$; $\Sigma F_y = 0$; $\Sigma F_z = 0$, we get the equilibrium equations

$$\left\{\begin{array}{l} \dfrac{\partial\sigma_x}{\partial x} + \dfrac{\partial\tau_{xy}}{\partial y} + \dfrac{\partial\tau_{xz}}{\partial z} + f_x = 0 \\ \dfrac{\partial\tau_{xy}}{\partial x} + \dfrac{\partial\sigma_y}{\partial y} + \dfrac{\partial\tau_{yz}}{\partial z} + f_y = 0 \\ \dfrac{\partial\tau_{xz}}{\partial x} + \dfrac{\partial\tau_{yz}}{\partial y} + \dfrac{\partial\sigma_z}{\partial z} + f_z = 0 \end{array}\right\} \qquad \text{...[8]}$$

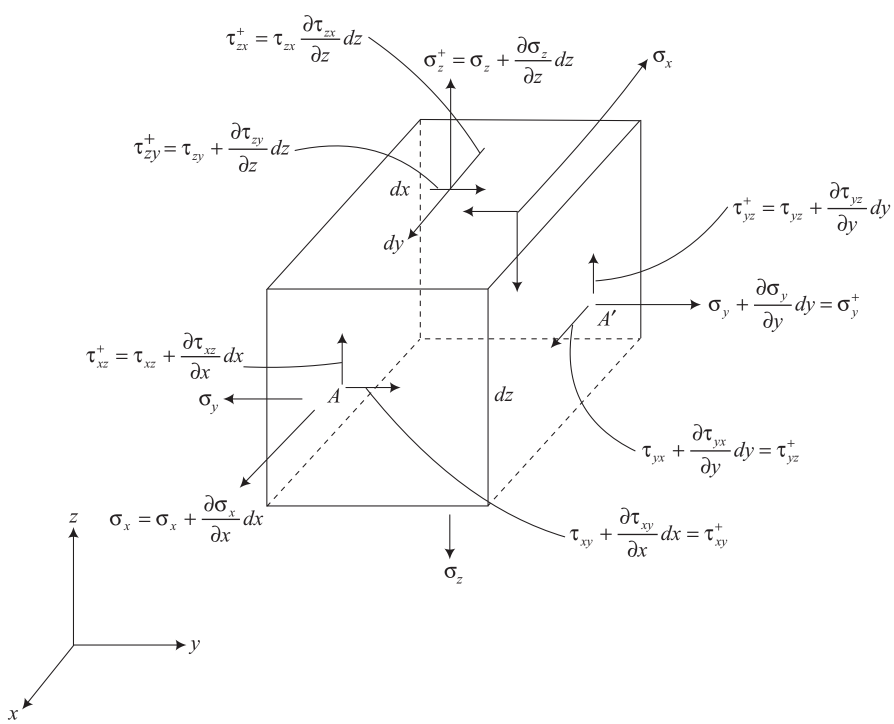


Fig. 1.7(b) Equilibrium of elemental volume, $u = 0$ on S_u

Sign convention: We shall employ the following sign convention for stresses.

A normal stress is positive when it produces tension; therefore, if it produces compression it is negative. Shear stresses shown in Fig. 1.8(a) is accompanied with two subscripts, the first indicating the plane given by its normal on which it acts and the second gives the direction of the component of the stress. To explain it further, if we consider the two shearing stress components acting on the plane which is perpendicular to x-axis, then the component which is in the direction of y-axis will be τ_{xy} and that in the z-direction will be τ_{xz}. The positive directions of the components of shearing stress on any side of the element are taken as the positive directions of the coordinate axes if a tensile stress on the same would have the positive direction of the corresponding axis. Thus if on any side, the tensile stress is opposite to the positive direction of the reference axis, the positive directions of the shearing stress components will then be reversed. All the stress components shown in Fig. 1.8(a) are positive, so that we can say that a stress is positive when it is on positive face in positive direction or on negative face in negative direction [Figs. 1.8(b) and (c)]. From the Fig. 1.8(a) it is clear that there are three components of normal stress—σ_x, σ_y and σ_z and six components of shear stress: τ_{xy}, τ_{yz}, τ_{zx}, τ_{yx}, τ_{zy} and τ_{xz}. Further it is known that $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$ and $\tau_{yz} = \tau_{zy}$. Therefore, to describe the stress at a point, the information of six quantities σ_x, σ_y, σ_z, σ_{xy}, σ_{yz}, σ_{zx} are all that is needed.

Boundary Conditions: There are two types of boundary conditions namely surface loading conditions and displacement boundary conditions, in addition to equations of equilibrium compatibility and constitutive law boundary conditions, to completely specify a problem in solid mechanics. If u is specified on part of the boundary denoted by S_u,

we have, $u = 0$ on S_u. ...[1]

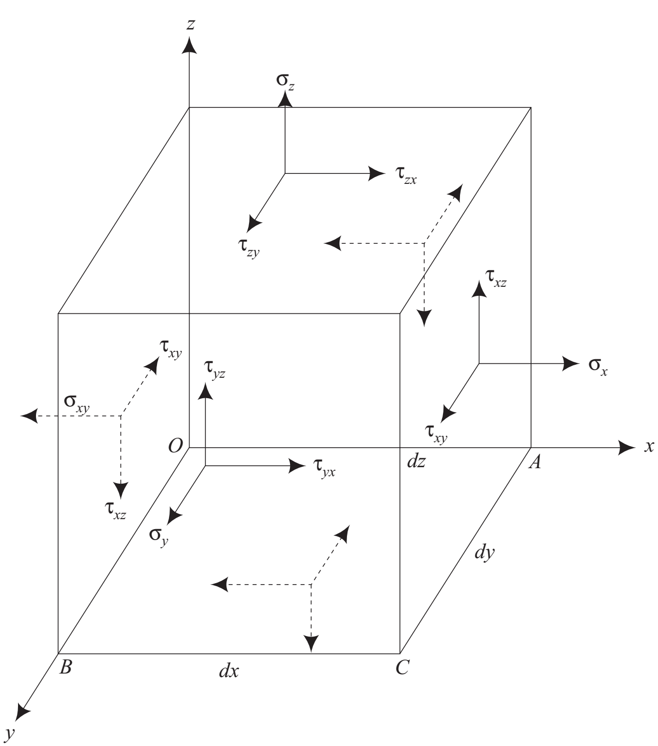


Fig. 1.8(a) Stressed element

Table 1.1

Face	*Stress on -ve Face*	*Stress on +ve Face*	*Remarks*
x	σ_x	$\sigma_x^+ = \sigma_x + \frac{\partial \sigma_x}{\partial x} dx$	Stress on positive face is equal to the stress on negative face plus rate of change of that stress multiplied by the distance between the faces. (See Fig. 1.9)
	τ_{xy}	$\tau_{xy}^+ = \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx$	
	τ_{xz}	$\tau_{xz}^+ = \tau_{xz} + \frac{\partial \tau_{xz}}{\partial x} dx$	
y	σ_y	$\sigma_y^+ = \sigma_y + \frac{\partial \sigma_y}{\partial y} dy$	
	τ_{yx}	$\tau_{yx}^+ = \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy$	
	τ_{yz}	$\tau_{yz}^+ = \tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} dy$	
z	σ_z	$\sigma_z^+ = \sigma_z + \frac{\partial \sigma_z}{\partial z} dz$	
	τ_{zx}	$\tau_{zx}^+ = \tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz$	
	τ_{zy}	$\tau_{zy}^+ = \tau_{zy} + \frac{\partial \tau_{zy}}{\partial z} dz$	

