ENERGY THEOREMS AND STRUCTURAL ANALYSIS

A Generalised Discourse with Applications on Energy Principles of Structural Analysis Including the Effects of Temperature and Non-Linear Stress-Strain Relations.

by

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PREFACE

THE present work was originally published as a series of articles in *Aircraft Engineering* between October 1954 and May 1955. The purpose of these papers was two-fold. Firstly to generalize and extend but at the same time also to unify the fundamental energy principles of analysis of elastic structures. Although much of the corresponding theory has been available for a number of years, to the best of the author's knowledge it has not been given before in such generality. As an example, whilst keeping within the small deflection theory the arguments have been developed *ab initio* to include non-linear elasticity and arbitrary initial strains e.g. thermal strains. The first assumption introduces naturally the twin concepts of work and complementary work first put forward by Engesser. The author has attempted in this connexion to refer to all relevant and historically important papers. Since the appearance of the present articles, a few papers have been published which touch upon the same subject but suffer, unfortunately, from a rather incomplete list of references.

Secondly, the writer developed in considerable detail practical methods of analysis of complex structures—in particular for aeronautical engineering applications. The most important contributions are the matrix methods of analysis. Since they are only cursorily referred to in the Introduction, it may be appropriate here to describe their use and origin in greater detail. The matrix formulation besides providing an elegant and concise expression of the theory of such structures, is ideally suited for modern automatic computation because of the systematic ordering of numerical operation which the matrix calculus affords. The necessary programming for the digital computer is simplified since it can be preprogrammed to carry out matrix operations with only simple orders as to location and size of the matrix concerned and the operation to be performed. The specific programming for a particular problem may therefore be written comparatively quickly and easily and, moreover, follows closely the algebraic analysis.

As developed here, the matrix methods of analysis follow from particular forms of the two fundamental energy principles applicable to structures made up as an assembly of discrete elements. The one principle leads to an analysis in terms if displacements as unknowns (displacement method), while the second leads to an analysis in terms of forces (force method). Besides revealing more clearly the duality of the two methods, this derivation shows also the close connexion between the aproximate methods (like the Rayleigh-Ritz method) for continuous systems and the matrix methods for finite assemblies. This is particularly valuable in providing suitable techniques for establishing the basic properties—stiffness and flexibility—of the individual elements of a complex structure where these elements have to be assigned simplified stress or strain patterns.

But in stressing the advantages of a unified approach to these diverse problems, a word of caution is necessary against carrying over into the modern methods too many ideas associated with practical calculations by the established or classical methods. The ability to tackle successfully problems in which the number of unknowns is measured in hundreds carries with it the necessity of rethinking one's practical approach if maximum advantage is to be gained from modern computational techniques. In the force method of analysis the choice of basic system and of the redundant forces must be governed primarily by the requirements of simplicity and standardization, in order to reduce the manual preparation of data to a minimum, and reduce the probability of errors.

At the time of publication of the original articles it was intended to reprint them as a single volume and to follow up the Parts I and II, contained here, with further parts dealing specifically with the practical application of the matrix methods. Unfortunately it was not possible, for a number of reasons, to complete this plan and the articles have for some time been unavailable. Since there appears to be a persistent interest in them the present reprint has been produced to meet the deficiency. Grateful thanks and acknowledgment are due to the Editor of *Aircraft Engineering* for permission to reprint the articles in this form. The method of reproduction has not permitted complete rearrangement of the text into book form, so that the divisions into monthly instalments are still marked by blank spaces. However, errors in the text have been corrected as far as possible, and the pages have been renumbered consecutively to make for easier reference. Grateful thanks are due to Miss J. A. Bergg for her care and skill in effecting these changes. The author would also like to thank here those correspondents who have written to point out textual errors and misprints.

A list of references to further work is also appended. These are all concerned with the matrix methods of analysis whose basic theory is developed here. In particular, Ref. 6 is an expanded and developed form of part of the work which was initially planned for the original series.

FURTHER REFERENCES TO RECENT WORK

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Part I. General Theory

By J. H. Argyris

1. INTRODUCTION

THE increasing complexity of aircraft structures and the many exact or approximate methods available for their analysis demand an integrated view of the whole subject, not only in order to simplify their applications but also to discover some more general truths and methods. There are also other reasons demanding a more comprehensive discussion of the basic theory. We mention only the increasing attention paid to temperature stresses and the realization of the importance of non-linear effects. When viewed from all these aspects the idea of presenting a unified analysis appears more than necessary.

With this present paper we set out to develop a comprehensive system for the determination of stresses and deformations in elastic structures based on two fundamental energy principles. Although much of the theory given has naturally been known for many years we believe that some of the theorems and the generality of the results are new. The loading systems considered are of an arbitrary nature and include *ab initio* the effect of temperature or other initial strains. Neither do we restrict ourselves to elastic bodies obeying Hooke's law but take account of purely elastic non-linear stress-strain laws. This is possibly not of very great importance at present but may have wider applications in the future. No problems of stability will be touched upon in the present series of articles and any other considerations of large-deflexion theory are, in general, omitted. Thus the purpose is to investigate, within the small-deflexion theory, the stresses and deformations in elastic bodies not necessarily obeying a linear stress-strain law and under any load and temperature distribution. Dynamic effects are initially not considered and hence it is assumed for the present that the loads and temperature are of the quasi-static type. When investigating thermal strain effects we ought strictly to base the analysis on thermodynamic considerations. These are, however, only slightly touched upon here.

As in all theoretical work, we start by discussing the exact implications and equations derived from the initial assumptions, but we do not restrict ourselves here to this aspect. On the contrary, we pay close attention to approximate methods of analysis based on the physical concepts of work and strain energy. In particular we attempt to give upper and lower bounds to overall properties of the structure such as its stiffness. No attempt is made to estimate the error of stress and deformations at any particular point.

This series of papers originally arose[12, 13] from lectures given by the author since 1949-50 at the Imperial College, University of London. Naturally, the scope of the present work has grown beyond the narrower concept of undergraduate teaching, but the basis of the analysis dates back to that time. It is appropriate here to point out that certain of the basic ideas originate with Engesser[2] who unfortunately does not seem to have followed them up. We refer, of course, to the two complementary concepts of work and complementary work. If we consider an ordinary load displacement diagram, then, even if we restrict ourselves to small displacements, this may be curvilinear, if the material follows a non-linear stress-strain law. Work is the area between the displacement axis and the curve, while complementary work is that included between the force axis and the curve. Thus, the two areas complement each other in the rectangular area (force)×(displacement) which would be the work if the ultimate force were acting with its full intensity from the beginning of the displacement. Naturally, in the case of a body following Hooke's law, the two complementary areas are equal, but it is still useful for the purpose of analysis to keep them apart. Since writing a previous paper[12] on the subject the author has had the opportunity of consulting the most interesting latest book[9] of Stephen Timoshenko. There a reference is made to the work of Westergaard,[11] who indeed has developed further the basic ideas of Engesser, but not on quite such a general basis as here. Since approximate methods figure prominently in this paper reference ought to be made to the work of Prager and Synge. They too set out to develop systematically the determination of upper and lower limits to strain energy, restricting themselves, however, to Hooke's law and excluding temperature effects. Moreover, it appears that although many of their results are identical with existing ideas they clothed them in a language not too familiar to engineers. This discussion of past authors' work brings us to a few points which are preferably stated now. In much of present day structural analysis there seems to be an unfortunate tendency to over-emphasize certain methods of analysing redundant structures and to neglect more useful ideas readily available for many years. This refers particularly to Castigliano's principles which are so often set out as the basis of all considerations, not only in theory, but also in the actual methods of calculation. This is, in our opinion, unfortunate, even though all methods naturally lead to the same results if based on the same assumptions. For example, if we select forces as redundancies then much the best means of obtaining the basic equations for their determination is the long established δ_{ik} method of Mueller-Breslau based on the Unit Load idea. We do not need, in fact, even the concept of strain energy for this purpose. All we require is the idea of work and kinematics as used in rigid-body mechanics. From such ideas we can write down immediately our equations in the unknowns without bothering about strain energy. These methods have been in use by civil engineers for the past sixty years and it is surely time that we accepted them in the aeronautical world as standard analytical equipment. Actually, the basic principles go much farther back than Mueller-Breslau and were, in fact, developed independently by Maxwell[4] and Mohr[5] nearly a hundred years ago. The first systematic application of the δ_{ik} method to stressed skin structures was given in the classical investigations of Ebner.* Regrettably enough this lucid work was occasionally referred to in the past as obscure, a lack of comprehension, no doubt, at least partly due to the too narrow understanding of redundant structures arising from a concentration on Castigliano's methods. However, the limitations of Castigliano's formulation of the problem are being at last increasingly recognized in aeronautical circles due to the demands of calculations for highly redundant systems. Naturally, most of the alternative methods suggested are really nothing more than a transcription of the Mueller-Breslau and Ebner technique.

We start our investigation in Section 3 with a discussion on work and complementary work in the presence of temperature effects and for non-linear stress-strain laws. With this basic knowledge we then proceed to the standard principle of virtual displacements or virtual work in Section 4. This is very similar to the currently used principle in rigid-body mechanics. Thus, we consider a state of equilibrium, apply virtual displacements to it and develop hence the classical principle of virtual work which substitutes, of course, for the equations of equilibrium. Since virtual displacements are kinematically possible ones this theorem starts from the assumption of inherent compatibility to find the necessary and sufficient condition for equilibrium. It is, of course, well known that the theorem applies also to large displacements but this aspect is ignored here. However, temperature effects and an arbitrary law of elasticity are considered as long as the latter is monotonically increasing. Having established this principle we deduce easily some important theorems and applications.

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Additional references are given as footnotes.

* See e.g. H. Ebner and H. Koeller, 'Zur Berechnung des Kraftverlaufes in versteiften Zylinderschalen.' *Luftfahrtforschung*, Vol. 14, No. 12, December 1937.

Firstly the principle of virtual displacements may always be used to derive, for any particular structural problem, the governing differential equations and the appropriate static boundary conditions in terms of the displacements. This method, however, is not recommended in general as a substitute for the derivation from consideration of equilibrium and elastic compatibility.

Next the principle of virtual work is used to derive Castigliano's theorem Part I, generalized for thermal effects. As is well known, this principle applies not only for non-linear stress-strain laws but also for large displacements. Our line of argument leads us then naturally to the principle of minimum strain energy for a fixed set of displacements and a given temperature distribution. This theorem applies also for non-linear stress-strain laws and is of great interest for approximate calculations in terms of assumed forms of displacements. It shows us that, while the strain energy is for a given set of displacements a minimum when the compatible state is also one of equilibrium, it is on the other hand a maximum for a given set of forces under the same conditions. These theorems were first developed for linearly elastic bodies by Lord Rayleigh more than seventy-five years ago. They are shown to apply also in the presence of thermal strain and for non-linear elasticity. In the remainder of the chapter we investigate in more detail approximate methods of analysis using the Rayleigh-Ritz procedure and it is in such applications that the principle of virtual displacements shows its greatest power. The particular form of the Rayleigh-Ritz procedure known as the Galerkin method is also discussed. It is of importance when the assumed deformations satisfy all boundary conditions. The methods indicated apply again in the presence of thermal strains and non-linear stress-strain laws. The next, Section 5, gives simple illustrations to the method of virtual displacements.

The second fundamental principle is developed in Section 6. We call it the principle of virtual forces or complementary virtual work. Here we consider a state of equilibrium, apply a statically consistent and infinitely small virtual force and stress system and find, by using the idea of complementary work, the second principle. This is a necessary and sufficient condition that the position of equilibrium is also one of elastic compatibility. Again this theorem may be used to derive the differential equations of any particular problem, this time in terms of stresses or stress resultants. However, our comments on the parallel method in the case of the virtual displacements are equally applicable here. It should never be used as a substitute for more physical and geometric reasoning.

Next, we derive what is essentially a generalization of Castigliano's Part II theorem. Contrary to what is generally believed this theorem does apply for non-linear stress-strain laws as long as we replace strain energy by complementary strain energy, which is defined in the same way as complementary work. It is extended to include temperature effects. We proceed then with the generalization of Castigliano's principle of minimum strain energy (or least work) for non-linear stress-strain laws and thermal strains. Some interesting developments derive from this and are given in the form of maximum and minimum theorems complementary to those developed under the virtual displacement method. They do not seem to have been given previously in this form and provide a useful background to approximate methods. They show us that any assumed statically equivalent stress distribution must always under-estimate the stiffness. This is most valuable for practical purposes and is exactly opposite to the effect of assumed displacement distributions which always overestimate the stiffness. The two in conjunction give us hence lower and upper bounds to overall characteristics of the structure such as its stiffness. In this section we discuss also the Unit Load Method which, as mentioned previously, provides the basis for one of the more convenient methods for the calculation of displacements and of redundant forces. It is shown to be applicable to structures with non-linear stress-strain laws. Section 7 presents some simple illustrations of the principle of virtual forces.

In the last section we develop a slightly more generalized version of the δ_{ik} method of Mueller-Breslau. These equations lend themselves readily to presentation in matrix form. Next we obtain the corresponding equations when displacements and not forces are introduced as the unknowns.

A Note on the Mathematics

The mathematics used in this paper is, in general, elementary and should be familiar to any university graduate. We have avoided the more formal application of the calculus of variations which can be singularly unattractive to those more physically inspired. Chapter 3 and parts of Chapters 4 and 6 may prove, at first, rather difficult for a student. However, it is always possible to gain an understanding of the basic ideas by substituting simple examples (e.g. frameworks) for the necessarily more general proofs given here.

The later parts of this series of papers will present a number of applications of the basic methods developed here.

2. BASIC EQUATIONS AND NOTATION

$\omega_x, \omega_y, \omega_z$	body forces (e.g. gravity forces) per unit volume	Parallel to a Cartesian co-ordinate system Ox, Oy, Oz (see FIGS. 1 and 2)
ϕ_x, ϕ_y, ϕ_z	surface forces per unit surface	
l, m, n	direction cosines of external normal to surface	
u, v, w	displacements	

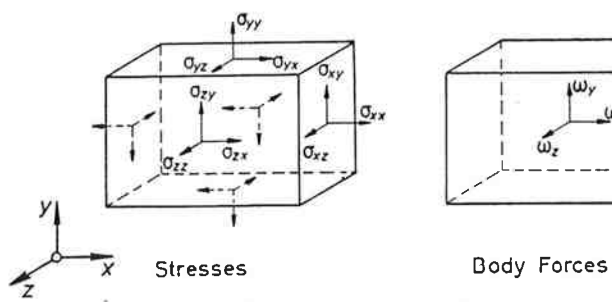


Fig. 1.—Stresses and body forces

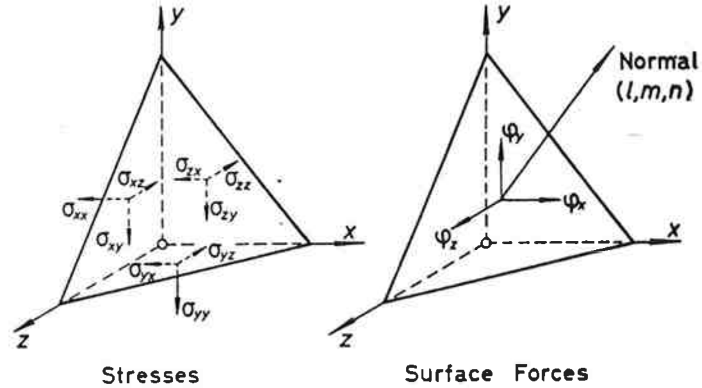


Fig. 2.—Stresses and surface forces

$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$	direct stresses	As shown in FIGS. 1 and 2
$\sigma_{xy}=\sigma_{yx}, \sigma_{yz}=\sigma_{zy}, \sigma_{zx}=\sigma_{xz}$	shear stresses	

$$\left.\begin{aligned} &\gamma_{xx}=\frac{\partial u}{\partial x},\quad \gamma_{yy}=\frac{\partial v}{\partial y},\quad \gamma_{zz}=\frac{\partial w}{\partial z} \quad \text{total direct strains}\\ &\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\quad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y},\quad \gamma_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z} \quad \text{total shear strains}\\ &g=\gamma_{xx}+\gamma_{yy}+\gamma_{zz} \end{aligned}\right\} \qquad (1)$$

$\eta_{xx}, \eta_{yy}, \eta_{zz}$	initial direct strains (e.g. thermal strains)
$\eta_{xy}, \eta_{yz}, \eta_{zx}$	initial shear strains
$\epsilon_{xx}=\gamma_{xx}-\eta_{xx}$ etc.	elastic direct strains
$\epsilon_{xy}=\gamma_{xy}-\eta_{xy}$ etc.	elastic shear strains
$e\ =\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz}$	

(2)

$dV=dxdydz$	element of volume
dS	element of surface
α	linear coefficient of thermal expansion (may vary with Θ)
Θ	rise of temperature
E	Young's modulus (May vary with Θ)
G	shear modulus (May vary with Θ)
ν	Poisson's ratio (May vary with Θ)

$$\sigma\gamma=\sigma_{xx}\gamma_{xx}+\sigma_{yy}\gamma_{yy}+\sigma_{zz}\gamma_{zz}+\sigma_{xy}\gamma_{xy}+\sigma_{yz}\gamma_{yz}+\sigma_{zx}\gamma_{zx} \qquad (3)$$

The corresponding explicit expressions for $\sigma\eta$ and $\sigma\epsilon$ are obtained by substituting the strains η_{xx} etc. and ϵ_{xx} etc. respectively for γ_{xx} etc.

W	work of external forces
$U_e=-W+\text{const.}$	potential (energy) of external forces
U_i	strain energy (or potential energy of elastic deformation)
W^*, U_e^*, U_i^*	complementary work, complementary potential of external forces and complementary potential energy of elastic deformation
U_d^*	complementary potential energy of total deformation

From a consideration of equilibrium on an element $dV=dxdydz$, illustrated for the x-direction in FIG. 3

$$\left.\begin{aligned} &\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}+\omega_x=0\\ &\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zy}}{\partial z}+\frac{\partial\sigma_{xy}}{\partial x}+\omega_y=0\\ &\frac{\partial\sigma_{zz}}{\partial z}+\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\omega_z=0 \end{aligned}\right\} \qquad (4)$$

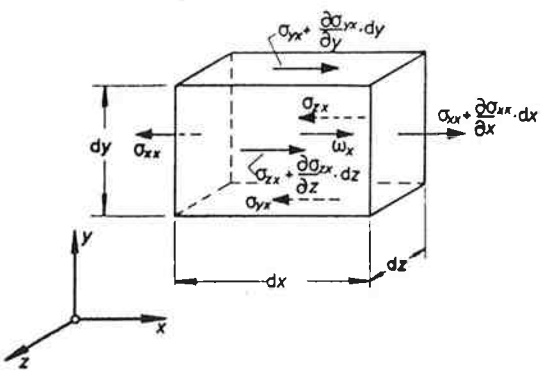


Fig. 3.—Internal equilibrium conditions

From a consideration of equilibrium on the surface (see FIG. 2)

$$\left.\begin{aligned} l\sigma_{xx}+m\sigma_{yx}+n\sigma_{zx}&=\phi_x\\ m\sigma_{yy}+n\sigma_{zy}+l\sigma_{xy}&=\phi_y\\ n\sigma_{zz}+l\sigma_{xz}+m\sigma_{yz}&=\phi_z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Over part of the surface the boundary conditions may be expressed in terms of stresses or forces (static boundary conditions) and over the remainder in terms of displacements or strains (kinematic or geometric boundary conditions). Naturally, the boundary conditions may be of both types over the same part of the surface. Consider, for example, the tube shown in FIG. 4. It is assumed fully built in at the root ($z=0$) and free at the tip ($z=l$). Ribs rigid in their own plane but freely flexible to deflexions out of their plane are assumed at $z=0$ and $z=l$. The boundary conditions are: at $z=0$, $u=v=w=0$, i.e. pure kinematic conditions; at $z=l$, $\sigma_{zz}=0$, $\left(\frac{\partial v}{\partial y}\right)_{z=l}=0$ for the vertical walls and $\left(\frac{\partial u}{\partial x}\right)_{z=l}=0$ for the horizontal walls, i.e. both static and kinematic conditions.

To denote infinitesimal elements of geometric properties of the structure (e.g. co-ordinates, area, volume) we use the standard symbol d.

To denote infinitesimal increments of forces, stresses, displacements, strains and work we use the symbol δ.

Thus, dV=infinitesimal element of volume$=dxdydz$, δP infinitesimal increment of force P.

The symbols

$$\int_V(\ldots\ldots)dV, \quad \int_S(\ldots\ldots)dS$$

denote integrations over a volume and surface respectively.

The formal mathematical proof of some of the basic theorems in this paper is shortened by using Green's theorem.* Let ϕ and ψ be two continuous functions and let also the first partial derivatives of ϕ and the first and second partial derivatives of ψ be also continuous. Green's theorem states:

$$\int_V\left[\frac{\partial\phi}{\partial x}\ \frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\ \frac{\partial\psi}{\partial y}+\frac{\partial\phi}{\partial z}\ \frac{\partial\psi}{\partial z}\right]dV$$

$$=-\int_V\phi\Delta\psi dV+\int_S\phi\left[l\frac{\partial\psi}{\partial x}+m\frac{\partial\psi}{\partial y}+n\frac{\partial\psi}{\partial z}\right]dS \quad \ldots\ldots\ldots\ldots \quad (6)$$

* See Courant, *Differential and Integral Calculus*, translated by J. E. McShane, Blackie and Son Ltd., London and Glasgow, 1949, Vol. II.

where

$$\Delta\psi=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}$$ and l, m, n are the direction cosines on the surface.

This theorem can be proved by integration by parts.

Examples of application:

take $\phi=\sigma_{xx}$, $\frac{\partial\psi}{\partial x}=\delta u$, $\frac{\partial\psi}{\partial y}=\frac{\partial\psi}{\partial z}=0$

Then

$$\int_V\frac{\partial\sigma_{xx}}{\partial x}\cdot\delta u\cdot dV=-\int_V\sigma_{xx}\cdot\delta\gamma_{xx}\cdot dV+\int_S l\sigma_{xx}\cdot\delta u\cdot dS$$

where $\delta\gamma_{xx}=\delta\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\delta u$ is the increment of the strain γ_{xx} corresponding to δu.

Similarly take $\phi=\sigma_{xy}$, $\frac{\partial\psi}{\partial x}=\delta v$, $\frac{\partial\psi}{\partial y}=\delta u$, $\frac{\partial\psi}{\partial z}=0$

Then

$$\int_V\left[\frac{\partial\sigma_{xy}}{\partial x}\delta v+\frac{\partial\sigma_{xy}}{\partial y}\delta u\right]dV=-\int_V\sigma_{xy}\delta\gamma_{xy}dV+\int_S\sigma_{xy}\left[m\delta u+l\delta v\right]dS$$

where

$$\delta\gamma_{xy}=\delta\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]=\frac{\partial\delta u}{\partial y}+\frac{\partial\delta v}{\partial x}$$

Note that although Green's theorem is helpful for the mathematical understanding of the present theory it is not really necessary for the physical understanding; a reader unfamiliar with these aspects of the integral calculus may omit the relevant parts.

3. WORK AND COMPLEMENTARY WORK—STRAIN ENERGY AND COMPLEMENTARY STRAIN ENERGY

The analysis of the present paper is restricted to small strains which can be expressed by the linear formulae given in the notation. Such displacements and corresponding strains are obviously additive (algebraically). Thus, if u_1, γ_1, and u_2, γ_2 are displacements and strains in a deformed state 1 and 2 respectively, then u_1+u_2, $\gamma_1+\gamma_2$ represent also a compatible state of deformation of the body. Our assumption does not impose, however, a linear stress-strain relationship; hence if P_1, σ_1 and P_2, σ_2 are the forces and stresses corresponding to the above two states of deformation of the body, the forces and stresses corresponding to the deformed state u_1+u_2 are not P_1+P_2 and $\sigma_1+\sigma_2$ except in the case of a linearly elastic body. In all cases, however, the stress-strain law is assumed to increase monotonically as shown in FIG. 5. In conclusion we can state that the law of superposition is assumed to hold for strains and displacements but not necessarily for the stresses.

In general, we assume also that the displacements are so small that the equilibrium conditions can be written down for the undeformed body. It follows then that the question of stability or instability of equilibrium does not enter in the analysis of this paper and there is a unique solution to every problem.

Consider a three-dimensional deformable body (not necessarily elastic) in equilibrium subjected to a self-equilibrating system of body forces ω_x etc., surface forces ϕ_x etc. and a temperature Θ. These forces and temperature may vary with time but the variations are assumed so slow that the dynamic effects are negligible. Let, in a time interval δt, the forces increase by $\delta\omega_x$, $\delta\phi_x$ etc. and the temperature by $\delta\Theta$. The displacements increase at the same time by δu etc. There arises hence an increment of work (see FIG. 6)

$$\delta W=\int_V[(\omega_x\delta u+\omega_y\delta v+\omega_z\delta w)dV$$

$$+\int_S[\phi_x\delta u+\phi_y\delta v+\phi_z\delta w]dS$$

+ terms of higher order.

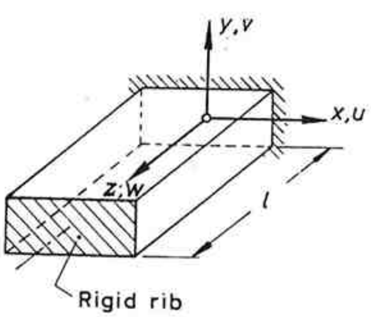


Fig. 4.—Kinematic and static boundary conditions

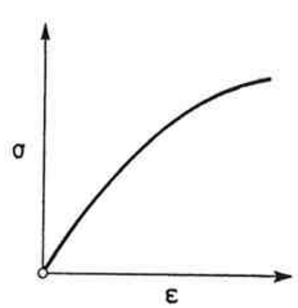


Fig. 5.—Stress-strain diagram

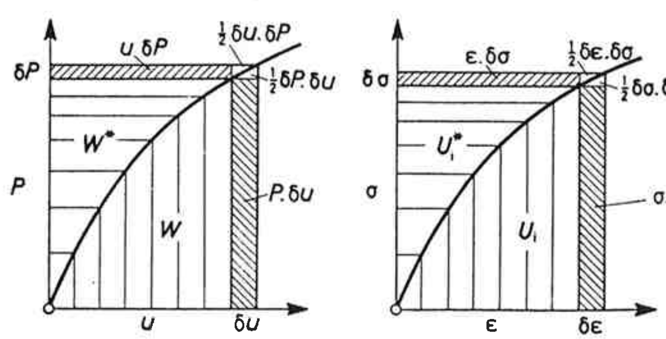


Fig. 6.—Work and complementary work; strain energy and complementary strain energy

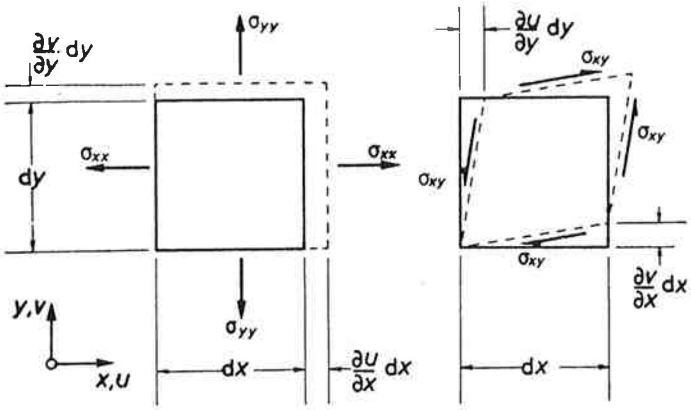


Fig. 7.—Work of direct and shear stresses

The terms of higher order involve expressions $\frac{1}{2}\delta\omega_x\cdot\delta u$, $\frac{1}{2}\delta\phi_x\cdot\delta u$ etc, and may be neglected to the first order of magnitude considered here. Thus.

$$\left.\begin{aligned}\delta W=&\int_V[\omega_x\delta u+\omega_y\delta v+\omega_z\delta w]dV\\&+\int_S[\phi_x\delta u+\phi_y\delta v+\phi_z\delta w]dS\end{aligned}\right\} \quad (7)$$

Note that Eq. (7) does not presume any specific force-displacement law, be it elastic or non-elastic.

It is simple to derive an alternative expression by considering the additive effect of the work done by the stress resultants on each volume element dV. A perusal of FIG. (7) shows that the deformation δu, δv, δw gives rise to an increment of work for an element dV

$$(\sigma_{xx}\delta\gamma_{xx}+\sigma_{yy}\delta\gamma_{yy}+\sigma_{zz}\delta\gamma_{zz}+\sigma_{xy}\delta\gamma_{xy}+\sigma_{yz}\delta\gamma_{yz}+\sigma_{zx}\delta\gamma_{zx})dV=\sigma\delta\gamma\cdot dV$$

again neglecting terms of higher order.

The incremental (infinitesimal) strains $\delta\gamma_{xx}$ etc. are those due to the displacements δu etc. Thus,

$$\delta\gamma_{xx}=\delta\frac{\partial u}{\partial x}\text{ etc., }\delta\gamma_{xy}=\delta\left\{\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right\}=\left\{\frac{\partial\delta u}{\partial y}+\frac{\partial\delta v}{\partial x}\right\} \quad (8)$$

It follows that the increment of work δW may also be expressed as

$$\delta W=\int_V(\sigma\delta\gamma)dV \quad (9)$$

The formal equivalence of Eqs. (7) and (9) may be proved without difficulty by using Green's theorem. To this effect multiply each of the internal equilibrium conditions (4) by $\delta u\cdot dV$, $\delta v\cdot dV$, $\delta w\cdot dV$ respectively, and integrate over the body. We obtain,

$$\int_V\left\{\left[\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}+\omega_x\right]\delta u+\left[\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zy}}{\partial z}+\frac{\partial\sigma_{xy}}{\partial x}+\omega_y\right]\delta v+\left[\frac{\partial\sigma_{zz}}{\partial z}+\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\omega_z\right]\delta w\right\}dV=0$$

Applying now Green's theorem to the terms involving the stresses, as shown in the previous section, we find, using the surface equilibrium conditions, Eqs. (5),

$$\int_V[\sigma\delta\gamma]dV=\int_V[\omega_x\delta u+\omega_y\delta v+\omega_z\delta w]dV+\int_S[\phi_x\delta u+\phi_y\delta v+\phi_z\delta w]dS=\delta W$$

where the $\delta\gamma$'s satisfy Eqs. (8) and are hence compatible strains. Note that where ϕ_x is unknown but the displacement (say u) fixed, the corresponding δu is zero and hence the relevant terms in the last relation vanish. Integrating Eq. (9) from the initial unstressed state O to a final state I we find the total work

$$W=\int_O^I\left\{\int_V[\omega_x\delta u+\omega_y\delta v+\omega_z\delta w]dV+\int_S[\phi_x\delta u+\phi_y\delta v+\phi_z\delta w]dS\right\}=\int_V\bar{W}dV \quad (10)$$

where

$$\bar{W}=\int_O^I\{\sigma_{xx}\delta\gamma_{xx}+\sigma_{yy}\delta\gamma_{yy}+\sigma_{zz}\delta\gamma_{zz}+\sigma_{xy}\delta\gamma_{xy}+\sigma_{yz}\delta\gamma_{yz}+\sigma_{zx}\delta\gamma_{zx}\} \quad (11)$$

Note that, in general, the work W done to reach a state I starting from a state O depends on the path chosen due to say plasticity, viscous effects, etc. In such cases δW is not the total differential* of the right-hand side of Eq. (10).

In what follows we assume that the body is fully elastic and isotropic† and that it is subjected both to external loads and thermal effects. In view of our initial assumption about the smallness of the strains we can write

$$\left.\begin{aligned}\delta\gamma_{xx}&=\delta\epsilon_{xx}+\delta\eta\\\delta\gamma_{yy}&=\delta\epsilon_{yy}+\delta\eta\\\delta\gamma_{zz}&=\delta\epsilon_{zz}+\delta\eta\end{aligned}\right\} \quad (12)$$

where $\delta\epsilon_{xx}$ . . . ., $\delta\eta$ are the increments of the true elastic strains and the thermal strains

$$\eta_{xx}=\eta_{yy}=\eta_{zz}=\eta=\int_O^\Theta\alpha\delta\Theta \quad (13)$$

α is the linear coefficient of thermal expansion which may vary with Θ. In view of the local isotropy of the body, thermal expansion does not give rise to any angular displacements.

Hence

$$\delta\gamma_{xy}=\delta\epsilon_{xy},\ \delta\gamma_{yz}=\delta\epsilon_{yz},\ \delta\gamma_{zx}=\delta\epsilon_{zx} \quad (14)$$

Substituting Eqs. (13) and (14) into (11) we obtain

$$\delta\bar{W}=\delta\bar{U}_i+s\delta\eta \quad (15)$$

and

$$\bar{W}=\bar{U}_i+\int_O^\Theta s\delta\eta \quad (16)$$

where

$$\bar{U}_i=\int_O^\epsilon\sigma\delta\epsilon \quad (17)$$

and

$$\sigma\delta\epsilon=\sigma_{xx}\delta\epsilon_{xx}+\ \dots\dots\ +\sigma_{xy}\delta\epsilon_{xy}+\ \dots\ +\sigma_{zx}\delta\epsilon_{zx}=\delta\bar{U}_i \quad (17a)$$

Note that the second term in (16) and thus also the work depend on the sequence of application of loads and temperature.

We assume now that the elastic properties of the material depend only on the instantaneous temperature Θ but not on the previous history of deformation, e.g. if adiabatic or isothermal; the error involved in this assumption is indeed negligible. Thus, the expression $\bar{U}_i$ is only a single valued function of the instantaneous state of elastic straining. In a closed cycle of deformation $\bar{U}_i$ is zero and $\delta\bar{U}_i=d\bar{U}_i$ is a total differential. The function $\bar{U}_i$ is commonly called the strain energy per unit volume; other names are strain energy function or density of strain energy.

It follows from Eq. (17a) that

$$\delta\bar{U}_i=\sigma_{xx}\delta\epsilon_{xx}+\ \dots\ +\sigma_{zx}\delta\epsilon_{zx}=\frac{\partial\bar{U}_i}{\partial\epsilon_{xx}}\delta\epsilon_{xx}+\ \dots\ +\frac{\partial\bar{U}_i}{\partial\epsilon_{zx}}\delta\epsilon_{zx}$$

Hence

$$\sigma_{xx}=\frac{\partial\bar{U}_i}{\partial\epsilon_{xx}}\ \dots,\ \sigma_{xy}=\frac{\partial\bar{U}_i}{\partial\epsilon_{xy}} \quad (18)$$

We conclude that the stress-strain law is uniquely determined by the strain-energy function and vice versa. Note that the law of elasticity is arbitrary in Eqs. (12) to (18).

Integrating Eq. (16) over the body we find,

$$\int_V\bar{W}dV=\int_V\bar{U}_idV+\int_V\left[\int_O^\Theta s\delta\eta\right]dV$$

or

$$W=U_i+\int_V\left[\int_O^\Theta s\delta\eta\right]dV \quad (19)$$

where

$$U_i=\int_V\bar{U}_idV=\int_V\left[\int_O^\epsilon\sigma\delta\epsilon\right]dV \quad (19a)$$

For $\Theta=0$ Eq. (19) reduces to the well-known equality

$$W=U_i \quad (20)$$

U_i is the strain energy or the internal elastic potential (or potential energy of elastic deformation) of the body.

If we now assume that the law of elasticity is a linear one then

$$\epsilon_{xx}=\gamma_{xx}-\eta=\frac{1}{E}[\sigma_{xx}-\nu(\sigma_{yy}+\sigma_{zz})] \quad (21)$$

and similar expressions for ϵ_{yy} and ϵ_{zz}. Also

$$\sigma_{xy}=G\gamma_{xy} \quad (22)$$

Expressing (21) in terms of stresses,

$$\sigma_{xx}=2G\left[\epsilon_{xx}+\frac{\nu}{1-2\nu}e\right]=2G\left[\gamma_{xx}+\frac{\nu}{1-2\nu}g\right]-\frac{E\eta}{1-2\nu} \quad (23)$$

and two similar expressions for σ_{yy} and σ_{zz}. The moduli E and G and the ratio ν may vary with Θ.

Substitution of the stresses given by Eqs. (22) and (23) into Eq. (15) yields

$$\delta\bar{W}=\delta\bar{U}_i+\frac{Ee\delta\eta}{1-2\nu} \quad (24)$$

and if temperature and loads are increased together from zero

* We say that $\delta W=dW$ is a total differential of W if $\int_c dW=O$, where the integration is taken around a closed curve; if this applies W is obviously independent of the path of deformation taken between states O and I.

† The isotropy need only be assumed at each point; the properties of the body may vary from point to point.

$$\bar{W}=\bar{U}_i+\frac{Ee\eta}{2(1-2\nu)} \quad \ldots\ldots \quad (25)$$

where

$$\bar{U}_i=G\left\{\frac{1-\nu}{1-2\nu}e^2-2[\epsilon_{xx}\epsilon_{yy}+\epsilon_{yy}\epsilon_{zz}+\epsilon_{zz}\epsilon_{xx}-\tfrac{1}{4}(\epsilon_{xy}{}^2+\epsilon_{yz}{}^2+\epsilon_{zx}{}^2)]\right\} \quad (26)$$

is the strain energy as a function of the elastic strains.

The following relations hold only for linear elasticity,

$$\frac{\partial \bar{U}_i}{\partial \sigma_{xx}}=\epsilon_{xx}=\gamma_{xx}-\eta \ldots\ldots, \frac{\partial \bar{U}_i}{\partial \sigma_{xy}}=\epsilon_{xy} \ldots\ldots \quad (27)$$

For two-dimensional stress distributions substitute the factor $1/(1-\nu)$ for $1/(1-2\nu)$ in Eqs. (24), (25) and (26). The corresponding strain energy function $\bar{U}_i$ is,

$$\bar{U}_i=G\left[\frac{1}{(1-\nu)}(\epsilon_{xx}+\epsilon_{yy})^2-2\epsilon_{xx}\epsilon_{yy}+\tfrac{1}{2}\epsilon_{xy}{}^2\right] \quad \ldots\ldots \quad (26a)$$

Parallel to the conceptions of work and strain energy two further ideas essentially due to Engesser[2], are of particular importance to our investigations. Consider to that effect a one-dimensional force-displacement and a stress-elastic strain diagram (FIG. 6). The vertically shaded areas are obviously those of work and strain energy respectively. It is natural to inquire if the horizontally shaded areas complementing the previous areas in the rectangular areas Pu and $\sigma\epsilon$ respectively are of any importance (see FIG. 6). In fact, as is shown farther on, the introduction of these new conceptions is proved a particularly happy one when generalizing some theorems, currently assumed to be valid only for linear elasticity, to bodies with non-linear stress-strain relations. Although the complementary areas are equal for linear elasticity it is still useful in such cases to differentiate between them.