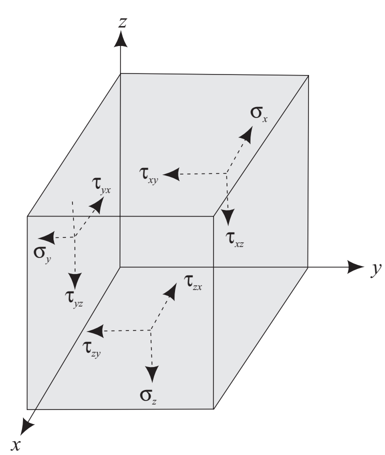


Fig. 1.8(b)

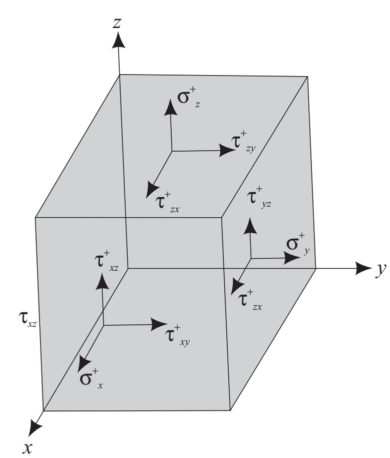


Fig. 1.8(c)

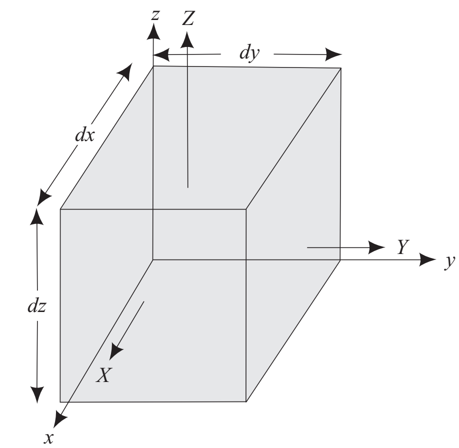


Fig. 1.9

Let us consider the equilibrium of an orthogonal tetrahedron element ABCD shown in Fig. 1.10 where *DA*, *DB* and *DC* are parallel to the x, y and z coordinate axes, respectively and dA = area of *ABC*, lies on the surface.

If $n = [n_x, n_y, n_z]^T$ is the outward drawn normal vector to dA, then

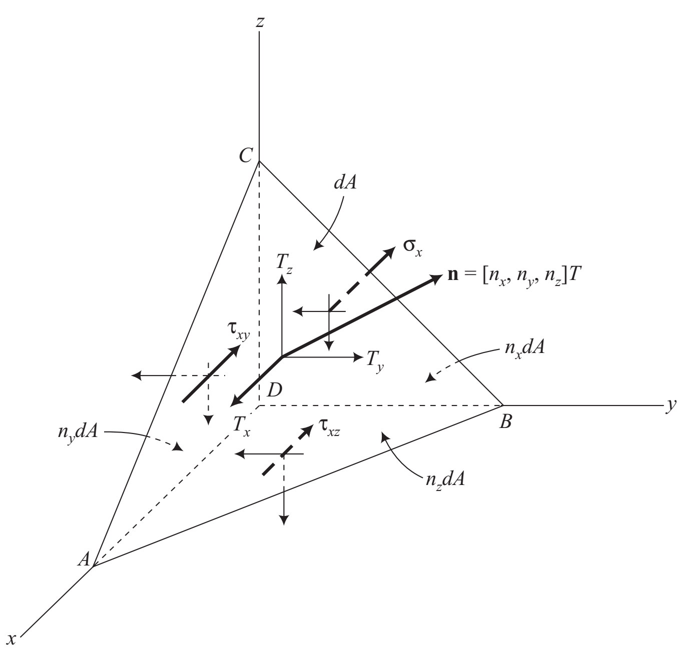


Fig. 1.10 An elemental volume at surface

$$\left\{\begin{aligned} \text{area } BDC &= n_x dA \\ \text{area } ADC &= n_y dA \\ \text{and} \quad \text{area } ADB &= n_z dA \end{aligned}\right\} \qquad ...[2]$$

In equilibrium, along the three axes direction yields

$$\left\{\begin{aligned} \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z &= T_x \\ \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z &= T_y \\ \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z &= T_z \end{aligned}\right\} \qquad ...[3]$$

The conditions [1], [2] and [3] must be satisfied on the boundary, S_T, where the tractions are applied.

Strain–Displacement Relations

The relations between the components of strain and the displacement components *u*, *v* and *w* at a point are

$$\varepsilon = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}]^T \qquad ...[1]$$

where ε_x, ε_y, and ε_z are normal strains and γ_{yz}, γ_{xz} and γ_{xy} are the engineering shear strains.

Figure 1.11 yields the deformation of the $dx - dy$ face for small deformations, which we consider here, and also considering the other faces, we get

$$\varepsilon = \left[\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}, \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]^T \qquad ...[2]$$

where the strain and displacement components are given by:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]$$

$$\varepsilon_z = \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \qquad ...[3]$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial x}$$

$$\left.\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x};\ \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\ \varepsilon_y &= \frac{\partial v}{\partial y};\ \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \\ \varepsilon_z &= \frac{\partial w}{\partial z};\ \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{aligned}\right\} \qquad \text{...[4]}$$

Stress–Strain Relations

The basic law of proportionality of stress and strain published by Robert Hooke in 1678 expressed in the form of law known as **Hooke's Law** which states that ''within elastic limits of materials the elongation produced by the tensile force is proportional to the tensile force''.

If the elongation is denoted by e and the tensile stress by T, then $T = Ee$, where E is a constant depending on the properties of the material. The constant E is called the modulus of tension (or Young's modulus).

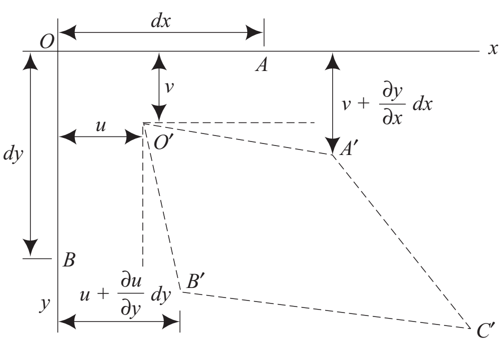


Fig. 1.11 Deformation of the element

Generalized Hooke's Law

The constitutive equations for a linear, elastic solid which relates all the stresses to all the strains is called the generalized Hooke's law. It has the form

$$\sigma_{ij} = C_{ijkm}\,\varepsilon_{km} \qquad \text{...[1]}$$

where C_{ijkm} are the *elastic constants* and in eq. [1] we have 81 such constants since C_{ijkm} is a tensor of fourth rank. The above equation is meant for an anisotropic material and the only requirement is that the material should be stressed within the elastic limits. However, due to symmetry of both stress and strain tensor there are utmost 36 distinct elastic constants only and the Hooke's law can be written as

$$\left\{\begin{array}{l}\sigma_{11} = C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33} + C_{14}\varepsilon_{23} + C_{15}\varepsilon_{31} + C_{16}\varepsilon_{12} \\ \sigma_{22} = C_{21}\varepsilon_{11} + C_{22}\varepsilon_{22} + C_{23}\varepsilon_{33} + C_{24}\varepsilon_{23} + C_{25}\varepsilon_{31} + C_{26}\varepsilon_{12} \\ \sigma_{33} = C_{31}\varepsilon_{11} + C_{32}\varepsilon_{22} + C_{33}\varepsilon_{33} + C_{34}\varepsilon_{23} + C_{35}\varepsilon_{31} + C_{36}\varepsilon_{12} \\ \sigma_{23} = C_{41}\varepsilon_{11} + C_{42}\varepsilon_{22} + C_{43}\varepsilon_{33} + C_{44}\varepsilon_{23} + C_{45}\varepsilon_{31} + C_{46}\varepsilon_{12} \\ \sigma_{31} = C_{51}\varepsilon_{11} + C_{52}\varepsilon_{22} + C_{53}\varepsilon_{33} + C_{54}\varepsilon_{23} + C_{55}\varepsilon_{31} + C_{56}\varepsilon_{12} \\ \sigma_{21} = C_{61}\varepsilon_{11} + C_{62}\varepsilon_{22} + C_{63}\varepsilon_{33} + C_{64}\varepsilon_{23} + C_{65}\varepsilon_{31} + C_{66}\varepsilon_{12}\end{array}\right\} \qquad ...[2]$$

which can be written as

$$\sigma_i = C_{ij}\,\varepsilon_j \qquad (i, j = 1, 2, 3, 4, 5, 6) \qquad ...[3]$$

This way it is understood that

$$\sigma_{11} = \sigma_1;\ \sigma_{22} = \sigma_2;\ \sigma_{33} = \sigma_3;\ \sigma_{23} = \sigma_{32} = \sigma_4;\ \sigma_{31} = \sigma_{13} = \sigma_5;\ \sigma_{12} = \sigma_{21} = \sigma_6 \qquad ...[4]$$

and

$$\varepsilon_{11} = \varepsilon_1;\ \varepsilon_{22} = \varepsilon_2;\ \varepsilon_{33} = \varepsilon_3;\ 2\varepsilon_{23}, = 2\varepsilon_{32} = \varepsilon_4;\ 2\varepsilon_{31} = 2\varepsilon_{13} = \varepsilon_5;\ 2\varepsilon_{12} = 2\varepsilon_{21} = \varepsilon_6 \qquad ...[5]$$

Constitutive Matrix: The system of equations [2] can be written in matrix form, for a linear elastic, anisotropic and homogeneous material.

$$\begin{Bmatrix}\sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx}\end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{16} \\ c_{21} & c_{22} & \dots & c_{26} \\ \dots & \dots & \dots & \\ \dots & \dots & \dots & \\ \dots & \dots & \dots & \\ c_{61} & c_{62} & \dots & c_{66}\end{bmatrix} \begin{Bmatrix}\varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx}\end{Bmatrix}$$

or $\{\sigma\} = [C]\,\{\varepsilon\}$ ...[1]

where $[C]$ is called the constitutive matrix and its inverse relation for strain is given by

$$\begin{Bmatrix}\varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx}\end{Bmatrix} = \begin{bmatrix} d_{11} & d_{12} & \dots & d_{16} \\ d_{21} & d_{22} & \dots & d_{26} \\ \dots & \dots & \dots & \\ \dots & \dots & \dots & \\ \dots & \dots & \dots & \\ d_{61} & d_{62} & \dots & d_{66}\end{bmatrix} \begin{Bmatrix}\sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx}\end{Bmatrix}$$

Thus, from Hooke's law, for linear elastic materials, we find stress–strain relationship in three dimensions (and later we will reduce it to two dimensions).

$$\left.\begin{aligned} \varepsilon_x &= \frac{\sigma_x}{E} - \frac{\gamma\sigma_y}{E} - \frac{\gamma\sigma_z}{E} \\ \varepsilon_y &= \frac{\sigma_y}{E} - \frac{\gamma\sigma_x}{E} - \frac{\gamma\sigma_z}{E} \\ \varepsilon_z &= \frac{\sigma_z}{E} - \frac{\gamma\sigma_x}{E} - \frac{\gamma\sigma_z}{E} \end{aligned}\right\} \qquad \text{...[1]}$$

where ε_x, ε_y and ε_z are linear strains. Now, the shearing strains are:

$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

Now,

$$G = \frac{E}{2(1+\gamma)} \qquad \text{...[2]}$$

where γ is the Poisson's ratio of the material
E is the Young's modulus of elasticity
G is the shear modulus of elasticity

From [1], we have

$$\boxed{\varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{(1-2\nu)}{E}(\sigma_x + \sigma_y + \sigma_z), \quad \text{(From Hooke's law)}} \qquad \text{...[3]}$$

Substituting for $(\sigma_y + \sigma_z)$ and so on into equation [1],
we get,

$$\sigma = D\,\varepsilon \qquad \text{...[4]}$$

where

$$\sigma = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xz} \end{Bmatrix} = [\sigma_x\, \sigma_y\, \sigma_z\, \tau_{xz}]^T$$

and

$$D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-\nu \end{bmatrix} \qquad \text{...[5]}$$

Particularly any element spreading in a single direction is known as one-dimensional element. The normal stress σ and the corresponding normal strain ε are related by $\sigma = E\varepsilon$, which is the stress-strain relation in one dimension (Hooke's law).

If u be the displacement, then the linear strain is given by $\varepsilon_x = \dfrac{\partial u}{\partial x}$, which is called the strain–displacement relation. The stress is given by

$$\boxed{\sigma_x = E\varepsilon_x;\ \ \varepsilon_x = D\varepsilon_x}$$

where $D = E$, (Young's modulus).

1.5 POTENTIAL ENERGY AND EQUILIBRIUM

The variational or energy methods are of primary importance in the finite element analysis. In solid mechanics, we come across the problem of determining the displacement u of the body shown in Fig. 1.12, satisfying the equilibrium equations [8]. It may be noted that the stresses are related to strain, which, in turn, are related to displacements. Approximate solution methods normally employ variational or energy methods. The energy a body possesses by virtue of its position is known as potential energy.

POTENTIAL ENERGY: (Π) (a term applicable to conservative fields of force only).

The potential energy of the system arises because of two causes:

(i) Due to position of external loads

(ii) Due to deformation of the structure (i.e., energy stored within the body due to physical deformation known as strain energy. The concept of potential energy is fundamental and exceedingly important in the study of structural mechanics and FEM. The total potential energy is denoted by the Greek letter Π which is a function of displacements.

Definition: (Potential energy) is defined as the negative of the work done in displacing a particle from its standard position to any other position, and the total potential energy, Π of an elastic body is defined as the sum of total strain energy (U) and the work potential (W_p).