It is interesting to note that in thermodynamics two similar complementary functions are used: the free energy function H of von Helmholtz and the function G of Gibbs.* In what follows we call the horizontally shaded areas complementary work and complementary strain energy and denote them by W^* and U_i^* respectively.

We generalize next our new conceptions by considering the three-dimensional case. Let the actual displacements in a body subjected to body forces ω, surface forces ϕ and temperature Θ be u, v, w. The increment of the complementary work as these displacements increase to $u+\delta u$, $v+\delta v$, $w+\delta w$ due to load increments $\delta\phi_x \ldots$, $\delta\omega_x \ldots$ etc. is given by

$$\delta W^*=\int_V[u\delta\omega_x+v\delta\omega_y+w\delta\omega_z]dV+\int_S[u\delta\phi_x+v\delta\phi_y+w\delta\phi_z]dS$$

+terms of higher order

or

$$\delta W^*=\int_V[u\delta\omega_x+v\delta\omega_y+w\delta\omega_z]dV+\int_S[u\delta\phi_x+v\delta\phi_y+w\delta\phi_z]dS \quad \ldots \quad (28)$$

since terms of higher order like $\frac{1}{2}\delta\phi_x\cdot\delta u$ can be neglected. It is simple, as in the case of work, to derive an alternative expression to Eq. (28) in terms of stresses and total strains. To find it, note that the increments $\delta\sigma_{xx}$ etc. of the stresses must be in equilibrium with the corresponding increments of the body forces $\delta\omega_x \ldots$ and surface forces $\delta\phi_x \ldots$. There are thus six relations of the type of Eqs. (4) and (5). We write here only the two for equilibrium in the x-direction

$$\frac{\partial(\delta\sigma_{xx})}{\partial x}+\frac{\partial(\delta\sigma_{yx})}{\partial y}+\frac{\partial(\delta\sigma_{zx})}{\partial z}+\delta\omega_x=0 \quad \ldots\ldots \quad (29)$$

and

$$l\delta\sigma_{xx}+m\delta\sigma_{yx}+n\delta\sigma_{zx}=\delta\phi_x \quad \ldots\ldots \quad (30)$$

Multiplying now each of the first set of equations by the displacements u, v, w respectively, summing and integrating over the body we obtain by applying Green's Theorem similarly to when we derived Eq. (9), and using Eq. (30)

$$\int_V[u\delta\omega_x+v\delta\omega_y+w\delta\omega_z]dV+\int_S[u\delta\phi_x+v\delta\phi_y+w\delta\phi_z]dS$$

$$=\int_V[\gamma_{xx}\delta\sigma_{xx}+\gamma_{yy}\delta\sigma_{yy}+\gamma_{zz}\delta\sigma_{zz}+\gamma_{xy}\delta\sigma_{xy}+\gamma_{yz}\delta\sigma_{yz}+\gamma_{zx}\delta\sigma_{zx}]dV \quad \ldots\ldots \quad (31)$$

where γ_{xx} etc. are the total strains. Note that where the forces, for example, ϕ_x, have fixed values $\delta\phi_x=0$. Thus ultimately

$$\delta W^*=\int_V\gamma\delta\sigma dV \quad \ldots\ldots \quad (32)$$

where

$$\gamma\delta\sigma=\gamma_{xx}\delta\sigma_{xx}+\ldots\ldots+\gamma_{xy}\delta\sigma_{xy}+\ldots\ldots \quad (32a)$$

Integrating Eq. (32) between the initial unstressed state O and a final state I we find the total complementary work

$$W^*=\int_V\bar{W}^*dV \quad \ldots\ldots \quad (33)$$

where

$$\bar{W}^*=\int_O^I\gamma\delta\sigma \quad \ldots\ldots \quad (34)$$

Note that as in the case of work W the complementary work depends, in general, on the chosen path of stressing between O and I.

We assume now the body is fully elastic and isotropic as described previously and find from Eq. (34)

$$\delta\bar{W}^*=\delta\bar{U}_i^*+\eta\delta s \quad \ldots\ldots \quad (35)$$

and

$$\bar{W}^*=\bar{U}_i^*+\int_O^I\eta\delta s \quad \ldots\ldots \quad (35a)$$

where

$$\bar{U}_i^*=\int_O^\sigma\epsilon\delta\sigma \quad \ldots\ldots \quad (36)$$

and

$$\epsilon\delta\sigma=\epsilon_{xx}\delta\sigma_{xx}+\ldots\ldots+\epsilon_{xy}\delta\sigma_{xy}+\ldots\ldots=\delta\bar{U}_i^*$$

Note that the second term in (35a) depends on the sequence of application of loads and temperature.

Under the same assumptions as for the internal potential energy, U_i^* depends now only on the instantaneous state of stress. In a closed cycle of deformation U_i^* is zero and δU_i^* is a total differential of the right-hand side of Eq. (36). Hence

$$\epsilon_{xx}=\frac{\partial\bar{U}_i^*}{\partial\sigma_{xx}} \ldots\ldots, \epsilon_{xy}=\frac{\partial\bar{U}_i^*}{\partial\sigma_{xy}} \ldots\ldots \quad (37)$$

(see also Eqs. (18)). Note that the law of elasticity is arbitrary in Eqs. (37). If $\bar{U}_i^*$ is given in terms of the stresses Eqs. (37) show that this determines uniquely the strain-stress laws and vice versa. It is natural hence to call $\bar{U}_i^*$ the complementary strain energy function.

Integrating over the body we find

$$W^*=U_i^*+\int_V\left[\int_O^I\eta\delta s\right]dV \quad \ldots\ldots \quad (38)$$

where

$$U_i^*=\int_V\bar{U}_i^*dV=\int_V\left[\int_O^\sigma\epsilon\delta\sigma\right]dV \quad \ldots\ldots \quad (38a)$$

where U_i^* is called the complementary strain energy or complementary elastic potential energy. When $\Theta=0$ then

$$W^*=U_i^* \quad \ldots\ldots \quad (39)$$

In the case of linear elasticity $W^*=W$ and $U_i^*=U_i$, but it is useful to differentiate still between them. For the linear stress-strain laws of Eqs. (21) and (22), Eq. (35a) becomes, if temperature and loads are increased together from zero,

$$\bar{W}^*=\bar{U}_i^*+\frac{s\eta}{2} \quad \ldots\ldots \quad (38b)$$

and

$$\bar{U}_i^*=\frac{1}{4G}\left[\frac{s^2}{1+\nu}-2(\sigma_{xx}\sigma_{yy}+\sigma_{yy}\sigma_{zz}+\sigma_{zz}\sigma_{xx}-\sigma_{xy}{}^2-\sigma_{yz}{}^2-\sigma_{zx}{}^2)\right] \quad (40)$$

which for two-dimensional stress distributions reduces to

$$\bar{U}_i^*=\frac{1}{4G}\left[\frac{(\sigma_{xx}+\sigma_{yy})^2}{1+\nu}-2(\sigma_{xx}\sigma_{yy}-\sigma_{xy}{}^2)\right] \quad \ldots\ldots \quad (40a)$$

The above considerations on strain energy and complementary strain energy may be used to derive all Castigliano theorems as generalized for non-linear stress-strain relations. We postpone, however, these investigations to subsequent chapters.

When the body obeys linear stress-strain relations the principle of superposition applies also to forces and corresponding stresses. Some important theorems derive from this property.

Thus, if the forces on a body are increased from zero to their final value then

$$\left.\begin{aligned}W=W^*&=\tfrac{1}{2}\int_V[\omega_xu+\omega_yv+\omega_zw]dV+\tfrac{1}{2}\int_S[\phi_xu+\phi_yv+\phi_zw]dS\\U=U^*&=\tfrac{1}{2}\int_V(\sigma\epsilon)dV\end{aligned}\right\} \quad (41)$$

where all symbols in the brackets refer to final values. Eqs. (41) are known as Clapeyron's theorem. Another very useful theorem is due to Betti.* Assume that the body is subjected to two force systems P_1 and P_2. Let the deflexions due to system P_1 alone be u_1 and due to P_2 alone u_2. By applying first system P_1 and subsequently P_2 and then reversing the process we prove easily—noting that the final state is in each case the same—that

$$W_{12}=W_{21} \quad \ldots\ldots \quad (42)$$

where

$$W_{12}=\Sigma P_1u_2 \text{ and } W_2=\Sigma P_2u_1 \quad \ldots\ldots \quad (42a)$$

* The sign of the latter function is taken as that fixed by the Committee of the International Union of Physics.

* See *Nuovo Cimento* (2), Vols. 7, 8, 1872.

are the work done by the system of forces $P_1(P_2)$ over the displacements $u_2(u_1)$ respectively. Relation (42) is known as the generalized reciprocal theorem of Betti.

A special form of Eq. (42) is Maxwell's reciprocal theorem. Thus, if systems 1 and 2 consist each of one force (or moment) only then

$$1.\ u_{12}=1.\ u_{21} \quad \ldots\ldots \quad (43)$$

where $u_{12}(u_{21})$ is the displacement or rotation in the direction of force or moment $P_1(P_2)$ due to force or moment $P_2(P_1)$.

4. THE PRINCIPLE OF VIRTUAL DISPLACEMENTS OR VIRTUAL WORK

We assumed in the previous section when discussing work and strain energy that the displacements δu, δv, δw arise from an actual variation of the applied forces and/or temperature distribution. However, this is an unnecessary restriction. We need only remember that to the first order of magnitude considered δW and δU_i are independent of the δP's and corresponding $\delta\sigma$'s since we ignore terms of the order $\delta P\cdot\delta u$. Also for the purpose in hand no variation in the temperature is called for. Hence, when finding δW and δU_i we can assume that forces, stresses and temperature remain constant while the displacements are varied to $u+\delta u$, $v+\delta v$, $w+\delta w$. It is only necessary that the δu, δv, δw's are compatible infinitesimal displacements (see Section 3, p. 4); thus they must be piece-wise continuous in the interior and satisfy the kinematic boundary conditions. For example, if the u, v, w are prescribed on part of the boundary then the selected variations δu, δv, δw must be zero there too. Similar arguments apply if the derivatives of any of the displacements are fixed. However, where the forces and stresses are prescribed the variation of the δu's is necessarily free. Note that there are cases when it is useful to relax even the kinematic boundary conditions when selecting the δu's.

Such geometrically possible infinitesimal displacements are used extensively in rigid body mechanics and are called virtual displacements. Noting that the temperature and hence thermal strains remain constant we can restate now Eqs. (15), (19) and (19*a*) more generally as follows:

An elastic body is in equilibrium under a given system of loads and temperature distribution if for any virtual displacements δu, δv, δw from a compatible state of deformation considered

$$\delta W=\delta U_i \quad \ldots\ldots \quad (44)$$

which is the standard principle of virtual displacements or virtual work. As stated here it is valid for an elastic body subjected both to loads and temperature effects.

Note that Eq. (44) is also valid for large displacements but the formulation of U_i is in such cases more complicated since the strain expressions are not any longer linear in the displacements and the equilibrium conditions have to be considered on the deformed element.

The point made above that the virtual displacements are arbitrary as long as they are infinitesimal and satisfy the internal compatibility conditions and kinematic boundary conditions is worth emphasizing. To fix ideas consider the statically determinate framework shown in FIG. (8) subjected to a transverse force P. We apply to the system a virtual displacement δu in the form shown in the figure which allows only an elastic deformation of the upper flange 1, 2. This virtual displacement satisfies the kinematic boundary conditions

$\delta u=0$ at A and B

but bears obviously no relation to the actual displacements of the framework due to the force P.

We find easily:

force in member 1, 2: $N_{12}=-\dfrac{Pab_2}{lh}$

virtual displacement of force P: $\delta u^1=\delta u\cdot a/b_1$

virtual elongation of member 1, 2: $\Delta l_{12}=-\dfrac{\delta uh}{b_1}-\dfrac{\delta uh}{b_2}=-\delta u\dfrac{lh}{b_1b_2}$

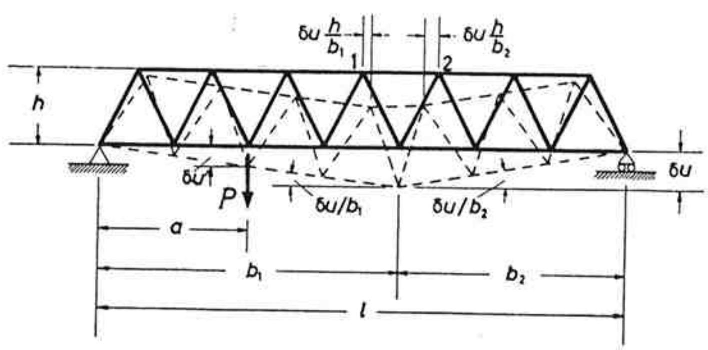


Fig. 8.—Example of an arbitrary virtual displacement

Thus,

$$\delta W=P\delta u\frac{a}{b_1} \text{ and } \delta U_i=\left(-\frac{Pab_2}{lh}\right)\times\left(-\frac{\delta ulh}{b_1b_2}\right)=P\delta u\frac{a}{b_1}$$

Hence

$$\delta W=\delta U_i$$

as indicated by the principle of virtual displacements or virtual work.

In order to realize best some of the implications of the principle of virtual displacements let us consider again the derivation of Eq. (44) which applies when $\delta\eta=0$. In accordance with the analysis of p. 5, if we multiply the internal equilibrium Eqs. (4) with the virtual displacements δu, δv, δw, sum the three expressions, integrate over the body, apply Green's theorem Eq. (6), and note the boundary conditions (5) we obtain Eq. (44). Again if we start from Eq. (44) we can apply Green's theorem in the opposite direction and are led to

$$\int_V\left\{\left[\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}+\omega_x\right]\delta u+[\ldots\ldots]\delta v+[\ldots\ldots]\delta w\right\}dV$$

$$=\int_S\left\{[l\sigma_{xx}+m\sigma_{yx}+n\sigma_{zx}-\phi_x]\delta u+[\ldots\ldots]\delta v+[\ldots\ldots]\delta w\right\}dS$$

For arbitrary virtual displacements δu, δv, δw this relation can only be true if each of the brackets vanishes separately; thus, starting from the principle of virtual work we have re-established the conditions of equilibrium. An exception occurs, of course, where a displacement (say u) is fixed on the surface and δu is automatically zero there.

We conclude that the principle of virtual displacements Eq. (44) is a necessary and sufficient condition for the existence of equilibrium of an elastic body. Or otherwise we can state that by using virtual, i.e. kinematically possible compatible displacements, we substitute Eq. (44) for the internal and external equilibrium conditions. Note that the idea of strain energy is not necessary to the establishment of the principle of virtual work.

Since the forces are assumed to be applied on the undistorted system and to remain constant during the virtual displacements we can regard δW as the variation of a potential $-U_e$. Thus

$$\delta W=-\delta U_e \quad \ldots\ldots \quad (45)$$

and

$$U_e=-\int_V[\omega_x u+\omega_y v+\omega_z w]dV-\int_S[\phi_x u+\phi_y v+\phi_z w]dS \quad \ldots\ldots \quad (46)$$

Thus

$$\omega_x=-\frac{\partial U_e}{\partial u}$$

which is the usual definition of a potential of forces. Note that δU_e is a total differential of the elastic displacement increments δu. U_e is denoted as potential of external forces.

We can write now Eq. (44) in the concise form

$$\delta_\epsilon U=0 \quad \ldots\ldots \quad (47)$$

where the suffix ϵ indicates that only elastic strains and displacements are varied.

$$U=U_i+U_e+\text{const.} \quad \ldots\ldots \quad (47a)$$

is the total potential energy of the system. Eq. (47) states that a position of elastic compatibility of an elastic body is also one of equilibrium (i.e. the body is at the true position of equilibrium) if any virtual variation of the displacements and strains whilst forces, stresses and temperature remain constant does not give rise to any (first order) variation of the total potential energy. The particular form (47) is known as the principle of a stationary value of total potential energy, if the latter is expressed in terms of displacements. Note that U_i itself may be calculated from formula (19*a*).

Actually the stationary value of U is in our case always a minimum and this confirms our previous assertion that with the assumptions of our analysis all systems are stable. The mathematical proof that U is a minimum at the true position of equilibrium is straightforward*; the point is discussed in greater detail under (C) below.

We have assumed until now that the initial strains η are due to a temperature variation. However, this is an unnecessary restriction and there may be strains arising from any source of self-straining. For example, in a framework they may be due to manufacturing errors in the lengths of the bars. In the more general cases of self-straining not only may the η_{xx}, η_{yy} and η_{zz} be different, but there may also arise initial shear strains η_{xy}, η_{yz} and η_{zx}. In such problems substitute

$$\sigma\delta\eta=\sigma_{xx}\delta\eta_{xx}+\sigma_{yy}\delta\eta_{yy}+\sigma_{zz}\delta\eta_{zz}+\sigma_{xy}\delta\eta_{xy}+\sigma_{yz}\delta\eta_{yz}+\sigma_{zx}\delta\eta_{zx} \quad \ldots \quad (48)$$

for $s\delta\eta$ in Eqs. (15) and (19) and in the other related expressions.

Equations (44) or (47) may be used to derive the results which follow.

(A) **The differential equations of the theory of elasticity** for arbitrary loading and temperature distribution or any particular structural problem in terms of the displacements; the appropriate static boundary conditions

* See Biezeno and Grammel (1), p. 74.

B

in terms of the displacements follow also from this analysis. It is important to note that in all applications it is best to form directly

$$\delta U_i = \int_V [\sigma \delta\epsilon] dV$$

and not to evaluate first U_i and then to take its increment δ.

(B) Castigliano's* theorem Part I generalized for thermal effects

$$\left[\frac{\partial U_i}{\partial u_r}\right]_{\Theta=\text{const.}} = P_r \quad \ldots\ldots \quad (49)$$

where P_r is the force (moment) applied in the direction of the deflexion (rotation) u_r. This relation may be obtained immediately if we apply a virtual elastic displacement δu_r solely to one external load P_r. Note that Eq. (49) applies also for non-linear stress-strain laws and may also be generalized for large displacements.

(C) The Principle of Minimum Strain Energy when U_i is expressed in terms of the displacements and the temperature is not varied.

We arrive immediately at this theorem if we select only such virtual displacements δu which are zero at the applied forces. Then

$$\delta W = 0$$

and we conclude from Eq. (44) that

$$\delta_e U_i = 0 \text{ and } U_i = \text{min.} \quad \ldots\ldots \quad (44a)$$

at true position of equilibrium if only such virtual deformations are allowed that no external work is done.

Hence, if we compare all possible compatible states of deformation of a body associated with a given set of displacements (not sufficient by themselves to fix completely the deformed shape of the body) then the true position of equilibrium has the minimum strain energy. This is still true if the body is subjected to temperature loading.

The point is of sufficient interest to warrant some elaboration. First it may be helpful to point out that when we state that at the position of equilibrium the total potential energy has a stationary value and that this is a minimum we do not compare physically possible adjoining states. For, in stating that the potential energy has a stationary value, i.e. $\delta U = 0$, we compare the true position u, with a position $u + \delta u$ assuming in both cases that forces and stresses are the same. This can obviously not be true for the second position since for given forces there is a unique position of equilibrium. In fact, we mentioned that this arises due to our legitimate neglect of the higher order terms in $\delta\sigma$ and δP. Also, when we go a step farther and state that the stationary value is a minimum we prove this by considering the influence of terms like $\frac{1}{2}\delta\sigma_{zz} \cdot \delta\epsilon_{zz}$ in U_i arising from the variation $\delta\sigma_{zz}$ associated with $\delta\epsilon_{zz}$, but we still keep the forces constant—although this cannot, in general, be true.

Having pointed out these aspects of the virtual displacements approach we shall, in what follows, discuss the question of the extremum of U_i from a more physical point of view. Again we prescribe certain displacements on the body and do not allow any forces P other than those arising due to and in the direction of the given displacements. The structure takes up its natural position of equilibrium from which we can deduce the value of the forces P. If we want to force the body to assume a position $u + \delta u$, $v + \delta v$, $w + \delta w$ while keeping the set of prescribed displacements constant we must apply certain additional body and surface forces to push the system away from its natural configuration. The work done by these constraint forces ($\frac{1}{2}\Sigma\delta P \cdot \delta u$, obviously positive), produces by reason of equilibrium in the new position an equal increase in the strain energy stored. Thus, the strain energy in any neighbouring compatible configuration is greater than that for the unconstrained original position and hence the strain energy there is a minimum.

An alternative way of producing a state of equilibrium different from the natural one in an elastic body under a prescribed set of displacements is the introduction (prior to the imposition of the displacements) of internal or external constraints that do no work. For example, in a shell or plate analysis, we may assume that the middle surface is inextensible and the transverse shear strains are zero; thus, in this case we impose infinite values for E and G in the middle surface and an infinite value for G in the

* See A. Castigliano, *Théorie de l'équilibre des systèmes élastiques*, Turin 1879.

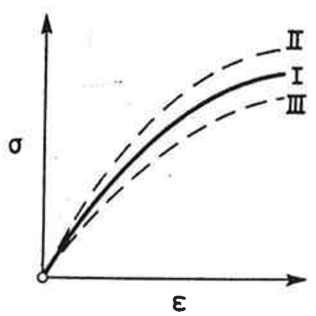


Fig. 9.—Stiffening and relaxation of stress-strain curve

transverse direction. Another type of constraint may be achieved by the introduction of a rigid support. Also the stress-strain diagram may be taken locally to be *II* instead of *I* (FIG. 9). In all such cases the arguments of the previous paragraph show that the strain energy of the constrained body for the given prescribed set of displacements is greater than for the unconstrained body. Conversely, if we relax any existing constraints whilst again keeping constant the prescribed displacements the strain energy is decreased. The relaxation of constraints may take the form of a stress-strain line *III* instead of *I*. Alternatively we may introduce a hinge in the structure. Another example is the case of a shell where we ignore the bending stiffness and admit only a membrane state of stress; a current procedure in wing stressing.

Thus we conclude: *the strain energy of an elastic body for a given set of displacements is increased (reduced) by the imposition (relaxation) of constraints that do no work.* An exception occurs if the effect of imposition or removal of a constraint is nil. For example, in an infinitely long, thin circular shell under internal pressure it is immaterial whether we take account of or ignore the bending stiffness of the walls.

Also since the constraints do no work it follows that the increase (reduction) of the strain energy can only be produced by the forces P. Thus: *the force system P which is set up at the and in the direction of the prescribed displacements and hence also the stiffness of the structure is increased (reduced) by the imposition (relaxation) of constraints that do no work.*

If we consider now the case of an elastic body under a given set of forces instead of displacements then we conclude immediately from the last theorem: *the displacements and hence also the strain energy in a body under a given set of forces are reduced (increased) by the imposition (relaxation) of constraints that do no work.*

Both the last two theorems illustrate two complementary effects of the action of constraints on the stiffness of a structure. The last theorem may also be expressed as follows: *the strain energy of an elastic body under a given set of forces is a maximum when it is subjected to the least number of constraints that do no work.*

The immediate application of the above considerations is, of course, to the effect of actual constraints on elastic structures as illustrated in the examples mentioned. A more important application appears in connexion with approximate analyses of deformations. Thus, if we reduce the freedom of deformation as we do in the Rayleigh-Ritz and related methods we always over-estimate the stiffness of the structure. Hence for a given set of forces (displacements) we under-estimate (over-estimate) the corresponding displacements and strain energy (forces and strain energy).

The above theorems on the effects of constraints on strain energy and stiffness appear to have been given first by Rayleigh* in 1875 for linearly elastic bodies and no temperature effects. Our arguments indicate, however, that they apply also to elastic bodies with non-linear stress-strain relationship and under thermal loading. The original principles are occasionally referred to as the static analogues of Bertrand's and Kelvin's theorems in dynamics.

(D) The Unit Displacement method. This method will be developed in Section 8.

(E) Approximate methods of displacement analysis, using the Rayleigh-Ritz† procedure. In this method we assume for the displacements approximate functions or series of functions satisfying the geometric but not necessarily the static boundary conditions. For example, in a three-dimensional elastic continuum we may express the total displacements u, v, w in a finite series as follows:

$$\left.\begin{aligned} u &= u_0(x, y, z) + \sum_{r=1}^{n} a_r u_r(x, y, z) \\ v &= v_0(x, y, z) + \sum_{r=1}^{n} b_r v_r(x, y, z) \\ w &= w_0(x, y, z) + \sum_{r=1}^{n} c_r w_r(x, y, z) \end{aligned}\right\} \quad \ldots\ldots \quad (50)$$

where u_0, v_0, w_0 satisfy the kinematic conditions where these are prescribed and u_r, v_r, w_r are linearly independent functions which vanish there. a_r, b_r, c_r are unknown constants to be determined by the Rayleigh-Ritz procedure. The elastic strains corresponding to (50) are (see Eqs. (1) and (2))

$$\left.\begin{aligned} \epsilon_{xx} &= \frac{\partial u}{\partial x} - \eta, \quad \epsilon_{yy} = \frac{\partial v}{\partial y} - \eta, \quad \epsilon_{zz} = \frac{\partial w}{\partial z} - \eta \\ \epsilon_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \epsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \epsilon_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{aligned}\right\} \quad \ldots\ldots \quad (51)$$

The chosen series satisfy the displacement boundary conditions and the infinitesimal deformations

$$\delta u = \delta a_r \cdot u_r, \ \delta v = \delta b_r \cdot v_r, \ \delta w = \delta c_r \cdot w_r \quad \ldots\ldots \quad (52)$$

*See Ref. 3, Vol. II, p. 94, and also 'General Theorems Relating to Equilibrium and Initial and Steady Motion', *Phil. Mag.*, March 1875. They have been discussed more recently by D. Williams in *Phil. Mag.* Ser. 7, Vol. 26, 1938, p. 617.

† W. Ritz, 'Theorie der Transversalschwingungen einer quadratischen Platte,' *Ann. d. Physik*, Vol. 28, p. 737, 1909; see also *J. reine u. angew. Math.*, Vol. 135, p. 1, and *Gesammelte Werke*, Paris, 1911, p. 192.

obtained by variation of a_r, b_r, c_r while the other coefficients and the temperature are kept constant are hence virtual elastic displacements. Note again that the chosen u, v, w functions need not satisfy the given static boundary conditions. Naturally, the accuracy of the analysis is enhanced if the latter are also satisfied.

To determine the coefficients a_r, b_r, c_r we use the condition of the stationary value of the total potential energy in the forms

$$\delta U=0 \text{ or } \delta U_i=\delta W$$

There are also cases when the theorem of minimum strain-energy, $U_i=\text{min.}$, is useful, see for example Part II of this series example 4. The application of $\delta U=0$ ensures the average satisfaction of the equilibrium conditions.

If we evaluate U in terms of (50) then for a linearly elastic body we obtain a quadratic function in a_r, b_r, c_r. Condition $\delta U=0$ which in this case becomes

$$\left.\begin{aligned}&\frac{\partial U}{\partial a_1}=\ldots\ldots=\frac{\partial U}{\partial a_r}=\ldots\ldots=\frac{\partial U}{\partial a_n}=0\\&\frac{\partial U}{\partial b_1}=\ldots\ldots=\frac{\partial U}{\partial b_r}=\ldots\ldots=\frac{\partial U}{\partial b_n}=0\\&\frac{\partial U}{\partial c_1}=\ldots\ldots=\frac{\partial U}{\partial c_r}=\ldots\ldots=\frac{\partial U}{\partial c_n}=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots \quad (53)$$

leads to a set of $3n$ linear equations in the $3n$ unknowns a_r, b_r, c_r. It is, however, superfluous to evaluate first U and then to differentiate with respect to the unknown coefficients. We can obtain directly the final equations by forming the $3n$ expressions

$$\delta_{ar}(U_i+U_e)=0,\ \delta_{br}(U_i+U_e)=0,\ \delta_{cr}(U_i+U_e)=0 \quad\ldots\ldots\ldots\ldots \quad (54)$$

where the suffices ar, br, cr indicate that the virtual displacements are chosen respectively as in Eqs. (52).

Using the first of Eqs. (52) in the first of Eqs. (54) we obtain the more explicit form

$$\{\int_V \sigma\epsilon_r dV-\Sigma P\bar{u}\}\delta a_r=0$$

or since δa_r is arbitrary

$$\int_V \sigma\epsilon_r dV-\Sigma P\bar{u}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (55)$$

where ϵ_r are the elastic strains due to u_r, σ the stress due to the elastic strains given by Eqs. (51), P the applied forces and $\bar{u}$ the displacements in their directions due to the set (u_r). There are in all $3n$ equations in a_r, b_r, c_r which are non-linear if the stress-strain relations are non-linear.

By a judicious choice of the u, v, w functions it is possible to obtain very good approximations to the deformations of the body. The number of necessary functions for a good estimate depends on the problem and on the choice of these component functions. The proof of the convergence to the exact solution with increasing n is a difficult question which cannot be considered here, see Trefftz[10], p. 130. We note only, referring to the previous paragraph, that the Rayleigh-Ritz method always over-estimates the stiffness.

Whilst the Rayleigh-Ritz method can provide a good approximation to the deformations, the accuracy of the associated stresses is, in general, not as good. This is obvious if we remember that the accuracy of any approximate function is decreased with every differentiation.

In two- or three-dimensional problems it is possible to improve upon the original Rayleigh-Ritz procedure by adopting a mixed technique of (A) and this paragraph. Thus, in a two-dimensional problem it is often possible to guess accurately the variation of the displacements parallel to one axis, say the x-axis, while it is much more difficult to make an intelligent assumption about the variation parallel to the other direction. We may then write the displacements u and v in the form

$$\left.\begin{aligned}u&=u_1(x)\cdot\phi(y)\\v&=v_1(x)\cdot\psi(y)\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots \quad (56)$$

where u_1, v_1 are assumed crosswise distributions of the u, v displacements and ϕ, ψ are unknown non-dimensional functions of y. Substituting Eqs. (56) into (55) one obtains after some transformations the differential equations in ϕ and ψ together with the necessary boundary conditions. Such an analysis can yield a very accurate result. It will be illustrated on a number of examples of some complexity in Part II to this report.

Note that the Rayleigh-Ritz procedure as presented here is also valid for non-linear stress-strain relations.

(F) Galerkin's method of approximate stress analysis. Consider the internal equilibrium Eqs. (4) which we write here in the by now familiar form

$$\left.\begin{aligned}&\int_V\Big[\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}+\omega_x\Big]\delta u\cdot dV=0\\&\int_V\Big[\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}+\frac{\partial\sigma_{zv}}{\partial x}+\omega_v\Big]\delta v\cdot dV=0\\&\int_V\Big[\frac{\partial\sigma_{zz}}{\partial z}+\frac{\partial\sigma_{xv}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\omega_z\Big]\delta w\cdot dV=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots \quad (57)$$

where the δu's are virtual, i.e. kinematically possible infinitesimal displacements. Eqs. (57) lead to the principle of virtual work Eq. (44) if the σ's satisfy not only the internal but also the boundary equilibrium conditions Eqs. (5).

Assume now that the displacements u, v, w are written in the approximate form of Eqs. (50) where the a_r, b_r, c_r are unknown coefficients. However, contrary to what we assumed in paragraph (4), series (50) are taken here to satisfy not only the kinematic conditions where prescribed on the surface but also by substitution into the stress-strain relations the equilibrium conditions where prescribed.

Expressing now the stresses in the brackets of (57) in terms of the displacements and temperature Θ and selecting as virtual elastic displacements one of the $3n$ possibilities (52) we obtain the n equations in the $3n$ unknowns a_r, b_r, c_r

$$\left.\begin{aligned}&\int_V\Big[\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}+\omega_x\Big]u_r dV=0\\&\int_V\Big[\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zy}}{\partial z}+\frac{\partial\sigma_{xy}}{\partial x}+\omega_y\Big]v_r dV=0\\&\int_V\Big[\frac{\partial\sigma_{zz}}{\partial z}+\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial x}+\omega_z\Big]w_r dV=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots \quad (58)$$

This is the Galerkin*† procedure usually only given for linear elasticity. To fix ideas take the case of linear elasticity and write σ_{xx} etc. in terms of u, v, w. We find easily three equations, the first of which is

$$\int_V\Big[\Delta u+\frac{1}{1-2\nu}\,\frac{\partial g}{\partial x}+\frac{\omega_x}{G}-\frac{2(1+\nu)}{1-2\nu}\,\frac{\partial\Theta}{\partial x}\Big]u_r dV=0 \quad\ldots\ldots\ldots\ldots \quad (59)$$

where

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} \text{ and } g=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (60)$$

The bracket in Eq. (59) is, of course, the equilibrium equation in the x-direction expressed in terms of displacements.

The procedure in any particular structural problem is to form the equilibrium conditions in the stresses or stress resultants and express each in terms of the displacements. Next we multiply each by the corresponding u_r (which may be a deflexion, slope, twist) and then integrate over the body. Thus, for a beam subjected to a distributed loading p in the y-direction, the equilibrium equation in terms of the deflexion v is, assuming engineers' theory of bending to hold

$$\frac{d^2}{dz^2}\Big(EI\frac{d^2v}{dz^2}\Big)-p=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (61)$$

and the Galerkin form of the virtual work equation is

$$\int_0^l\Big\{\frac{d^2}{dz^2}\Big(EI\frac{d^2v}{dz^2}\Big)-p\Big\}\delta v\cdot dz=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (61a)$$

It is easy to see that for displacement functions (50) satisfying *all* boundary conditions the Galerkin and Rayleigh-Ritz methods must yield the same equations for a_r, b_r, c_r and hence also the same deformations. We need only realize that in this case Eqs. (57) are indeed equivalent to the principle of virtual work. Hence substitution of u, v, w in Eqs. (57) must give the same result as substitution into

$$\delta U=0$$

The advantage of Galerkin's method lies in a more direct derivation of the equations in a_r, b_r, c_r. However, contrary to what is usually assumed, this advantage is small if we calculate δU directly. We note also that Galerkin's method allows only such approximate functions as satisfy all boundary conditions, while the Rayleigh-Ritz procedure requires only the satisfaction of the kinematic boundary conditions.

* Timoshenko[8] mentions on p. 159 that equations of this type appear already in W. Ritz's work. See references on p. 353 and also *Gesammelte Werke*, p. 228.

† B. G. Galerkin, 'Series Solutions of Some Problems of Elastic Equilibrium of Rods and Plates,' *Wjestnik Ingenerow Petrograd* (1915), No. 19, p. 879; see also W. J. Duncan, 'Application of the Galerkin Method to the Torsion and Flexure of Cylinders and Prisms,' *Phil. Mag.*, Ser. 7, Vol. 25, 1938, p. 636.

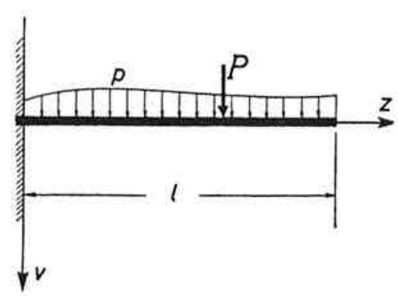


Fig. 10.—Cantilever beam

Consider the cantilever under transverse load shown in FIG. 10. To obtain an approximate expression for the deflexions v by the Rayleigh-Ritz procedure we need only select a function or sequence of functions giving

$$v=\frac{dv}{dz}=0 \text{ at } z=0$$

However, when applying the Galerkin method Eq. (61a) must satisfy also the static conditions at $z=l$, i.e.

Shear force=Bending moment=0

$$\text{or } \left(\frac{d^3v}{dz^3}\right)_{z=l}=\left(\frac{d^2v}{dz^2}\right)_{z=l}=0$$

5. ILLUSTRATIONS OF THE METHOD OF VIRTUAL DISPLACEMENTS

In this section we present a number of applications of the principle of virtual work. These are not meant to give the shortest possible solutions to the problems considered but merely to illustrate the way in which the method can be applied in some simple cases. More complicated problems are investigated in Part II

(a) Continuous Beam with Non-Linear Spring Support

The uniform beam of bending stiffness EI shown in FIG. 11 carries a uniformly distributed load p and is simply supported at the ends. At the centre an additional support consists of a spring with the load displacement law

$$P=kV\left[1+\frac{a}{1-(V/V_o)}\right] \quad \text{(a1)}$$

Thus k is the initial spring stiffness and V_o is the displacement at which the spring becomes solid.

With the assumption of the Engineers' Theory of Bending, establish by means of the Principle of Virtual Displacements the differential equation for the deflexion v of the beam and the boundary conditions. Find also the deflexion V at the spring.

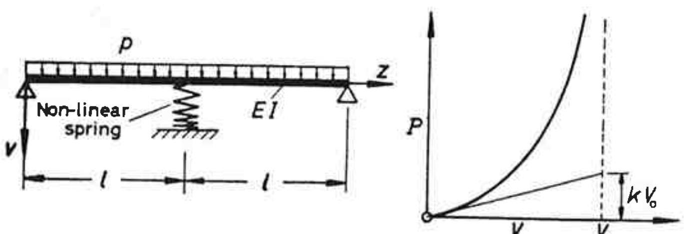


Fig. 11.—Virtual displacements: Example (a) Simply supported beam with non-linear spring

The first part of the problem is, of course, trivial and the result known to any undergraduate, but we want to show here how the Virtual Displacements method can be applied in such a case.

We consider virtual displacements consisting of small arbitrary additional deflexions (δv) of the beam from its equilibrium position. The Principle of Virtual Displacements then expresses the equilibrium condition in the form

$$\delta U_i-\delta W=0 \quad \text{(a2)}$$

The virtual displacements must satisfy certain kinematic boundary conditions, namely:

$$\delta v=0 \text{ for } z=0 \text{ and } z=2l \quad \text{(a3)}$$

and since both structure and loading are symmetrical we need only consider symmetrical virtual displacements. Hence

$$d(\delta v)/dz=(\delta v)'=0 \text{ for } z=l \quad \text{(a4)}$$

Strains and stresses in beam due to bending:

$$\epsilon=\epsilon_{zz}=-\bar{y}v'',\ \sigma=\sigma_{zz}=-E\bar{y}v'' \quad \text{(a5)}$$

and therefore the virtual strain due to δv is

$$\delta\epsilon=-\bar{y}\delta(v'')=-\bar{y}(\delta v)'' \quad \text{(a6)}$$

The increment of strain energy in the beam due to bending is thus

$$2\int_0^l\int_A\sigma\cdot\delta\epsilon\cdot dA\cdot dz=2E\int_0^l\left\{\int_A\bar{y}^2\cdot dA\right\}v''(\delta v)''dz=2EI\int_0^l v''(\delta v)''dz \quad \text{(a7)}$$

which becomes, on integrating twice by parts,

$$2EI\{v''(\delta v)'\Big|_0^l-v'''(\delta v)\Big|_0^l+\int_0^l v^{\mathrm{iv}}\cdot\delta v\cdot dz\} \quad \text{(a8)}$$

The increment of strain energy in the spring is

$$P\delta V \quad \text{(a9)}$$

and the increment of work done by the distributed load is

$$\delta W=2\int_0^l p\cdot\delta v\cdot dz \quad \text{(a10)}$$

Hence the complete virtual work Eq. (a2) becomes

$$2\int_0^l\{EIv^{\mathrm{iv}}-p\}\delta v\cdot dz+2EIv''(\delta v)'\Big|_0^l-2EIv'''(\delta v)\Big|_0^l+P\delta V=0 \quad \text{(a11)}$$

But

$$(\delta v)_l'=0,\ (\delta v)_o=0 \text{ and } (\delta v)_l=\delta V$$

Since otherwise δv is arbitrary we conclude that to satisfy Eq. (a11) we must have

$$(EIv'')_{z=o}=0,\ 2(EIv''')_{z=l}-P=0 \quad \text{(a12)}$$

and v must satisfy the differential equation

$$EIv^{\mathrm{iv}}-p=0 \quad \text{(a13)}$$

Eqs. (a12) and (a13) together with the kinematic conditions

$$(v)_{z=o}=0 \text{ and } (v')_{z=l}=0 \quad \text{(a14)}$$

give all the necessary information for the determination of v.