The potential energy of the external load P is

$$W_p = \text{(Load) (Displacement from zero potential state).}$$

$$= -Pu \qquad \text{...[1]}$$

where the zero potential state corresponds to $u = 0$. Hence the total potential energy π is

$$\boxed{\Pi = U + W_P} \qquad \text{...[2]}$$

where U is the strain energy, W_p is the potential of the applied loads (or work potential). The strain energy density (strain energy per unit volume in the body) for a linear elastic body is defined as

$$\boxed{dU = \frac{1}{2}\sigma^{T}\varepsilon} \quad \Rightarrow \quad \boxed{\text{total energy} = U = \frac{1}{2}\int_V \sigma^T \varepsilon\, dV}$$

and the potential of the applied loads (or work potential) is given by (W_p).

$$W_p = -\int_V U^T f\, dV - \int_s U^T T.ds - \sum_i U_i^T P_i \quad \text{...[3]}$$

Hence the total potential becomes

$$\Pi = \frac{1}{2}\int_V \sigma^T \varepsilon\, dV - \int_V U^T f\, dV - \int_s U^T T\, ds - \sum_i U_i^T P_i \quad \text{...[4]}$$

Principle of virtual work: Since energy has to balance itself we equate the work done by the system to the work done by the system in opposition.

The application of forces displaces various points of the system deforming it. The internal stresses moving through the internal strains do another work.

Strain energy stored in the linear electric spring:

The spring constant (stiffness of the spring) is a constant denoted by *k* which is merely the force per unit displacement. Here, force and displacement are general terms.

If *D* be the displacement of the spring, then energy stored in the spring is

$$U = \left(\frac{1}{2}\right) \mathrm{k}\, \mathrm{D}^2$$

In case of flexural systems, the strain energy is expressed in two ways:

$$U = \int \frac{1}{2}\, \mathrm{M}^2 \frac{dx}{EI}$$

Alternatively,

$$U = \frac{EI}{2} \int \left[\frac{d^2y}{dx^2}\right]^2 dx$$

Potential of a constant force : Each and every activeforce has a capacity to perform work. For example, it may be observed that from Fig. 1.12, a force is acting at a point A which is the reference point.

Let the potential or potential energy of the force at point *A* be zero. When the force *P* moves through a distance *D* along the line of action to a point *B* it loses some of its capacity to do work. Now the potential of the force at *B* is defined as

Potential at $B = 0 - (+P)(+D)$

$= -PD$

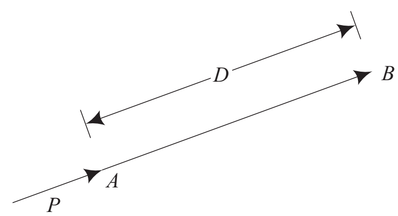


Fig. 1.12 Potential of a force

We may note two points:

(i) Both force and displacement are in the same direction. Hence (+) sign is attached to both of them.

(ii) When the force moves, it loses some of the energy. Hence (–) sign is attached in the beginning.

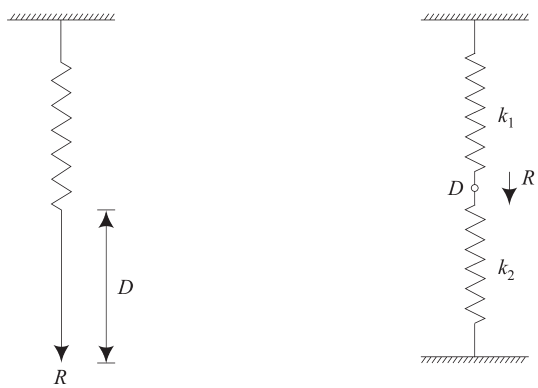
Potential Energy Function

The potential energy function is given by the equation

$$\Pi = U + W_P \qquad ...[1]$$

where U is the strain energy of the material and W_p is the work potential.

Two spring systems are shown in Fig. 1.13.



(a) Single spring (b) Springs in parallel with same displacement

Fig. 1.13

$$U = \frac{1}{2kD^2}$$

$$W_p = -RD$$

$$\pi = \frac{1}{2kD^2} - \frac{1}{2RD} \qquad ...[2]$$

The P.E. function for spring system shown in Fig. 1.13 becomes

$$\pi = \frac{1}{2k_1D^2} + \frac{1}{2k_2D^2} - \mathrm{RD} \qquad ...[3]$$

Equations [1] and [2] are functions which depend on the displacement D.

Hence, they are called potential energy function.

Potential energy functional

Let us consider the potential energy for the beam shown in Fig. 1.14 below:

$$\Pi = U + W_p$$

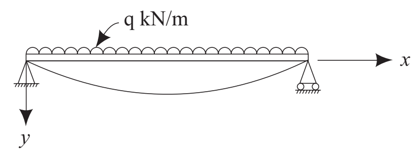


Fig. 1.14

$$U = \int_0^l \frac{EI}{2}\left[\frac{d^2y}{dx^2}\right]^2 dx$$

$$W_p = \int_0^l qy\, dx$$

$$\therefore \qquad \pi = \int_0^l \left[\frac{EI}{2}\left(\frac{d^2y}{dx^2}\right)^2 - qy\right] dx \qquad ...[1]$$

($\because\ \pi = U + W_p$)

[1] is known as a functional.

Note: Π depends not only on y and its derivatives at a point but upon their integrated effect over a region of interest. π becomes a functional when the system has infinite number of degrees of freedom.

Derivation of Principle of Stationary Potential Energy

It is known that the principle of virtual work assumes the following two systems of quantities:

(i) Equilibrium system:
   σ with F in W_p
   and T in S
(ii) A system of virtual displacements u such that $\delta u = 0$ ...[1]

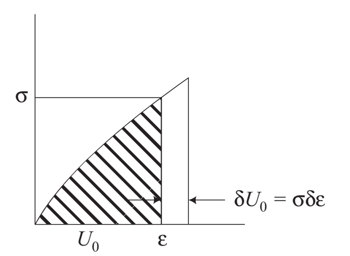


Fig. 1.15 Increment in strain energy

Another set of displacements u such that $u = \bar{u}$ on S_{ij} and u produce (by the principle of stationary potential energy) a set of ε which results in the stress σ through a single valued constitute relation.

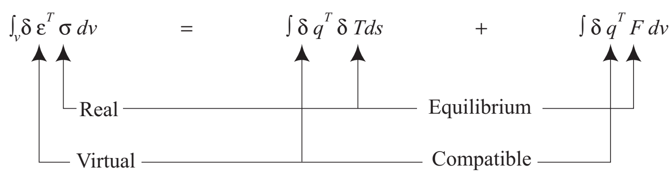
If a unique strain energy density, "$U_0(\varepsilon)$" is associated with every set of strain ε, the material is said to be elastic. For such material

$$U = \int_0^{\varepsilon} \sigma \, \delta \varepsilon \qquad ...[2]$$

energy in an elastic body is conserved and recovered and then

$$\frac{\partial U}{\partial \varepsilon} = \sigma \qquad ...[3]$$

Now, if we consider the set of displacements, u, which produce the set of strains ε producing a set of stresses σ and if we now consider δ be infinitesimal increment in u resulting in incremental strain δε then the left hand side of the virtual work equation



becomes

$$\int_V \sigma \delta \varepsilon \, dv = \int_V \frac{\partial U}{\partial \varepsilon} \delta \varepsilon \, dv = \int_V \delta U_0 \, dv$$

$$= \delta \int U_0 \, dv$$

Therefore, the principle of virtual work becomes

$$\boxed{\delta U = \int_V \delta q^T F \, dv + \int_s \delta q^T T \, ds} \qquad ...[4]$$

which is now the equilibrium condition for stresses σ arising from the displacements q which are subjected to incremental displacements q, and which produces δu.