Integrating Eq. (a13) and using the boundary conditions (a12) and (a14) we find finally for V the quadratic equation

$$(1+\psi)(V/V_o)^2-[1+\psi(1+a)+V_1/V_o](V/V_o)+V_1/V_o=0 \quad \text{(a15)}$$

where

$$\psi=kl^3/6EI,\ V_1=5pl^4/24$$

(b) Plane Redundant Framework

A plane framework consisting of a single joint connected by a number of hinged bars to a rigid foundation is loaded by forces X and Y along the axes Ox, Oy respectively (FIG. 12). In addition, the bars are heated to arbitrary temperatures and have also initial strains due to errors in manufacture.

Find by application of the Principle of Virtual Displacements the forces in the bars.

Let u, v be the displacements of the loaded hinge, measured from the position for which all bars have the correct length and are at zero temperature. Then the total direct strain in the rth bar for these displacements is

$$\gamma_r=\frac{u\cos\theta_r+v\sin\theta_r}{l_r} \quad \text{(b1)}$$

The total strain is made up of the elastic strain ϵ_r together with the thermal strain $\eta_r=\alpha\Theta$ and the initial strain η_{or}, where the initial strain is

$$\eta_{or}=\Delta l_r/l_r \quad \text{(b2)}$$

Δl_r being the additional length of bar (in excess of the correct length) due to manufacturing errors or other causes. Hence the elastic strain due to u, v is

$$\epsilon_r=\gamma_r-\eta_r-\eta_{or}$$
$$=\frac{u\cos\theta_r+v\sin\theta_r}{l_r}-(\eta_r+\eta_{or}) \quad \text{(b3)}$$

and the direct stress in the bar is

$$\frac{N_r}{A_r}=\sigma_r=E\epsilon_r=E\left[\frac{u\cos\theta_r+v\sin\theta_r}{l_r}-(\eta_r+\eta_{or})\right] \quad \text{(b4)}$$

If we now impose on the joint the virtual displacements δu, δv there arises an increment of strain energy δU_i, and an increment of work δW of the applied forces, where

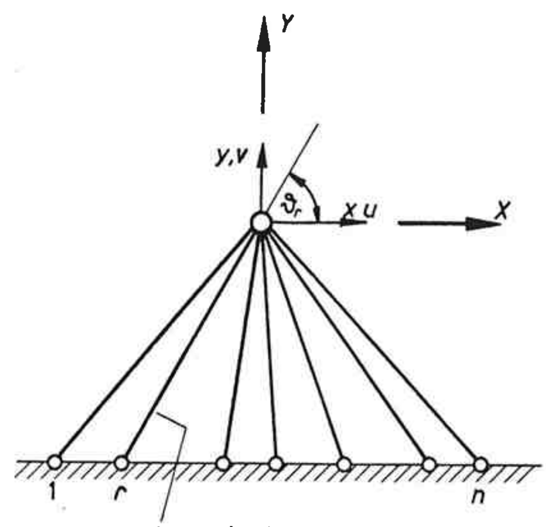


Fig. 12.—Virtual displacements: Example (b) Redundant system of bars

$$\delta U_i=\sum_{r=1}^{n}A_r\sigma_r l_r\delta\gamma_r$$

$$=\sum_{r=1}^{n}EA_r l_r\left[\frac{u\cos\theta_r+v\sin\theta_r}{l_r}-(\eta_r+\eta_{or})\right]\left[\frac{\delta u\cos\theta_r+\delta v\sin\theta_r}{l_r}\right] \quad \text{(b5)}$$

$$\delta W=X\delta u+Y\delta v \quad \text{(b6)}$$

By the Principle of Virtual Displacements

$$\delta U_i-\delta W=0$$

and therefore

$$\delta u\left[u\sum_{r=1}^{n}\frac{EA_r}{l_r}\cos^2\theta_r+v\sum_{r=1}^{n}\frac{EA_r}{l_r}\sin\theta_r\cos\theta_r-\sum_{r=1}^{n}EA_r(\eta_r+\eta_{or})\cos\theta_r-X\right]+$$

$$\delta v\left[u\sum_{r=1}^{n}\frac{EA_r}{l_r}\sin\theta_r\cos\theta_r+v\sum_{r=1}^{n}\frac{EA_r}{l_r}\sin^2\theta_r-\sum_{r=1}^{n}EA_r(\eta_r+\eta_{or})\sin\theta_r-Y\right]=0 \quad \text{(b7)}$$

If Eq. (b7) is to be satisfied, since δu, δv are arbitrary, the two expressions in brackets must separately be equal to zero and hence we have two equations in the displacements u, v. Solving for u, v the stresses and forces in the bars can be calculated from Eq. (b4).

The two equations are, of course, the equilibrium conditions in the x and y directions and could be derived directly by statics without recourse to the Principle of Virtual Displacements.

Note that there are always only two unknowns in this approach, regardless of the number n of bars. Hence it is preferable to operate with the displacements u, v as unknowns than with forces in the bars when $n>4$.

(c) The Open Tube Under Torque

A uniform, open tube of length l is subjected to a distributed torque m_z per unit length and end torques T_o and T_l.

With the assumption that shear strains due to restrained warping are zero (Wagner assumption) establish the differential equation for the angle of twist θ and the static boundary conditions. Give also a series solution for the case when the ends $z=0$, $z=l$ are prevented from twisting but are free to warp.

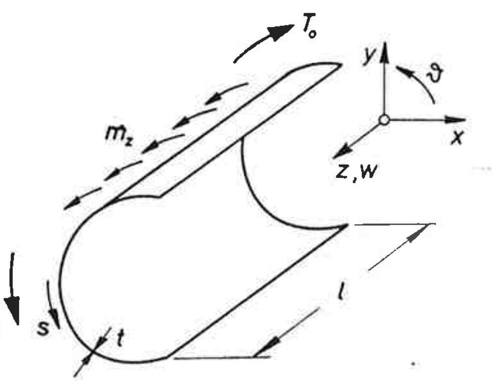


Fig. 13.—Virtual displacements: Example (c) The open tube

The warping displacement w of the cross-section is given by*

$$w=\bar{w}(d\theta/dz)=\bar{w}\theta' \quad \text{(c1)}$$

where $\bar{w}$ is the warping per unit rate of twist. The direct strain ϵ_{zz} and stress σ_{zz} due to non-constant rate of twist are

$$\epsilon_{zz}=\frac{\partial w}{\partial z}=\bar{w}\theta'',\ \sigma_{zz}=E\epsilon_{zz}=E\bar{w}\theta'' \quad \text{(c2)}$$

The torque due to the St Venant torsional shear stresses is

$$T_J=GJ\theta' \quad \text{(c3)}$$

We obtain now virtual displacements by giving to the twist θ an increment $\delta\theta$. The corresponding virtual strain $\delta\epsilon$ is given by

$$\delta\epsilon_{zz}=\bar{w}\delta(\theta'')=\bar{w}(\delta\theta)'' \quad \text{(c4)}$$

and the increment of strain energy due to the direct (torsion-bending) stresses is

$$\int_0^l\left[\int_0^S\sigma_{zz}\delta\epsilon_{zz}t\,ds\right]dz=\int_0^l E\theta''(\delta\theta)''\left[\int_0^S\bar{w}^2t\,ds\right]dz=\int_0^l E\Gamma\theta''(\delta\theta)''dz \quad \text{(c5)}$$

where Γ is the torsion-bending constant.*

The increment of strain energy due to the St Venant shear stresses is

$$\int_0^l GJ\theta'\delta(\theta')dz=\int_0^l GJ\theta'(\delta\theta)'dz \quad \text{(c6)}$$

* See Argyris, 'The Open Tube', Aircraft Engineering, Vol. XXVI, No. 302, April 1954, p. 102 et seq.

and the work done by the distributed torque and the end torques

$$\delta W=\int_0^l m_z\delta\theta dz+T_l(\delta\theta)_l-T_o(\delta\theta)_o \quad \text{(c7)}$$

Applying the Principle of Virtual Work

$$\delta U_i-\delta W=0=\int_0^l[E\Gamma\theta''(\delta\theta)''+GJ\theta'(\delta\theta)'-m_z(\delta\theta)]dz-T\delta\theta\Big|_0^l \quad \text{(c8)}$$

Integrating by parts the first term twice and the second once, we finally obtain

$$\int_0^l[E\Gamma\theta^{\mathrm{iv}}-GJ\theta''-m_z]\delta\theta dz+[GJ\theta'-E\Gamma\theta'''-T]\delta\theta\Big|_0^l+E\Gamma\theta''(\delta\theta)'\Big|_0^l=0 \quad \text{(c9)}$$

For the integral to vanish, since $\delta\theta$ is arbitrary, θ must satisfy the differential equation

$$E\Gamma\theta^{\mathrm{iv}}-GJ\theta''-m_z=0 \quad \text{(c10)}$$

which is recognized as the usual Wagner equation differentiated with respect to z.

If the twist at both ends of the tube is specified then $\delta\theta$ must there be taken zero and the first bracketed term in (c9) vanishes also. If in addition the warping (and hence θ') is specified (e.g. built-in end) then $(\delta\theta)'$ is also zero at $z=0$ and $z=l$ and the remaining term also vanishes.

If, however, the end $z=l$ is free to twist and warp, $(\delta\theta)_l$ and $(\delta\theta)_l'$ are arbitrary and we have as further conditions from (c9)

$$\left.\begin{aligned}&GJ(\theta')_l-E\Gamma(\theta)'''_l-T_l=0\\ &\text{and}\\ &E\Gamma(\theta'')_l=0\end{aligned}\right\} \quad \text{(c11)}$$

which are the necessary static boundary conditions at the free end. The first is of course the condition for equality of external and internal torque and the second the condition for zero direct stress.

For the series solution, we represent the twisted shape by the Fourier series

$$\theta=\sum_{n=1}^{\infty}a_n\sin\left(\frac{n\pi z}{l}\right) \quad \text{(c12)}$$

and take for virtual displacements the increments of twist produced by a small variation δa_m of the coefficient a_m.

Using Eq. (c8) we find

$$\left.\begin{aligned}\delta U_i-\delta W=\delta a_m\Bigg[&GJ\sum_{n=1}^{\infty}a_n\frac{nm\pi^2}{l^2}\int_0^l\cos\frac{m\pi z}{l}\cos\frac{n\pi z}{l}dz\\ &+E\Gamma\sum_{n=1}^{\infty}a_n\left(\frac{nm\pi^2}{l^2}\right)^2\int_0^l\sin\frac{m\pi z}{l}\sin\frac{n\pi z}{l}dz-\int_0^l m_z\sin\frac{m\pi z}{l}dz\Bigg]\end{aligned}\right\} \quad \text{(c13)}$$

which gives on integrating

$$a_m=\frac{\int_0^l m_z\sin\frac{m\pi z}{l}dz}{\frac{m^2\pi^2}{2l}\left[GJ+\left(\frac{m\pi}{l}\right)^2E\Gamma\right]} \quad \text{(c14)}$$

If m_z is constant

$$\left.\begin{aligned}&a_m=\frac{4l^2}{m^3\pi^3}\,\frac{m_z}{\left[GJ+\left(\frac{m\pi}{l}\right)^2E\Gamma\right]}\quad\text{for } m \text{ odd}\\ &\text{and}\\ &a_m=0\quad\text{for } m \text{ even}\end{aligned}\right\} \quad \text{(c15)}$$

which gives for the twist distribution

$$\theta=\frac{4l^2m_z}{GJ\pi^3}\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\frac{\sin(n\pi z/l)}{n^3\left[1+\left(\frac{n\pi}{l}\right)^2\frac{E\Gamma}{GJ}\right]} \quad \text{(c16)}$$

In this case, since the assumed form of solution satisfies also the static boundary conditions $\theta''=0$ for $z=0$ and l we can alternatively use the Galerkin form of the Virtual Work equation, which is in this case

$$\int_0^l[E\Gamma\theta^{\mathrm{iv}}-GJ\theta''-m_z]\delta\theta dz=0 \quad \text{(c17)}$$

and is given immediately from Eq. (c9).

If we approximate our solution for θ by retaining only the first term in (c16) we underestimate the average angle of twist. This ties up with our statements on p. 8; for, by putting $\theta=a_n\sin(\pi z/l)$ we apply constraints on the tube and hence overestimate the stiffness.

6. THE PRINCIPLE OF VIRTUAL FORCES OR COMPLEMENTARY VIRTUAL WORK

It is natural in reviewing the developments of Sections 3 and 4 to inquire if it is possible to enlarge upon the conception of complementary work and strain energy in a similar way as accomplished for work and strain energy by the introduction of virtual displacements. In fact, if we consider Eqs. (28) and (36) we realize immediately that the functions δW^* and δU_i^* are independent of the variations δu and $\delta\epsilon$ associated with the force and stress increments, just as δW and δU_i are independent of the variations δP and $\delta\sigma$ associated with the chosen δu and $\delta\epsilon$'s. Hence, when finding δW^* and δU_i^* we may assume that displacements and strains remain constant. Also, the infinitesimal increments $\delta\sigma$, $\delta\omega$, $\delta\phi$ are arbitrary as long as they satisfy the equilibrium conditions in the interior and, where such are prescribed, on the surface. Thus, if we fix that the surface forces are not to be varied over part of the boundary we must have there $\delta\phi=0$; however, where kinematic conditions are prescribed on the boundary the $\delta\phi$ variation cannot be assigned. It is apparent that our incremental stress system need not even be an elastically compatible one. It is only restricted by the condition that it must be statically equivalent to the load increments $\delta\omega$ and $\delta\phi$. While these increments are applied it is assumed as in Section 4 that the temperature remains constant.

Such infinitesimal variations of forces and stresses which are arbitrary as long as they satisfy the prescribed equilibrium conditions we call virtual forces and stresses.

Before restating theorem (35) for the more general conceptions introduced here let us consider again its derivation in the light of our new ideas. Thus, if we multiply the *true* displacements u, v, w by the internal equilibrium conditions (29) which the virtual stresses $\delta\sigma$ and $\delta\omega$ must satisfy, sum, integrate over the body, apply Green's Theorem and note the boundary conditions (30) we obtain Eq. (35) where γ_{xx} etc. are the total strains associated with the displacements u, v, w. Next let us apply Green's Theorem in the opposite direction by starting from the right-hand side of Eq. (35). We find that this function can only be equal to the left-hand side if the terms γ_{xx} etc. are indeed the expressions for the strains (3) and (12) and satisfy the kinematic boundary conditions.

Thus, we conclude that an elastic body is in an elastically compatible state under a given system of forces and temperature distribution if for any virtual increments of forces and stresses from a position of equilibrium

$$\delta W^*=\delta U_i^*+\int_V \eta\delta s dV \quad\quad (62)$$

where

$$\delta U_i^*=\int_V \epsilon\delta\sigma\cdot dV \text{ and } \eta\delta s=\eta(\delta\sigma_{xx}+\delta\sigma_{yy}+\delta\sigma_{zz}) \quad\quad (63)$$

See also Eqs. (35) and (38*a*).

Eq. (62) is, in fact, a necessary and sufficient condition for elastic compatibility of the equilibrium.

Theorem (62) we call the principle of virtual forces or virtual complementary work for elastic bodies subjected to loads and temperature distribution. Note that (62) applies for non-linear stress-strain laws.

The above discussion indicates that there is a close parallel between the principles of virtual displacements and virtual forces. Thus, by substituting virtual forces (stresses) for virtual elastic displacements (strains), actual total displacements (strains) for forces (stresses), and invariant state of straining for invariant state of forces we obtain Eq. (62) from Eq. (44). However, this duality is only complete for continuous structures which are infinitely redundant. If, on the other hand, we consider a statically determinate structure we find that while it is still possible to describe an infinite set of virtual displacements δu associated with a prescribed set of certain of the displacements, only one stress system can exist for given external forces; hence no $\delta\sigma$ can be assigned in the latter case and the principle of virtual forces has no application. A more fundamental limitation of the principle of virtual forces appears if we want to extend our theorems to finite displacements. Here we find that it is, in general, impossible to achieve this for the principle of virtual forces while, as mentioned in Section 4, no basic difficulty arises in the case of the principle of virtual displacements. However, for the usual analyses of redundant systems involving small displacement theory the principle of virtual forces with its many particular forms is the most useful one since the standard procedure introduces forces as unknowns. Naturally, there are many cases, especially in multi-redundant structures, where it is advantageous to introduce displacements as unknowns; here the principle of virtual displacements is the indicated method as shown in Example (*b*) of Section 5.

We return now to Eq. (62) and will illustrate its validity on a very simple example. Consider to that effect the redundant beam of uniform flexural stiffness EI built-in at $z=0$, simply supported at $z=l$, and subjected to a uniform load p (see FIG. 14). Under the assumption that the ordinary engineers' theory of bending holds and that the shear deflexions are negligible the deflexion v is given by

$$v=\frac{pl^4}{48EI}\left(\frac{z}{l}\right)^2\left[3-5\left(\frac{z}{l}\right)+2\left(\frac{z}{l}\right)^2\right]$$

As a virtual force we select δP at the centre of the beam as shown in FIG. 14(*a*). However, since we require only a statically equivalent stress system to equilibrate the applied virtual load we may eliminate the one redundancy and select a statically determinate beam. The two alternative choices leading to a simply supported beam and a cantilever are seen in FIGS. 14(*b*) and 14(*c*). Denoting the true deflexion at the centre v_1 we have

$$\delta W^*=\delta P\cdot v_1=\delta P\cdot\frac{pl^4}{192EI}$$

Also since the Engineers' theory of bending applies ($\gamma_{yz}=0$)

$$\delta U_i^*=\int_0^l\left[\int_A \epsilon_{zz}\delta\sigma_{zz}dA\right]dz=-\int_0^l\frac{d^2v}{dz^2}\delta M dz$$

where the integral in the square bracket refers to the integration over the cross-section. For the case shown in FIG. 14(*c*).

$$\delta M=-\delta P\left(\frac{l}{2}-z\right) \text{ for } 0\leqslant z\leqslant l/2$$

$$\delta M=0 \qquad \text{for } l/2<z<l$$

Also since,

$$\frac{d^2v}{dz^2}=\frac{pl^2}{8EI}\left[1-5\frac{z}{l}+4\left(\frac{z}{l}\right)^2\right]$$

$$\delta U_i^*=\delta P\frac{pl^2}{8EI}\int_0^{l/2}\left(\frac{l}{2}-z\right)\left[1-5\frac{z}{l}+4\left(\frac{z}{l}\right)^2\right]dz=\delta P\frac{pl^4}{192EI}=\delta W^* \text{ q.e.d.}$$

If we apply now a force $-\frac{5}{16}\delta P$ at the free end our virtual system (*c*) is transformed into (*a*). No additional δW^* arises since $v=0$ at $z=l$. The additional bending moment δM produced by $-\frac{5}{16}\delta P$ is

$$\frac{5}{16}\left(1-\frac{z}{l}\right)\delta P.l$$

and this is easily found not to create an additional δU_i^*. By relaxing the moment restraint at A we may finally prove without difficulty that $\delta U_i^*=\delta W^*$ applies also for the virtual system (*b*).

We return now to Eq. (62) and note that since the displacements are assumed constant when the virtual forces are applied we may regard δW^* as the variation of a potential $-U_e^*$ where

$$U_e^*=-\int_V[u\omega_x+v\omega_y+w\omega_z]dV-\int_S[u\phi_x+v\phi_y+w\phi_z]dS \quad\quad (64)$$

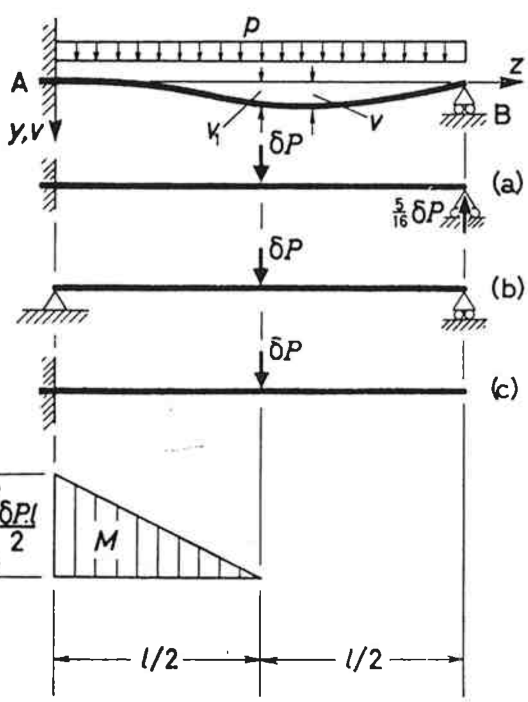
Thus, $\delta W^*=-\delta U_e^*$ and U_e^* may be termed the complementary potential of the external forces. Note, however, that W^* is *not* $-U_e^*$ since in obtaining W^* from δW^* we must, naturally, perform the integration for displacements varying with load. In fact, for a linear system and no temperature effects $W^*=-U_e^*/2$; compare also Eqs. (64) and (46).

Also, since the thermal strains are kept constant we may write the right-hand side of (62) as

$$\delta U_i^*+\int_V\eta\delta s dV=\delta\left(U_i^*+\int_V\eta s dV\right)=\delta U_a^* \quad\quad (65)$$

where

$$\delta U_a^*=\int_V\gamma\delta\sigma dV \quad\quad (65a)$$



14.—Example of an arbitrary virtual force

but

$$U_d^* = U_i^* + \int_V \eta s dV - \int_V [\int_O^\sigma \epsilon \delta\sigma] dV + \int_V \eta s dV \quad \ldots\ldots (66)$$

since Θ=const. U_d^* we term the complementary potential energy of total deformation. Note that it is always simpler to calculate directly δU_d^* from Eq. (65). Particular care is necessary in evaluating U_d^* for as Eq. (66) shows in the first integral ϵ is taken to vary with σ from the initial to the final state while η, s in the second integral refer only to the final values. Physically speaking we may consider U_d^* as the complementary work necessary to reach the final true state of deformation from an initial state in which we allowed free thermal expansion and destroyed compatibility.

Formulae (65) and (66) may be extended immediately to the case of arbitrary initial straining by substituting

$$\eta\sigma - \eta_{xx}\sigma_{xx} + \ldots\ldots + \eta_{xy}\sigma_{xy} + \ldots\ldots + \eta_{zx}\sigma_{zx} \quad \text{for} \quad \eta s$$

Eq. (62) can now be written more concisely

$$\delta_\sigma U^* = \delta_\sigma(U_d^* + U_r^*) = 0 \quad \ldots\ldots (67)$$

where the suffix σ indicates that only forces and stresses are varied and

$$U^* = U_d^* + U_r^* \quad \ldots\ldots (67a)$$

is defined as the total complementary potential energy of the system. Eq. (67) states that a state of equilibrium of an elastic body is also one of elastic compatibility (i.e. the body is at the true position of equilibrium) if any virtual variation of the stresses and forces, while displacements remain constant, does not give rise to any (first order) variation of the total complementary potential energy. This theorem we call the principle of a stationary value of total complementary potential energy if the latter is expressed in terms of forces and stresses. Actually the stationary value of U^* is a minimum as we may prove without difficulty.* This point is discussed in more detail under (C) below.

Eqs. (62) or (67) may be used to derive the results which follow.

(A). The differential equations of the theory of elasticity (for arbitrary loading and temperature distribution), or any particular structural problem, in terms of stresses or stress-resultants; the appropriate kinematic conditions in terms of forces and stresses follow also from this analysis. It is important to note that in all applications it is best to form directly

$$\delta U_d^* = \int_V [\gamma \delta\sigma] dV$$

and not to evaluate first U_d^* and then to take its increment δU_d^*.

(B). Castigliano's† Theorem Part II generalized for Thermal Effects and non-linear elasticity

$$\left[\frac{\partial U_d^*}{\partial P_r}\right]_{\Theta=\text{const.}} = u_r \quad \ldots\ldots (68)$$

where u_r is the deflexion (rotation) in the direction of the force (moment) P_r. This relation may be obtained immediately if we apply one virtual external force δP_r in the direction of the displacement u_r.

(C). The Principle of Stationary (Minimum) value of Complementary potential energy of total deformation for internally redundant structures

This may be derived from (62) if we do not apply any external virtual forces, i.e.

$$\delta\phi_x = \delta\phi_y = \delta\phi_z = \delta\omega_x = \delta\omega_y = \delta\omega_z = 0$$

while varying the stresses σ.

Then

$$\delta_\sigma(U_d^*) = 0 \quad \ldots\ldots (69)$$

which is our generalization of the standard principle of Castigliano of Minimum Strain Energy to include temperature effects and non-linear stress-strain laws. Note again that Principle (69) necessarily applies only to internally redundant structures, since for given external loads only one stress distribution can exist in statically determinate structures.

Eq. (69) itself only indicates that U_d^* has a stationary value in that particular state of equilibrium in which all the elastic and kinematic compatibility conditions are satisfied. Note, as mentioned before, that with the limitations of the present assumptions, i.e. small displacements and monotonically increasing stress-strain diagram, there is only one position possible where both the equilibrium and compatibility conditions are satisfied. We now investigate the nature of the extremum of U_d^*, which naturally requires the consideration of second order terms as in Section 4.

Consider an elastic body under given loads and temperature distribution in its compatible equilibrium position. We make a series of cuts in the body but at the same time apply stresses σ_c acting across and along the cuts of the same magnitude as in the uncut body; these are obviously the stresses required to maintain the compatibility condition of perfect fit at the cuts. If we impose the *virtual* stresses $\delta\sigma_c$ it is apparent that since these produce corresponding deformations δu_c on the cut faces the latter are not any longer compatible. It is important to realize that the $\delta\sigma_c$ systems are self-equilibrating since the external loads P remain constant. Thus, in a framework we may obtain a system $\delta\sigma_c$ by cutting a redundant bar and applying a variation δN to the true force N in the bar.

We now investigate the differences in complementary work W^* and U_d^* between the original equilibrium position of the uncut body and the new enforced equilibrium position of the cut body. Comparing Eqs. (38) and (66) we find

$$W^* = U_d^* - \int_V (\eta s - \int_O^I \eta \delta s) dV \quad \ldots\ldots (70)$$

In moving from the uncut (compatible) equilibrium state to the cut one we note first that the integral does not vary since η is constant in this step. Also the first order increment, δU_d^*, of U_d^* is zero since this is the condition for compatible equilibrium of the original body and no first order increment δW^* can arise since the loads P remain constant. We are then left only with second order increments.

For the complementary work this is

$$\delta^2 W^* = \tfrac{1}{2}\int_C \delta u_c \cdot \delta\sigma_c \cdot dS \quad \ldots\ldots (70a)$$

(where the integral is taken over the cut faces) which is the work of the virtual stresses $\delta\sigma_c$ over the displacements δu_c they themselves produce. This is clearly positive. The second order increment of U_d^* is merely

$$\delta^2 U_d^* = \tfrac{1}{2}\int_V \delta\sigma \cdot \delta\epsilon \cdot dV \quad \ldots\ldots (70b)$$

since η remains constant; $\delta\sigma$ and $\delta\epsilon$ are the stresses and strains due to $\delta\sigma_c$. Terms (70*a*) and (70*b*) are equal and both positive.

We conclude that *the complementary potential energy of total deformation U_d^* and the complementary work W^* have for given forces and temperature distribution a minimum at that position of equilibrium of the uncut body at which compatibility is satisfied.*

It follows that if U_d^* is overestimated by assuming a statically equivalent stress system which does not satisfy all compatibility conditions and we, ignoring the latter fact, equate U_d^* to W^* of the applied loads P alone we cannot but overestimate the magnitude of the displacement system under the loads P. Conversely to achieve a given displacement system our calculations based on a non-compatible stress system must underestimate the corresponding load system P. Thus, the latter has its maximum for the unique equilibrium position which is also truly compatible. This may be expressed also as follows:

For given displacements and temperature distribution the complementary potential energy of total deformation has a maximum when the state of equilibrium satisfies also the compatibility conditions.

The above theorems may be combined to give:

The stiffness of an elastic body in which the equilibrium conditions are satisfied is a maximum when the elastic compatibility conditions are all met.

Thus we see that the effect of introducing assumed forms of stress distribution for the purpose of approximate solutions is the opposite to that of the method of Virtual Displacements and therefore application of both methods to a given problem yields upper and lower bounds to such aggregate quantities as stiffness. No general conclusion as to bounds can, of course, be drawn for the details of the stress distribution.

The above theorems which apply also in the presence of initial strains other than those due to temperature do not appear to have been given before with this degree of generality.

(D). The Unit Load Method

Assume that we require the deformation (deflexion or slope) u_r at a given point and direction of an elastic redundant body subjected to given forces and thermal effects. Let the *actual total* strains in the structure be known and given by*

$$\gamma_{xx} = \epsilon_{xx} + \eta, \; \gamma_{xy} = \epsilon_{xy}$$

Applying a load (force or moment), δP_r in the direction of u_r and using Eq. (62) we find

$$\delta P_r \cdot u_r = \int_V \gamma \delta\sigma dV$$

$$= \int_V [\gamma_{xx}\delta\sigma_{xx} + \ldots\ldots + \gamma_{xy}\delta\sigma_{xy} + \ldots + \gamma_{zx}\delta\sigma_{zx}] dV \quad \ldots\ldots (71)$$

where $\delta\sigma_{xx} \ldots \delta\sigma_{xy} \ldots$ are the virtual stresses due to δP_r. In a linearly elastic system $\delta\sigma_{xx}$ etc., are proportional to δP_r and Eq. (69) can be written

$$1 \cdot u_r = \int_V [\gamma_{xx}\bar\sigma_{xx} + \gamma_{yy}\bar\sigma_{yy} + \gamma_{zz}\bar\sigma_{zz} + \gamma_{xy}\bar\sigma_{xy} + \gamma_{yz}\bar\sigma_{yz} + \gamma_{zx}\bar\sigma_{zx}] dV \quad \ldots\ldots (71a)$$

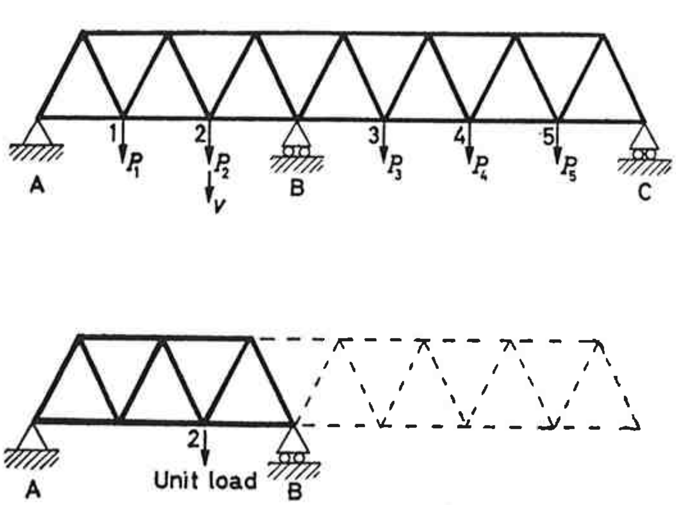
where $\bar\sigma_{xx} \ldots \bar\sigma_{xy} \ldots$ are the stresses due to a unit load. Since $\bar\sigma_{xx} \ldots \bar\sigma_{xy} \ldots$ need only satisfy the internal equilibrium conditions and the external one for $\delta P_r = 1$ it is obviously advantageous to determine $\bar\sigma_{xx} \ldots \bar\sigma_{xy} \ldots$ in the most simple statically determinate basic system.

For a non-linear system Eq. (71*a*) is still applicable as long as $\bar\sigma_{xx}$ etc. are calculated in a statically determinate basic system. For only in the

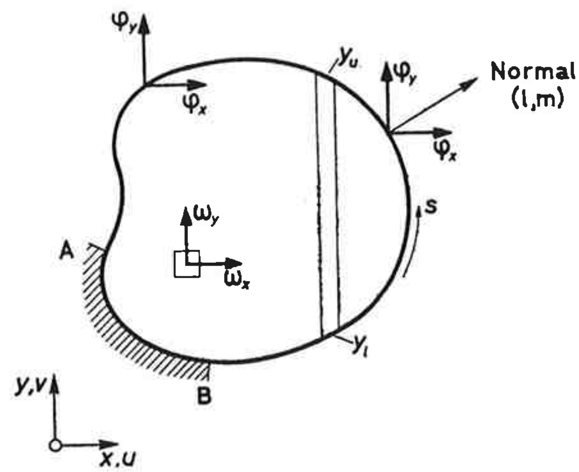
* See Biezeno and Grammel[1], p. 75.

† A. Castigliano, *Théorie de l'équilibre des systèmes élastiques*, Turin 1879.

* Naturally, this method may also be applied in the case of an initial strain system with shear strains.



15.—Unit load method for displacement of redundant framework



16.—Two-dimensional stress case, static and kinematic boundary conditions

latter case will

$\delta\sigma_{xx}/\delta P_r = \partial\sigma_{xx}/\partial P_r$

be the stresses corresponding to a unit load.

The Unit Load Method is the most suitable tool in the calculation of structures with a finite number of redundancies expressed as stresses or stress resultants.

This will be shown in some detail in Section 8.

Example of the application of Eq. (71a).

Consider the plane framework with a redundant support as shown in FIG. 15(*a*). We seek the deflexion v at joint 2 for the loading case shown. Let the actual elongation of the members due to loads $P_1 \ldots P_5$, temperature and manufacturing errors be denoted by Δl. Next we apply a unit load at 2 in the direction of v and find the forces $\bar{N}$ in the bars. Since we need only consider a statically determinate case we ignore the support at C and are left with the very simple problem of finding $\bar{N}$ in the left-hand span only. Application of Eq. (71*a*) yields the simple formula

$$1 \cdot v = \Sigma \bar{N} \Delta l \qquad (71b)$$

where the summation extends only over the continuously drawn bars. The formula given is due essentially to Maxwell* and Mohr† who applied it to statically determinate frameworks. Actually Mohr derived this type of equation by using the principle of virtual displacements with the actual elongations taken as virtual ones and the unit load system as the actual one. Although such a procedure is in the present case of small displacements permissible, it should nevertheless be avoided since Eq. (71*b*) follows more naturally from the principle of virtual forces.

(E). Approximate method of stress analysis

Consider an elastic body‡ subjected to external loads (body and surface forces) and a temperature distribution Θ. The boundary conditions are assumed to be both of the static and kinematic kind; however, where the latter are prescribed they are taken to be of the rigid kind, e.g. rigidly built-in or sliding in a rigid groove (see FIG. 16). To limit the present analysis we restrict our investigation to a state of plane stress. The solution of such problems is often expedited by the use of the Airy stress function F. Then, the stresses are given by§

$$\sigma_{xx} = \frac{\partial^2 F}{\partial y^2} - \int^x \omega_x dx, \quad \sigma_{xy} = -\frac{\partial^2 F}{\partial x \partial y}, \quad \sigma_{yy} = \frac{\partial^2 F}{\partial x^2} - \int^y \omega_y dy \qquad (72)$$

which satisfy automatically the internal equilibrium conditions (4).

Eliminating the displacements from the strain expressions (1) we obtain the compatibility condition for the strains,

$$\frac{\partial^2 \gamma_{xx}}{\partial y^2} + \frac{\partial^2 \gamma_{yy}}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \qquad (73)$$

where γ_{xx} etc. are the *total* strains $\epsilon_{xx} + \eta$ etc. For a given stress-strain law we can express γ_{xx} in terms of the stresses (72) and the temperature Θ. Hence by substituting into (73) we obtain the differential equation in the unknown F, which will, in general, be non-linear. However, in the case of bodies obeying Hooke's law we obtain for α=const. the simple linear result

$$\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = -(1-\nu)\left(\frac{\partial \omega_x}{\partial x} + \frac{\partial \omega_y}{\partial y}\right) - \alpha E\left(\frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2}\right) \qquad (74)$$

F must, of course, also satisfy the given boundary conditions. In fact, where surface forces are prescribed we have (see Eqs. (5) and FIG. 16)

$$\left.\begin{aligned} \frac{\partial}{\partial y}\left(l\frac{\partial F}{\partial y} - m\frac{\partial F}{\partial x}\right) &= \phi_x \\ \frac{\partial}{\partial x}\left(m\frac{\partial F}{\partial x} - l\frac{\partial F}{\partial y}\right) &= \phi_y \end{aligned}\right\} \qquad (75)$$

An approximate method may proceed as follows. Assume that the stress function F is expressed in the form of a finite series

$$F = F_0 + \sum_{r=1}^{n} b_r F_r \qquad (76)$$

where $F_0, F_1, \ldots F_n$ are known functions of which F_0 satisfies the static boundary conditions (75) where these are prescribed and the functions F_r vanish there. b_1 to b_n are constants to be determined by the virtual forces principle. The system (76) satisfies by definition all given equilibrium conditions and the increment

$$\delta F = \delta b_r \cdot F_r \qquad (77)$$

may be regarded as a virtual stress system corresponding to zero increments of external loads where the latter are fixed.

Since $\delta W^* = 0$ (either forces are given or displacements are zero) the principle of minimum complementary potential energy of total deformation is applicable here and takes the form

$$\delta U_d^* = \int_A [\gamma_{xx}\delta\sigma_{xx} + \gamma_{yy}\delta\sigma_{yy} + \gamma_{xy}\delta\sigma_{xy}]dA = 0 \qquad (78)$$

We use here $\int_A (\ldots)dA$ to denote the integration $\iint(\ldots\ldots)\,dxdy$ over the area of the two dimensional continuum.