Eq. [4] is valid for an elastic body with non-conservative external forces. But it is more usual to consider the case of conservative forces for which it is possible to define the potential.

The total potential energy of the external forces V can be written as

$$W_p = -\left(\int_v q^T F \, dv + \int_s q^T T \, ds \right) \qquad ...[5]$$

and the variation in potential is

$$\delta W_p = -\left(\int_v \delta q^T F\, dv + \int_s \delta q^T T\, ds\right) \quad \text{...[6]}$$

Hence Eq. 6.17 may be written as

$$\delta U = -\delta W_p \quad \text{or,} \quad \delta (U + W_p) = 0, \quad \text{...[7]}$$

Defining the total potential π as $U + W_p$

Then the equilibrium $\delta\Pi = 0$ for Principle of Stationary Potential energy

$$\boxed{\delta\Pi = \sum_{i=1}^{n} \frac{\partial \pi}{\partial q_i}\, \delta q_i = 0} \quad \text{...[8]}$$

"Of all possible system of deformations, that are compatible and satisfy the displacement boundary conditions that system which also satisfies equilibrium makes the total potential energy stationary." That is,

$$\frac{\partial \pi}{\partial q_i} = 0 \text{ gives the condition of equilibrium.}$$

Principle of Minimum Potential Energy: It states that among all displacement fields that satisfy the prescribed constraint condition of the structure, those that satisfy the equilibrium condition make the total potential energy a minimum.

It may be observed that the structure (Fig. 1.16) is subjected to the generalized forces in the following two states *A* and *B* as q_i, $q_i + \delta q_i$ $\pi(q_i)$ and $\pi(q_i + \delta q_i)$ refer to the total potential in states *A* and *B* respectively. State *B* is in the vicinity of state *A* and is obtained by applying virtual displacement δq perturbing from states *A* to *B*. Total variation in potential energy in going from states *A* to *B* may be written as

$$\Delta\pi = \pi(q_i + \delta q_i) - \pi(q_i)$$

$$= \sum \frac{\partial \pi}{\partial q_i} \partial q_i + \frac{1}{2} \sum \sum \frac{\partial^2 \pi}{\partial q_i\, \partial q_j} \delta q_i \cdot \delta q_j \ldots. \quad \text{...[1]}$$

$$\Delta\pi = \delta\pi + \frac{1}{2} \delta^2 \pi + \ldots..$$

For equilibrium $\delta\pi = 0$. If in addition we assume linear stress–strain relation (π is quadratic) hence $\delta^3 \pi$ ... and higher order terms are zero

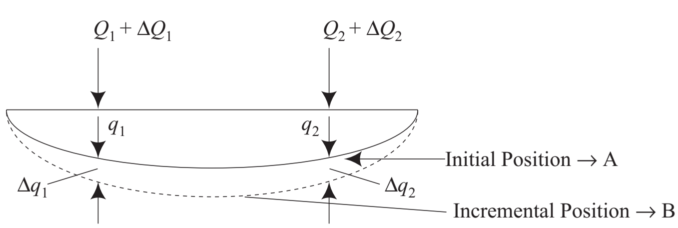


Fig. 1.16 Initial and deformed configurations of structure

$$\Delta\pi = \frac{1}{2} \sum \sum \frac{\partial^2 \pi}{\partial q_i \, \partial q_j} \delta q_i \ \delta q_j \qquad ...[2]$$

Moreover, $\delta^2 \pi$ is not a function of Q because the potential of external forces is first order function in terms of displacement q

$$\Delta\pi = \frac{1}{2} \delta^2 \pi = \frac{1}{2} \delta^2 W_p \qquad [3]$$

Equation [3] is of course a familiar requirement that the slope of a function be zero at stationary point, from differential calculus.

A starting point must meet this condition but this is not sufficient to ascertain if we are at maximum, minimum or neutral points. In order to establish this the sign of curvature must be considered.

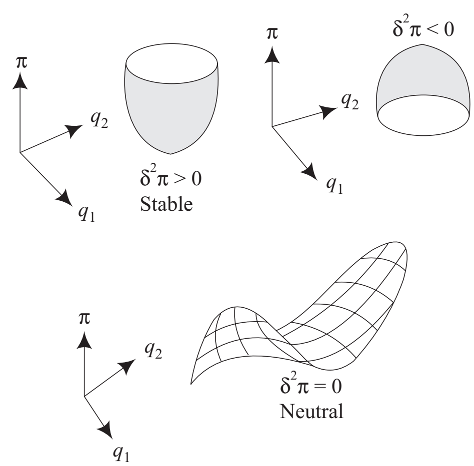


Fig. 1.17 States of equilibrium for two degree of freedom system

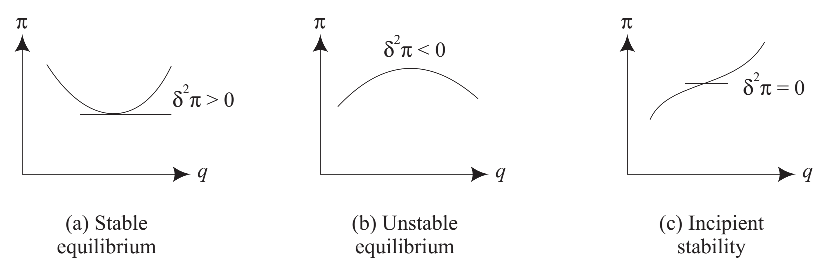


Fig. 1.18

$$\left\{\begin{array}{ll} \delta^2\pi > 0 & \text{then } \pi \text{ is minimum} \\ \delta^2\pi = 0 & \text{then } \pi \text{ is neutral} \\ \delta^2\pi < 0 & \text{then } \pi \text{ is maximum} \\ \text{This is very clear from Figs. 1.17 and 1.18.} & \end{array}\right\}$$

Note: For a linear elastic structure, we have

$$\pi = (U + W_p) \qquad ...(1)$$

where U is always positive and W_p is always negative as shown in Fig. 1.19.

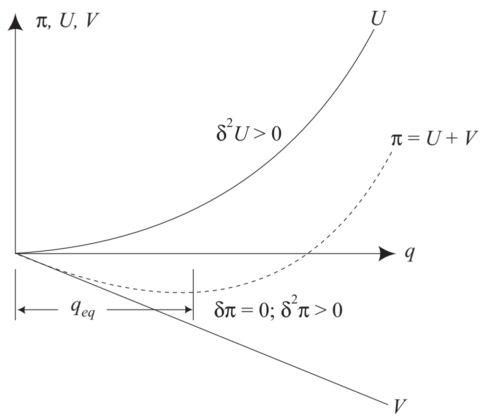


Fig. 1.19 Plot of π, U and V with respect to q

π is quadratic form and $\pi = 0$ locates a minimum point on the plot π against q.

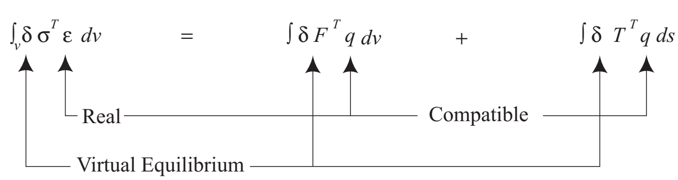
Under these conditions principle of stationary potential energy becomes the principle of minimum potential energy.

It is seen that

(a) Minimum potential energy principle applies only to elastic systems with conservative forces.

(b) $\delta^2 \pi$ is positive (if π is positive definite)—the theorem of stationary potential energy becomes the theorem of minimum potential energy.

Principle of Virtual Forces: The principle of virtual forces may be written as



and the principle of stationary complementary potential may be stated as

$\delta\pi^* = 0$ for a compatible system

$$\delta\,(U^* + V^*) = 0 \qquad ...[2]$$

Derivation of potential energy (Π) for the beam shown in Fig. 1.20.
(The system has a finite number of degrees of freedom.)

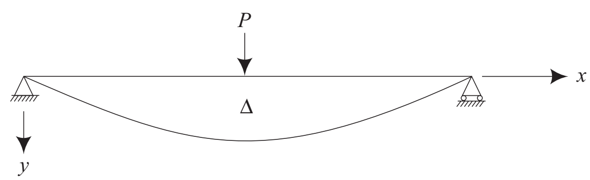


Fig. 1.20 Beam with a central load

Since
$$\pi = U + W_p$$
$$U = \int_0^l \frac{EI}{2}\left(\frac{d^2y}{dx^2}\right)^2 dx$$
$$W_p = -P\Delta$$
∴
$$\boxed{\pi = \int_0^l \frac{EI}{2}\left(\frac{d^2y}{dx^2}\right)^2 dx - P\Delta}$$

Derivation of potential energy functional (Π) for a truss element subjected to uniform traction as shown in Fig. 1.21.

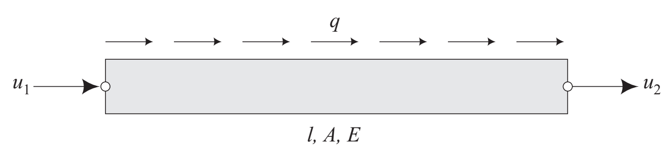


Fig. 1.21 Truss member with uniform traction

Since the system has got infinite degrees of freedom, π expression results in a functional.

Solution:
$$\pi = U + W_p \qquad ...[1]$$
$$U = \int \frac{1}{2}\{\varepsilon\}^T\{\sigma\}\, dv \qquad ...[2]$$
We know
$$\sigma = E\varepsilon$$
$$dv = A\, dx$$
Potential Energy
∴
$$U = \int_0^l \frac{AE}{2}\varepsilon^2\, dx \qquad ...[3]$$

But $$\varepsilon = \frac{du}{dx}$$

$\therefore$ $$U = \int_0^l \frac{AE}{2}\left(\frac{du}{dx}\right)^2 dx \qquad ...[4]$$

$$W_p = -\int_0^l (q\,dx)\,u$$

$$W_p = -\int_0^l qu\,dx \qquad ...[5]$$

Substituting Eqs. (4) and (5) in Eq. (1), we get

$$\Pi = \int_0^l \left[\frac{AE}{2}\left(\frac{du}{dx}\right)^2 - qu\right] dx$$

Example: Establish the P.E. functional for a beam column subjected to a uniformly distibuted load *q* kN/metre as shown in Fig. 1.22.

(Infinite degrees of freedom, hence, P.E. expression results in a functional.)

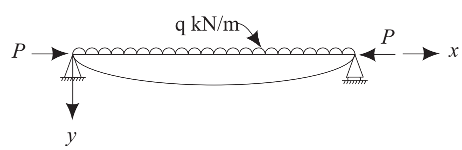


Fig. 1.22 A Beam Column

Solution: Since $$\pi = U + W_P \qquad ...[1]$$

$$U = \int_0^l \frac{EI}{2}\left(\frac{d^2y}{dx^2}\right)^2 dx \qquad ...[2]$$

$$W_p = -P\,\lambda - qy \qquad ...[3]$$

where

$$\lambda = \int_0^l \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx$$

Substituting Eqs. (2) and (3) in Eq. (1), we get

$$\pi = \int_0^1 \left[\frac{EI}{2}\left(\frac{d^2y}{dx^2}\right) - \frac{P}{2}\left(\frac{dy}{dx}\right)^2 - qy\right] dx$$

Computation of λ: From Fig. 1.23(b) originally straight *ds* has become curved *ds*. During this process a descent $d\lambda$ occurs as seen in Fig. 1.23(b).

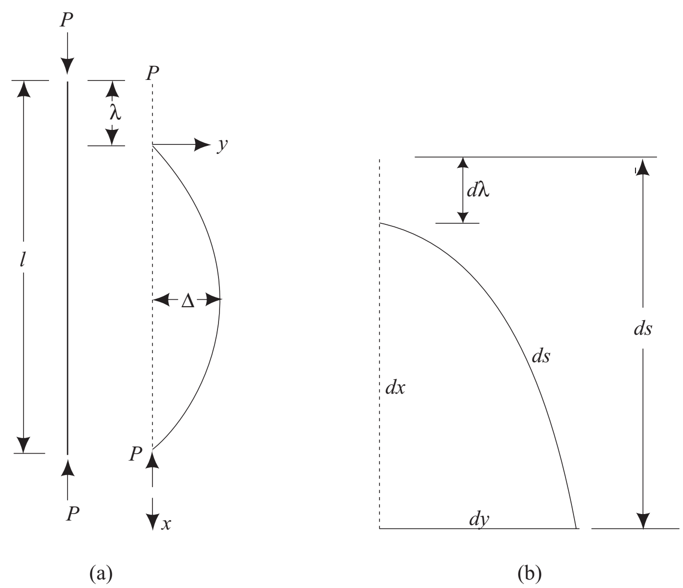


Fig. 1.23 (a) Two equilibrium shapes
(b) Originally straight *ds* has become curved *ds*

$$\therefore \qquad d\lambda = ds - dx$$

$$= \sqrt{dx^2 + dy^2} - dx$$

$$= dx\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} - dx$$

For small values of $\frac{dy}{dx}$, we know

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} = 1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

$$d\lambda = dx\left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right] - dx$$

$$d\lambda = \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx \qquad \text{...[1]}$$

Therefore, for the entire column Eq. (1) is integrated on both sides. Then

$$\lambda = \frac{1}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx$$

Example: Figure 1.24 shows a system of springs. Assemble equations of equilibrium by direct approach. Prove that minimization of potential energy also yields the same result.

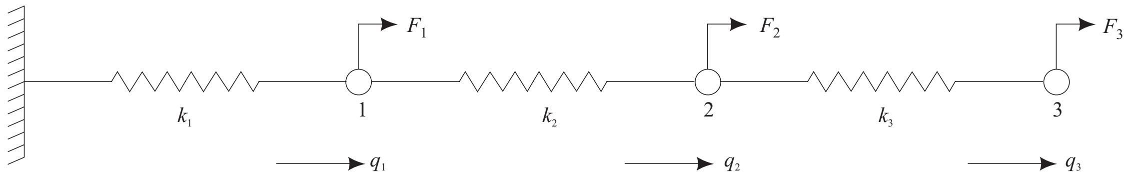


Fig. 1.24

Solution: (1) ***Direct approach:*** Let the nodes ①, ② and ③ be as shown in Fig. 1.25.