Substituting $\delta\sigma_{xx}$ etc. in terms of (77) we find

$$\int_A \left[\gamma_{xx}\frac{\partial^2 F_r}{\partial y^2} + \gamma_{yy}\frac{\partial^2 F_r}{\partial x^2} - \gamma_{xy}\frac{\partial^2 F_r}{\partial x \partial y}\right]\delta b_r dA = 0 \qquad (79)$$

and since δb_r is arbitrary.

$$\int_A \left[\gamma_{xx}\frac{\partial^2 F_r}{\partial y^2} + \gamma_{yy}\frac{\partial^2 F_r}{\partial x^2} - \gamma_{xy}\frac{\partial^2 F_r}{\partial x \partial y}\right]dA = 0 \qquad (79a)$$

If the total strains are expressed in terms of the stresses (72) and temperature distribution Θ we obtain from (79*a*) n equations for the unknowns b_1 to b_n. These are only linear if the body follows a linear stress strain law. In the latter case and for constant body forces Eq. (74) shows that the solution must be independent of the Poisson's ratio ν if all boundary conditions are of the static type. Hence we may take $\nu=0$ and (79*a*) becomes

$$\int_A \left\{\left[\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F_r}{\partial y^2} + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 F_r}{\partial x^2} + 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 F_r}{\partial x \partial y}\right] + E\eta\left[\frac{\partial^2 F_r}{\partial x^2} + \frac{\partial^2 F_r}{\partial y^2}\right]\right\}dA = 0 \qquad (79b)$$

from which we may obtain without difficulty the n linear equations for b_1 to b_n.

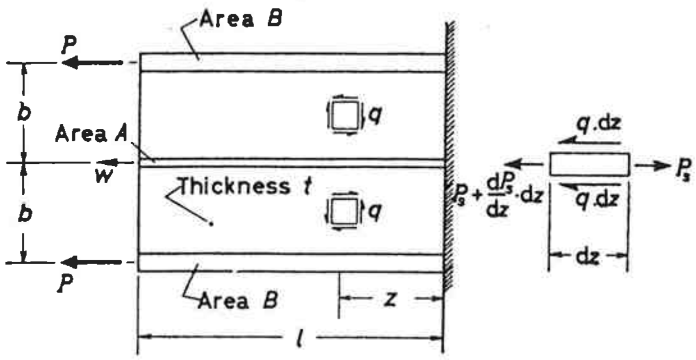
The case when all prescribed boundary conditions are static is interesting for all surface conditions are then exactly satisfied by (76). This indicates that it should be possible to express (79*a*) in a form similar to that given as Galerkin's method under the virtual displacement principle. In fact, if we integrate (79*a*) twice by parts, or better, if we apply Green's Theorem

* J. C. Maxwell, *Phil. Mag.*, vol. 27, p. 294, 1864

† O. Mohr, *Zeit. Architek. u. Ing. Ver. Hannover*, 1874, p. 509; 1875, p. 17.

‡ The presentation is restricted to singly connected domains.

§ See Timoshenko 8, p. 26.



17.—Virtual Forces: Example (a) Diffusion problem

we find

$$\int_A\left[\gamma_{xx}\frac{\partial^2F_r}{\partial y^2}+\gamma_{yy}\frac{\partial^2F_r}{\partial x^2}-\gamma_{xy}\frac{\partial^2F_r}{\partial x\partial y}\right]dxdy=\int_A\left[\frac{\partial^2\gamma_{xx}}{\partial y^2}+\frac{\partial^2\gamma_{yy}}{\partial x^2}-\frac{\partial^2\gamma_{xy}}{\partial x\partial y}\right]F_r dA$$

$$+\int_C\left[l\frac{\partial\gamma_{yy}}{\partial x}+m\frac{\partial\gamma_{xx}}{\partial y}-\tfrac{1}{2}\left(l\frac{\partial\gamma_{xy}}{\partial y}+m\frac{\partial\gamma_{xy}}{\partial x}\right)\right]F_r ds$$

$$+\int_C\left[\left(m\gamma_{xx}-\frac{l}{2}\gamma_{xy}\right)\frac{\partial F_r}{\partial y}+\left(l\gamma_{yy}-\frac{m}{2}\gamma_{xy}\right)\frac{\partial F_r}{\partial x}\right]ds$$

$\int_C(\ldots.)ds$ denotes the line integral along the boundary of the two-dimensional continuum.

But, on the boundary $F_r=0$ and also

$$\frac{\partial F_r}{\partial x}=0 \text{ and } \frac{\partial F_r}{\partial y}=0$$

(see Eqs. (75)).

Hence Eq. (79*a*) reduces to the slightly simpler form,

$$\int_A\left[\frac{\partial^2\gamma_{xx}}{\partial y^2}+\frac{\partial^2\gamma_{yy}}{\partial x^2}-\frac{\partial^2\gamma_{xy}}{\partial x\partial y}\right]F_r dA=0 \text{ for } r=1 \text{ to } n \quad \ldots\ldots\ldots\ldots \quad (80)$$

which shows clearly how the method of virtual forces satisfies in the average the compatibility condition (73). When the body is linearly elastic Eq. (80) may be written as (α=const.)

$$\int_A\left[\left(\frac{\partial^4F}{\partial x^4}+\frac{\partial^4F}{\partial y^4}+2\frac{\partial^4F}{\partial x^2\partial y^2}\right)+E\alpha\left(\frac{\partial^2\Theta}{\partial x^2}+\frac{\partial^2\Theta}{\partial y^2}\right)\right.$$

$$\left.+(1-\nu)\left(\frac{\partial\omega_x}{\partial x}+\frac{\partial\omega_y}{\partial y}\right)\right]F_r dA=0 \text{ for } r=1 \text{ to } n \quad \ldots\ldots \quad (80a)$$

the expression in the bracket being of course Eq. (74). Note again the independence of the solution from ν when body forces are constant.

The above application of the principle of virtual forces is a generalization of a method developed by Timoshenko[8], p. 167. A thermal stress example of the above analysis is given in Part II.

Naturally, the method can be extended to three-dimensional cases.

W. Ritz* proposed as early as 1908 a similar procedure for the solution of St. Venant's torsion problem; this method is illustrated on an example of considerable complexity in Part II.

A slightly more refined approach than that shown above may be adopted when it is possible to estimate accurately the variation f of F parallel to one co-ordinate say y while the distribution parallel to the other co-ordinate is more difficult to guess. Then, we may set

$$F=f(y)\cdot\phi(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (81)$$

where $\phi(x)$ is an unknown function of x. It is, of course, possible to formulate the analysis in any other suitable co-ordinate system. Substituting (81), with $\delta F=f\delta\phi$ in place of F_r, into (79*a*) or (80) (or related expressions), we obtain after some simple integrations the differential equation in ϕ; when there are also kinematic boundary conditions the corresponding boundary expressions for ϕ follow also from (79*a*).

Consider, for example, the case of linear elasticity, zero body forces and pure static boundary conditions. Eq. (80*a*) takes here the simple form,

$$\int_A[\Delta^2F+E\alpha\Delta\Theta]f\delta\phi dA=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (80b)$$

Using (81) in (80*b*), integrating with respect to y and noting that (80*b*) must be true for any virtual variation $\delta\phi$, we obtain the differential equation in ϕ

$$\frac{d^4\phi}{dx^4}\cdot\int_{y_l}^{y_u}f^2dy+2\frac{d^2\phi}{dx^2}\cdot\int_{y_l}^{y_u}f\frac{d^2f}{dy^2}dy+\phi\int_{y_l}^{y_u}f\frac{d^4f}{dy^4}dy+E\alpha\int_{y_l}^{y_u}\Delta\Theta f dy=0 \quad \ldots \quad (80c)$$

where y_l and y_u are the extreme (boundary) values of y corresponding to the same x (see FIG. 16). Thus, the coefficients of the homogeneous part of (80*c*) are only constants in the case of a rectangular field.

This method can yield very accurate results and is actually the one adopted in Part II.

7. ILLUSTRATIONS OF THE PRINCIPLE OF VIRTUAL FORCES

In this section we present a number of applications of the principle of virtual forces to quite simple problems. Again, it is not necessarily suggested that the method is the most suitable one for the problems considered. It is only intended to show how it can be applied in these simple cases. In subsequent parts of the paper some rather more complex problems will be dealt with. In all the examples of this section linear elasticity is assumed.

(a) Diffusion Problem

The panel shown in FIG. 17 is subjected to loads P applied at the free ends of the edge members. Assuming that the sheet carries only shear stress which is constant across the width b of each half (usual diffusion assumption) obtain by application of the principle of virtual forces the differential equation for the load P_s in the central stringer. Find also the displacement w of the free end of the stringer.

From the equilibrium of an element of the stringer, we find for the shear flow ($q=\sigma_{yz}t$) in the sheet

$$q=-\frac{1}{2}\frac{dP_s}{dz}=-\frac{1}{2}P_s' \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (a1)$$

and from the equilibrium of the free end of the panel we find for the load P_B in the edge members

$$P_B=P-\frac{P_s}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (a2)$$

For the virtual forces we consider a variation δP_s in the stringer load. The applied forces P are maintained constant and hence to satisfy the equilibrium conditions on the free end ($P_s=0$) we must take δP_s to be zero there

$$\text{i.e. } (\delta P_s)_{z=l}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (a3)$$

Otherwise the variation δP_s is arbitrary.

The virtual shear flow in the sheet is thus

$$\delta q=-\tfrac{1}{2}\delta(P_s')=-\tfrac{1}{2}(\delta P_s)' \quad \ldots\ldots\ldots\ldots\ldots \quad (a4)$$

and the virtual load in the edge members

$$\delta P_B=-\frac{\delta P_s}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (a5)$$

Since the applied forces are not varied, the virtual forces principle (Eq. (62) for $\Theta=0$) reduces to

$$\delta U_i^*=0$$

The virtual complementary energy due to δP_s is

$$\delta U_i^*=\int_0^l\left[\frac{P_s}{EA}\delta P_s+2\frac{P_B}{EB}\delta P_B+2\frac{q}{Gt}b\delta q\right]dz \quad \ldots\ldots\ldots\ldots \quad (a6)$$

Substituting for P_B, δP_B, q, δq in terms of P_s, δP_s and integrating the last term by parts we find

$$\delta U_i^*=\int_0^l\left[\frac{P_s}{E}\left(\frac{1}{A}+\frac{1}{2B}\right)-\frac{b}{2Gt}\frac{d^2P_s}{dz^2}-\frac{P}{EB}\right]\delta P_s\cdot dz+\frac{1}{Gt}\left[\frac{dP_s}{dz}\delta P_s\right]_0^l=0 \quad \ldots\ldots\ldots\ldots \quad (a7)$$

and therefore, since δP_s is arbitrary, we must have

$$\frac{d^2P_s}{dz^2}-\frac{2Gt}{Eb}\left(\frac{1}{A}+\frac{1}{2B}\right)P_s+\frac{2Gt}{EbB}P=0 \quad \ldots\ldots\ldots\ldots \quad (a8)$$

or

$$\frac{d^2P_s}{dz^2}-\mu^2P_s=-\mu^2P_{so} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (a9)$$

where

$$\mu^2=\frac{2Gt}{Eb}\left(\frac{1}{A}+\frac{1}{2B}\right),\ P_{so}=\frac{2A}{A+2B}P \quad \ldots\ldots\ldots\ldots \quad (a10)$$

which is the required differential equation in P_s.

Since δP_s is zero for $z=l$, the remaining term in Eq. (a7) vanishes for the upper limit $z=l$. At the lower limit $z=0$, however, δP_s is arbitrary and hence

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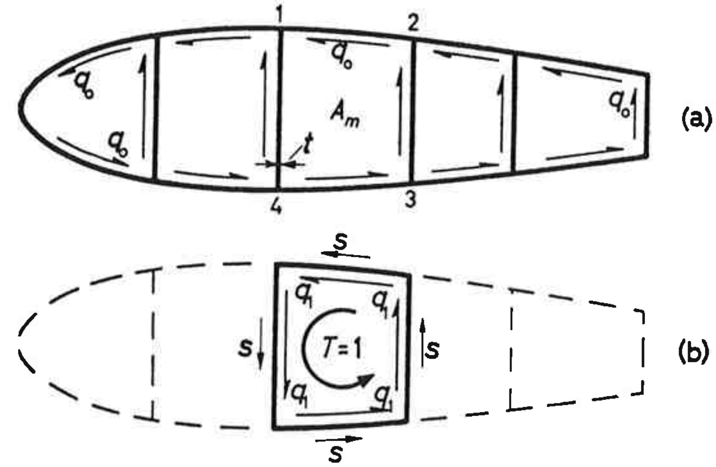
* W. Ritz, *J. reine angew. Math.*, vol. 135, 1908, and *Ann. d. Physik*, series 4, vol. 28, p. 737, 1909.



18.—Virtual Forces: Example (*b*) Unit load method for twist of multicell tube

$$\frac{dP_s}{dz}=0 \text{ for } z=0 \quad \ldots\ldots \quad \text{(a11)}$$

which with the equilibrium conditions on the free end,

$$P_s=0 \text{ for } z=l \quad \ldots\ldots \quad \text{(a12)}$$

gives the necessary boundary conditions.

The differential equation (a9) is, of course, alternatively derived by a direct consideration of the deformations. In such case the kinematic boundary condition (a11) appears as the condition of zero shear strain at the built-in end.

Solution of (a9) with the boundary conditions (a11) and (a12) gives

$$P_s=P_{so}\left[1-\frac{\cosh \mu(l-z)}{\cosh \mu l}\right] \quad \ldots\ldots \quad \text{(a13)}$$

To determine the displacement w of the free end of the stringer we apply there a unit force and since we need only consider a statically determinate system we assume the stringer alone loaded by the unit force. We find then (see Eq. (71))

$$1\cdot w=\int_o^l \bar{\sigma}_{zz}\epsilon_{zz}A\,dz$$

and since

$$\bar{\sigma}_{zz}A=\text{unit load}=1,\ \epsilon_{zz}=P_s/EA$$

$$w=\int_o^l \frac{P_s}{EA}dz \quad \ldots\ldots \quad \text{(a14)}$$

which is, of course, merely the extension of the stringer under the varying end load P_s.

(*b*) *Rate of Twist of Multi-cell Tube*

In the uniform thin-walled tube whose cross-section is shown in FIG. 18(*a*), q_o is the known shear flow distribution in the walls of the tube due to a given loading. Using the unit load method, find the rate of twist $d\theta/dz$ of the tube.

We consider unit length of the tube and apply a unit torque $T=1$. Since we only need consider a statically determinate system for the unit load stresses we select the single cell (1, 2, 3, 4) shown by full lines in FIG. 18(*b*). The unit torque gives then merely a constant shear flow

$$q_1=\frac{1}{2A_m} \quad \ldots\ldots \quad \text{(b1)}$$

around the single cell and the rate of twist is given immediately as

$$\frac{d\theta}{dz}=\oint_C \frac{q_o}{Gt}q_1 ds=\frac{1}{2A_m}\oint_C \frac{q_o}{Gt}ds \quad \ldots\ldots \quad \text{(b2)}$$

where the integral is obviously taken only around the single cell (1, 2, 3, 4).

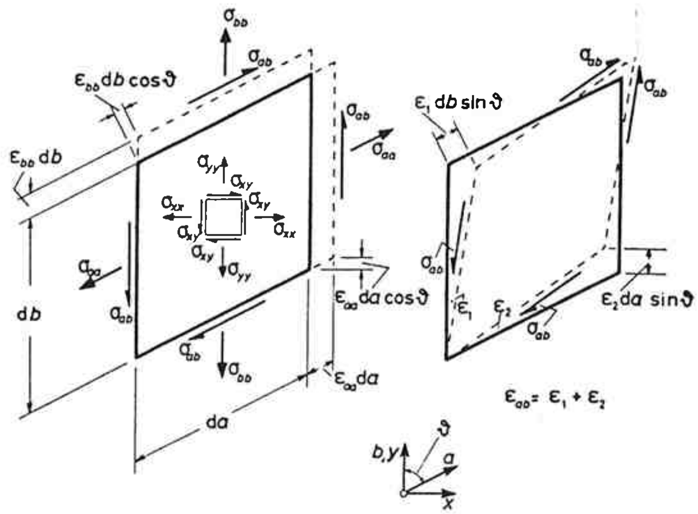
Eq. (b2) is, of course, a well-known result in the theory of closed thin-walled tubes.

(*c*) *Plan. Stress-Strain Relations for Oblique Co-ordinates*

In a uniform isotropic plate, the stresses σ_{aa}, σ_{bb}, $\sigma_{ab}=\sigma_{ba}$ are referred to the oblique co-ordinates Oa, Ob (FIG. 19). Using the principle of virtual forces and assuming the stress-strain relations for rectilinear stresses, find expressions for the strains ϵ_{aa}, ϵ_{bb} and ϵ_{ab} in terms of the stresses.

The oblique strains ϵ_{aa}, ϵ_{bb}, ϵ_{ab} are defined as the elongations in the directions Oa and Ob and the decrease in the angle θ respectively of the unit parallelogram (see FIG. 19).

For the stresses σ_{xx}, σ_{yy}, σ_{xy} equivalent to the oblique stresses, we find easily from statics



19.—Virtual Forces: Example (c) Stresses and strains for oblique coordinates

$$\left.\begin{aligned}\sigma_{xx}&=\sigma_{aa}\sin\theta\\ \sigma_{xy}&=\sigma_{ab}+\sigma_{aa}\cos\theta\\ \sigma_{yy}&=(\sigma_{bb}+\sigma_{aa}\cos^2\theta+2\sigma_{ab}\cos\theta)/\sin\theta\end{aligned}\right\} \quad \ldots\ldots \quad \text{(c1)}$$

The rectilinear strains are

$$\epsilon_{xx}=\frac{1}{E}(\sigma_{xx}-\nu\sigma_{yy}),\ \epsilon_{yy}=\frac{1}{E}(\sigma_{yy}-\nu\sigma_{xx}),\ \epsilon_{xy}=\frac{2(1+\nu)}{E}\sigma_{xy} \quad \ldots\ldots \quad \text{(c2)}$$

and hence the virtual complementary energy per unit thickness of the element $dxdy$ is

$$\begin{aligned}\delta U_i^*&=dxdy[\epsilon_{xx}\delta\sigma_{xx}+\epsilon_{yy}\delta\sigma_{yy}+\epsilon_{xy}\delta\sigma_{xy}]\\ &=\frac{dxdy}{E}[\sigma_{xx}\delta\sigma_{xx}+\sigma_{yy}\delta\sigma_{yy}-\nu(\sigma_{xx}\delta\sigma_{yy}+\sigma_{yy}\delta\sigma_{xx})+2(1+\nu)\sigma_{xy}\delta\sigma_{xy}]\end{aligned} \quad \ldots\ldots \quad \text{(c3)}$$

Substituting for σ_{xx}, σ_{yy}, σ_{xy} from (c1) we can now express δU_i^* in terms of the oblique stresses and virtual stresses. Thus for the virtual stress $\delta\sigma_{bb}$ we find for the virtual complementary energy

$$\delta U_i^*=\frac{dadb}{E\sin\theta}[\sigma_{bb}-\lambda\sigma_{aa}+2\sigma_{ab}\cos\theta]\delta\sigma_{bb} \quad \ldots\ldots \quad \text{(c4)}$$

where

$$\lambda=\nu\sin^2\theta-\cos^2\theta \quad \ldots\ldots \quad \text{(c5)}$$

From FIG. (19) the virtual complementary work of $\delta\sigma_{bb}$ is seen to be

$$\delta W^*=da\delta\sigma_{bb}\cdot\epsilon_{bb}db \quad \ldots\ldots \quad \text{(c6)}$$

and therefore from the Virtual forces principle

$$\delta W^*=\delta U_i^* \quad \ldots\ldots \quad \text{(c7)}$$

we find, using Eqs. (c4) and (c6) in (c7)

$$\epsilon_{bb}=\frac{1}{E\sin\theta}[\sigma_{bb}-\lambda\sigma_{aa}+2\sigma_{ab}\cos\theta] \quad \ldots\ldots \quad \text{(c8)}$$

Applying a virtual stress $\delta\sigma_{aa}$ in the same way we obtain for the strain ϵ_{aa} the corresponding expression

$$\epsilon_{aa}=\frac{1}{E\sin\theta}[\sigma_{aa}-\lambda\sigma_{bb}+2\sigma_{ab}\cos\theta] \quad \ldots\ldots \quad \text{(c9)}$$

Consider now the virtual shear stress $\delta\sigma_{ab}$. From Eq. (c3) we find for the virtual complementary strain energy due to $\delta\sigma_{ab}$

$$\delta U_i^*=\frac{da\cdot db}{E\sin\theta}\{2[(1+\nu)\sin^2\theta+2\cos^2\theta]\sigma_{ab}+2(\sigma_{aa}+\sigma_{bb})\cos\theta\}\delta\sigma_{ab} \quad \ldots\ldots \quad \text{(c10)}$$

Calculating the complementary work of the virtual shear stress σ_{ab} we find (see FIG. 19)

$$\delta W^*=\delta\sigma_{ab}[\epsilon_{aa}\cos\theta+\epsilon_{bb}\cos\theta+\epsilon_{ab}\sin\theta]da\cdot db \quad \ldots\ldots \quad \text{(c11)}$$

Thus the virtual shear stress $\delta\sigma_{ab}$ does work not only due to the shear strain ϵ_{ab}, but also due to the strains ϵ_{aa} and ϵ_{bb}.

Substituting from Eqs. (c8) and (c9) for the strains ϵ_{aa}, ϵ_{bb} and equating δU_i^* and δW^* of Eqs. (c10) and (c11) respectively we finally obtain for the shear strain:

$$\epsilon_{ab}=\frac{2(1+\nu)}{E}\left[\sigma_{ab}+\frac{\cos\theta}{2\sin^2\theta}(\sigma_{aa}+\sigma_{bb})\right] \quad \ldots\ldots \quad \text{(c12)}$$

Note that with the strains defined as above the increment of complementary energy is

$$\delta U_i^*=\{\epsilon_{aa}\delta\sigma_{aa}+\epsilon_{bb}\delta\sigma_{bb}+[(\epsilon_{aa}+\epsilon_{bb})\cos\theta+\epsilon_{ab}\sin\theta]\delta\sigma_{ab}\}dadb \quad \ldots \quad \text{(c13)}$$

as compared with the simple result for rectilinear axes in Eq. (c3).

8. METHODS OF ANALYSIS OF STRUCTURES WITH A FINITE NUMBER OF REDUNDANCIES

THE general theorems given in Sections 4 and 6 include, from the fundamental point of view, all that is required for the analysis of redundant structures. However, to facilitate practical calculations it is helpful to develop more explicit methods and formulae. To find these is the purpose of this Section.

A structure is by common definition redundant if there are not sufficient conditions of equilibrium to obtain all internal forces (stresses or stress-resultants) and reactions; the number of redundancies is the difference between the number of unknown forces (or stresses) and the number of independent equilibrium conditions. Strictly all actual structures are infinitely redundant but for practical purposes it is, in general, necessary and justified to simplify and idealize the structure and/or stress distribution in order to obtain a system with a finite (or even zero) number of redundancies. Such typical processes of simplification are, for example, the assumption of pin-joints in frameworks and the assumption of the engineers' theory of bending in the analysis of beams. Note, moreover, that the Rayleigh-Ritz procedure discussed in Section 6F amounts also, in fact, to the substitution of a finitely redundant structure for the actual elastic body.

All our considerations in this Section are restricted to linearly elastic bodies but Example 2 in Part II shows how the present methods may be extended to the analysis of non-linear redundant structures.

It is curious to note that, while the solution of problems in the theory of elasticity is derived very often from the differential equations in the displacements, the stress-deformation analysis of engineering structures was, until a few years ago, generally based on the concept of force-redundancies. Interestingly enough, Navier,* who was the first to evolve a general method for the analysis of redundant systems, when investigating problem (b) in Section 5 used also the displacement method. The analysis of indeterminate structures on the basis of redundant forces goes back to Clerk Maxwell[4] and Otto Mohr[5] and was ultimately developed by Mueller-Breslau.[6]† This technique is, as mentioned in the introduction, more concise and physically more illuminating than the Castigliano approach; it derives most naturally from the unit load method (see Section 6D, Eq. (71a)). Mueller-Breslau's technique is generalized here and presented also in matrix form. The effect of temperature or other initial strains is included *ab initio*.

Parallel to the rapid development of the force-redundant theory occasional practical problems were solved by selecting deformations as unknowns. Fundamentally this method is equivalent to the virtual displacement analysis given in Section 4. Mohr‡ was probably the first to use such an approach in engineering structures when finding the secondary bending stresses in frameworks of the type usually assumed to be pin-jointed.

* C. L. Navier. *Résumé des leçons sur l'application de la mécanique a l'établissement des constructions et des machines.* Paris 1826, 3 ed. par B. de St.-Venant 2 vols. Paris 1864. See also: A. Clebsch, *Theorie der Elasticitaet fester Koerper*. Leipzig, 1862; French edition by B. de St. Venant: A. Clebsch, *Theorie de l'élasticité des corps solides, avec de notes étendues de B. de St. Venant et A. Flamant*, Paris, 1883. W. Thomson and D. G. Tait, *Treatise on natural philosophy*; 1 ed. Oxford, 1867.

† See also H. Mueller-Breslau, *Die graphische Statik der Baukonstruktionen*, 2 ed., Koerner, Leipzig, 1886.

‡ O. Mohr, *Zivilingenieur*, Vol. 38, p. 577, 1892; see also, *Abhandlungen aus dem Gebiete der technischen Mechanik*, 2 ed., Berlin, 1914, p. 407.

Additional Notation

R, Q, P	single, generalized, orthogonal force (moment)
$\mathbf{R}\ \mathbf{Q}\ \mathbf{P}$	corresponding column matrices
r, q, p	single, generalized, orthogonal displacement (rotation)
$\mathbf{r}\ \mathbf{q}\ \mathbf{p}$	corresponding column matrices
$\sigma_i, \bar{\sigma}_i$	(true) stress and virtual (statically equivalent) stress due to unit load at i
$\mathbf{s}_i, \bar{\mathbf{s}}_i$	corresponding column matrices
ϵ_i	strain due to unit load at i
$\mathbf{e}_i$	corresponding column matrix
f_{jj}, f_{jh}	direct and cross-flexibility
$\mathbf{F}$	matrix of flexibilities f_{jh}
$\mathbf{B}$	transformation matrix for forces
$\mathbf{F}_q, \mathbf{F}_p$	generalized and orthogonal flexibility matrices
$\mathbf{b}$	rectangular transformation matrix for internal forces (stresses)
$\mathbf{S}$	column matrix of internal forces (stresses)
$\mathbf{v}$	column matrix of strains
$\mathbf{f}_g$	flexibility matrix of g element
$\mathbf{f}$	flexibility matrix of all elements
ϕ	flexibility of element of unit length
$\epsilon^i, \bar{\epsilon}^i$	true and virtual strain due to unit displacement at i
σ^i	stress due to unit displacement i
k_{jj}, k_{jh}	direct and cross-stiffnesses
$\mathbf{K}$	matrix of stiffnesses k_{jh}
$\mathbf{A}$	transformation matrix for displacements
$\mathbf{K}_q, \mathbf{K}_p$	generalized and orthogonal stiffness matrices
$\mathbf{a}$	rectangular transformation matrix for strains
$\mathbf{k}_g$	stiffness matrix of g element
$\mathbf{k}$	stiffness matrix of all elements
κ	stiffness of element of unit length
σ_o	stress system of basic structure
σ_s	self-equilibrating stress systems
X_i, Y_i, Z_i	redundant force (moment)
δ_{io}, ζ_{io}	relative displacement at cut i-redundancy in basic system due to external loads and initial strains
δ_{ik}, ζ_{ik}	influence (flexibility) coefficients of basic system for the directions of redundant forces
$\mathbf{D}$	matrix of δ_{ik}
$\mathbf{D}_o$	column matrix of δ_{io}
$\mathbf{T}$	triangular matrix
$\mathbf{M}$	elimination matrix
Δ	prescribed relative displacement (linear or angular) either inside the structure (e.g. lack of fit) or at the supports ('give' of foundations)
C_i	force or moment in the basic system due to $X_i=1$ acting on an element which experiences a Δ in the direction of this Δ
$N(n), S(s), M(m)$	normal force, shear force, bending moment
$\mathbf{O}, \mathbf{o}$	(rectangular) zero matrix
$\mathbf{I}$	unit matrix
$1 \ldots . j \ldots . . h \ldots . . m$	direction of external forces
$1 \ldots . . i \ldots . . k \ldots . . n$	redundancies
$a, b, \ldots . . g, \ldots . . s$	elements of structure
$\mathbf{A}', \mathbf{A}^{-1}$	transpose and inverted (reciprocal) matrix of $\mathbf{A}$
$\{\ \ldots . .\ \}$	column matrix

Following Mohr's analysis his ideas were applied to stiff-jointed frameworks, the first systematic work being that of the Danish engineer Axel Bendixen.* However, the great potentialities of the method were only discovered by Ostenfeld,† a compatriot of Bendixen. He was the first to point out the duality of the force and displacement approach. In fact, his equations for the unknown displacements in a structure complement Mueller-Breslau's equations for the redundant forces. It is regrettable that Timoshenko in his fascinating History[9] does not mention Ostenfeld's classical book. We give here a considerable generalization of Ostenfeld's ideas to include any structures under any load and temperature distribution.

The 'slope-deflexion' equations of Bendixen form the basis of the method of successive approximation due to Calisev‡ and developed by Hardy-Cross‖ as the well-known moment distribution method. The technique used is essentially a particular example of the relaxation method of Southwell§ which has been successfully applied to a wide range of problems. In its application to elasticity and structural problems this latter method is particularly representative of the modern tendency in making practical the numerical solution of highly redundant systems and has been used in conjunction with both forces or stresses and displacements as unknowns. Further discussion of this method is beyond the scope of the present work which is not concerned with iteration methods but the reader is referred to the original literature on the subject.

In this Section we make use, where appropriate, of the matrix notation. Although the complete analysis could be developed *ab initio* in this form it is thought preferable to give first most of the basic principles in the more familiar 'long-hand' notation. Only the most elementary properties of the matrix algebra like matrix partition, multiplication, transposition and inversion are necessary for the understanding of our theory. The reader may consult the classical work of Frazer, Duncan and Collar¶ for these and more advanced matrix operations. Another modern and readable account is given in the recent book of Zurmuehl.** The most comprehensive work to date on the formulation of aircraft structural analysis in matrix notation, anyhow on this side of the Atlantic, is that of B. Langefors.†† D. Williams‡‡ presented recently an interesting account of some aspects of matrix operations in static and dynamic elastic problems.

Before proceeding to a discussion of the general methods for the analysis of redundant structures we introduce some concepts helpful to the understanding of the following theories and their subsequent matrix formulation.

A. Flexibilities

Consider a cantilever beam with a plane of symmetry yz consisting of three connected segments a, b and c with bending stiffnesses for deflexions in the yz-plane $(EI)_a$, $(EI)_b$ and $(EI)_c$ respectively (see FIG. 20). Let the corresponding shear stiffnesses‖‖ be $(GA')_a$, $(GA')_b$ and $(GA')_c$. Transverse forces R_1, R_2 and R_3 are applied in the yz plane at the joints B, C and D. Since the system is assumed to be linear the principle of superposition holds and we can express the deflexions r_1, r_2 and r_3 in terms of the loads as follows:

$$\left.\begin{aligned} r_1 &= f_{11}\,R_1 + f_{12}\,R_2 + f_{13}\,R_3 \\ r_2 &= f_{21}\,R_1 + f_{22}\,R_2 + f_{23}\,R_3 \\ r_3 &= f_{31}\,R_1 + f_{32}\,R_2 + f_{33}\,R_3 \end{aligned}\right\} \qquad (82)$$

where f_{jj}, f_{jh} are, of course, the well-known influence coefficients.§§ In fact, f_{jh} is the displacement in the j-direction due to a unit force $R_h = 1$ in the h-direction. We call also f_{jj} and f_{jh} the direct- and cross-flexibilities respectively and deduce immediately from Maxwell's reciprocality theorem (Eq. (43), Section 3) that

$$1\cdot f_{jh} = 1\cdot f_{hj} \qquad (83)$$

To find the flexibilities f in any linearly elastic body we may use the unit load method developed in Section 6D. Thus, from Eq. (71a),

$$\left.\begin{aligned} 1\cdot f_{jj} &= \int_V \sigma_j \epsilon_j dV \\ 1\cdot f_{jh} &= \int_V \sigma_j \epsilon_h dV = \int_V \sigma_h \epsilon_j dV = 1\cdot f_{hj} \end{aligned}\right\} \qquad (84)$$

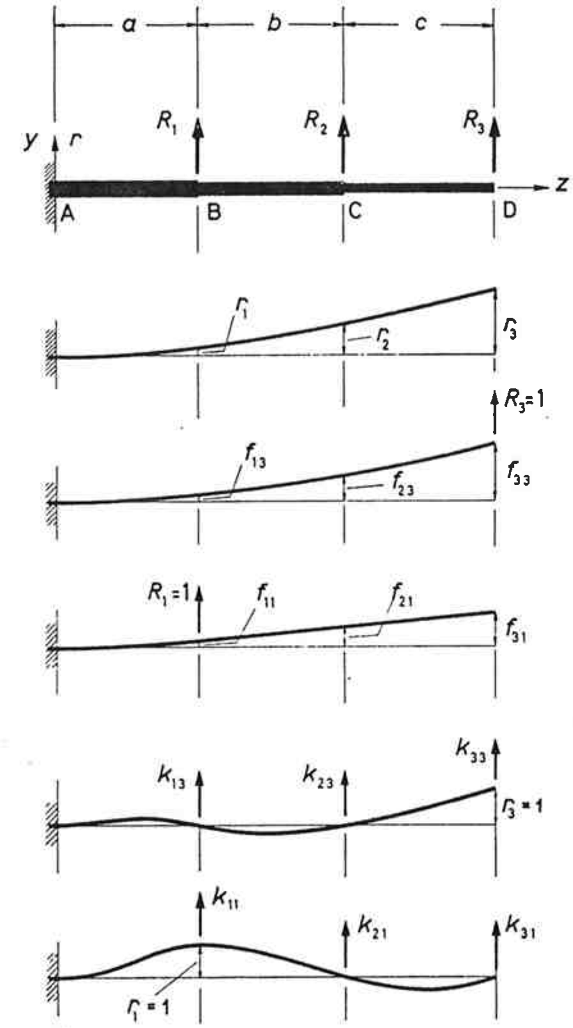


Fig. 20.—Flexibilities and stiffnesses of a cantilever

where (see also Eq. (3))

$$\left.\begin{aligned} \sigma_j \epsilon_j &= \sigma_{xxj}\epsilon_{xxj} + \ldots\ldots\ldots\ldots + \sigma_{zxj}\epsilon_{zxj} \\ \sigma_j \epsilon_h &= \sigma_{xxj}\epsilon_{xxh} + \ldots\ldots\ldots\ldots + \sigma_{zxj}\epsilon_{zxh} \end{aligned}\right\} \qquad (84a)$$

ϵ_j, σ_j (ϵ_h, σ_h) are the strains and stresses corresponding to a unit load at and in the direction of j (h). Under load we understand either force or moment. Similarly the flexibilities may represent either displacements or rotations. Naturally, formulae (84) yield also linear (angular) flexibilities due to moments (forces) respectively. Note that while ϵ_j, ϵ_h *must* be the *true strains* due to unit loads at j and h respectively, σ_j, σ_h *need* only be *virtual, i.e. statically equivalent, stresses* due to the same loads. This is of great importance in redundant structures. Thus, denoting by $\bar{\sigma}_j$, $\bar{\sigma}_h$ any statically equivalent stress system due to unit loads at j and h respectively in a redundant structure we can write Eqs. (84) also in the form,

$$\left.\begin{aligned} 1\cdot f_{jj} &= \int_V \bar{\sigma}_j \epsilon_j dV \\ 1\cdot f_{jh} &= \int_V \bar{\sigma}_j \epsilon_h dV = \int_V \bar{\sigma}_h \epsilon_j dV = 1\cdot f_{hj} \end{aligned}\right\} \qquad (84b)$$

It is, of course, possible to substitute in the above formulae true stresses and virtual strains for true strains and virtual stresses but for reasons of logical consistency this is best avoided.*

Assuming in the case of the beam shown in FIG. 20 that the Engineers' theory of bending stresses is true we find, noting that the system is statically determinate and hence $\bar{\sigma} = \sigma$,

$$1\cdot f_{jh} = \int_0^l \left[\frac{M_j M_h}{EI} + \frac{S_j S_h}{GA'}\right] dz = 1\cdot f_{hj} \qquad (85)$$

* Axel Bendixen, *Die Methode der Alpha-Gleichungen zur Berechnung von Rahmenkonstruktionen*, Springer, Berlin, 1914.

† A. Ostenfeld, *Die Deformationsmethode*, Springer, Berlin, 1926.

‡ Calisev, K. A., *Techniski List* No. 1-2, 1922, Nos. 17-21, 1923. See also Timoshenko and Young, *Theory of Structures*, McGraw-Hill, New York, 1945.

‖ Hardy Cross, 'Analysis of continuous frames by distributing fixed end moments', Paper No. 1793, Vol. 96, *Trans. A.S.C.*, 1932, pp. 1-10.

§ R. V. Southwell, *Relaxation methods in engineering science*, Oxford Univ. Press, 1940. R. V. Southwell, *Relaxation methods in theoretical physics*, Oxford Univ. Press, 1946.

¶ R. A. Frazer, W. J. Duncan, A. R. Collar, *Elementary Matrices*, Cambridge Univ. Press, Cambridge, 1938.

** R. Zurmuehl, *Matrizen*, Springer, Berlin, 1950.

†† B. Langefors, 'Analysis of Elastic Structures by Matrix Transformation with special regard to Monocoque Structures'. *Journ. of Aero. Sci.*, Vol. 19, No. 7, 1952. *Structural Analysis of Swept-Back Wings by Matrix Transformation*, Saab, T.N. 3, August, 1951.

‡‡ D. Williams, 'Recent Developments in the Structural Approach to Aeroelastic Problems', *J.R.Ae.S.* Vol. 58, No. 522, June, 1954.

‖‖ These shear stiffnesses in bending are commonly based on the assumption of the Engineers' theory of bending shear stresses. See Argyris and Dunne, *Structural Analysis (Handbook of Aeronautics*, Vol. 1), Pitman 1952, for a derivation of the area A'.

§§ The influence coefficients were discovered independently by E. Winkler, *Mitt. Architek. u. Ing. Ver. Boehmen* 1868, p. 6 and O. Mohr, *Zeit. Architek. u. Ing. Ver. Hannover*, 1868, p. 19.

* See Section 6D, p. 14.

where M_j, S_j (M_h, S_h) are the moments and shear forces corresponding to $R_j=1$ ($R_h=1$). Eqs. (85) yield easily the following set of influence coefficients,

$$f_{33}=\frac{(a+b+c)^3-(b+c)^3}{3(EI)_a}+\frac{(b+c)^3-c^3}{3(EI)_b}+\frac{c^3}{3(EI)_c}$$
$$+\frac{a}{(GA')_a}+\frac{b}{(GA')_b}+\frac{c}{(GA')_c}$$
$$f_{32}=f_{23}=\frac{1}{6(EI)_a}\Big[(a+b)^2(2a+2b+3c)-b^2(2b+3c)\Big]+\frac{b^2}{6(EI)_b}(2b+3c)$$
$$+\frac{a}{(GA')_a}+\frac{b}{(GA')_b}$$
$$f_{31}=f_{13}=\frac{a^2}{6(EI)_a}\Big[2a+3(b+c)\Big]+\frac{a}{(GA')_a} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (86)$$

To obtain f_{22} and f_{11} from the expression for f_{33} omit in the latter the terms c and b, c respectively. Also to find $f_{12}=f_{21}$ omit the terms involving c in the last of Eqs. (86). Naturally, we can also derive the influence coefficients (86) by direct integration of the differential equation for the deflected beam when shear deformations are included. A systematic method for deriving the flexibility coefficients for compound engineering structures is given later.

Influence or flexibility coefficients are of great importance in the static and dynamic analysis of linearly elastic engineering structures. In this connexion it is most appropriate to make use of the matrix notation not only for conciseness of presentation but also for the systematic programming of the considerable computational work usually involved in practical problems. The matrix algebra is, in fact, ideally suited for the automatic digital computers now available.

The matrix form of Eqs. (82) is,

$$\mathbf{r}=\mathbf{F}\mathbf{R} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (87)$$

where $\mathbf{r}$ and $\mathbf{R}$ are the column matrices of the displacements and forces

$$\mathbf{r}=\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}=\{r_1\ r_2\ r_3\}, \qquad \mathbf{R}=\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}=\{R_1\ R_2\ R_3\} \quad \ldots \quad (88)$$

and

$$\mathbf{F}=\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (89)$$

is the so-called flexibility matrix; note that $\mathbf{F}$ is a symmetrical square matrix. The relation (87) is, of course, valid for any number m of displacements and rotations in any linearly elastic body. To each displacement or rotation r_j there corresponds a force or moment R_j. Thus, in such a case the matrices are

$$\mathbf{r}=\{r_1\, r_2 \ldots\ldots r_j \ldots\ldots r_m\}, \qquad \mathbf{R}=\{R_1\, R_2 \ldots\ldots R_j \ldots\ldots R_m\} \quad (90)$$

and

$$\mathbf{F}=\begin{bmatrix} f_{11}\ f_{12} \ldots\ldots f_{1j} \ldots\ldots f_{1m} \\ \hdashline f_{j1}\ f_{j2} \ldots\ldots f_{jj} \ldots\ldots f_{jm} \\ \hdashline f_{m1}\ f_{m2} \ldots\ldots f_{mj} \ldots\ldots f_{mm} \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (91)$$

where the f_{jh} can be calculated always from Eqs. (84). The flexibilities in (91) need not necessarily refer to m different points. For example, we may choose three directions x, y, z at a particular point of a three dimensional body and define six flexibilities

$$f_{xx}, f_{yy}, f_{zz}$$
$$f_{xy}=f_{yx}, f_{yz}=f_{zy}, f_{zx}=f_{xz}$$

corresponding to the three forces $R_x=1$, $R_y=1$, $R_z=1$ at the *same* point. Similarly for a beam in which we assume that the engineers' theory of bending is true we may require the slope and the deflexion at a cross-section under transverse force and moment applied there. Three flexibilities are required for this information; note, however, that if shear deformations are included we must specify that the bending moment is applied as engineers' theory direct stresses at the particular cross-section. A characteristic property of the influence coefficients is that any f_{jh} in a given elastic body depends only on the points and directions j and h but not on any other directions selected for the calculation of a flexibility matrix (91).

A perusal of the flexibilities of Eqs. (86) shows that it is possible to split the complete flexibility matrix $\mathbf{F}$ into two additive and distinctly different matrices. Thus,

$$\mathbf{F}=\mathbf{F}_B+\mathbf{F}_S \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (92)$$

where $\mathbf{F}_B$ and $\mathbf{F}_S$ are the flexibilities corresponding to pure bending and shear deformations respectively. The first contains only terms involving EI and the second only terms involving GA'. For example,

$$f_{11B}=\frac{a^3}{3(EI)_a}, f_{23S}=\frac{a}{(GA')_a}+\frac{b}{(GA')_b} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (92a)$$

Such a splitting of the flexibility matrix is extremely useful in numerical calculations, particularly when obtaining first approximations in which we neglect certain flexibilities. Thus, in a first approximate wing analysis we may neglect the rib deformability; at a later stage we can ascertain its influence by adding the corresponding flexibility matrix to the original flexibility (see Example (b) of Section 9). The method is, of course, quite general as Eqs. (84) and (84*b*) show, for it is always possible to write

$$1\cdot f_{jh}=\int_V [(\sigma_{xxj}\epsilon_{xxh}+\sigma_{yyj}\epsilon_{yyh}+\sigma_{zzj}\epsilon_{zzh})+(\sigma_{xyj}\epsilon_{xyh}+\sigma_{yzj}\epsilon_{yzh}+\sigma_{zxj}\epsilon_{zxh})]dV \quad \ldots\ldots\ldots \quad (93)$$

where the first expression in round brackets gives the contribution of the direct strains to f_{jh} whilst the second expression gives the contribution of the shear strains. Eq. (93) shows also that the splitting may be carried a step farther by considering separately, for instance, the effect of the strains ϵ_{yy} or of the three shear strains ϵ_{xy}, ϵ_{yz}, ϵ_{zx}. Similarly, in fuselage ring analysis where we usually neglect the deformations due to shear and normal forces we can check their influence by adding the corresponding flexibility $\mathbf{F}_S$ and $\mathbf{F}_N$ to the matrix $\mathbf{F}_B$ for pure bending deformations. These matrices are,

$$\mathbf{F}_B=\Big[\int\frac{M_jM_h}{EI}ds\Big];\ \mathbf{F}_N=\Big[\int\frac{N_jN_h}{EA}ds\Big];\ \mathbf{F}_S=\Big[\int\frac{S_jS_h}{GA'}ds\Big] \quad \ldots\ldots\ldots \quad (94)$$

where M_j, N_j, S_j (M_h, N_h, S_h) are the bending moment, normal and shear forces due to a unit load at j (h).

It is often convenient not to operate in single loads (or moments) but in groups of loads (or moments), which are known as generalized forces. To fix ideas, consider that in the example of FIG. 20 we select as applied generalized forces the three loads Q_1, Q_2 and Q_3 given by,

$$\left.\begin{aligned} Q_1&=G_{11}R_1+G_{12}R_2+G_{13}R_3 \\ Q_2&=G_{21}R_1+G_{22}R_2+G_{23}R_3 \\ Q_3&=G_{31}R_1+G_{32}R_2+G_{33}R_3 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (95)$$

or in matrix form

$$\mathbf{Q}=\mathbf{G}\mathbf{R} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (96)$$

where $\mathbf{G}$ is the square matrix, in general not symmetrical, of the coefficients G_{jh}. We call $\mathbf{G}$ a load transformation matrix and assume that it is non-singular, i.e. that the determinant $|\,\mathbf{G}\,|$ of the coefficients is different from zero. We may solve Eq. (96) for $\mathbf{R}$ by premultiplying with $\mathbf{G}^{-1}$ and obtain

$$\left.\begin{aligned} &\mathbf{R}=\mathbf{B}\mathbf{Q} \\ \text{where} \quad & \\ &\mathbf{B}=\mathbf{G}^{-1} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (97)^*$$

is the so-called reciprocal or inverse matrix of $\mathbf{G}$. Its determination is equivalent to solving Eqs. (95) and therefore involves considerable numerical labour if the number of equations is large. In such cases approximate methods may have to be used. However, with the advent of the automatic digital computers this difficulty is no longer insuperable. We give later in this Section a systematic procedure suitable for punch-card machines for computing the reciprocal matrix but hope to return in greater detail to this and similar questions in Part III. Next we have to determine the generalized displacements $\mathbf{q}$ corresponding to the generalized forces $\mathbf{Q}$. By definition $\mathbf{q}$ are obtained from the equality of the two expressions for work in the two sets of variables $\mathbf{R}$, $\mathbf{r}$ and $\mathbf{Q}$, $\mathbf{q}$. Thus, in matrix notation

$$W=\tfrac{1}{2}\mathbf{r}'\mathbf{R}=\tfrac{1}{2}\mathbf{q}'\mathbf{Q} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (98)$$

where $\mathbf{r}'$ and $\mathbf{q}'$ are the transposed matrices of the column matrices $\mathbf{r}$ and $\mathbf{q}$ and are hence the row matrices

$$\mathbf{r}'=[r_1\ r_2\ r_3],\ \mathbf{q}'=[q_1\ q_2\ q_3] \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (99)$$

Using the first of Eqs. (97) in (98) we obtain

$$\mathbf{q}'=\mathbf{r}'\mathbf{B}$$

or

$$\mathbf{q}=\mathbf{B}'\mathbf{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (100)$$

where $\mathbf{B}'$ is the transpose of the matrix $\mathbf{B}$, i.e.

$$\mathbf{B}'=\begin{bmatrix} B_{11} & B_{21} & B_{31} \\ B_{12} & B_{22} & B_{32} \\ B_{13} & B_{23} & B_{33} \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (101)$$

Substituting (87) into (100) we find,

$$\mathbf{q}=\mathbf{B}'\mathbf{F}\mathbf{R}=\mathbf{B}'\mathbf{F}\mathbf{B}\mathbf{Q}=\mathbf{F}_q\mathbf{Q} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (102)$$

Eq. (102) shows that

$$\mathbf{F}_q=\mathbf{B}'\mathbf{F}\mathbf{B} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (103)$$

is the flexibility matrix corresponding to the generalized forces and displacements $\mathbf{Q}$, $\mathbf{q}$†.

We illustrate now the application of generalized forces on a simple

* In practice the generalized forces are more naturally defined directly by the matrix **B** of Eq. (97).

† Formula (103) is also given by W. J. Duncan, *Mechanical Admittances and their Applications to Oscillation Problems*, A.R.C., R. & M. 2000, 1947.

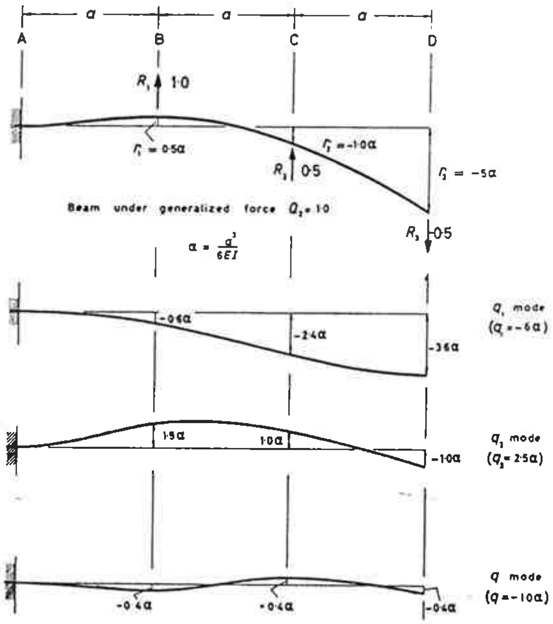


Fig. 21.—Generalized force and displacements

example. FIG. (21) shows a uniform cantilever of bending stiffness EI and negligible shear deformability. The flexibility $\mathbf{F}$ for forces R_1, R_2, R_3 at the equidistant points B, C, D is easily found from formulae (86) to be,

$$\mathbf{F}=\alpha\begin{bmatrix}2 & 5 & 8\\ 5 & 16 & 28\\ 8 & 28 & 54\end{bmatrix} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (104)$$

where

$$\alpha=\frac{a^3}{6EI} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (104a)$$

We seek now the components of forces and the generalized displacements corresponding to the generalized force,

$$Q_2=1$$

for the assumed load transformation matrix,

$$\mathbf{B}=\begin{bmatrix}0 & 1 & 1\\ 1 & 0{\cdot}5 & -1\\ 1 & -0{\cdot}5 & 0{\cdot}5\end{bmatrix} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (105)$$

Using Eq. (97),

$$\mathbf{R}=\mathbf{BQ}=\mathbf{B}\ \{0\ \ 1\ \ 0\}$$

we deduce immediately,

$$\mathbf{R}=\{R_1\ R_2\ R_3\}=\{1\ \ 0{\cdot}5\ \ -0{\cdot}5\}$$

Hence,

$$\mathbf{r}=\mathbf{FR}=\{\alpha/2\quad -\alpha\quad -5\alpha\}$$

The values of R and r are shown in FIG. (21).

Next we find the generalized flexibility $\mathbf{F}_q$ by straightforward matrix multiplication from Eq. (103),

$$\mathbf{F}_q=\mathbf{B}'\mathbf{FB}=\alpha\begin{bmatrix}126 & -6 & 10\\ -6 & 2{\cdot}5 & -1\\ 10 & -1 & 1{\cdot}5\end{bmatrix} \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (106)$$

We calculate now the generalized displacements from Eq. (102) as,

$$\mathbf{q}=\mathbf{F}_q\mathbf{Q}=\mathbf{F}_q\{0\ \ 1\ \ 0\}$$

or

$$\mathbf{q}=\{-6\alpha\quad 2{\cdot}5\alpha\quad -\alpha\}$$

Finally, we analyse the three generalized displacements q in their r-components. From Eq. (100),

$$\mathbf{r}=\mathbf{G}'\mathbf{q}=(\mathbf{B}^{-1})'\mathbf{q}=(\mathbf{B}^{-1})'\ \{-6\alpha\quad 2{\cdot}5\alpha\quad -\alpha\}$$

For the inverted and transposed matrix of $\mathbf{B}$ we obtain from Eq. (105),

$$(\mathbf{B}^{-1})'=\mathbf{G}'=\begin{bmatrix}0{\cdot}1 & 0{\cdot}6 & 0{\cdot}4\\ 0{\cdot}4 & 0{\cdot}4 & -0{\cdot}4\\ 0{\cdot}6 & -0{\cdot}4 & 0{\cdot}4\end{bmatrix} \quad\ldots\ldots\ldots\ldots\ldots \quad (107)$$

and therefore

$$\mathbf{r}=\begin{bmatrix}-0{\cdot}6\alpha+1{\cdot}5\alpha-0{\cdot}4\alpha\\ -2{\cdot}4\alpha+1{\cdot}0\alpha+0{\cdot}4\alpha\\ -3{\cdot}6\alpha-1{\cdot}0\alpha-0{\cdot}4\alpha\end{bmatrix}=\begin{bmatrix}\alpha/2\\ -\alpha\\ -5\alpha\end{bmatrix}$$

in agreement with the previously given values of r. Each of the three columns of the intermediate expressions represents obviously the r-components of the corresponding q-coordinate. FIG. (21) illustrates in detail the three q-modes.

Naturally, Eqs. (97), (100) and (103) are valid for any linearly elastic body and any number m of forces (moments) R and displacements (rotations) r. The load transformation matrix takes then the form

$$\mathbf{B}=\mathbf{G}^{-1}=\begin{bmatrix}B_{11}\ \ldots\ldots\ B_{1j}\ \ldots\ldots\ B_{1m}\\ \text{-------------------}\\ B_{j1}\ \ldots\ldots\ B_{jj}\ \ldots\ldots\ B_{jm}\\ \text{-------------------}\\ B_{m1}\ \ldots\ldots\ B_{mj}\ \ldots\ldots\ B_{mm}\end{bmatrix} \quad\ldots\ldots\ldots \quad (108)$$

and is not, in general, symmetrical. However, the transformation (103), called a congruent transformation, ensures that the flexibility matrix $\mathbf{F}_q$ is still symmetrical.

Attention is drawn to the dual relationship (97) and (100). Thus, if we transform a load system $\mathbf{R}$ by the transformation

$$\mathbf{R}=\mathbf{BQ}$$

the corresponding displacements $\mathbf{r}$ are transformed as

$$\mathbf{q}=\mathbf{B}'\mathbf{r} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (109)$$

It is often required to find the set of forces and displacements $\mathbf{P}$, $\mathbf{p}$ for which all cross-flexibilities f_{pjh} (when $j\neq h$) are zero. These are, of course, the elastic eigenmodes corresponding to the set of displacements r_1 to r_m. The load-displacements law is then given by

$$\mathbf{p}=\mathbf{F}_p\mathbf{P}$$

where $\mathbf{F}_p$ is the diagonal matrix,

$$\mathbf{F}_p=\begin{bmatrix}f_{p11} & 0 & \ldots\ldots 0 & \ldots\ldots 0\\ 0 & f_{p22} & \ldots\ldots 0 & \ldots\ldots 0\\ & & \ddots & \\ 0 & 0 & \ldots\ldots f_{pjj} & \ldots\ldots 0\\ & & & \ddots\\ 0 & 0 & \ldots\ldots 0 & \ldots\ldots f_{pmm}\end{bmatrix} \quad\ldots\ldots\ldots\ldots \quad (110)$$

Thus,

$$p_1=f_{p11}P_1,\ \ldots\ldots,\ p_j=f_{pjj}P_j,\ \ldots\ldots,\ p_m=f_{pmm}P_m \quad\ldots\ldots \quad (111)$$

It is always possible to find the unique load transformation matrix $\mathbf{B}_p$ which transforms our system $\mathbf{R}$, $\mathbf{r}$ into the orthogonal system $\mathbf{P}$, $\mathbf{p}$. We do not enter here into its detailed derivation since the reader can consult a number of textbooks on this subject.*

Our above considerations and in particular Eqs. (84) and (84b) show that the flexibilities are particularly simple to derive for a statically determinate structure, e.g. the beam of FIG. 20. For a redundant structure we must first find the forces or stresses in the redundant members before we can obtain the true strains ϵ_j for the unit loads. The necessary analysis is developed later but it is helpful to give here a formal matrix derivation of the flexibility $\mathbf{F}$ of an engineering structure, the stress distribution of which is known. To this effect we use again the unit load method given by Eq. (71a). We denote by $\mathbf{e}_h$, $\mathbf{s}_h$ two column matrices for the true strains and stresses respectively, corresponding to a unit load $R_h=1$ at the point and direction h. Thus,

$$\left.\begin{aligned}\mathbf{e}_h&=\{\epsilon_{xxh}\epsilon_{yyh}\epsilon_{zzh}\epsilon_{xyh}\epsilon_{yzh}\epsilon_{zxh}\}\\ \mathbf{s}_h&=\{\sigma_{xxh}\sigma_{yyh}\sigma_{zzh}\sigma_{xyh}\sigma_{yzh}\sigma_{zxh}\}\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots \quad (112)$$

where the elements of these matrices may, of course, vary with x, y, z.

It is always possible to write

$$\mathbf{e}_h=\mathbf{f}_V\mathbf{s}_h \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (113)$$

where $\mathbf{f}_V$ is the flexibility matrix of a unit cube at the point x, y, z. Thus for an isotropic body

* See Frazer, Duncan, Collar, *loc. cit.* Zurmuehl, *loc. cit.*

$$\mathbf{f}_V = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix} \quad \ldots\ldots\ldots (114)$$

Let also $\bar{\mathbf{s}}_j$ be a column matrix for any statically equivalent stress system corresponding to a unit load $R_j = 1$ at the point and direction j. Thus,

$$\bar{\mathbf{s}}_j = \{\bar{\sigma}_{xxj}\bar{\sigma}_{yyj}\bar{\sigma}_{zzj}\bar{\sigma}_{xyj}\bar{\sigma}_{yzj}\bar{\sigma}_{zxj}\} \quad \ldots\ldots\ldots (115)$$

For a statically determinate system only one $\mathbf{s}_j$ system can be given—the true one, $\mathbf{s}_j$. We derive now the flexibility coefficients f_{jh} from the unit load method as,

$$\left.\begin{aligned} 1\cdot f_{jh} &= \int_V \bar{\mathbf{s}}_j'\mathbf{e}_h dV = \int_V \bar{\mathbf{s}}_j'\mathbf{f}_V\mathbf{s}_h dV = \int_V \bar{\mathbf{s}}_h'\mathbf{f}_V\mathbf{s}_j dV = 1\cdot f_{hj} \\ 1\cdot f_{jj} &= \int_V \bar{\mathbf{s}}_j'\mathbf{f}_V\mathbf{s}_j dV \end{aligned}\right\} \quad \ldots\ldots\ldots (116)$$

Hence the total flexibility matrix $\mathbf{F}$ for m points and directions is,

$$\mathbf{F} = [f_{jh}] = \int_V \bar{\mathbf{s}}'\mathbf{f}_V\mathbf{s}\,dV \quad \ldots\ldots\ldots (117)$$

where $\mathbf{s}$, $\bar{\mathbf{s}}$ are the partitioned row matrices

$$\left.\begin{aligned} \mathbf{s} &= [\mathbf{s}_1 \ldots\ldots \mathbf{s}_j \ldots\ldots \mathbf{s}_m] \\ \bar{\mathbf{s}} &= [\bar{\mathbf{s}}_1 \ldots\ldots \bar{\mathbf{s}}_j \ldots\ldots \bar{\mathbf{s}}_m] \end{aligned}\right\} \quad \ldots\ldots\ldots (118)$$

We shall apply now formula (117) to an engineering structure consisting of any number s of simple elements joined together at their ends or boundaries. These elements may be plates, flanges, beams, rods, pin-jointed trusses, etc., and take in such a structure the place of the volume element dV in a continuum. Let the structure be subjected to the force (and/or moment-) system

$$\mathbf{R} = \{R_1 \ldots\ldots R_j \ldots\ldots R_m\} \quad \ldots\ldots\ldots (119)$$

where R_j itself need not necessarily be a single force or moment but may be a generalized force. Due to these loads the typical g member is subjected at its ends or boundaries to a loading expressed as a column matrix $\mathbf{S}_g$ whose elements are direct and shear stresses, or stress resultants, e.g. torque, bending moment, shear force, normal force, etc. Now $\mathbf{S}_g$ is obviously linear in the R's. Thus,

$$\mathbf{S}_g = \mathbf{b}_g\mathbf{R} \quad \ldots\ldots\ldots (119a)$$

where $\mathbf{b}_g$ is a rectangular matrix with m columns, and corresponds, of course, to the stress matrix $\mathbf{s}$ at a point (x, y, z) of a three-dimensional continuum. If our structure is redundant $\mathbf{b}_g$ cannot be determined by statics alone. However, for the present we assume that $\mathbf{b}_g$ is known. It is obvious that the relative displacements (shear angle, elongation, deflexion, slope, twist, etc.) $\mathbf{v}_g$ at the ends or boundaries of the g element can be written as a column matrix

$$\mathbf{v}_g = \mathbf{f}_g\mathbf{S}_g = \mathbf{f}_g\mathbf{b}_g\mathbf{R} \quad \ldots\ldots\ldots (120)$$

where $\mathbf{f}_g$ is the flexibility matrix of the g element and has as many rows as $\mathbf{v}_g$. Each element being assumed to have a simple geometry it is usually easy to write down—often merely by inspection—the matrix $\mathbf{f}_g$. Since there are, in general, alternative but equivalent ways of expressing the loading $\mathbf{S}_g$ on the element, there are also correspondingly alternative expressions for the flexibility $\mathbf{f}_g$. This aspect is elaborated on in an example at the end of our main argument. Note that $\mathbf{f}_g$ represents all deformations that are necessary to ensure the compatibility of the g element within the complete structure.

The internal stress and deformation matrix of the aggregate structure may now be expressed as

$$\mathbf{S} = \{\mathbf{S}_a\mathbf{S}_b \ldots\ldots \mathbf{S}_g \ldots\ldots \mathbf{S}_s\} = \mathbf{bR} \quad \ldots\ldots\ldots (121)$$

$$\mathbf{v} = \{\mathbf{v}_a\mathbf{v}_b \ldots\ldots \mathbf{v}_g \ldots\ldots \mathbf{v}_s\} = \mathbf{fbR} \quad \ldots\ldots\ldots (122)$$

where $\mathbf{S}$ and $\mathbf{v}$ are single column partitioned matrices and

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}_a \\ \mathbf{b}_b \\ \vdots \\ \mathbf{b}_g \\ \vdots \\ \mathbf{b}_s \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} \mathbf{f}_a & \mathbf{o} & \ldots & \mathbf{o} & \ldots & \mathbf{o} \\ \mathbf{o} & \mathbf{f}_b & \ldots & \mathbf{o} & \ldots & \mathbf{o} \\ & & \ddots & & & \\ \mathbf{o} & \mathbf{o} & \ldots & \mathbf{f}_g & \ldots & \mathbf{o} \\ & & & & \ddots & \\ \mathbf{o} & \mathbf{o} & \ldots & \mathbf{o} & \ldots & \mathbf{f}_s \end{bmatrix} \quad \ldots\ldots\ldots (123)$$

The matrix $\mathbf{b}$ has submatrices with m columns. $\mathbf{f}$ is a partitioned diagonal matrix whose elements are the flexibility matrices $\mathbf{f}_g$.

We denote now by $\bar{\mathbf{b}}$ a matrix whose m columns are loading systems on the s members statically equivalent to the external loads $R_1 = 1$, $R_2 = 1$, $\ldots\ldots$, $R_m = 1$ respectively. If we choose these systems to be also elastically compatible then, of course,

$$\bar{\mathbf{b}} = \mathbf{b} \quad \ldots\ldots\ldots (124)$$

Applying now the unit load method and using Eq. (122) and the transpose matrix $\bar{\mathbf{b}}'$ we find by an argument similar to that leading to Eq. (117) that the deflexions $\mathbf{r}$ at the points of application and in the directions of the loads $\mathbf{R}$ are given by

$$\mathbf{r} = \bar{\mathbf{b}}'\mathbf{fbR} \quad \ldots\ldots\ldots (125)$$

Therefore the flexibility matrix $\mathbf{F}$ for the prescribed m directions in the complete structure is

$$\mathbf{F} = \bar{\mathbf{b}}'\mathbf{fb} \quad \ldots\ldots\ldots (126)$$

The matrix operations in (126) are again congruent and thus $\mathbf{F}$ is indeed symmetrical. Eqs. (123) show that Eq. (126) can also be written in the form

$$\mathbf{F} = \Sigma_g\bar{\mathbf{b}}'_g\mathbf{f}_g\mathbf{b}_g \quad \ldots\ldots\ldots (126a)$$

Eq. (126) is the general expression for obtaining the flexibility of a complete structure from the flexibilities of the constituent elements. The configuration of the elements is said to be in series since the assembly condition is expressed by the matrix $\mathbf{b}$ which derives from conditions of equilibrium. *Thus,* Eq. (126) *may be regarded as the most general formulation of the flexibility matrix of a structure consisting of elastic elements in series.*

It is also clear why Eq. (103) for the flexibility matrix of generalized forces has the same form as Eq. (126). In the first case we derive generalized forces from single forces and in the second internal forces from external forces but in both cases this entails a linear transformation matrix $\mathbf{B}$ or $\mathbf{b}$. Note also that $\mathbf{F}$ is in the first instance the flexibility matrix of the complete structure for the single forces and $\mathbf{f}$ in the second instance the flexibility matrix of the individual elements. It is seen, however, that whereas $\mathbf{B}$ is always a square matrix $\mathbf{b}$ is, in general, rectangular.

Before illustrating the application of Eq. (126) we draw attention to an interesting dual relationship (see also p. 20). Thus, Eqs. (121) and (125) prove that if the internal loads $\mathbf{S}$ are derived from the external load system $\mathbf{R}$ with the relationship

$$\mathbf{S} = \mathbf{bR} \quad \ldots\ldots\ldots (121)$$

the deflexions $\mathbf{r}$ at the points of application of the $\mathbf{R}$-loads are found from the internal relative displacements (strains) $\mathbf{v}$ from the relationship

$$\mathbf{r} = \bar{\mathbf{b}}'\mathbf{v} = \mathbf{b}'\mathbf{v} \quad \ldots\ldots\ldots (125a)$$

Naturally, Eq. (125a) merely restates the unit load theorem. We stress again the fact that $\bar{\mathbf{b}}$ need only be the matrix of statically equivalent stress systems.

Illustration of Eq. (126).

We observe first that Eq. (126) includes as a particular case Eq. (92) for the splitting of a flexibility matrix. This may be seen as follows. Splitting the flexibility matrix is equivalent to considering the combined effect of two or more geometrically identical structures (elements) to each of which is assigned only part of the complete flexibility of the structure (e.g. flexibility in bending or shear, or normal force). Thus, the constituent elements are in this case geometrically identical and hence the load transference matrices $\mathbf{b}_a$ etc. are merely unit matrices $\mathbf{I}$. If then, flexibilities of each of the elements are written as $\mathbf{f}_a$, $\mathbf{f}_b$ etc., we find

$$\mathbf{F} = [\mathbf{I}\ \mathbf{I} \ldots \mathbf{I} \ldots \mathbf{I}] \cdot \begin{bmatrix} \mathbf{f}_a & \mathbf{o} & \ldots & \mathbf{o} & \ldots & \mathbf{o} \\ \mathbf{o} & \mathbf{f}_b & \ldots & \mathbf{o} & \ldots & \mathbf{o} \\ & & \ddots & & & \\ \mathbf{o} & \mathbf{o} & \ldots & \mathbf{f}_g & \ldots & \mathbf{o} \\ & & & & \ddots & \\ \mathbf{o} & \mathbf{o} & \ldots & \mathbf{o} & \ldots & \mathbf{f}_s \end{bmatrix} \cdot \begin{bmatrix} \mathbf{I} \\ \mathbf{I} \\ \vdots \\ \mathbf{I} \\ \vdots \\ \mathbf{I} \end{bmatrix}$$

$$= \mathbf{f}_a + \mathbf{f}_b + \ldots + \mathbf{f}_g + \ldots + \mathbf{f}_s \quad \ldots\ldots\ldots (127)$$

q.e.d. The order of the unit matrices and the $\mathbf{f}$ matrices is m, the number of assigned directions in the structure.

Consider next a beam built-up by two uniform component beams a and b as shown in FIG. 22. We seek the flexibility matrix for the transverse forces R_1, R_3 and moments R_2, R_4 under the assumption that the E.T.B. holds and shear deflexions are negligible. We analyse first each beam separately as a cantilever built-in at the L.H.S. and subjected to transverse force and bending moment at the tip, the signs of which are taken to be those of the

* The usual presentation of the strain and stress matrix as a 3 × 3 square matrix or tensor is not so suitable for our considerations here. See also Section [illegible]. There too attention is drawn to the fact that $\bar{\mathbf{s}}_j$ need only be determined in the most suitable statically determinate system.

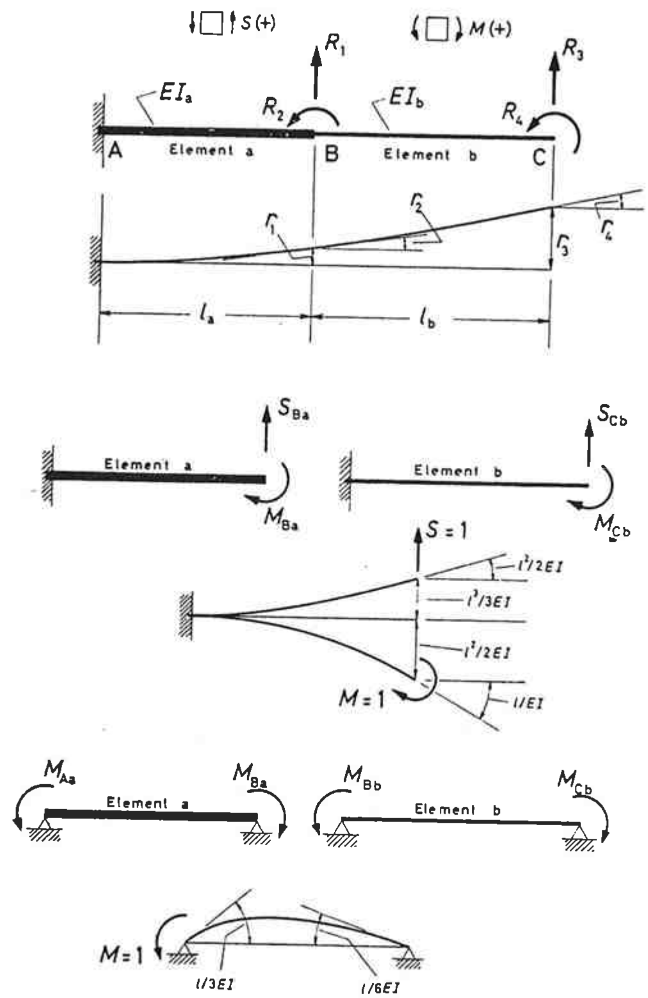


Fig. 22.—Break-down of cantilever for calculation of flexibility matrix

corresponding $+S$ (shear force in the beam) and $+M$ (bending moment in the beam). The tip deflexion and slope are fixed as positive if they are in the direction of the positive applied shear and end moment respectively. With this sign convention,

$$\left.\begin{array}{ll} S_{Ba}=R_1+R_3 & S_{Cb}=R_3 \\ M_{Ba}=-R_2-R_4-R_3l_b & M_{Cb}=-R_4 \end{array}\right\} \qquad (128)$$

The loading matrices $\mathbf{S}_a$ and $\mathbf{S}_b$ of the two elements are accordingly

$$\left.\begin{array}{l} \mathbf{S}_a=\{S_{Ba}M_{Ba}\}=\mathbf{b}_a\{R_1\ R_2\ R_3\ R_4\}=\mathbf{b}_a\mathbf{R} \\ \mathbf{S}_b=\{S_{Cb}M_{Cb}\}=\mathbf{b}_b\{R_1\ R_2\ R_3\ R_4\}=\mathbf{b}_b\mathbf{R} \end{array}\right\} \qquad (129)$$

where

$$\mathbf{b}_a=\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & -l_b & -1 \end{bmatrix} \qquad \mathbf{b}_b=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad (130)$$

and hence

$$\mathbf{b}=\begin{bmatrix} \mathbf{b}_a \\ \mathbf{b}_b \end{bmatrix} \qquad (130a)$$

The flexibility matrices $\mathbf{f}_a$, $\mathbf{f}_b$ follow immediately from FIG. (22),

$$\mathbf{f}_{a,b}=\begin{bmatrix} \frac{l^3}{3}\phi & -\frac{l^2}{2}\phi \\ -\frac{l^2}{2}\phi & l\phi \end{bmatrix}_{a,b} \qquad (131)$$

where

$$\phi_{a,b}=\left(\frac{1}{EI}\right)_{a,b} \qquad (132)$$

gives the flexibility per unit length of the cantilevers. The negative sign in the cross-flexibilities arises from the sign convention. The total flexibility follows as

$$\mathbf{f}=\begin{bmatrix} \mathbf{f}_a & \mathbf{o} \\ \mathbf{o} & \mathbf{f}_b \end{bmatrix} \qquad (131a)$$

Applying now Eq. (126) the flexibility $\mathbf{F}$ of the complete structure is

$$\mathbf{F}=\mathbf{b}'\mathbf{f}\mathbf{b}=\mathbf{b}'_a\mathbf{f}_a\mathbf{b}_a+\mathbf{b}'_b\mathbf{f}_b\mathbf{b}_b \qquad (133)$$

In the present case $\bar{\mathbf{b}}=\mathbf{b}$ since the system is statically determinate. The deformations of the structure may finally be obtained from

$$\mathbf{r}=\mathbf{F}\mathbf{R} \qquad (87)$$

The expressions for the deflexions r_1 and r_3 derived from Eq. (133) agree with those of Eq. (86) when the shear deformations are neglected.

An alternative approach to the problem is to express the loading on the component beam by the end moments (see FIG. 22b). The internal loading matrix is now,

$$\mathbf{S}_1=\begin{bmatrix} \mathbf{S}_a \\ \mathbf{S}_b \end{bmatrix}=\begin{bmatrix} M_{Aa} \\ M_{Ba} \\ M_{Bb} \\ M_{Cb} \end{bmatrix}=\mathbf{b}_1\mathbf{R} \qquad (134)$$

where

$$\mathbf{b}_1=\begin{bmatrix} -l_a & -1 & -(l_a+l_b) & -1 \\ 0 & -1 & -l_b & -1 \\ 0 & 0 & -l_b & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (135)$$

The internal flexibilities $\mathbf{f}_a$ and $\mathbf{f}_b$ derive in this case solely from the end bending moments and to find them we have to consider only the end slopes of the simply supported beams shown in FIG. 22b; thus, taking end slopes positive in the direction of positive moments we find

$$\mathbf{f}_{1a,b}=\begin{bmatrix} \frac{l}{3}\phi & \frac{l}{6}\phi \\ \frac{l}{6}\phi & \frac{l}{3}\phi \end{bmatrix} \qquad (136)$$

Hence,

$$\mathbf{F}=\mathbf{b}'_1\mathbf{f}_1\mathbf{b}_1 \qquad (137)$$

where,

$$\mathbf{f}_1=\begin{bmatrix} \mathbf{f}_{1a} & \mathbf{o} \\ \mathbf{o} & \mathbf{f}_{1b} \end{bmatrix} \qquad (136a)$$

It is easily seen that the flexibility matrix $\mathbf{F}$ of Eq. (137) is identical with that of (133). We observe that the cantilever position of the constituent elements is derived merely by a rigid body rotation from the simply supported beams shown in FIG. 22b. No additional total virtual work is associated with such a linear transformation and the flexibility $\mathbf{F}$ is, hence the same.

The form (136) of the flexibility of the element is important for it applies to a linear variation of any deforming stress, force or moment as long as we substitute for ϕ the appropriate unit flexibility. Thus, for a normal force varying linearly in a beam with constant direct stiffness EA, we can use Eq. (136) with $\phi=1/EA$.