If q_1, q_2, q_3 be the displacements of nodes, then the extensions of springs 1, 2 and 3 are:

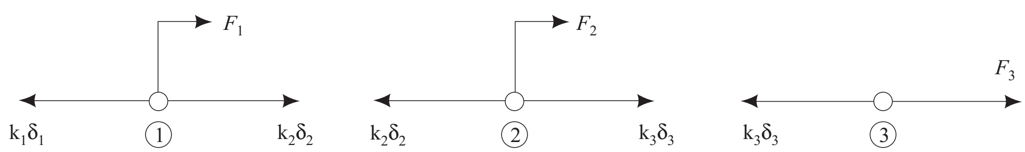


Fig. 1.25

$$\left\{\begin{aligned} \delta_1 &= q_1; \\ \delta_2 &= q_2 - q_1; \\ \delta_3 &= q_3 - q_2; \end{aligned}\right\} \qquad \text{...[1]}$$

The equilibrium equations are:

$$\left\{\begin{aligned} -k_1\delta_1 + k_2\delta_2 + F_1 &= 0 \\ -k_2\delta_2 + k_3\delta_3 + F_2 &= 0 \\ -k_3\delta_3 + F_3 &= 0 \end{aligned}\right\} \qquad \text{...[2]}$$

From first and second equations of eq. [2], we get

$$\left\{\begin{aligned} -k_1q_1 + k_2(q_2 - q_1) + F_1 &= 0 \\ -k_2(q_2 - q_1) + k_3(q_3 - q_2) + F_2 &= 0 \\ -k_3(q_3 - q_2) + F_3 &= 0 \end{aligned}\right\} \qquad \text{...[3]}$$

$$\left\{\begin{aligned} (k_1 + k_2)q_1 - k_2q_2 &= F_1 \\ -k_2q_1 + (k_2 + k_3)q_2 - k_3q_3 &= F_2 \\ -k_3q_2 + k_3q_3 &= F_3 \end{aligned}\right\} \qquad \text{...[4]}$$

Matrix form of system [4] is

$$\begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \qquad ...[5]$$

(2) ***Potential energy approach:***

$$\pi = \frac{1}{2} k_1 \delta_1^2 + \frac{1}{2} k_2 \delta_2^2 + \frac{1}{2} k_3 \delta_3^2 - F_1 q_1 - F_2 q_2 - F_3 q_3$$

$$= \frac{1}{2} k_1 q_1^2 + \frac{1}{2} k_2 (q_2 - q_1)^2 + \frac{1}{2} k_3 (q_3 - q_2)^2 - F_1 q_1 - F_2 q_2 - F_3 q_3$$

$$\therefore \qquad \frac{\partial \pi}{\partial q_1} = 0 \Rightarrow \begin{Bmatrix} k_1 q_1 + k_2 (q_2 - q_1)(-1) - F_1 = 0 \\ k_1 q_1 - k_2 (q_2 - q_1) - F_1 = 0 \\ (k_1 + k_2) q_1 - k_2 q_2 = F_1 \end{Bmatrix} \qquad ...[6]$$

$$\left.\begin{aligned} &\frac{\partial \pi}{\partial q_2} = 0 \Rightarrow k_2(q_2 - q_1) + k_3(q_3 - q_2)(-1) = F_2 \\ &\qquad -k_2 q_1 + (k_2 + k_3) q_2 - k_3 q_3 = F_2 \\ \text{and} \quad &\frac{\partial \pi}{\partial q_3} = 0 \Rightarrow k_3(q_3 - q_2) - F_3 = 0 \\ &\qquad -k_3 q_2 + k_3 q_3 = F_3 \end{aligned}\right\} \qquad ...[7]$$

Matrix form is given by

$$\begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \qquad ...[8]$$

Equations [5] and [8] are same. This completes the proof.

Example: The potential energy for the linear elastic rod with body force neglected is

$$\pi = \frac{1}{2} \int_0^L EA \left(\frac{du}{dx} \right) dx - 2u_1$$

where $u_1 = u(1)$.

Compute the stress in the bar.

Solution: We assume a quadratic trial function

$$u = C_1 + C_2 x + C_3 x^2 \qquad ...[1]$$

Now, when $x = 0$, $u(0) = 0$ yields $C_1 = 0$ and when $x = 2$, $u(2) = 0$ yields

$$0 = C_1 + 2C_2 + 4C_3$$

$$\Rightarrow C_2 = -2\,C_3$$

$\therefore$ [1] becomes $u = C_3\,(-2x + x^2)$

$$\Rightarrow u_1 = -\,C_3 \quad (\text{at } x = 1)$$

$\therefore$ $$\frac{du}{dx} = 2C_3\,(-1 + x) \text{ and } \pi = \frac{1}{2}\int_0^2 4C_3^2\,(-1 + x)^2\,dx - 2\,(-\,C_3)$$

$$= 2C_3^2 \int_0^2 (1 - 2x + x^2)\,dx + 2C_3$$

$$= 2C_3^2 \left(\frac{2}{3}\right) + 2C_3$$

$\therefore$ $$\frac{\partial \pi}{\partial C_3} = 4C_3\left(\frac{2}{3}\right) + 2 = 0.$$

$$\Rightarrow C_3 = -\,0.75;$$

$$u_1 = -C_3 = 0.75$$

Hence, the stress in the bar is given by

$$\sigma = E\,\frac{du}{dx}$$

$$= 2\,(-0.75)\,(-1 + x)$$

$$\Rightarrow \quad \boxed{\sigma = 1.5\,(1 - x)}$$

1.6 RAYLEIGH–RITZ'S METHOD

The Rayleigh–Ritz method was proposed independently by Lord Rayleigh (1842–1919) and Walter Ritz (1878–1909). Stated simply, "a function is assumed in terms of some unknown coefficients that would represent a solution of equation $\frac{d^2T}{dx^2} = -\,\frac{Q}{K}$. An approximate solution can be obtained for a differential equation using the Rayleigh–Ritz method whereby an approximating function is substituted into the variational function. The approximating function must satisfy the boundary conditions for the problem being studied.

The Rayleigh–Ritz method and the Ritz method are entirely different. The distinction is that the Ritz method always refers to the nonvariational integral formulation whereas the Rayleigh–Ritz method makes use of variational approach.

Illustration:

Let $$I = \int_a^b F(x, y, \dot{y}, \ddot{y})\,dx \qquad \text{be a functional.} \qquad ...[1]$$

Choosing $$F = F(x, y, \dot{y}, \ddot{y}) = -\frac{1}{2}\left(\frac{dy}{dx}\right)^2 + 1000\,x^2y \qquad \text{...[2]}$$

$\therefore$ Equation [1] becomes

$$I = \int_0^1 \left(1000\,x^2y - \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right)dx \qquad \text{...[3]}$$

[3] maximizes or minimizes I.

In Rayleigh–Ritz method for continuous system we deal with the functional :

$$I = \int_a^b F(x, y, \dot{y})\,dx \qquad \text{...[4]}$$

Discription of the Method

Rayleigh–Ritz method is a method for determining approximate solutions of functional equations through the expedient of replacing them by finite systems of equations.