B. Stiffnesses

We return now to Eqs. (87) and solve them for R_1 to R_m to find equations of the type

$$\left.\begin{array}{l} R_1=k_{11}r_1+k_{12}r_2+\ldots\ldots+k_{1j}r_j+\ldots\ldots+k_{1m}r_m \\ R_2=k_{21}r_1+k_{22}r_2+\ldots\ldots+k_{2j}r_j+\ldots\ldots+k_{2m}r_m \\ \text{-----------------------} \\ R_j=k_{j1}r_1+k_{j2}r_2+\ldots\ldots+k_{jj}r_j+\ldots\ldots+k_{jm}r_m \\ \text{-----------------------} \\ R_m=k_{m1}r_1+k_{m2}r_2+\ldots\ldots+k_{mj}r_j+\ldots\ldots+k_{mm}r_m \end{array}\right\} \qquad (138)$$

The coefficients k_{jj} and k_{jh} are known as direct- and cross-stiffnesses in the directions of the selected m displacements. In fact, it follows directly from Eqs. (138) that the general stiffness k_{jh} is the force (or moment) applied in the direction j if we displace the body by $r_h=1$ whilst keeping the remaining $(m-1)$ r's at zero. Using matrix notation the solution (138) of Eqs. (87) may be written as

$$\mathbf{R}=\mathbf{F}^{-1}\mathbf{r}=\mathbf{K}\mathbf{r} \qquad (139)$$

where **R** and **r** are the column matrices defined by Eqs. (90) and **K** is the $m \times m$ stiffness matrix

$$\mathbf{K}=\begin{bmatrix} k_{11}\ldots\ldots & k_{1j}\ldots\ldots & k_{1m} \\ \hline k_{j1}\ldots\ldots & k_{jj}\ldots\ldots & k_{jm} \\ \hline k_{m1}\ldots & k_{mj}\ldots\ldots & k_{mm} \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (140)$$

The stiffness matrix may be determined either directly or from the identity

$$\mathbf{K}=\mathbf{F}^{-1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (140a)$$

by inversion of the flexibility matrix **F**. Eq. (140*a*) shows that **K** is symmetrical, i.e.

$$k_{jh}=k_{hj} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (141)$$

This may be seen also as follows:

Let σ^j, ϵ^j and σ^h, ϵ^h be the stresses and strains corresponding to unit displacements $r_j=1$ and $r_h=1$ respectively while all other r displacements are kept at zero. Applying now the principle of virtual work or displacements to the true state j (h) and virtual state $h(j)$ we obtain

$$1\cdot k_{jh}=\int_V \sigma^h\epsilon^j dV=\int_V \sigma^j\epsilon^h dV=1\cdot k_{hj} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (142)$$

Where $\sigma^h\epsilon^j$ etc. stands for an expression analogous to Eq. (84*a*). This application of the pr nciple of virtual displacements, by an obvious analogy with Eq. 84, is called the 'unit displacement method'.

We remarked on page 19 that the direct- or cross-flexibilities depend for a given structure only on the points and directions to which they refer. This is not so for the stiffnesses which by definition depend on the complete set of points and directions selected to describe the stiffness of the body. Thus, if we choose an additional direction $m+1$ to augment our description of the elastic behaviour of the structure all the original k_{jh} will in general change whilst the f_{jh} remain unaffected.

Consider again now the example of FIG. 20. A study of Eqs. (86) and (140*a*) shows that the stiffnesses k_{jh} corresponding to unit deflexions at B, C and D are considerably more complicated than the expressions for the flexibilities. However, this is not always the case. Naturally, we can calculate the stiffnesses directly. For example, the k_{33} may be obtained by analysing a continuous beam built-in at A and simply supported at B, C and D at which last support there is a fixed 'give' of unity. We may solve this thrice redundant problem either with the three-moment equation or by the slope-deflexion method.

Assume now that not only transverse forces but also moments are applied at the junctions of the component beams and at the tip (FIG. 23). To simplify the argument we ignore, moreover, the effect of shear deformability. The modes and stiffnesses corresponding to unit displacements in the directions, 1, 2, 3, 4 can now be determined very easily. For example, for the modes $r_1=1$ and $r_4=1$ shown in FIG. 23 we find respectively,

$$\left.\begin{aligned} k_{11}&=12\left(\frac{EI}{l^3}\right)_a+12\left(\frac{EI}{l^3}\right)_b\ , & k_{31}&=-12\left(\frac{EI}{l^3}\right)_b \\ k_{21}&=-6\left(\frac{EI}{l^2}\right)_a+6\left(\frac{EI}{l^2}\right)_b\ , & k_{41}&=6\left(\frac{EI}{l^2}\right)_b \end{aligned}\right\} \qquad \ldots\ldots (143)$$

$$\left.\begin{aligned} k_{44}&=4\left(\frac{EI}{l}\right)_b\ , & k_{24}&=2\left(\frac{EI}{l}\right)_b \\ k_{14}&=6\left(\frac{EI}{l^2}\right)_b\ , & k_{34}&=-6\left(\frac{EI}{l^2}\right)_b \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots (144)$$

The important point about this example is that it shows how easy the determination of the stiffnesses can be *once we consider all possible modes of deformation of joints connecting simple component elements of a structure.* Another example will help to clarify the argument further. Consider the symmetrical framework of FIG. 24 and assume that we seek the flexibility or stiffness at the central point 2 for vertical displacements. In the first case we must solve a thrice redundant problem and in the second a four times redundant problem with a central unit "give". If, on the other hand, we select the complete set of stiffnesses corresponding to vertical and horizontal displacements at all movable joints then the calculations are most simple. In fact, for the typical cases shown in FIG. 24 we find by inspection,

$$\left.\begin{aligned} &k_{2,2}=\frac{\kappa_h}{h}+2\frac{\kappa_d}{d}\left(\frac{h}{d}\right)^2\ , \quad k_{4,2}=-\frac{\kappa_h}{h} \\ &k_{7,2}=-k_{11,2}=\frac{\kappa_d}{d}\frac{ha}{d^2}\ , \quad k_{8,2}=k_{12,2}=-\frac{\kappa_d}{d}\left(\frac{h}{d}\right)^2 \\ &k_{1,2}=k_{3,2}=k_{5,2}=k_{6,2}=k_{9,2}=k_{10,2}=0 \\ &\text{and,} \\ &k_{1,1}=2\frac{\kappa_a}{a}+2\frac{\kappa_d}{d}\left(\frac{a}{d}\right)^2\ , \quad k_{5,1}=k_{9,1}=-\frac{\kappa_a}{a} \\ &k_{7,1}=k_{11,1}=\frac{\kappa_d}{d}\left(\frac{a}{d}\right)^2\ , \quad k_{8,1}=-k_{12,1}=\frac{\kappa_d}{d}\frac{ha}{d^2} \\ &k_{2,1}=k_{3,1}=k_{4,1}=k_{6,1}=k_{10,1}=0 \end{aligned}\right\} \qquad \ldots (145)$$

where

$$\kappa=EA \qquad \ldots\ldots\ldots\ldots\ldots\ldots (145a)$$

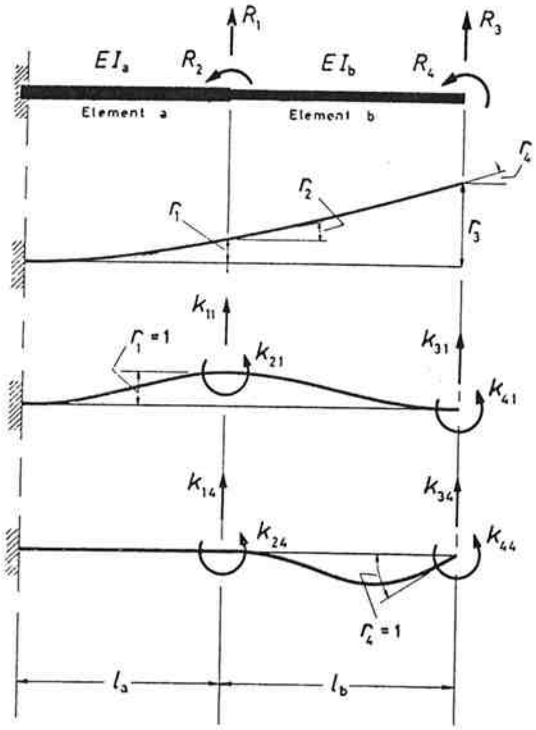
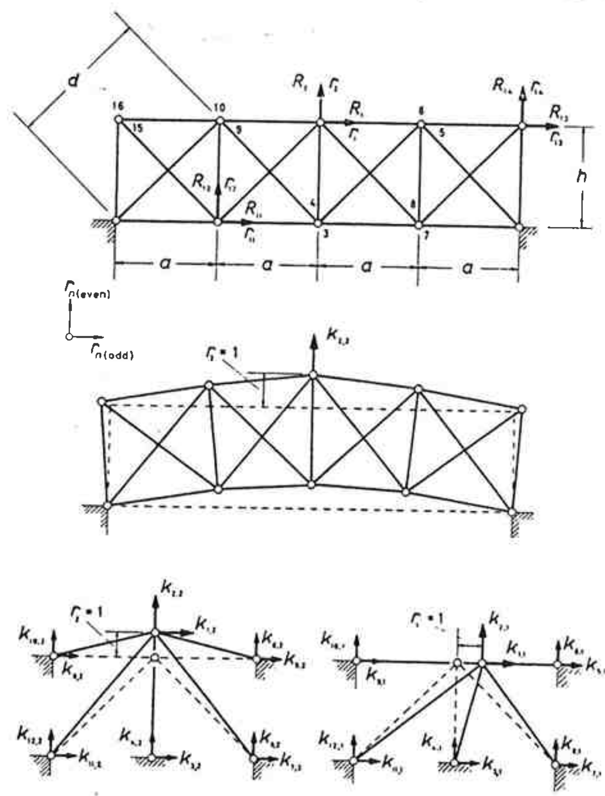


Fig. 23.—Stiffnesses of a cantilever

is the stiffness per unit length of a bar.

The stiffnesses at a point associated with the unit displacements at the same point are, in fact, already derived by the method of virtual displacements in Example 5*b* where they appear as the coefficients to the displacements u, v. Naturally, the problem of deriving the single stiffness k_{22} at the point 2 from the set (145) still remains. A general method for solving



Unit stiffness of all horizontal bars $\varkappa_a$
Unit stiffness of all vertical bars $\varkappa_h$
Unit stiffness of all diagonal bars $\varkappa_d$

Fig. 24.—Stiffnesses of pin-jointed framework

C

this and related problems is given further below. It is characteristic that the direct calculation of the flexibilities corresponding to (145) is as complicated as that for the single flexibility at 2.

In both examples we see that the stiffnesses are determined most straightforwardly, in fact, practically by inspection, once we find the set of unit deformations for which it is simple to calculate the strains and hence stresses and forces. The advantage of first deriving the stiffnesses may be particularly marked in highly redundant structures but it requires, as the example of FIG. 24 shows, the consideration of many degrees of freedom which may also have its disadvantages. On the other hand flexibilities are always easier to calculate if the stresses corresponding to unit forces can be found without difficulty as in statically determinate structures.

We gave in Eq. (142) a general formula for the determination of the stiffnesses. Let us consider it again in more detail. Observing first that, while σ^j and σ^h must be the true stresses corresponding to $r_j=1$ and $r_h=1$ respectively, the strains ϵ^h and ϵ^j need only be virtual strains $\bar{\epsilon}^h$ and $\bar{\epsilon}^j$ (i.e., compatible but not necessarily statically consistent strains*), corresponding to $r_h=1$ and $r_j=1$ respectively; this may contribute to a considerable simplification of the calculations. However, the actual practical use of Eqs. (142) rewritten here in the form

$$\left.\begin{aligned} k_{jj} &= \int_V \sigma^j \bar{\epsilon}^j dV \\ k_{jh} &= \int_V \sigma^h \bar{\epsilon}^j dV = \int_V \sigma^j \bar{\epsilon}^h dV = k_{hj} \end{aligned}\right\} \qquad (146)$$

is somewhat limited. This follows from the previous discussion which shows that stiffnesses are best found either by considering all possible degrees of freedom at the joints, in which case the determination of the k's is usually performed by inspection, or by inverting the flexibility matrix **F**. Moreover, even if we calculate the stiffnesses k for a restricted total number of degrees of freedom at the joints (e.g. example of FIG. 20) Eqs. (146) are really superfluous. Thus, in the example of FIG. 20 we have to find the true stresses σ for a four times redundant structure, the analysis of which includes the derivation of the forces k_{jh} and the use of Eq. (146) is hence unnecessary. Nevertheless, Eq. 146 is of considerable value when the elements into which the structure is broken down are characterized, not by simple loading systems (e.g. beam elements or bars) for which the k's are determinable by inspection, but by simple (assumed) displacement patterns. An example of this application is given in D of this section. Also given later is the matrix formulation of Eq. 142, which is most useful in practical cases.

We now find a generalization of the concept of stiffness corresponding to the generalized flexibility given on page 19. Thus, following the argument there we introduce the generalized displacements **q** and forces **Q** defined by

$$\mathbf{r} = \mathbf{A}\mathbf{q} \qquad \mathbf{Q} = \mathbf{A}'\mathbf{R} \qquad (147)$$

where

$$\mathbf{A} = \mathbf{G}_1' = (\mathbf{B}^{-1})' \qquad (147a)$$

see also Eqs. (96), (97) and (100). Substituting the expressions for **r** and **R** in Eq. (139) we find immediately

$$\mathbf{Q} = \mathbf{K}_q \mathbf{q} \qquad (148)$$

where

$$\mathbf{K}_q = \mathbf{A}'\mathbf{K}\mathbf{A} \qquad (149)$$

is the generalized stiffness corresponding to the m generalized displacements q_j. Eq. (149) may naturally also be derived by inversion from Eq. (103), i.e.

$$\mathbf{K}_q = (\mathbf{F}_q)^{-1} \qquad (149a)$$

The particular linear transformation **B** (or the corresponding matrix **A**) which reduces the cross-flexibilities f_{jh} to zero nullifies also the corresponding cross-stiffness k_{jh}. In fact, we obtain from Eq. (109)

$$\mathbf{P} = \mathbf{K}_p \mathbf{p} \qquad (150)$$

where $\mathbf{K}_p$ is the diagonal matrix

$$\mathbf{K}_p = \begin{bmatrix} k_{p11} & 0 & \ldots\ldots 0 & \ldots\ldots 0 \\ 0 & k_{p22} & \ldots\ldots 0 & \ldots\ldots 0 \\ & & \ddots & \\ 0 & 0 & \ldots\ldots k_{pjj} & \ldots\ldots 0 \\ & & & \ddots \\ 0 & 0 & \ldots\ldots 0 & \ldots\ldots k_{pmm} \end{bmatrix} \qquad (151)$$

and

$$k_{p11} = \frac{1}{f_{p11}}, \quad k_{p22} = \frac{1}{f_{p22}}, \ldots\ldots, \quad k_{pjj} = \frac{1}{f_{pjj}}, \ldots\ldots, \quad k_{pmm} = \frac{1}{f_{pmm}} \qquad (152)$$

* See Section 4.

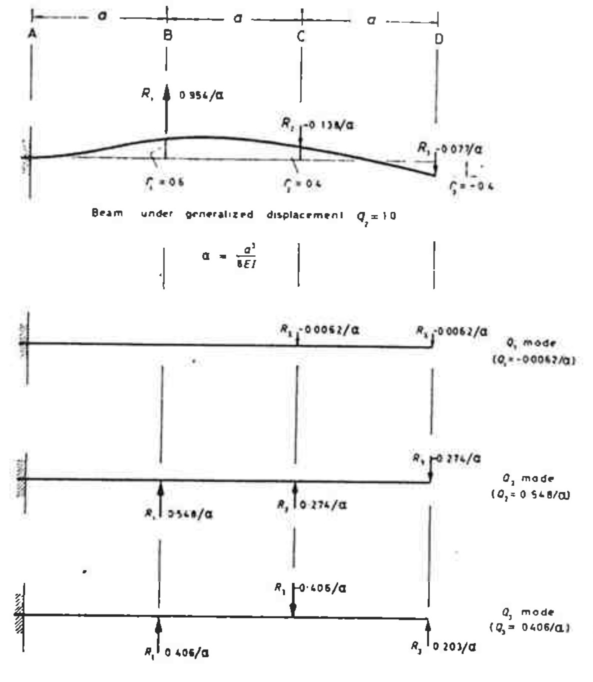


Fig. 25.—Generalized displacement and forces

We illustrate now in FIG. (25) the application of formulae (147)-(149) on the simple example of FIG. (21) and for the same load transformation matrix **B** of Eq. (105), but seek here the components of forces corresponding to a generalized displacement

$$q_2 = 1$$

The displacement transformation matrix **A** of Eq. (147a) is given by Eq. (107)

$$\mathbf{A} = \mathbf{G}' = (\mathbf{B}^{-1})' = \begin{bmatrix} 0{\cdot}1 & 0{\cdot}6 & 0{\cdot}4 \\ 0{\cdot}4 & 0{\cdot}4 & -0{\cdot}4 \\ 0{\cdot}6 & -0{\cdot}4 & 0{\cdot}4 \end{bmatrix} \qquad (107a)$$

Hence

$$\mathbf{r} = \mathbf{A}\mathbf{q} = \mathbf{A}\{0 \ 1 \ 0\} = \{0{\cdot}6 \ 0{\cdot}4 \ -0{\cdot}4\}$$

The generalized stiffness matrix $\mathbf{K}_q$ is best obtained by inversion of $\mathbf{F}_q$ in Eq. (106) and is

$$\mathbf{K}_q = \frac{1}{\alpha}\begin{bmatrix} 0{\cdot}0169 & -0{\cdot}00615 & -0{\cdot}117 \\ -0{\cdot}00615 & 0{\cdot}548 & 0{\cdot}406 \\ -0{\cdot}117 & 0{\cdot}406 & 0{\cdot}218 \end{bmatrix} \qquad (106a)$$

where α is given by Eq. (104a). Hence,

$$\mathbf{Q} = \mathbf{K}_q\mathbf{q} = \mathbf{K}_q\{0 \ 1 \ 0\}$$

or

$$\mathbf{Q} = \frac{1}{\alpha}\left\{-0{\cdot}00615 \ 0{\cdot}548 \ 0{\cdot}406\right\}$$

We analyse finally the generalized forces Q in their R-components. From Eq. (97) or (147)

$$\mathbf{R} = \mathbf{B}\mathbf{Q} = \frac{1}{\alpha}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0{\cdot}5 & -1 \\ 1 & -0{\cdot}5 & 0{\cdot}5 \end{bmatrix}\begin{bmatrix} -0{\cdot}00615 \\ 0{\cdot}548 \\ 0{\cdot}406 \end{bmatrix}$$

and carrying out the multiplication,

$$\mathbf{R} = \frac{1}{\alpha}\begin{bmatrix} 0 & +0{\cdot}548 & +0{\cdot}406 \\ -0{\cdot}006 & +0{\cdot}274 & -0{\cdot}406 \\ -0{\cdot}006 & -0{\cdot}274 & +0{\cdot}203 \end{bmatrix} = \begin{bmatrix} +0{\cdot}954/\alpha \\ -0{\cdot}138/\alpha \\ -0{\cdot}077/\alpha \end{bmatrix}$$

Each of the three columns of the intermediate expression represents obviously the R components of Q_1, Q_2, Q_3 respectively. We can check now the previously given result for **r** from

$$\mathbf{r} = \mathbf{F}\mathbf{R}$$

where **F** is given in Eq. (104).

Next we derive a general formula for the stiffness **K** of a structure consisting of a finite number of simple elements. The expression given is the matrix formulation of Eqs. (146) and corresponds to the flexibility matrix

F of Eq. (126). Since the analysis follows closely the arguments on page 21 we need present here only the outlines of the proof.

Consider again an assembly of s structural elements joined together at their ends or boundaries. m displacements r are selected to describe the stiffness **K** of the complete structure. Let $\mathbf{k}_g$ be the stiffness matrix of the g element due to the characteristic strains of the element arising from the displacements $\mathbf{v}_g$ at the boundaries. Naturally, there are usually several different but equivalent possible ways of expressing the straining of the element. Let $\mathbf{a}_g$ be the matrix, in general rectangular, which transforms the displacements **r** into the true strains $\mathbf{v}_g$ of the element. i.e.

$$\mathbf{v}_g=\mathbf{a}_g\mathbf{r} \quad \ldots\ldots\ldots\ldots (153)$$

Then

$$\mathbf{S}_g=\mathbf{k}_g\mathbf{v}_g=\mathbf{k}_g\mathbf{a}_g\mathbf{r} \quad \ldots\ldots\ldots\ldots (154)$$

is the matrix for the forces (moments, etc.) applied on the element due to the displacements **r**. The internal force (or stress) matrix **S** of the aggregate structure is now given by

$$\mathbf{S}=\mathbf{kar} \quad \ldots\ldots\ldots\ldots (155)$$

where

$$\mathbf{S}=\{\mathbf{S}_a\mathbf{S}_b\ldots\ldots\mathbf{S}_g\ldots\ldots\mathbf{S}_s\} \quad \ldots\ldots\ldots\ldots (156)$$

and

$$\mathbf{a}=\{\mathbf{a}_a\mathbf{a}_b\ldots\ldots\mathbf{a}_g\ldots\ldots\mathbf{a}_s\} \quad \ldots\ldots\ldots\ldots (157)$$

k is the symmetrical diagonal partitioned matrix,

$$\mathbf{k}=\begin{bmatrix} \mathbf{k}_a & \mathbf{o} & \ldots\ldots & \mathbf{o} & \ldots\ldots & \mathbf{o} \\ \mathbf{o} & \mathbf{k}_b & \ldots\ldots & \mathbf{o} & \ldots\ldots & \mathbf{o} \\ & & \ddots & & & \\ \mathbf{o} & \mathbf{o} & \ldots\ldots & \mathbf{k}_g & \ldots\ldots & \mathbf{o} \\ & & & & \ddots & \\ \mathbf{o} & \mathbf{o} & \ldots\ldots & \mathbf{o} & \ldots\ldots & \mathbf{k}_s \end{bmatrix} \quad \ldots\ldots\ldots\ldots (158)$$

Applying now the principle of virtual work, taking the internal forces **S** and external forces **R** as the true state and selecting as virtual state the internal strains corresponding to unit displacements $r_1=1$, $r_2=1$, ..., $r_m=1$ respectively we find

$$\mathbf{R}=\mathbf{a'kar} \quad \ldots\ldots\ldots\ldots (159)$$

where **a′** is the transpose of **a**. Thus, the stiffness matrix **K** of the compound structure is

$$\mathbf{K}=\mathbf{a'ka} \quad \ldots\ldots\ldots\ldots (160)$$

Eq. (160) may also be written as

$$\mathbf{K}=\sum_g \mathbf{a}'_g\mathbf{k}_g\mathbf{a}_g \quad \ldots\ldots\ldots\ldots (160a)$$

Since the virtual strains need only satisfy the compatibility but not necessarily the equilibrium conditions we may select for the virtual states a simpler matrix **a** which satisfies only the former. Eq. (160) becomes then

$$\mathbf{K}=\bar{\mathbf{a}}'\mathbf{ka} \quad \ldots\ldots\ldots\ldots (160b)$$

However, the application of a possibly simplified matrix **a** is really not required in practice. As mentioned on page 23 the stiffness matrix **K** is best calculated for all degrees of freedom at the joints, yielding very simple matrices.

The configuration of the elements of the compound structure is said to be in parallel in Eq. (160) since the assembly condition is expressed by the matrix **a** which derives from conditions of compatibility. *Thus* Eq. (160) *may be regarded as the most general formulation of the stiffness matrix for a structure with constituent elements in parallel.* It is immediately apparent why Eq. (149) which expresses the stiffness matrix for generalized displacements must have the same form as Eq. (160). In the first case we derive generalized displacements from single displacements and in the second, internal strains from external displacements. In both applications this entails a linear transformation matrix which, however, is a square matrix in the former case. Also **K** is the stiffness of the complete structure for the single displacements while **k** is the stiffness matrix of the individual members.

Eqs. (153) and (159) show that there is a most illuminating parallel development to Eqs. (121) and (125*a*). Thus, if the internal relative displacements (strains) **v** derive from the external displacements **r** with the relationship

$$\mathbf{v}=\mathbf{ar} \quad \ldots\ldots\ldots\ldots (153a)$$

Then the external forces **R** derive from the internal forces (stresses) **S** with the relationship

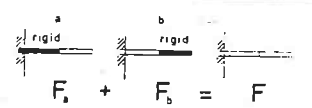
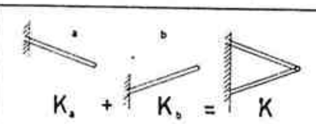
$$\mathbf{R}=\bar{\mathbf{a}}'\mathbf{S}=\mathbf{a}'\mathbf{S} \quad \ldots\ldots\ldots\ldots (159a)$$

Eq. (159*a*) restates, of course, the principle of Virtual Work.

TABLE I

Duality of Force and Displacement Methods

(it is always possible to substitute **a**, **b** for $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ respectively)

Method of Forces		Method of Displacements
Force R → Flexibility F → Displacement r	$FK = I = KF$	Displacement r → Stiffness K → Force R
Generalized Force Q: $R = BQ$		Generalized Displacement q: $r = Aq$
Generalized Flexibility: $F_q = B'FB$	$A'B = I = B'A$; $F_q K_q = I = K_q F_q$	Generalized Stiffness: $K_q = A'KA$
Generalized Displacement: $q = B'r = F_qQ$		Generalized Force: $Q = A'R = K_qq$
Generalized Series Assembly		**Generalized Parallel Assembly**
Stress on elements S: $S = bR$		Strain of elements V: $V = ar$
Strain of elements V: $r = \bar{b}'v$		Stress on elements S: $R = \bar{a}'S$
Flexibility of elements f (for stresses S)		Stiffness of elements k (for strains V)
Flexibility of complete structure: $F = \bar{b}'fb$		Stiffness of complete structure: $K = \bar{a}'ka$
Addition of Flexibilities (Special series assembly)		Addition of Stiffnesses (Special parallel assembly)
		

Before illustrating applications of Eq. (160) we draw attention to the by now all too apparent complete parallel between the flexibility and stiffness approach in the analysis of structures. We may express this concisely by the tabular arrangement under the two headings: 'Methods of Forces' and 'Method of Displacements'.

The analogy between the two methods is developed considerably in what follows and is shown in greater detail in TABLE II.

Illustrations to Eq. (160).

Consider the beams I and II of FIG. (26) joined by inextensional bars which connect the set of points B, C, D and B′, C′, D′ respectively. Let $\mathbf{K}_I$ and $\mathbf{K}_{II}$ be the stiffness matrices of the upper and lower beam respectively defined for vertical displacements r_1, r_3 and r_5. From the definition of the stiffness it follows immediately that the stiffness **K** for the displacements r_1, r_3 and r_5 in the compound structure is given by

$$\mathbf{K}=\mathbf{K}_I+\mathbf{K}_{II} \quad \ldots\ldots\ldots\ldots (161)$$

This simple result may also be derived from the general Eq. (160). For in this special case the joint displacements **r** etc. of the complete structure and the straining displacements $\mathbf{v}_I$ and $\mathbf{v}_{II}$ of the component beams are the same. Thus,

$$\mathbf{v}_I=\mathbf{a}_I\mathbf{r},\quad \mathbf{v}_{II}=\mathbf{a}_{II}\mathbf{r} \quad \ldots\ldots\ldots\ldots (162)$$

where

$$\mathbf{r}=\{r_1\,r_3\,r_5\}$$

and

$$\mathbf{a}_I=\mathbf{a}_{II}=\mathbf{I}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \ldots\ldots\ldots\ldots (163)$$

We conclude,

$$\mathbf{K}=[\mathbf{I}\ \ \mathbf{I}]\begin{bmatrix} \mathbf{K}_I & \mathbf{o} \\ \mathbf{o} & \mathbf{K}_{II} \end{bmatrix}\begin{bmatrix} \mathbf{I} \\ \mathbf{I} \end{bmatrix}=\mathbf{K}_I+\mathbf{K}_{II} \quad \ldots\ldots (161a)$$

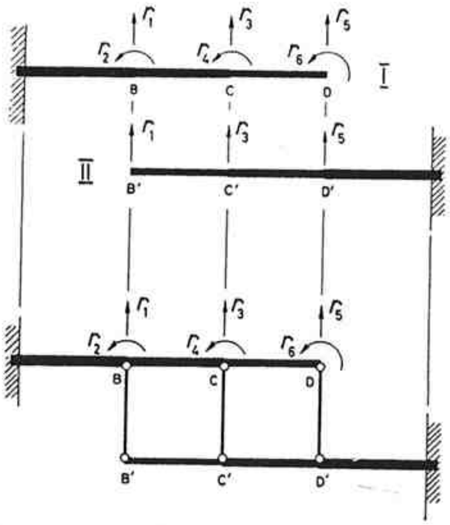


Fig. 26.—Parallel combination of cantilever. Addition of stiffnesses

Eq. (161) applies to any compound structure in which the stiffnesses are defined for *at least* all common degrees of freedom associated with the joining of structures I and II. Thus, in the example of FIG. (26) the common degrees of freedom are the vertical displacements r_1, r_3 and r_5. Formula (161) is, however, still true if we define the stiffnesses of the upper beam *and* the complete structure for both the vertical displacements r_1, r_3 and r_5 and the slopes r_2, r_4, r_6. Then $\mathbf{K}_I$ can be calculated by the methods given previously. $\mathbf{K}_{II}$ is still only definable for vertical displacement, the corresponding entries associated with r_2, r_4 and r_6 being zero. Naturally, we can define the stiffness matrix $\mathbf{K}$ and say $\mathbf{K}_I$ for points not connected to *II*. Again the corresponding terms of $\mathbf{K}_{II}$ are zero. FIG. (27) shows the joining of two arbitrary structures to give $\mathbf{K}=\mathbf{K}_I+\mathbf{K}_{II}$. Note that at a joint point like (2) we must define the stiffnesses for two displacements, say the x and y-directions.

Formula (161) may, of course, also be applied in obtaining the stiffness matrix of the compound cantilever consisting of elements a and b, FIG. (23). Again, we must define the stiffness for all common deflexions and slopes at the joints, assuming the E.T.B. to be true and the shear deflexions zero. The total stiffness $\mathbf{K}$ is then

$$\mathbf{K}=\mathbf{K}_a+\mathbf{K}_b \quad \ldots\ldots\ldots\ldots \quad (161b)$$

where the elements of the split matrices may be found from Eqs. (143) and (144) for the displacements r_1 and r_4 and similar equations for r_2 and r_3. Thus,

$$\mathbf{K}_a=\begin{bmatrix} 12\frac{EI}{l^3} & 0 & -6\frac{EI}{l^2} & 0 \\ 0 & 0 & 0 & 0 \\ -6\frac{EI}{l^2} & 0 & 4\frac{EI}{l} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_a \quad \ldots\ldots\ldots\ldots \quad (164)$$

$$\mathbf{K}_b=\begin{bmatrix} 12\frac{EI}{l^3} & -12\frac{EI}{l^3} & 6\frac{EI}{l^2} & 6\frac{EI}{l^2} \\ -12\frac{EI}{l^3} & 12\frac{EI}{l^3} & -6\frac{EI}{l^2} & -6\frac{EI}{l^2} \\ 6\frac{EI}{l^2} & -6\frac{EI}{l^2} & 4\frac{EI}{l} & 2\frac{EI}{l} \\ 6\frac{EI}{l^2} & -6\frac{EI}{l^2} & 2\frac{EI}{l} & 4\frac{EI}{l} \end{bmatrix}_b \quad \ldots\ldots\ldots\ldots \quad (164a)$$

where the columns and rows refer to displacements r_1, r_3, r_2, r_4 respectively. Formula (161*b*) may be generalized for any numbers of component beams, and for any structure in which the joint displacements $\mathbf{r}$ express also the straining displacements $\mathbf{v}$ of the elements (i.e. $\mathbf{a}=\mathbf{I}$). In such cases the stiffness matrix $\mathbf{K}$ can be written

$$\mathbf{K}=\mathbf{K}_a+\mathbf{K}_b+\ldots\ldots+\mathbf{K}_g+\ldots\ldots+\mathbf{K}_s \quad \ldots\ldots\ldots \quad (165)$$

Note that the only non-zero coefficients in $\mathbf{K}_g$ are the stiffnesses k corresponding to the displacements at the ends or boundaries of the g-element.

The flexibility matrix $\mathbf{F}$ corresponding to (165) is

$$\mathbf{F}=\mathbf{K}^{-1}=(\mathbf{K}_a+\mathbf{K}_b+\ldots\ldots+\mathbf{K}_s)^{-1}=(\mathbf{F}_a^{-1}+\mathbf{F}_b^{-1}+\ldots\ldots+\mathbf{F}_s^{-1})^{-1} \quad (166)$$

The parallel between the displacement and force method is underlined

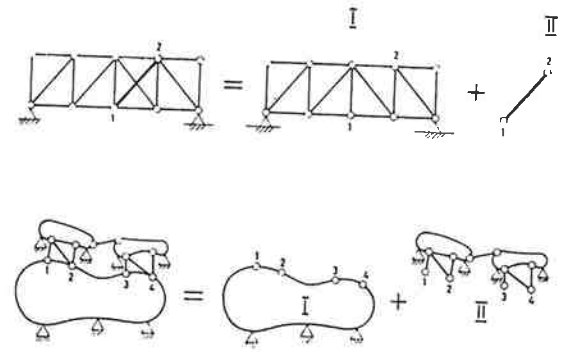


Fig. 27.—Addition of stiffnesses for arbitrary structures

further by comparison of Eqs. (92) and (161). The first shows the case of additive partial flexibilities for series assembly and the second, the case of additive partial stiffnesses for parallel assembly. A very simple application of Eq. (160) is given by the pin-jointed framework shown in FIG. 12 of example 5b.* Thus, the stiffness $\mathbf{k}_r$ of the bar corresponding to unit elongation $\Delta l_r=1$ is,

$$\mathbf{k}_r=k_r=\frac{(EA)_r}{l_r}=\frac{\kappa_r}{l_r} \quad \ldots\ldots\ldots\ldots \quad (167)$$

where $\kappa_r=(EA)_r$ is the stiffness per unit length of the rth component bar. The transformation matrix $\mathbf{a}_r$ for displacements in the x- and y-directions is,

$$\mathbf{a}_r=\begin{bmatrix}\cos\theta_r, \sin\theta_r\end{bmatrix} \quad \ldots\ldots\ldots\ldots \quad (168)$$

Hence

$$\mathbf{a}=\{\mathbf{a}_1\mathbf{a}_2\ldots\ldots\mathbf{a}_r\ldots\ldots\mathbf{a}_n\} \quad \ldots\ldots\ldots\ldots \quad (168a)$$

and

$$\mathbf{K}=\mathbf{a}'\begin{bmatrix} \kappa_1/l_1 & 0 & \ldots\ldots 0 & \ldots\ldots 0 \\ 0 & \kappa_2/l_2 & \ldots\ldots 0 & \ldots\ldots 0 \\ & & \ddots & \\ 0 & 0 & \ldots\ldots \kappa_r/l_r \ldots\ldots & 0 \\ & & & \ddots \\ 0 & 0 & \ldots\ldots 0 & \ldots\ldots \kappa_n/l_n \end{bmatrix}\mathbf{a}$$

$$\mathbf{K}=\begin{bmatrix} \sum\frac{\kappa_r}{l_r}\cos^2\theta_r & \sum\frac{\kappa_r}{l_r}\cos\theta_r\sin\theta_r \\ \sum\frac{\kappa_r}{l_r}\cos\theta_r\sin\theta_r & \sum\frac{\kappa_r}{l_r}\sin^2\theta_r \end{bmatrix}=\begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix} \quad \ldots\ldots \quad (169)$$

in agreement with Eq. (67).

We mentioned previously that the simplest method of calculating stiffnesses is to define them for as many degrees of freedom as are necessary to obtain simple deformation patterns of the elements of the structure. Having calculated such a stiffness matrix it will become necessary to 'condense' it—i.e. to refer it to the smaller number of displacements in which we may be interested. This changes, of course, all the stiffness coefficients, but the necessary analysis is easily arranged in matrix form for automatic computation. Let the original stiffness matrix be of the order $m\times m$ and denoted by $\mathbf{K}_o$. We want to find the matrix $\mathbf{K}$ referred to p-directions only, where $p<m$. We have,

$$\mathbf{R}=\mathbf{K}_o\mathbf{r}$$

where

$$\left.\begin{aligned} \mathbf{R}&=\{R_1\ldots\ldots R_p\,R_{p+1}\ldots\ldots R_m\}=\{\mathbf{R}_I\,\mathbf{R}_{II}\} \\ \mathbf{r}&=\{r_1\ldots\ldots r_p\,r_{p+1}\ldots\ldots r_m\}=\{\mathbf{r}_I\,\mathbf{r}_{II}\} \end{aligned}\right\} \quad \ldots\ldots \quad (171)$$

in which we write first the p-directions required for the condensed matrix $\mathbf{K}$. $\mathbf{K}_o$ may be expressed as a partitioned matrix as follows,

* See p. 10.

$$\mathbf{K}_n = \begin{bmatrix} \mathbf{K}_I & \mathbf{K}'_{III} \\ \mathbf{K}_{III} & \mathbf{K}_{II} \end{bmatrix} \qquad (172)$$

where $\mathbf{K}_I$ and $\mathbf{K}_{II}$ are square matrices of order p and m-p respectively. Eq. (170) can now be transformed to

$$\left.\begin{aligned} \mathbf{R}_I &= \mathbf{K}_I\mathbf{r}_I + \mathbf{K}'_{III}\mathbf{r}_{II} \\ \mathbf{R}_{II} &= \mathbf{K}_{III}\mathbf{r}_I + \mathbf{K}_{II}\mathbf{r}_{II} \end{aligned}\right\} \qquad (173)$$

Also, by definition, in the structure with stiffness defined in p directions only

$$\left.\begin{aligned} \mathbf{R}_I &= \mathbf{K}\mathbf{r}_I \\ \mathbf{R}_{II} &= \mathbf{O} \end{aligned}\right\} \qquad (173a)$$

Putting $\mathbf{R}_{II} = \mathbf{O}$ in Eq. (173) and eliminating $\mathbf{r}_{II}$ we find

$$\mathbf{R}_I = (\mathbf{K}_I - \mathbf{K}'_{III}\mathbf{K}_{II}^{-1}\mathbf{K}_{III})\mathbf{r}_I \qquad (174)$$

and hence comparing with Eq. (173a)

$$\mathbf{K} = \mathbf{K}_I - \mathbf{K}'_{III}\mathbf{K}_{II}^{-1}\mathbf{K}_{III} \qquad (175)$$

Eq. (175) gives naturally the solution to the particular problem of FIG. (24) discussed on page 23. Another example illustrating the application of Eq. (175) is discussed under D of this Section.

The above method, is, of course, the basis of the solution of partly homogeneous equations. A parallel relationship exists also in the 'force-method' investigation of structures. Thus, in this case, we have to find the flexibility $\mathbf{F}$ of a redundant structure in which we know the flexibility $\mathbf{F}_n$ of the basic structure. The analysis is given under C below.

C. The Calculation of Redundant Structures by the Force-Method

We develop now a generalization of the Mueller-Breslau[6]* technique for the calculation of linearly elastic redundant structures. Following our investigations under (A) we could easily formulate immediately the complete analysis in matrix notation. However, since the basic ideas do not appear to be generally known we think it preferable to develop them first in the more standard form.

Consider a structure subject to arbitrary external loads R, temperature strains $\alpha\Theta$ and any other initial strains η. We assume that the system has n internal or external redundancies

$$X_1, X_2, \ldots\ldots, X_i, \ldots\ldots X_n$$

which may be stresses, forces, moments or linear combinations of such (generalized forces). By including the supporting body—assumed rigid—in our structure we can denote all redundancies as internal. The stress distribution in the body remains statically indeterminate until an elastic analysis yields the n unknown X's. If, irrespective of compatibility we assume the X_i to be zero, we obtain the 'basic' (principal or null) system which is statically determinate. This procedure of obtaining the basic structure may sometimes be identified with the process of an actual physical cut of redundant members (e.g. of bars in a redundant pin-jointed framework). However, the simple idea of a cut is not always applicable to continuous structures typical of aircraft. We discuss this point later but for the sake of linguistic simplicity continue to use the expression 'cut redundancy'.

Let the stress-system in the basic system be denoted by

$$\sigma_0$$

It must obviously be in equilibrium with the applied loads. We describe it as a 'statically equivalent stress system', thus drawing attention to the fact that in its determination only statical conditions enter. We find also in the basic structure the stress systems

$$\sigma_1, \sigma_2, \ldots\ldots, \sigma_i, \ldots\ldots \sigma_n$$

due to

$$X_1 = 1,\ X_2 = 1,\ \ldots\ldots,\ X_i = 1,\ \ldots\ldots,\ X_n = 1$$

respectively. The systems $\sigma_1, \sigma_2, \ldots\ldots, \sigma_i, \ldots\ldots, \sigma_n$ are obviously self-equilibrating. Since our structure is by definition linearly elastic the true stresses σ in the uncut original structure can be expressed as

$$\sigma = \sigma_0 + \sum_{i=1}^{n} \sigma_i X_i \qquad (176)$$

Similar equations may be written down for stress resultants (forces or moments). Thus, the problem reduces to the determination of the X's, which as already mentioned, need not be simple forces or moments but can be linear combinations of such (generalized forces).

The Equations in the unknown X.

We define the following set of deformations in the basic system.

- δ_{io} Relative movement of ends of cut ith redundancy due to all external causes, i.e. loads, temperature changes, lack of fit, 'give' at the supports, etc.; $i = 1$ to n.
- $\delta_{ik} = \delta_{ki}$ relative movement of ends of cut ith (kth) redundancy due to the self-equilibrating load system $X_k = 1$ ($X_i = 1$); i and k take values 1 to n. The δ-coefficients are taken positive if the relative movements are in the positive direction of the X's.

* See also footnote, p. 17.

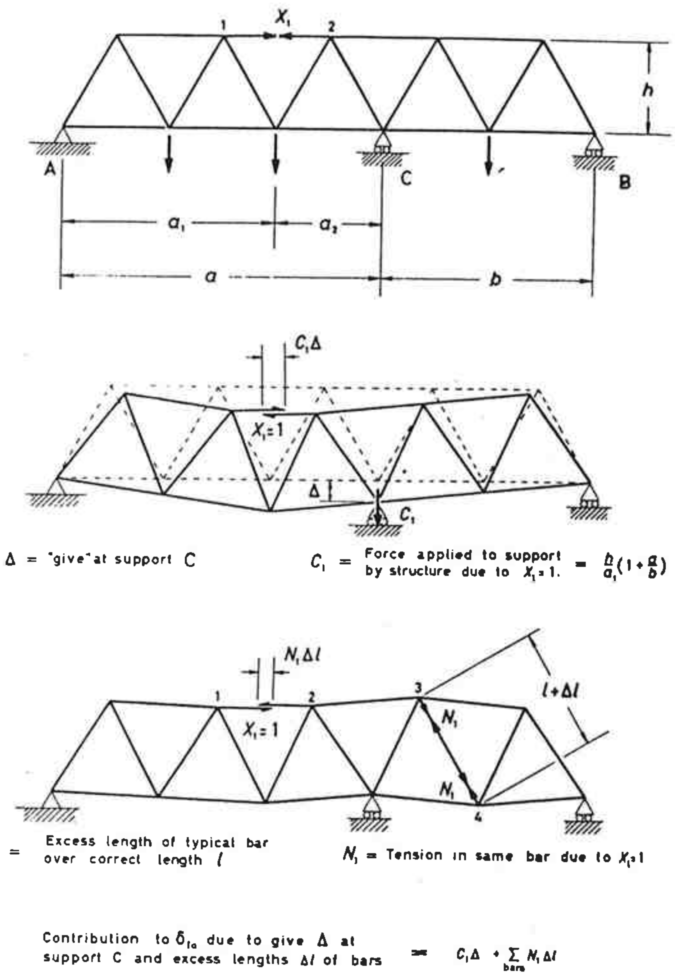


Fig. 28.—Singly redundant, pin-jointed framework. Contribution tu δ_{10} from sinking or 'give' of support and excess lengths of bars

The δ_{ik} are, of course, the influence or flexibility coefficients of the basic structure for the directions of the redundant forces. We use here the symbol δ for these flexibilities since it is standard in the literature. To calculate the δ's we apply the unit load method of Section 6D.

Thus, using again the abbreviations

$$\left.\begin{aligned} \sigma\epsilon &= \sigma_{xx}\epsilon_{xx} + \ldots\ldots + \sigma_{zx}\epsilon_{zx} \\ \sigma\eta &= \sigma_{xx}\eta_{xx} + \ldots\ldots + \sigma_{zx}\eta_{zx} \end{aligned}\right\} \qquad (3)$$

we find from Eq. (71a),

$$\delta_{io} = \int_V \sigma_i(\epsilon_o + \eta)dV \qquad (177)$$

$$\delta_{ii} = \int_V \sigma_i\epsilon_i dV,\ \delta_{ik} = \int_V \sigma_i\epsilon_k dV = \int_V \sigma_k\epsilon_i dV = \delta_{ki} \qquad (178)$$

where σ_i, ϵ_i (σ_k, ϵ_k) are the stresses and strains corresponding to $X_i = 1$ ($X_k = 1$) and σ_o, ϵ_o are the stresses and strains due to the applied loads. Eqs. (178) reproduce, of course, merely Eqs. (84) for the flexibility coefficients. The total initial strains η imposed upon the basic system may be separated into thermal and other strains

$$\eta_{xx} = \alpha\Theta + \eta_{xxo},\ \ldots\ldots,\ \eta_{xy} = \eta_{xyo},\ \ldots\ldots \qquad (179)$$

where η_{xxo}, etc. are initial strains due to say lack of fit, 'give' at the supports. The effect of the latter upon δ_{io} is best considered separately and expressed in terms of the imposed changes of length (rotations) and 'gives'. Consider, for example, the singly redundant framework of FIG. (28) and assume that the manufactured length of the bars exceeds the correct length l by Δl. Let also each bar be subjected to a different thermal straining $\alpha\Theta$. We assume furthermore that the intermediate support gives or sinks by the amount Δ. As redundancy we select the force X_1 in the bar (1, 2) and denote by N_0 and N_1 the (tension) forces in the bars of the basic system due to the applied loads and $X_1 = 1$ respectively. The loading case $X_1 = 1$, with the corresponding force applied to support C by the structure is shown in FIG. (28).

We find immediately

$$\delta_{11} = \Sigma\frac{N_1^2 l}{AE} \qquad (180)$$

Applying now the unit load method to the state $X_1=1$ and the true total strains $(\epsilon_o+\eta)$ and displacements in the basic system we derive (see FIG. (28) and Eq. (71*a*)),

$$1\cdot\delta_{10}=\Sigma N_1(\epsilon_o+\eta)l=\Sigma\frac{N_1N_o}{AE}l+\Sigma N_1(\alpha\Theta l+\Delta l)+\frac{h}{a_1}\left(1+\frac{a}{b}\right)\Delta \quad \ldots\ldots (181)$$

Naturally, we can alternatively deduce by kinematical reasoning the contribution to δ_{10} of the initial elongations $\alpha\Theta$ and Δl and the 'give' Δ. However, the unit load method yields the results much more conveniently and systematically.

More general formulae for the δ-coefficients are given further below.

The condition of consistent deformations at the cut ends of the n redundancies in the actual structure or application of the unit load method yields the following n equations in the n unknown X.

$$\begin{array}{l}
\delta_{11}X_1+\delta_{12}X_2+\ldots\ldots+\delta_{1i}X_i+\ldots\ldots+\delta_{1n}X_n+\delta_{10}=0\\
\delta_{21}X_1+\delta_{22}X_2+\ldots\ldots+\delta_{2i}X_i+\ldots\ldots+\delta_{2n}X_n+\delta_{20}=0\\
-\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\\
\delta_{i1}X_1+\delta_{i2}X_2+\ldots\ldots+\delta_{ii}X_i+\ldots\ldots+\delta_{in}X_n+\delta_{i0}=0\\
-\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\\
\delta_{n1}X_1+\delta_{n2}X_2+\ldots\ldots+\delta_{ni}X_i+\ldots\ldots+\delta_{nn}X_n+\delta_{n0}=0
\end{array} \quad \ldots (182)$$

The solution of these equations determines the X and hence also the total stress distribution after substitution into Eqs. (176)

To solve Eqs. (182) by elimination* is particularly simple when they are of the three-moment or five moment type; see for example the tube analysis in Section 9(b). In general, however, all unknowns may appear in each equation and we need a systematic and mnemonic method for the determination of the X. The most convenient method for this purpose is the shortened elimination process of Gauss† (known also as Gaussian algorithm). This method is so well known that we need not discuss its formulation in the present 'long-hand' notation but may use immediately the matrix notation. Accordingly, we write the system of Eqs. (182) in the form

$$\mathbf{DX}+\mathbf{D}_0=\mathbf{O} \quad \ldots\ldots (182a)$$

where $\mathbf{D}$ is the symmetrical square matrix of the δ_{ik}-coefficients, $\mathbf{D}_o$ the column matrix of the δ_{io}-coefficients, $\mathbf{X}$ the column matrix of the unknown X_i and $\mathbf{O}$ a zero column matrix. The Gaussian elimination process reduces the system (182*a*) to

$$\mathbf{TX}=\mathbf{T}_0 \quad \ldots\ldots (183)$$

where $\mathbf{T}$ is a triangular matrix, i.e. a matrix whose elements above (or alternatively below) the principal diagonal are zero and $\mathbf{T}_0$ is again a column matrix. For example,

$$\mathbf{T}=\begin{bmatrix}
t_{11} & 0 & \ldots & \ldots & \ldots & 0\\
t_{12} & t_{22} & 0 & \ldots & \ldots & 0\\
\vdots & & \ddots & & & \vdots\\
t_{i1} & t_{i2} & \ldots & t_{ii} & 0\ldots & 0\\
\vdots & & & & \ddots & \vdots\\
t_{ni} & t_{n2} & \ldots & \ldots & \ldots & t_{nn}
\end{bmatrix}, \quad \mathbf{T}_0=\begin{bmatrix} t_{10}\\ t_{20}\\ \vdots\\ t_{i0}\\ \vdots\\ t_{n0}\end{bmatrix} \quad \ldots (184)$$

The solution of Eqs. (183) and (184) is straightforward by substitution, starting from the first of Eqs. (183) or in matrix language it is simple to find the inverse $\mathbf{T}^{-1}$ of $\mathbf{T}$ and write

$$\mathbf{X}=\mathbf{T}^{-1}\mathbf{T}_0 \quad \ldots\ldots (185)$$

For the automatic computation techniques now available and, in particular, for the punched card machines it is usually preferable to modify slightly the Gaussian elimination process and obtain directly the inverse of $\mathbf{D}$ and hence also the column of $\mathbf{X}$. This method is known as the Jordan technique and we restrict here our discussion to this process.

Consider a system of n independent equations

$$\mathbf{aX}=\mathbf{Z} \quad \ldots\ldots (186)$$

where $\mathbf{a}$ is a square matrix and $\mathbf{Z}$ is a column matrix. Thus,

$$\mathbf{a}=\begin{bmatrix}
a_{11} & \ldots\ldots & a_{1i} & \ldots\ldots & a_{1n}\\
- & - & - & - & -\\
a_{i1} & \ldots\ldots & a_{ii} & \ldots\ldots & a_{in}\\
- & - & - & - & -\\
a_{n1} & \ldots\ldots & a_{ni} & \ldots\ldots & a_{nn}
\end{bmatrix} \quad \ldots\ldots (187)$$

To eliminate in the first column of $\mathbf{a}$ all its elements but a_{11} and to reduce the latter to 1 we premultiply $\mathbf{a}$ with the matrix

$$\mathbf{M}_1=\begin{bmatrix}
1/a_{11} & 0 & \ldots & \ldots & \ldots & 0\\
-a_{21}/a_{11} & 1 & 0 & \ldots & \ldots & 0\\
\vdots & & \ddots & & & \\
-a_{i1}/a_{11} & 0\ldots\ldots 0 & & 1 & 0\ldots\ldots & 0\\
\vdots & & & & \ddots & \\
-a_{n1}/a_{11} & 0 & \ldots & \ldots & 0 & 1
\end{bmatrix} \quad \ldots (188)$$

which square matrix has, but for the first column, unit diagonal and zero cross-elements. We find,

$$\mathbf{b}=\mathbf{M}_1\mathbf{a}=\begin{bmatrix}
1 & b_{12} & \ldots\ldots & b_{1i} & \ldots\ldots & b_{1n}\\
0 & b_{22} & \ldots\ldots & b_{2i} & \ldots\ldots & b_{2n}\\
- & - & - & - & - & -\\
0 & b_{i2} & \ldots\ldots & b_{ii} & \ldots\ldots & b_{in}\\
- & - & - & - & - & -\\
0 & b_{n2} & \ldots\ldots & b_{ni} & \ldots\ldots & b_{nn}
\end{bmatrix} \quad \ldots\ldots (187a)$$

where

$$b_{1k}=a_{1k}/a_{11} \text{ and } b_{ik}=a_{ik}-\frac{a_{i1}a_{1k}}{a_{11}} \quad \text{for } i\neq 1 \quad \ldots\ldots (189)$$

Next we eliminate the elements of the second column of $\mathbf{b}$, except for b_{22} which we reduce to 1, by premultiplying with

$$\mathbf{M}_2=\begin{bmatrix}
1 & -b_{12}/b_{22} & 0 & \ldots & \ldots & 0\\
0 & 1/b_{22} & 0 & \ldots & \ldots & 0\\
\vdots & & \ddots & & & \\
0 & -b_{i2}/b_{22} & 0\ldots 0 & 1 & 0\ldots\ldots & 0\\
\vdots & & & & \ddots & \\
0 & -b_{n2}/b_{22} & 0 & \ldots & 0 & 1
\end{bmatrix} \quad \ldots (186a)$$

* Naturally we may also use iteration techniques but such methods are not discussed in this series of papers.

† See: Simultaneous Linear Equations and the determination of the eigenvalues, *National Bureau of Standards, Applied Math. Series* 29 (1953), and O. Heck: 'Ueber den Zeitaufwand fuer das Berechnen von Determinanten und fuer das Anfloesen von linearen Gleichungen.' *Dissert. Techn. Hochsch. Darmstadt*, 1946. Zurmuehl, *loc. cit.*

We obtain

$$\mathbf{c}=\begin{bmatrix} 1 & 0 & c_{13} \cdots\cdots\cdots c_{1n} \\ 0 & 1 & c_{23} \cdots\cdots\cdots c_{2n} \\ - - & - - & \\ 0 & 0 & c_{i3} \cdots\cdots\cdots c_{in} \\ - - & - - & \\ 0 & 0 & c_{n3} \cdots\cdots\cdots c_{nn} \end{bmatrix} \quad \ldots\ldots\ldots\ldots (187b)$$

where

$$c_{2k}=b_{2k}/b_{22} \text{ and } c_{ik}=b_{ik}-\frac{b_{i2}b_{2k}}{b_{22}} \qquad \text{for } i\neq 2 \ \ldots\ldots\ldots\ldots (189a)$$

The procedure will by now be clear. Thus, at the $(g-1)^{\text{th}}=f^{\text{th}}$ elimination step we obtain a matrix **g** of the form

$$\mathbf{g}=\begin{bmatrix} \mathbf{I} & \vdots & \\ \cdots\cdots & \vdots & \mathbf{G} \\ \mathbf{O} & \vdots & \end{bmatrix} \quad \ldots\ldots\ldots\ldots (187c)$$

where **I** is the unit matrix with f columns and **O** is a zero matrix with f columns and $(n-f)$ rows. **G** is a rectangular $n\times(n-f)$ matrix. For the next step, i.e. to obtain the **h** matrix, we premultiply with $\mathbf{M}_g$ which has a g^{th} column

$$\{-g_{1g}/g_{gg}-g_{2g}/g_{gg}\ldots\ldots\ldots 1/g_{gg}\ldots\ldots\ldots -g_{ng}/g_{gg}\} \qquad (188b)$$

and otherwise unit diagonal elements and zero cross-elements. The resulting **h**-matrix is of the form

$$\mathbf{h}=\mathbf{M}_g\mathbf{g}=\begin{bmatrix} 1 & 0\ldots\ldots 0 & h_{1h}\ldots\ldots h_{1n} \\ 0 & 1\ldots\ldots 0 & h_{2h}\ldots\ldots h_{2n} \\ & \ddots & \\ - - - - & \ddots & \\ & \ddots & \\ 0 & 0\ldots\ldots\ldots\ldots & h_{hh}\ldots\ldots h_{hn} \\ - - - - & & \\ 0 & 0\ldots\ldots\ldots\ldots & h_{nh}\ldots\ldots h_{nn} \end{bmatrix} \quad \ldots\ldots\ldots\ldots (187d)$$

where

$$h_{gm}=g_{gm}/g_{gg} \text{ and } h_{km}=g_{km}-\frac{g_{kg}g_{gm}}{g_{gg}} \qquad \text{for } k\neq g \ \ldots\ldots\ldots (189b)$$

and so on until the last premultiplication with

$$\mathbf{M}_n=\begin{bmatrix} & \vdots & -n_{1n}/n_{nn} \\ \mathbf{I} & \vdots & \vdots \\ & \vdots & -n_{in}/n_{nn} \\ \cdots\cdots & \vdots & \vdots \\ \mathbf{O} & \vdots & 1/n_{nn} \end{bmatrix}$$

(where **I** has $(n-1)$ rows) gives

$$\mathbf{M}_n\mathbf{M}_{n-1}\ldots\ldots\mathbf{M}_i\ldots\ldots\mathbf{M}_1\mathbf{a}=\mathbf{I} \quad \ldots\ldots\ldots\ldots\ldots (190)$$

Eq. (190) then yields the inverse matrix

$$\mathbf{a}^{-1}=\mathbf{M}_n\mathbf{M}_{n-1}\ldots\ldots\mathbf{M}_i\ldots\ldots\mathbf{M}_1 \quad \ldots\ldots\ldots\ldots\ldots (191)^*$$

From which we find our unknown

$$\mathbf{X}=\mathbf{a}^{-1}\mathbf{Z} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (192)$$

In practice it is usually preferable not to determine $\mathbf{a}^{-1}$ explicitly but to perform first the multiplication $\mathbf{M}_1\mathbf{Z}$ and continue then premultiplying to find **X** directly.

It is apparent that if in the above procedure any h_{hh} becomes zero the elimination process cannot be continued. An interchange of rows is indicated but this is obviously inconvenient for automatic computation. Clearly, if

$$a_{ii}>>a_{ik}$$

no h_{hh} can vanish. Moreover, the condition

$$a_{ii}>a_{ik} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (193)$$

* As a matter of interest we point out that the actual operations on the digital computer to obtain $\mathbf{a}^{-1}$ do not follow exactly the typical matrix multiplication rules as implied by formula (192).

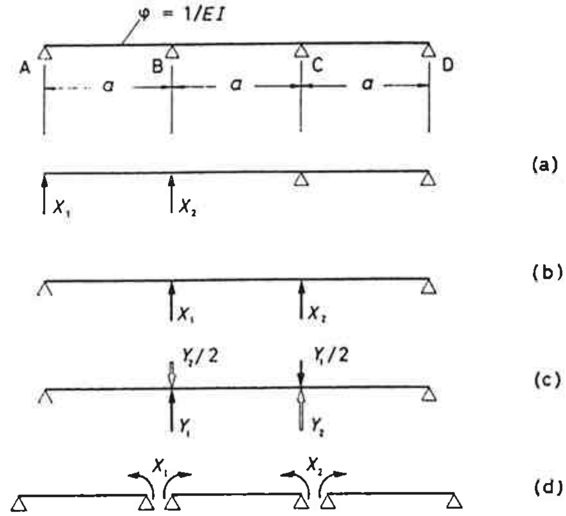


Fig. 29.—Continuous beam. Good and bad choices of basic system and redundancies

is necessary to avoid serious accumulation of round-off errors in the most important digits. For if $a_{ii}\leqslant a_{ik}$ the limited number of digits of the machine is not sufficient to ensure reliable computations. Thus, requirement (193) is seen to be essential for well conditioned equations.

Our Eqs. (186) are also ill-conditioned if two or more columns are nearly linearly dependent, e.g.

$$\{a_{1i}a_{2i}\ldots\ldots a_{ii}\ldots\ldots a_{ni}\}\approx C\{a_{1k}a_{2k}\ldots\ldots a_{kk}\ldots\ldots a_{nk}\} \ \ldots (194)$$

The one diagonal element g_{gg} will inevitably become very small and grave errors will again arise. Naturally, in actual structural problems two columns could never be exactly linearly dependent for otherwise this would indicate that we overestimated by one our number of redundancies. Nevertheless, a bad choice of redundant forces or moments may give an approximate linear dependence which would yield a result grossly in error.

If it is found that our initial choice of redundancies leads to an ill-conditioned set of equations then we can always obtain more suitable equations by introducing as unknowns appropriate linear combinations **Y** of the initial unknown **X**. Such a transformation may be represented always by

$$\mathbf{X}=\mathbf{B}\mathbf{Y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (195)$$

where **B** is a non-singular square matrix $n\times n$. If **X** are initially single forces or moments then **Y** represents groups of forces or generalized forces. Such groups of forces were first introduced by Mueller-Breslau* guided by pure physical reasoning and this is still the best method of finding them. The transformation (195) may be introduced directly into Eqs. (186a) yielding

$$\mathbf{DBY}+\mathbf{D}_o=\mathbf{O}$$

or

$$\bar{\mathbf{D}}\mathbf{Y}+\mathbf{D}=\mathbf{O} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (196)$$

where $\bar{\mathbf{D}}$ is determined by matrix multiplication from

$$\bar{\mathbf{D}}=\mathbf{DB} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (196a)$$

Physically, Eqs. (196) express the compatibility conditions at the cuts of the original unknowns **X** in terms of the new unknowns **Y**. If the transformation matrix **B** is unsymmetrical then the resulting matrix will also be unsymmetrical. Although, in general, the simple substitution of (195) into (186) can lead to a slight improvement of ill-conditioned equations the effect is usually small. The next obvious step is to express the compatibility condition (186) in terms of the generalized displacements at the cuts corresponding to the generalized forces **Y** of (195). Following our discussion on generalized displacements and flexibilities on p. 19 the generalized compatibility equations are derived by premultiplying Eqs. (196) by **B**′ as

$$\mathbf{B}'\mathbf{DBY}+\mathbf{B}'\mathbf{D}_o=\mathbf{O}$$

or

$$\mathbf{D}_y\mathbf{Y}+\mathbf{D}_{yo}=\mathbf{O} \quad \ldots\ldots\ldots\ldots\ldots\ldots (197)$$

where

$$\mathbf{D}_y=\mathbf{B}'\mathbf{DB} \text{ and } \mathbf{D}_{yo}=\mathbf{B}'\mathbf{D}_o \ \ldots\ldots\ldots\ldots\ldots\ldots (197a)$$

It is evident that the column matrix $\mathbf{D}_{yo}$ and the symmetrical matrix $\mathbf{D}_y$ would be obtained directly by selecting *ab initio* the generalized forces **Y** as redundancies and deriving the corresponding δ_{io} and δ_{ik} from the standard formulae given.

In many cases it is best, when the equations are ill conditioned, to select a different basic system and corresponding redundancies **Y**. Naturally, the latter are again statically related by a transformation matrix **B** with any previous choice **X** of redundancies.

* Mueller-Breslau, *loc. cit.* p. 17.

We illustrate the above considerations on ill-conditioned equations on a very simple structural example. FIG. (29) shows a uniform, continuous beam of three equal spans simply supported at A, B, C, D—a twice redundant system. We discuss four alternative choices for the two redundancies.

(a) X_1 and X_2 are taken as the reactions at the supports A and B. The **D** matrix for this system is

$$\mathbf{D}=\begin{bmatrix}\delta_{11} & \delta_{12}\\ \delta_{21} & \delta_{22}\end{bmatrix}=\frac{a^3\phi}{6}\begin{bmatrix}24 & 9\\ 9 & 4\end{bmatrix}$$

A remarkably bad choice since $\delta_{12}>>\delta_{22}$.

(b) X_1, X_2 taken as the reactions at the intermediate supports B and C. Then,

$$\mathbf{D}=\frac{a^3\phi}{18}\begin{bmatrix}8 & 7\\ 7 & 8\end{bmatrix}$$

Still a bad choice since all δ's are of the same order of magnitude.

(c) Y_1 and Y_2 are generalized redundancies formed from system (b) by the transformation matrix

$$\mathbf{B}=\begin{bmatrix}1 & -0{\cdot}5\\ -0{\cdot}5 & 1\end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(198)}$$

The $\mathbf{D}_y$ matrix may be obtained either directly or from Eq. (197*a*) and is

$$\mathbf{D}_y=\frac{a^3\phi}{24}\begin{bmatrix}4 & 1\\ 1 & 4\end{bmatrix}$$

The improvement over (b) and (a) is immediately apparent.

(d) X_1 and X_2 are the bending moments at supports B and C. This choice of unknowns is statically identical to Y_1 and Y_2 of (c), but the basic system is different. The **D** matrix is

$$\mathbf{D}=\frac{a\phi}{12}\begin{bmatrix}4 & 1\\ 1 & 4\end{bmatrix}$$

and clearly a scalar multiple of the **D** matrix of (c).

The final system (d) is recognized as a particular case of the well-known Three-Moment Equation of Clapeyron. Since the basic system approximates more closely to the actual system than that of (c) it is clearly the most suitable choice of all.

The differences between the above four systems become even more pronounced when the number of spans is increased. Moreover, the δ_{ik}-coefficients of choices (a) and (b) tend, for a large number of spans, to become linearly dependent.

This discussion shows how important the choice of the redundant forces is for the convenient numerical solution of a problem. An extreme case of simplicity is achieved if in all equations only one unknown appears. The particular set of redundancies

$$Y_1,\ Y_2,\ Y_3, \text{- - - -},\ Y_i, \text{- - - -}\ Y_n$$

for which this condition is satisfied is called orthogonal. Then all but the corresponding direct influence coefficients are zero, i.e.

$$\zeta_{ik}=0 \text{ if } i\neq k \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(199)}$$

where we introduce the symbol ζ for δ to emphasize the special nature of this system. Eqs. (182) take now the simple form,

$$\zeta_{ii}Y_i+\zeta_{io}=0 \text{ for } i=1 \text{ to } n \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(200)}$$

and hence

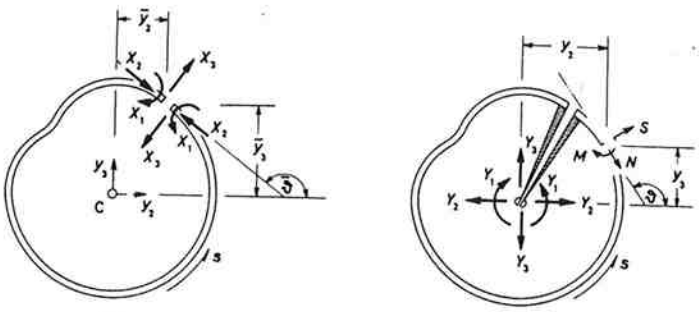
$$Y_i=-\frac{\zeta_{io}}{\zeta_{ii}} \text{ for } i=1 \text{ to } n \ldots\ldots\ldots\ldots\ldots\ldots \text{(200a)}$$

This system Y_i may always be obtained by a particular linear transformation

$$\mathbf{X}=\mathbf{B}_y\mathbf{Y} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(201)}$$

However, the computations involved in determining $\mathbf{B}_y$ are usually more laborious than the direct solution of Eqs. (182) if these are well conditioned. Nevertheless there are structures in which it is simple and hence advantageous to find the orthogonal set of redundancies. This is particularly so if physical and not mathematical considerations indicate how to find them.

For example, this is so in arches and singly connected rings where, if



(a) Non-orthogonal redundancies

(b) Orthogonal redundancies

System $Y_1=1$: $M_1=1$, $N_1=0$, $S_1=0$

System $Y_2=1$: $M_2=y_3$, $N_2=\cos\vartheta$, $S_2=-\sin\vartheta$

System $Y_3=1$: $M_3=y_2$, $N_3=-\sin\vartheta$, $S_3=-\cos\vartheta$

$$\left.\begin{aligned}\zeta_{12}=\zeta_{21}=\int\frac{y_3}{EI}\,ds\\ \zeta_{13}=\zeta_{31}=\int\frac{y_2}{EI}\,ds\end{aligned}\right\}=0 \text{ for elastic centre C.}$$

$$\zeta_{23}=\zeta_{32}=\int\frac{y_2y_3}{EI}\,ds+\int\sin\vartheta\cos\vartheta\left(\frac{1}{GA'}-\frac{1}{EA}\right)ds$$

$=0$ for "principal axes" Cy_2, Cy_3.

Fig. 30.—Orthogonal and non-orthogonal redundancies—elastic centre of singly connected ring

we restrict ourselves to bending deformations and assume the E.T.B. to hold, an orthogonal set of redundancies is obtained by referring them to the principal axes (Cy_2 Cy_3) of the ring neutral axis weighted with the ring flexibility $\phi=1/EI$ (see FIG. 30). The origin C of this system which is, of course, the centroid of the elastically weighted ring is known as the elastic centre. The transformation matrix **B** relating the orthogonal set of redundancies **Y** to a set **X** consisting of bending moment X_1, normal force X_2 and shear force X_3 at some point is in the notation of FIG. 30,

$$\mathbf{B}=\begin{bmatrix}1 & \bar{y}_3 & \bar{y}_2\\ 0 & \cos\bar{\theta} & -\sin\bar{\theta}\\ 0 & -\sin\bar{\theta} & -\cos\bar{\theta}\end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots \text{(202)}$$

In practice it is nearly always worthwhile to find the elastic centre and eliminate two of the cross-flexibilities but determination of the principal axes, unless possible by inspection, is not usually worth the trouble. A further point is that the elastic centre concept is still valid if deformations due to normal and shear forces are included whereas the principal axes requirement becomes more complicated.

Interestingly enough this solution was first given by Mohr* more than seventy years ago, but it appears not to be universally known, for otherwise it would not have been necessary to rediscover it so many times. A more recent derivation and application of an orthogonal set of redundancies (in general infinite) is the system of self-equilibrating eigenloads developed by Argyris and Dunne † for their general theory of tubes in bending and torsion.

Particular forms of the δ_{ik} *and* δ_{io} *coefficients*

We return now to Eqs. (177) and (178) for the δ-coefficients and give, following our expressions (180) and (181) for a pin-jointed framework, some further explicit formulae for more complex structures.

Stiff-jointed plane framework. We assume the Engineers' theory of bending stresses to be true and introduce the special notation:

N_o, S_o, M_o normal force, shear force, bending moment in basic system due to applied loads.

N_i, S_i, M_i normal force, shear force, bending moment in basic system due to $X_i=1$ where $i=1$ to n.

s coordinate along axis of beam.

Θ temperature at neutral line of cross-section.

$\Delta\Theta/h$ temperature gradient across depth h of beam; positive if giving rise to thermal bending strain of the same sign as that due to positive bending moment M.

* O. Mohr, *Z. Architek u. Ing. Ver. Hannover*, Vol. 27, 1881, p. 143. See also the generalization and tabular presentation of this method in: J. H. Argyris and P. C. Dunne, *Structural Analysis* (Vol. I of *Handbook of Aeronautics*), Pitman, 1952, Table 17.1. Both the deformations due to normal and shear forces are included in the latter analysis.

† See J. H. Argyris, P. C. Dunne, 'The General Theory', etc., *J.R.Ae.S.*, Vol. LI (1947), Feb., Sept., Nov. and Vol. LIII (1949), May, June.

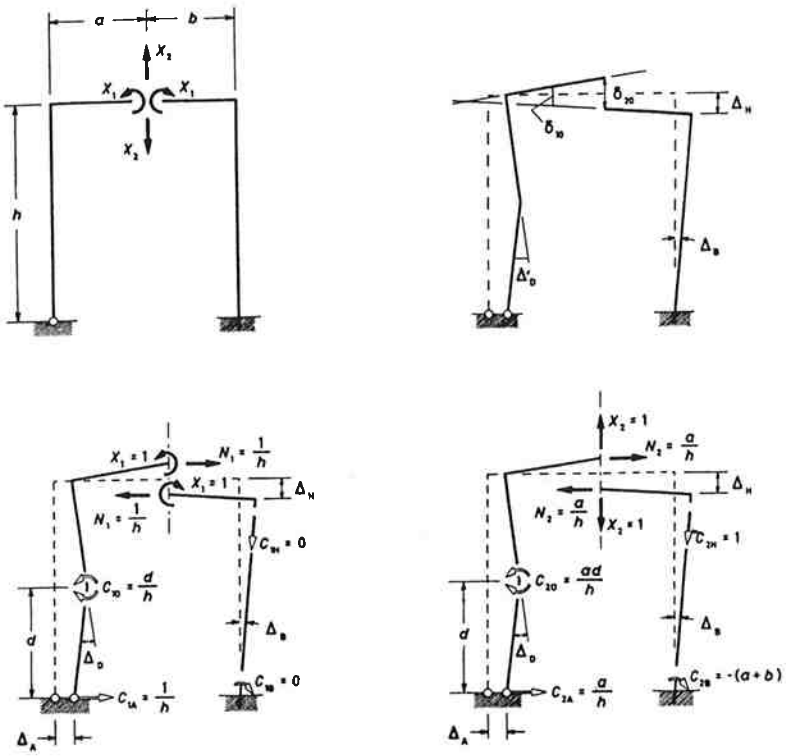


Fig. 31.—δ_{i0} due to initial strains in rigid jointed frame and manufacturing errors and 'give' at supports

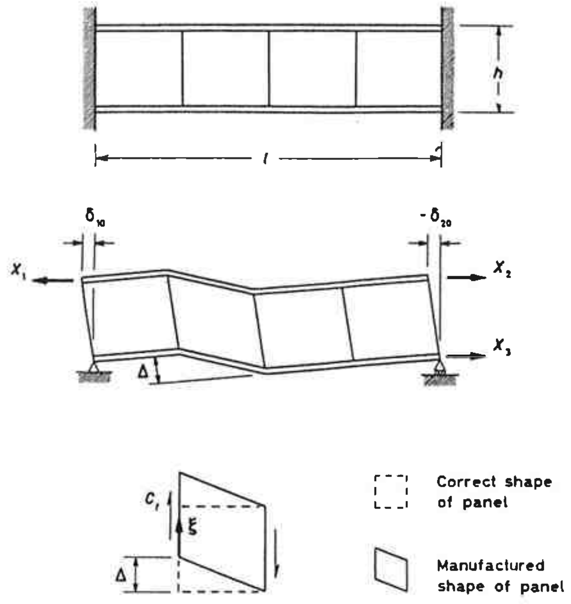


Fig. 32.—δ_{i0} due to initial shear strain arising from incorrect manufacture

Δ prescribed relative displacements (linear or angular) either inside the structure (e.g. lack of fit) or at the supports ('give' of foundations).

C_i force or moment in the basic system due to $X_i=1$ acting on an element which experiences a Δ in the direction of this Δ.

EA, GA', EI direct, shear, bending stiffness of beam.

We deduce immediately from Eq. (178), see also Eq. (94),

$$\delta_{ik}=\int\frac{N_iN_k}{EA}ds+\int\frac{S_iS_k}{GA'}ds+\int\frac{M_iM_k}{EI}ds \qquad (203)$$

Also from Eq. (177) or directly from the unit load method,

$$\delta_{io}=\int\frac{N_iN_o}{EA}+\int N_i a\Theta ds+\int M_i\frac{a\Delta\Theta}{h}+\Sigma C_i\Delta \qquad (204)$$

For pin-jointed frameworks we omit the terms involving M and S. On the other hand in stiff-jointed frameworks we can, in general, omit the terms involving S and often also the terms in N. FIG. (31) illustrates on a twice redundant system how the contribution of the prescribed displacements Δ to δ_{io} is calculated. X_1 and X_2 are the chosen redundancies and Δ_a, Δ_b, Δ_c and Δ_d are linear or angular imposed displacements arising from errors in the manufacture of the frame.

Two-dimensional stress distribution

We restrict ourselves here to a presentation in cartesian co-ordinates x, y but the formulae are not restricted to stress distributions in flat plates. They are applicable to stress-states in any curved membrane which, by definition, does not allow for any variation of the stress over the thickness. Hence, x, y are, in general, orthogonal curvilinear co-ordinates; for example, in a cylindrical membrane y may be measured along the generator and x along the periphery of the cross-section.

The following special notation applies.

Δ_n, Δ_t prescribed direct (n) and tangential (t) relative displacements either inside the membrane (lack of fit of parts, slip of rivets) or at the boundaries ('give' at supports).

ξ distance along part which experiences Δ_n and/or Δ_t

C_{ni}, C_{ti} direct and tangential forces per unit length ξ in the basic system due to $X_i=1$ acting on the element which experiences a Δ in the direction of this Δ.

t thickness.

We deduce from Eq. (178),

$$\delta_{ik}=\frac{1}{E}\int\int\Big[\sigma_{xxi}\sigma_{xxk}+\sigma_{yyi}\sigma_{yyk}-\nu(\sigma_{xxi}\sigma_{yyk}+\sigma_{yyi}\sigma_{xxk})+2(1+\nu)\sigma_{xyi}\sigma_{xyk}\Big]\,tdxdy \qquad (205)$$

Also from Eq. (177),

$$\delta_{io}=\frac{1}{E}\int\int\Big[\sigma_{xxi}\sigma_{xxo}+\sigma_{yyi}\sigma_{yyo}-\nu(\sigma_{xxi}\sigma_{yyo}+\sigma_{yyi}\sigma_{xxo})+2(1+\nu)\sigma_{xyi}\sigma_{xyo}\Big]\,tdxdy$$
$$+\int\int\Big[\sigma_{xxi}+\sigma_{yyi}\Big]a\Theta tdxdy+\int\Big[C_{ni}\Delta_n+C_{ti}\Delta_t\Big]d\xi \qquad (206)$$

The immediate application of the above formulae is to major aircraft components like wings and fuselages. Their matrix formulation is discussed further below.