It is one of the most useful approximate methods stemming from variational considerations, wherein, for the present we employ the following approximate displacement field components for expressing the total potential energy:

$$\left\{\begin{aligned} U_n &= \phi_0(x, y, z) + \sum_{i=1}^{l} a_i\phi_i(x, y, z) \\ V_n &= \psi_0(x, y, z) + \sum_{j=l+1}^{m} b_j\psi_j(x, y, z) \\ W_n &= \gamma_0(x, y, z) + \sum_{k=m+1}^{n} c_k\gamma_k(x, y, z) \end{aligned}\right\}_{n>m>l} \qquad \text{...[1]}$$

where a_i, b_j and c_k are $3n$ linearly independent parameters, which behave as generalised coordinates. In order to determine these coefficients $3n$ equations/conditions are sought: ϕ_i, ψ_j and γ_k are chosen as to satisfy kinematic boundary conditions.

Finite element technique may be thought of as an extension of Ritz's analysis of a continuous system. It tries to represent any continuous system with the help of parameters at some chosen points only. It was first taken up by Rayleigh and then elaborated by Ritz. The method is also known as Rayleigh–Ritz method.

For a most stable equilibrium, the total potential energy is minimized.

We may write the system of equations [1] as

$$\left\{\begin{aligned} U_n &= \sum a_i \, \phi_i \,; \\ V_n &= \sum b_j \, \psi_j \,; \\ W_n &= \sum c_k \, \gamma_k \,; \end{aligned}\right\} \qquad ...[2]$$

$$\therefore \qquad \left\{\begin{aligned} \delta U_n &= \sum \phi_i \,.\, \delta a_i \\ \delta V_n &= \sum \psi_j \,.\, \delta b_j \\ \delta W_n &= \sum \gamma_k \,.\, \delta c_k \,; \end{aligned}\right\} \qquad ...[3]$$

The total energy can be expressed as variations in a_i, b_j and c_k means variations in energy.

$$\therefore \qquad \delta U = \sum \left(\frac{\partial U}{\partial a_i} . \delta a_i + \frac{\partial U}{\partial b_j} . \delta b_j + \frac{\partial U}{\partial c_k} . \delta c_k \right) = 0 \qquad ...[4]$$

This gives $3n$ equations as:

$$\frac{\partial U}{\partial a_i} = 0 = \frac{\partial U}{\partial b_j} = \frac{\partial U}{\partial c_k} \qquad ...[5]$$

a_i, b_j and c_k can be solved from these equations. Figure 1.26 shows the type of approximations it follows:

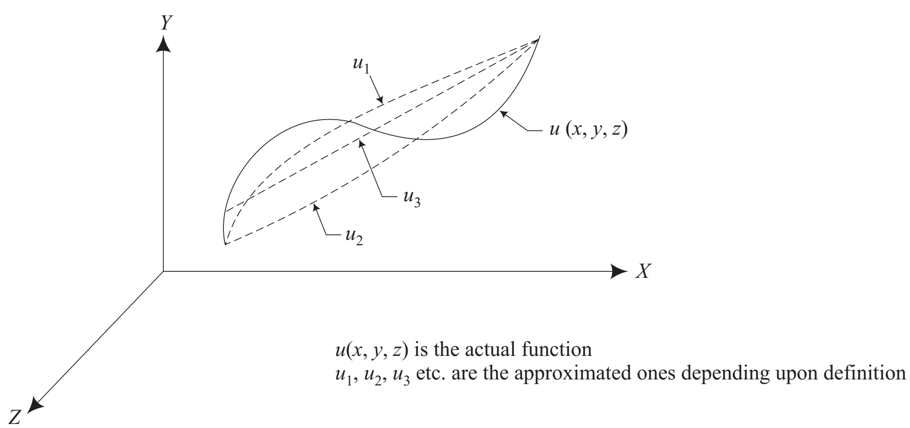


Fig. 1.26 Ritz's Approximation

Illustration:

We know that

$$\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + f(x, y) = 0 \qquad ...[1]$$

is a Poisson's equation. Its solution can be approximated in the form:

$$q(x, y) = \Sigma M_i(x, y)\, q_i \qquad ...[2]$$

where q_i are the parameters defined at only some selected points i (i = 1, 2, 3, ... n), M_i are the functions to be designed/chosen in a way to satisfy all the boundary conditions related to the problem.

The governing functional can be expressed as

$$j(q) = \frac{1}{2}\int\int\left[\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2 - 2 f(x, y)\, q\right] dx\, dy \qquad ...[3]$$

In order that $J(q)$ is minimum,

$$\frac{\partial J}{\partial q_i} = 0$$

Therefore, $$\frac{\partial J}{\partial q_i} = \int\int\left[\frac{\partial q}{\partial x}\cdot\frac{\partial}{\partial q_i}\left(\frac{\partial q}{\partial x}\right) + \frac{\partial q}{\partial y}\cdot\frac{\partial q}{\partial q_i}\left(\frac{\partial q}{\partial y}\right) - f(x, y)\cdot\frac{\partial q}{\partial q_i}\right] dx\, dy = 0 \qquad ...[4]$$

The equation [4] yields n equations for the unknowns giving approximate solution to $q\,(x, y)$ as $q'(x, y)$, if $M_i = N_i$

OR, $$\int\int\left[\frac{\partial}{\partial x}[N]\,(q)\cdot\frac{\partial}{\partial q_i}\left(\left[\frac{\partial N}{\partial x}\right]\{q\}\right)\right] + \frac{\partial}{\partial y}[N]\,(q)\cdot\frac{\partial}{\partial q_i}\left(\left[\frac{\partial N}{\partial y}\right]\{q\}\right) - \left[f(x, y)\cdot\frac{\partial}{\partial q_i}[N]\,\{q\}\right] dx\,.\,dy = 0 \qquad ...[5]$$

OR, $$\int\int\left(\left[\frac{\partial N}{\partial x}\right]\{q\}\cdot\frac{\partial N_i}{\partial x} + \left[\frac{\partial N}{\partial y}\right]\right)(q)\cdot\frac{\partial N_i}{\partial y} - \int(x, y)\,.\,N_i \Bigg) dx\,.\,dy = 0 \qquad ...[6]$$

OR, $$\int\int\left(\left[\frac{\partial N}{\partial x}\right]\cdot\frac{\partial N_i}{\partial x} + \left[\frac{\partial N}{\partial y}\right]\frac{\partial N_i}{\partial y}\right).\,dx\,.\,dy\,.\,\{q\} + \int\int - f(x, y)\,.\,N_i\,.\,dx\,.\,dy = 0 \qquad ...[7]$$

OR, $$[K]\,\{q\} - \{F\} = \{0\} \qquad ...[8]$$

where $$[K] = \int\int\left(\left[\frac{\partial N}{\partial x}\right]\cdot\frac{\partial N_i}{\partial x} + \left[\frac{\partial N}{\partial y}\right]\cdot\frac{\partial N_i}{\partial y}\right).\,dx\,.\,dy$$

$$\{F\} = \int\int f(x, y)\,.\,N_i\,.\,dx\,.\,dy$$