FIG. (32) illustrates how on a thrice redundant beam with shear carrying web the contribution of a prescribed displacement Δ to δ_{io} is calculated. Δ is in this case an initial shearing displacement of a panel due to error of manufacture.

It was assumed in all our above considerations that the basic or cut system is statically determinate. However, nothing in the theory so far given restricts us to such a choice. We can select in a structure with a total number of redundancies n a statically indeterminate basic system with $(n-r)(r<n)$ redundancies by 'cutting' only r redundant members. Equations of the type (182) can then be written down for the cut r unknowns, the corresponding δ_{ik}-coefficients being still defined as in our previous analysis in the basic system. In fact, to calculate the δ_{ik} we may apply again Eqs. (178). Similarly, for δ_{io} we may use Eq. (177) if we substitute ϵ_η for η where ϵ_η is the true strain in the basic system due to the prescribed initial strains. This modification is necessary since the basic system is now redundant and the imposed initial strains η are not free to develop. However, both formulae for δ_{ik} and δ_{io} may be simplified considerably if we remember that in the unit load method from which they derive (see Eq. (71a)) only the strains must be the true ones for the system considered—in the present case the redundant basic system. The stresses corresponding to the unit load may be any suitable statically equivalent stress system and may hence

be found in the simplest statically determinate system. Thus, if we introduce the notation

$\bar{\sigma}_i$ = statically equivalent stress system in redundant basic system due to $X_i=1$

we may write,

$$\delta_{io}=\int_V \bar{\sigma}_i(\epsilon_o+\epsilon_\eta)dV \quad \ldots\ldots\ldots\ldots\ldots\ldots (177a)$$

$$\delta_{ii}=\int_V \bar{\sigma}_i\epsilon_i dV, \qquad \delta_{ik}=\int_V \bar{\sigma}_i\epsilon_k dV=\int_V \bar{\sigma}_k\epsilon_i dV=\delta_{ki} \quad \ldots\ldots\ldots\ldots\ldots (178a)$$

The introduction of $\bar{\sigma}_i$ instead of σ_i in Eqs. (177) and (178) may shorten the analysis greatly. Naturally, Eqs. (178a) are again identical with formulae (93).

The above method presumes that the strains ϵ and ϵ_η in the redundant basic system are known. Such information may be available either from previous calculations or the literature. Alternatively, we may have to analyse first the basic system for the external loads (and/or imposed strains) and the r $X_i=1$ by the method given previously. From a mathematical point of view the selection of a redundant basic system means that we solve the problem of n equations with n unknowns in two steps involving respectively the solution of r equations with r unknowns and $(n-r)$ equations with $(n-r)$ unknowns. This method is particularly useful if we have available information on the stress distribution of the redundant basic system or if the number n is very high.

Consider, for example, FIG. (33) showing a fuselage ring with transverse beam under uniform load p. The loading is equilibrated by tangential shear flows q applied by the fuselage to the ring. The structure is six times redundant and as redundancies we select the two groups X_1, X_2, X_3 and X_4, X_5, X_6 at the intersection of the axis of symmetry with the upper part of the ring and the transverse beam. Due to symmetry of loading and structure

$$X_3=X_6=0$$

and hence the problem reduces to finding the remaining four redundancies. We may solve the system by direct application of Eqs. (182), which in the present case take the form,

$$\left.\begin{aligned}\delta_{11}X_1+\delta_{12}X_2+\delta_{14}X_4+\delta_{15}X_5+\delta_{10}=0\\ \delta_{21}X_1+\delta_{22}X_2+\delta_{24}X_4+\delta_{25}X_5+\delta_{20}=0\\ \delta_{41}X_1+\delta_{42}X_2+\delta_{44}X_4+\delta_{45}X_5+\delta_{40}=0\\ \delta_{51}X_1+\delta_{52}X_2+\delta_{54}X_4-\delta_{55}X_5+\delta_{50}=0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (207)$$

where the δ_{ik} and δ_{io} are calculated with formulae (203) and (204) in which the integrals extend over ring and transverse beam. Note that if the unknowns X_1, X_2, X_3 are referred to the elastic centre of the ring the coefficient δ_{12} vanishes. Having solved Eqs. (207) we find N, S, M in the actual structure from,

$$\left.\begin{aligned}N=N_0+N_1X_1+N_2X_2+N_4X_4+N_5X_5\\ S=S_0+S_1X_1+S_2X_2+S_4X_4+S_5X_5\\ M=M_0+M_1X_1+M_2X_2+M_4X_4+M_5X_5\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (208)$$

where M_0, M_i are defined on page 30. Alternatively to this standard method we may solve the problem by cutting only the beam at the axis of symmetry. Then the structure is only twice redundant (X_4 and X_5) with a basic system that is itself twice redundant. We denote by

$$m_o,\ n_o,\ s_o$$
$$m_i,\ n_i,\ s_i\ (i=4 \text{ or } 5)$$

the normal force, shear force and bending moment in the new basic system due to p and $X_i=1$ respectively and assume that they are known. The stress distribution in the actual structure is then found from

$$\left.\begin{aligned}M=m_0+m_4X_4+m_5X_5\\ N=n_0+n_4X_4+n_5X_5\\ S=s_0+s_4X_4+s_5X_5\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (209)$$

The equation of compatibility in the unknowns X_4 and X_5 are now of the form

$$\left.\begin{aligned}\zeta_{44}X_4+\zeta_{45}X_5+\zeta_{40}=0\\ \zeta_{54}X_4+\zeta_{55}X_5+\zeta_{50}=0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (210)$$

where we write ζ instead of δ to stress that these coefficients are different from the correspondtng δ's in Eqs. (207). To find the ζ's we apply Eqs. (177a) and (178a) in the new basic system and remember that the virtual stresses due to $X_i=1$ may be selected in a statically determinate system. Thus, omitting the contributions of the normal and shear forces for convenience of printing, we find

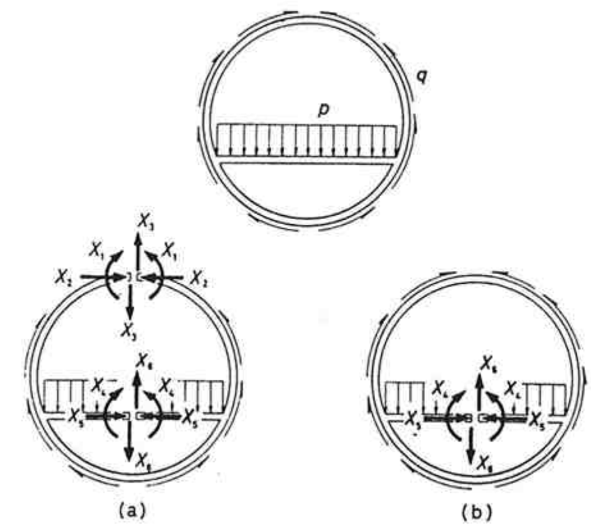
$$\zeta_{ik}=\int\frac{m_im_k}{EI}ds=\int\frac{M_im_k}{EI}ds=\int\frac{m_iM_k}{EI}ds=\zeta_{ki} \quad \ldots\ldots\ldots (211)$$

$$\zeta_{io}=\int\frac{m_im_o}{EI}ds=\int\frac{M_im_o}{EI}ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (212)$$

If, in addition, any initial strains η are imposed on our structure giving rise to moments m_η in the basic system the corresponding contribution to ζ_{io} becomes

$$\zeta_{io}=\int\frac{m_im_\eta}{EI}ds=\int\frac{M_im_\eta}{EI}ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (212a)$$

The effects of the normal and shear forces may be included without difficulty, the terms following immediately from Eqs. (203) and (204) and the arguments leading to the contribution of the moments.



(a) Statically determinate basic system

(b) Redundant basic system

Fig. 33.—Statically indeterminate basic system. Doubly connected ring

It is useful to write down the δ_{ik} and δ_{io} for a two-dimensional stress distribution when the basic system is redundant. Thus, with the definition of $\bar{\sigma}_i$ given previously we find,

$$\delta_{ik}=\frac{1}{E}\iint\Big[\sigma_{xxi}\sigma_{xxk}+\bar{\sigma}_{yyi}\sigma_{yyk}-\nu(\bar{\sigma}_{xxi}\sigma_{yyk}+\bar{\sigma}_{yyi}\sigma_{xxk})+2(1+\nu)\bar{\sigma}_{xyi}\sigma_{xyk}\Big]t\,dx\,dy \quad \ldots\ldots\ldots (205a)$$

and a similar formula with σ and $\bar{\sigma}$ interchanged. Also,

$$\delta_{io}=\frac{1}{E}\iint\Big[\bar{\sigma}_{xxi}\sigma_{xxo}+\sigma_{yyi}\sigma_{yyo}-\nu(\bar{\sigma}_{xxi}\sigma_{yyo}+\bar{\sigma}_{yyi}\sigma_{xxo})+2(1+\nu)\bar{\sigma}_{xyi}\sigma_{xyo}\Big]t\,dx\,dy \quad \ldots\ldots\ldots (206a)$$

where σ_{xxo} etc., may be here not only the stresses due to external loads but also due to the imposed initial strains η. (Remember that the latter cannot develop freely in the statically indeterminate basic system and give rise to some strain-stress state ϵ_η, σ_η.)

The use of a redundant basic system arises continuously in wing theory. Thus, following Ebner and Köller* and Argyris and Dunne† it is customary in wing analysis to express the actual stresses in the form,

$$\sigma=\sigma_o+\sigma_x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (213)$$

where the stress system σ_o—the choice of which is at our discretion—satisfies both the external and internal equilibrium condition and is therefore the statically equivalent stress system, σ_x are the self-equilibrating stress systems (in general, infinite in number), necessary to ensure external and internal compatibility. In our present terminology σ_o is the basic stress system and σ_x the redundant stress systems which for practical purposes are approximated to a finite number. In fact,

$$\sigma_x=\sigma_1X_1+\sigma_2X_2+\ldots\ldots+\sigma_nX_n=\sum_{i=1}^{n}\sigma_iX_i \quad \ldots\ldots\ldots (214)$$

It is advantageous in the selection of the basic system σ_o to try and satisfy the two, at times conflicting, requirements of simplicity and not too great difference from the exact system σ. For it is obvious that small σ_x-systems are highly desirable from both the theoretical and practical point of view. Now for wings with not too small an aspect ratio an excellent choice for the direct stresses of σ_o is given by the Engineers' theory of bending for beams since it combines simplicity with reasonable accuracy. If the wing forms a single cell tube we deduce the shear stresses from the boom load gradients, the undetermined constant of integration being found from the overall torque equilibrium; thus, in this case the basic system σ_o is statically determinate. If on the other hand the wing is an N-cell tube we see that whereas there is still only one torque equilibrium condition there are N undetermined constants of integration. To calculate them we must introduce conditions of deformation and those are the equality of rate of twist of all cells. Hence, our basic system is redundant, the degree of redundancy being $N-1$. The solution to this problem is reproduced in Example (a) of Section 9. General considerations on the calculation of the redundant self-equilibrating stress systems σ_x are given later in this Section in matrix form (see also Example (b) of Section 9).

The example of the tube is useful also to illustrate another point. We stated on p. 27 that the basic structure is obtained by cutting redundant members but mentioned that in continuous structures the idea of a physical

* *Loc. cit.* p. 1.
† *Loc. cit.* p. 30.

cut is not always applicable. Thus, in the case of the tube in the last paragraph, when obtaining the basic stresses we do not actually cut any redundant member but rather select the engineers' theory of bending direct stresses and the associated shear flows as statically equivalent to the applied bending moment. In general, all members are found to be load carrying. The stresses σ_o only exceptionally satisfy the elastic compatibility conditions—for example, due to warping varying parallel to the axis of the tube and to rib deformability. We may give some physical reality to the basic structure in which σ_o is true by releasing in the actual structure the warping restraints at every cross-section and by assuming the ribs as rigid; the former idealization does no doubt require a complicated mechanism for its realization. The idea of selecting σ_o as any suitable statically equivalent stress system without reference to actual cuts may, of course, also be applied to frameworks.

The use of a redundant basic structure is important also from a further point of view. Consider the wing of FIG. (34) the main portion I of which is swept and attached to some root structure II. It is assumed that the ribs of I are taken perpendicular to a longitudinal axis approximately parallel to the spars. The necessity may arise of investigating alternative angles of sweep obtained merely by changing the root structure II. Thus, in FIG. (34) we show two alternative arrangements. In such instances it is obviously advantageous to have as much as possible of the stress analysis in common in the two alternative calculations. To this purpose we release at the junction of I and II all redundant forces or groups of forces X_1 to X_n appearing there. The tube I is then connected to tube II by some statically determinate arrangement and this new structure is taken as the basic system. The scheme of the analysis is as follows. Analyse first Tube I for all external forces and also for $X_1=1$ to $X_n=1$ respectively. This investigation involves, of course, the solution of a highly redundant system. Irrespective, however, of the form tube II takes the analysis of tube I remains unaffected. Next we analyse tube II for the external forces (which include the statically determinate reactions P from tube I), and also for $X_1=1$ to $X_n=1$ respectively. Again this may involve the solution of a redundant problem. Finally we can write down equations of the type (182) for the unknowns $X_1=1$ to $X_n=1$ and note that the δ-coefficients are in each case

$$\delta=\delta_I+\delta_{II} \qquad \text{(215)}$$

Hence if we change structure II only δ_{II} but not δ_I is altered—an obvious advantage. The solution of Eqs. (182) yields ultimately the stress distribution in the actual wing.

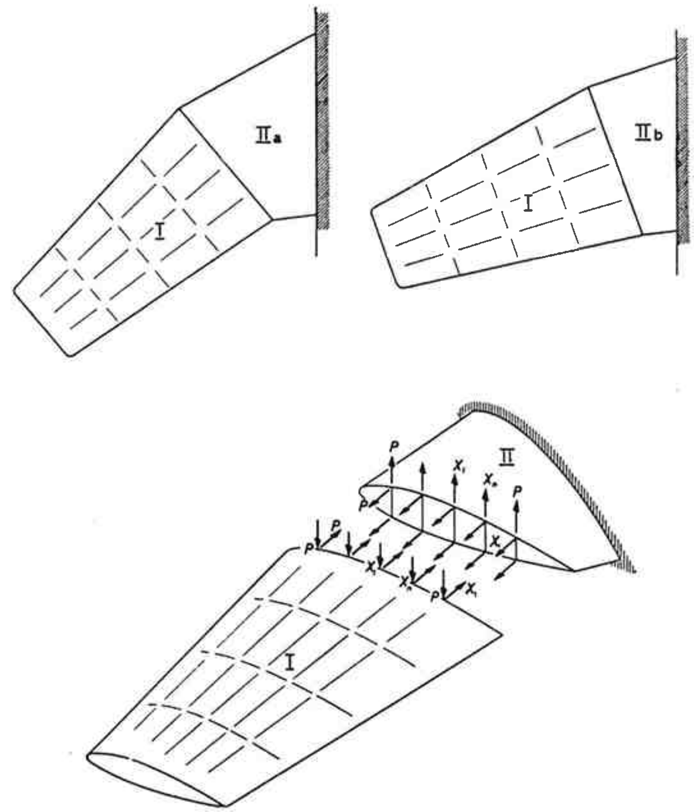


Fig. 34.—Alternative root structures for swept-back wing. Redundant basic systems

HAVING discussed in the standard longhand notation the main ideas and methods for the calculation of redundant structures on the basis of forces as unknowns we now turn our attention to the matrix formulation of the analysis. Consider a system consisting of s structural elements with a total number n of redundancies which may be forces (stresses), moments or any generalized forces. We select a basic system by 'cutting' a number r of redundancies where $r<n$. Thus, the simple idea of a statically determinate basic system ($r=n$) is but a particular case of our investigations.

The structure is assumed subjected to a system of m loads (generalized forces)

$$\mathbf{R}=\{R_1R_2 \ldots\ldots R_j \ldots\ldots R_m\} \quad \ldots\ldots\ldots\ldots \quad (90)$$

We denote by $\mathbf{X}$ the column matrix of the r cut (unknown) redundancies,

$$\mathbf{X}=\{X_1X_2 \ldots\ldots X_i \ldots\ldots X_r\} \quad \ldots\ldots\ldots\ldots \quad (216)$$

The column matrix $\mathbf{S}$ of the stresses and forces in the actual (uncut) structure can always be written in the simple form

$$\mathbf{S}=\mathbf{b}_0\mathbf{R}+\mathbf{b}_1\mathbf{X} \quad \ldots\ldots\ldots\ldots \quad (217)$$

where $\mathbf{b}_0$ and $\mathbf{b}_1$ are rectangular matrices with m and r columns respectively and the same number of rows as $\mathbf{S}$. In fact, the elements of $\mathbf{b}_1$ are or correspond to the stresses σ_i given previously (see Eq. (176)). If the basic system is statically determinate the two matrices $\mathbf{b}_0$ and $\mathbf{b}_1$ are found merely by statical reasoning.

For a redundant basic system we may obtain the necessary data either by analysing it first for the loads $\mathbf{R}$ and the r forces $X_i=1$ or in many cases by using existing standard information.

Additional Notation

$\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$	column matrices of X_i, Y_i, Z_i respectively.
$\mathbf{H}$	column matrix of initial strains (displacements).
$\mathbf{c}$	rectangular matrix of forces (moments) C_i (see p. 31).
B, C	areas of actual longitudinal and transverse flanges.
B_e, C_e	areas of effective longitudinal and transverse flanges.
l	length of longitudinal flanges between nodal points.
d	length of transverse flanges between nodal points.
h	height of web.
Ω	area enclosed by cell.
Φ, Φ_w	areas of cover panel and web panel respectively.
$\mathbf{L}_{jh}$, $\mathbf{l}_{jh}$	2×2 matrices.
$\mathbf{L}=[\mathbf{L}_{jh}]$, $\mathbf{l}=[\mathbf{l}_{jh}]$	
P, Q	effective longitudinal and transverse flange load respectively.
$\mathbf{k}_s$, $\mathbf{k}_d$, $\mathbf{k}_f$	Partial stiffnesses due to shear strains in sheet, direct strains in sheet and direct strains in flanges respectively.
$\mathbf{U}$, $\mathbf{W}$	column matrix of kinematically indeterminate joint displacements U, W respectively.
$\mathbf{a}_0$	column matrix of strain of elements due to unit r's and $\mathbf{U}=\mathbf{O}$.
$\mathbf{a}_1$, $\bar{\mathbf{a}}_1$	column matrix of actual and kinematically equivalent (virtual) strain of elements due to unit U's when $\mathbf{r}=\mathbf{O}$ respectively.
$\mathbf{C}=\bar{\mathbf{a}}_1'\mathbf{k}\mathbf{a}_1=\mathbf{a}_1'\mathbf{k}\mathbf{a}_1$	
$\mathbf{C}_0=\bar{\mathbf{a}}_1'\mathbf{k}\mathbf{a}_0\mathbf{r}=\mathbf{a}_1'\mathbf{k}\mathbf{a}_0\mathbf{r}$	
$\mathbf{J}$	column matrix of initial stresses.

Example for the $\mathbf{b}_0$ *and* $\mathbf{b}_1$ *matrices.*

FIG. (35a) shows a five times redundant structure, assumed symmetrical, subjected to the loads R_1 and R_2. Due to symmetry of loading and structure the system is effectively three times redundant. For the basic system we select the statically determinate structure of FIG. (35b). The $\mathbf{b}_0$ and $\mathbf{b}_1$ matrices for half the structure including the central vertical member (11) are found easily as

$$\mathbf{b}_0'=\begin{bmatrix} 0 & -a/h & a/h & a/h & 0 & 0 & -d/h & 0 & 0 & 1 & 0 \\ 0 & -a/2h & a/2h & a/h & 0 & 0 & -d/2h & -d/2h & 0 & 1/2 & 1 \end{bmatrix}$$
$$\qquad \textit{1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11}$$

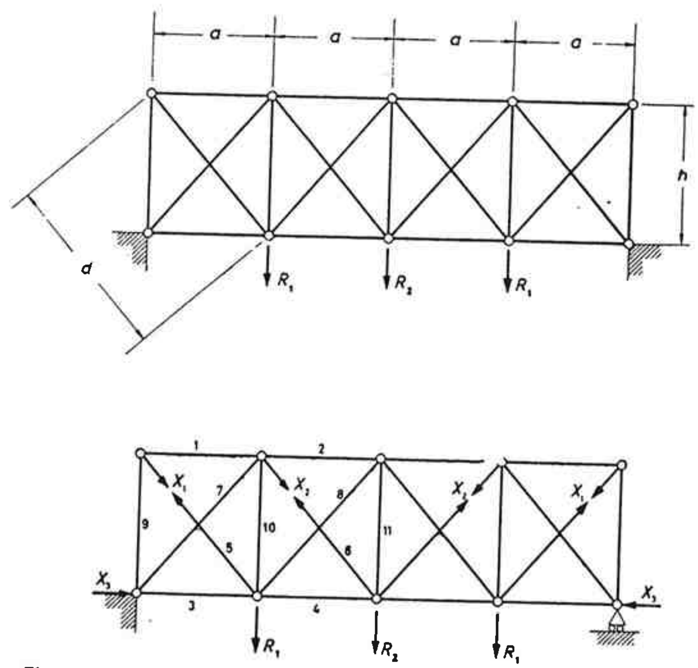
(218)

and

$$\mathbf{b}_1'=\begin{bmatrix} -a/d & 0 & -a/d & 0 & 1 & 0 & 1 & 0 & -h/d & -h/d & 0 \\ 0 & -a/d & 0 & -a/d & 0 & 1 & 0 & 1 & 0 & -h/d & -2h/d \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\qquad \textit{1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11}$$

(219)

where the numbers under the columns refer to the numbered bars of FIG. 35b.



Figs. 35 (a and b).—Statically indeterminate pin-jointed framework. Illustration of $\mathbf{b}_0$ and $\mathbf{b}_1$ matrices

Following our previous analysis of structures when the basic system is redundant we introduce now the matrix

$$\bar{\mathbf{b}}_1 \quad \dots\dots\dots\dots \quad (220)$$

to denote any suitable statically equivalent stress (force) matrix corresponding to

$$\mathbf{X}=\{1 \quad 1\ldots\ldots\ldots 1\ldots\ldots\ldots 1\}$$

Thus, $\bar{\mathbf{b}}_1$ corresponds to the stress system $\bar{\sigma}_i$ introduced before. When the basic system is statically determinate only one $\bar{\mathbf{b}}_1$ can be found,

$$\bar{\mathbf{b}}_1=\mathbf{b}_1 \quad \dots\dots\dots\dots \quad (220a)$$

We define also by

$$\bar{\mathbf{b}}_0 \quad \dots\dots\dots\dots \quad (221)$$

any suitable matrix statically equivalent to

$$\mathbf{R}=\{1 \quad 1\ldots\ldots\ldots 1\ldots\ldots\ldots 1\}$$

$\bar{\mathbf{b}}_0$ may, in fact, be determined in a different statically determinate system from $\bar{\mathbf{b}}_1$.

Next we derive the matrix equation for compatibility of deformation in the actual structure. Denoting the relative displacements at the r cuts of the basic system due to loads $\mathbf{R}$ and the r redundancies X_i by $\mathbf{v}_r$ the compatibility condition is

$$\mathbf{v}_r=\mathbf{O} \quad \dots\dots\dots\dots \quad (222)$$

where $\mathbf{v}_r$ is a column matrix with r elements. To express Eq (222) in terms of $\mathbf{R}$ and $\mathbf{X}$ we note from Eq. (122) that the relative deformations $\mathbf{v}$ (these may be elongations of bars or flanges, shearing angles of plates), at the ends or boundaries of the s elements are,

$$\mathbf{v}=\mathbf{fS}=\mathbf{fb}_0\mathbf{R}+\mathbf{fb}_1\mathbf{X} \quad \dots\dots\dots\dots \quad (223)$$

$\mathbf{f}$, the flexibility matrix of the s elements, is the partitioned diagonal matrix of Eq. (123). We find now $\mathbf{v}_r$ directly from the argument leading to Eq. (125) as

$$\mathbf{v}_r=\bar{\mathbf{b}}_1'\mathbf{v}=\mathbf{O}$$

and hence

$$\bar{\mathbf{b}}_1'\mathbf{fb}_1\mathbf{X}+\bar{\mathbf{b}}_1'\mathbf{fb}_0\mathbf{R}=\mathbf{O} \quad \dots\dots\dots\dots \quad (224)$$

These are the required equations in the r unknown X_i, and are, in fact, equivalent to formulae (182). The symmetrical square matrix

$$\mathbf{D}=\bar{\mathbf{b}}_1'\mathbf{fb}_1$$

(to use the notation of Eq. (182)) is the flexibility matrix for the directions of the r unknown X_i in the basic system. Also in the notation of Eq. (182a)

$$\mathbf{D}_0=\bar{\mathbf{b}}_1'\mathbf{fb}_0\mathbf{R} \quad \dots\dots\dots\dots \quad (225a)$$

Eqs. (224) are the most general formulation in matrix algebra of the equations for the r unknown X_i in a structure with a redundant basic system. Solving for $\mathbf{X}$ we find

$$\mathbf{X}=-(\bar{\mathbf{b}}_1'\mathbf{fb}_1)^{-1}\bar{\mathbf{b}}_1'\mathbf{fb}_0\mathbf{R} \quad \dots\dots\dots\dots \quad (226)$$

Substituting (226) in (217) we determine $\mathbf{S}$ solely as a function of the R's. Thus,

$$\mathbf{S}=[\mathbf{b}_0-\mathbf{b}_1(\bar{\mathbf{b}}_1'\mathbf{fb}_1)^{-1}\bar{\mathbf{b}}_1'\mathbf{fb}_0]\mathbf{R} \quad \dots\dots\dots\dots \quad (227)$$

Comparing (227) with Eq. (121) we can write

$$\mathbf{S}=\mathbf{bR}$$

where

$$\mathbf{b}=\mathbf{b}_0-\mathbf{b}_1(\bar{\mathbf{b}}_1'\mathbf{fb}_1)^{-1}\bar{\mathbf{b}}_1'\mathbf{fb}_0 \quad \dots\dots\dots\dots \quad (227a)$$

Naturally, it is always possible to substitute $\mathbf{b}_1'$ for $\bar{\mathbf{b}}_1'$ in Eqs. (224) to (227a). However, the introduction of the statically determinate matrix $\bar{\mathbf{b}}_1$ when the basic system is redundant simplifies the calculations, often considerably.

We can apply now Eq. (227a) to derive the flexibility

$$\mathbf{F}$$

of the actual structure for the m points and directions of the applied loads. Eq. (126) gives

$$\mathbf{F}=\bar{\mathbf{b}}'\mathbf{fb} \quad \dots\dots\dots\dots \quad (126)$$

For $\bar{\mathbf{b}}$ we may use

$$\bar{\mathbf{b}}=\mathbf{b}_0 \text{ or even simpler } \bar{\mathbf{b}}=\bar{\mathbf{b}}_0$$

We obtain

$$\mathbf{F}=\bar{\mathbf{b}}_0'\mathbf{f}[\mathbf{b}_0-\mathbf{b}_1(\bar{\mathbf{b}}_1'\mathbf{fb}_1)^{-1}\bar{\mathbf{b}}_1'\mathbf{fb}_0]$$

or

$$\mathbf{F}=\mathbf{F}_0-\bar{\mathbf{b}}_0'\mathbf{fb}_1(\bar{\mathbf{b}}_1'\mathbf{fb}_1)^{-1}\bar{\mathbf{b}}_1'\mathbf{fb}_0 \quad \dots\dots\dots\dots \quad (229)$$

where

$$\mathbf{F}_0=\bar{\mathbf{b}}_0'\mathbf{fb}_0=\mathbf{b}_0'\mathbf{fb}_0 \quad \dots\dots\dots\dots \quad (230)$$

is the flexibility of the basic system for the loads $\mathbf{R}$.

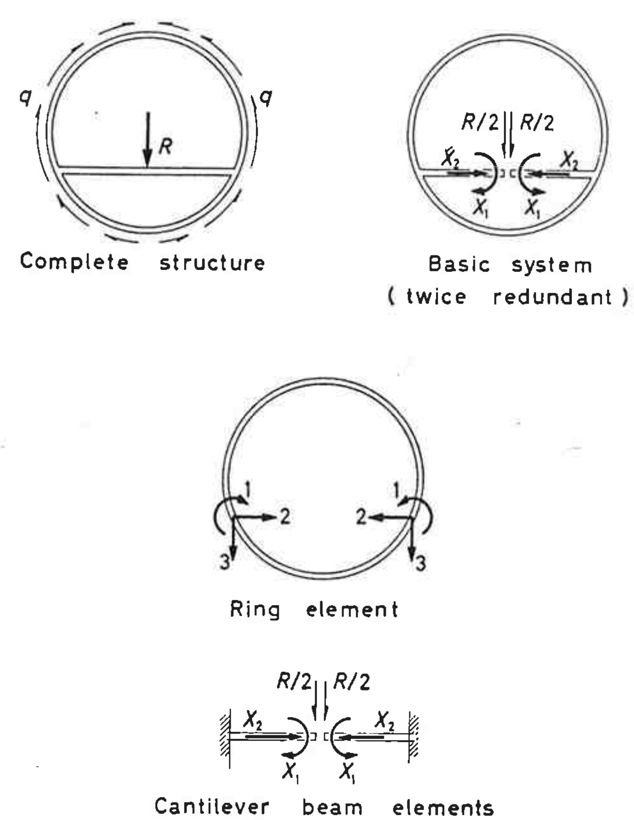


Fig. 36.—Doubly connected ring. Analysis with redundant basic system

Simple example of Eq. (224)

Consider the symmetrical fuselage ring with transverse beam and central load R shown in FIG. 36. As in page 32 we select as a basic system the structure with the beam cut at the centre. For the components s of the basic system we take the two statically determinate cantilever beams and the closed ring. It is assumed that we know the stress distribution and hence the flexibilities due to the pairs of loads applied to the ring (FIG. 36). The basic system is thrice redundant but due to symmetry $X_3=0$.

The load transformation matrices $\mathbf{b}_0$ and $\mathbf{b}_1$ are

$$\mathbf{b}_0=\begin{bmatrix}\mathbf{b}_{0B}\\ \mathbf{b}_{0R}\end{bmatrix} \quad \dots\dots\dots\dots \quad (231)$$

where

$$\mathbf{b}_{0B}=\begin{bmatrix}0\\ 1/2\\ 0\end{bmatrix}, \qquad \mathbf{b}_{0R}=\begin{bmatrix}1/2\\ 0\\ 1/2\end{bmatrix} \quad \dots\dots\dots\dots \quad (231a)$$

and

$$\mathbf{b}_1=\begin{bmatrix}\mathbf{b}_{1B}\\ \mathbf{b}_{1R}\end{bmatrix} \quad \dots\dots\dots\dots \quad (232)$$

where

$$\mathbf{b}_{1B}=\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix} \qquad \mathbf{b}_{1R}=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix} \quad \dots\dots\dots\dots \quad (232a)$$

Note that in the present case $\bar{\mathbf{b}}_1=\mathbf{b}_1$ since the loadings on the two elements of the structure are statically determinate.

The flexibilities of the elements for the forces and moments may be written as

$$\mathbf{f}_B=\begin{bmatrix} \frac{l}{EI} & -\frac{l^2}{2EI} & 0 \\ -\frac{l^2}{2EI} & \frac{l^3}{3EI} & 0 \\ 0 & 0 & \frac{l}{EA} \end{bmatrix} \qquad \mathbf{f}_R=\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \quad \ldots\ldots\ldots \quad (233)$$

and hence

$$\mathbf{f}=\begin{bmatrix} \mathbf{f}_B & \mathbf{o} \\ \mathbf{o} & \mathbf{f}_R \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (234)$$

It is now possible to write down the equation for $\mathbf{X}$,

$$\mathbf{b}_1'\mathbf{f}\mathbf{b}_1\mathbf{X}+\mathbf{b}_1'\mathbf{f}\mathbf{b}_0\mathbf{R}=\mathbf{O}$$

Returning to the general argument we investigate now in a system with a total number n of redundancies the effects of initial strains e.g. those due to temperature rise, excess length of bars due to manufacturing errors, 'give' of foundation at supports. Assume that the column matrix of the relative displacements at the ends or boundaries of the s elements due to η is

$$\mathbf{H} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (235)$$

We assume first that the basic system is statically determinate: then the elements of $\mathbf{H}$ are merely the integrated effect of the imposed η. For example, in a pin-jointed framework subjected to temperature rise the elements of $\mathbf{H}$ are

$$\mathbf{H}=\{(\alpha\Theta l)_a\ldots\ldots\ldots\ldots(\alpha\Theta l)_s\} \quad \ldots\ldots\ldots \quad (235a)$$

If the bars are of an excess length Δl due to inaccurate manufacture, then these form directly the elements of $\mathbf{H}$. Independently of the nature of $\mathbf{H}$, however, the corresponding relative displacements at the cut redundancies are simply

$$\mathbf{b}_1'\mathbf{H}$$

and the equation for the n unknowns X is

$$\mathbf{b}_1'\mathbf{f}\mathbf{b}_1\mathbf{X}+\mathbf{b}_1'\mathbf{H}=\mathbf{O} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (236)$$

Note that in the present case $\bar{\mathbf{b}}_1=\mathbf{b}_1$ since the basic system is taken to be statically determinate. When the deformations arise from p 'gives' Δ at the foundations it is advantageous to express $\mathbf{H}$ as

$$\mathbf{H}=\{\Delta_1\ldots\ldots\ldots\ldots\Delta_p\} \quad \ldots\ldots\ldots\ldots \quad (235b)$$

Then for $\mathbf{b}_1'$ we must substitute the matrix $\mathbf{c}'$ where

$$\mathbf{c}=\begin{bmatrix} C_{12}\ldots\ldots C_{1i}\ldots\ldots C_{1p} \\ \hline C_{i1}\ldots\ldots C_{ii}\ldots\ldots C_{ip} \\ \hline C_{n1}\ldots\ldots C_{ni}\ldots\ldots C_{np} \end{bmatrix} \quad \ldots\ldots\ldots\ldots \quad (237)$$

the n rows of which are the forces due to $X_i=1$, applied by the structure to the foundations in the directions of the gives Δ. Eq. (236) becomes now,

$$\mathbf{b}_1'\mathbf{f}\mathbf{b}_1\mathbf{X}+\mathbf{c}'\mathbf{H}=\mathbf{O} \quad \ldots\ldots\ldots\ldots\ldots \quad (236a)$$

If we select a redundant basic system we cannot derive the elements of $\mathbf{H}$ immediately from the prescribed initial strain since the latter are not free to develop in a redundant structure. In this case unless we have the necessary information from previous calculations we must first analyse the basic system by the method of the previous paragraph. Having found the column matrix $\mathbf{H}$ the r equations in the r unknowns take the form

$$\bar{\mathbf{b}}_1'\mathbf{f}\mathbf{b}_1\mathbf{X}+\bar{\mathbf{b}}_1'\mathbf{H}=\mathbf{O} \quad \ldots\ldots\ldots\ldots\ldots \quad (236b)$$

where we may write $\bar{\mathbf{b}}_1$ for $\mathbf{b}_1$ since the basic system is now redundant.

The systematic solution of (224) and related equations was discussed on page 28 but there are a few further points arising in practical calculations which are best investigated here. Thus, we mentioned on page 20 that it is often possible and justified to neglect certain part flexibilities of the elements; for example, in a ring analysis we can usually ignore the direct and shear flexibility. This applies not only to the evaluation of the external flexibility $\mathbf{F}$ but also to the determination of the internal redundancies $\mathbf{X}$. We write now the $\mathbf{D}$ and $\mathbf{D}_0$ matrices in the split form

$$\mathbf{D}=\mathbf{D}_a+\mathbf{D}_b \text{ and } \mathbf{D}_o=\mathbf{D}_{oa}+\mathbf{D}_{ob} \quad \ldots\ldots\ldots\ldots\ldots \quad (238)$$

where the suffices a and b refer to the two flexibilities into which we separate the total flexibility of each element. An approximate solution $\mathbf{X}_a$ to the unknown column $\mathbf{X}$ is then obtained by ignoring the flexibility b. Then,

$$\mathbf{D}_a\mathbf{X}_a+\mathbf{D}_{oa}=\mathbf{O} \text{ or } \mathbf{X}_a=-\mathbf{D}_a^{-1}\mathbf{D}_{oa} \quad \ldots\ldots\ldots\ldots \quad (239)$$

Occasionally we may require subsequently the correction $\mathbf{x}$ to $\mathbf{X}_a$ to find the true column $\mathbf{X}$,

$$\mathbf{X}=\mathbf{X}_a+\mathbf{x} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (239a)$$

Substituting (238) and (239a) into Eq. (224) we derive easily

$$\mathbf{x}=-\mathbf{D}^{-1}(\mathbf{D}_{ob}+\mathbf{D}_b\mathbf{X}_a)$$

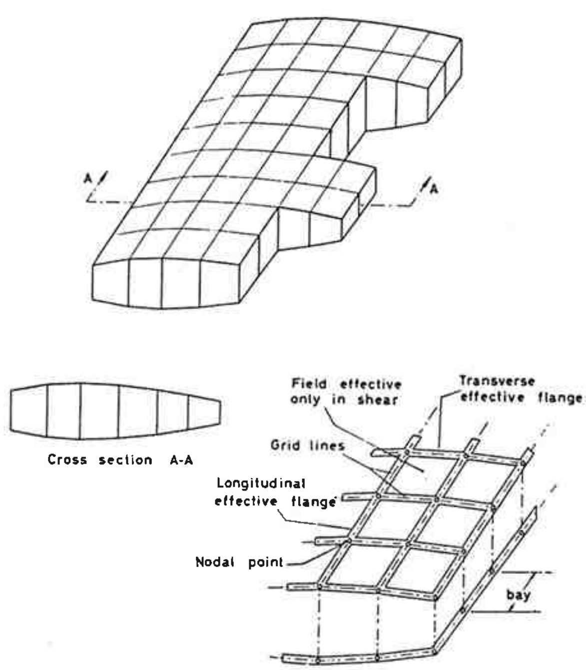


Fig. 37.—Aircraft wing structure. Geometry and definition

and since $\mathbf{x}$ is usually small in comparison with $\mathbf{X}$ approximate methods may often be used in the evaluation of the right hand side. This technique is immediately applicable to the correction of a completed stress analysis when subsequent small changes in the flexibilities of some elements are introduced by modifications to their cross-sectional dimensions.

When the number of equations is too large for performing the matrix operation on a digital computer then we may apply the following method which is basically identical with the idea of a redundant basic system. Assume that in a structure with a number n of redundancies we select first a basic system with $t=n-r$ redundancies and that we write down the r equations in X_i in the standard form (224),

$$\mathbf{D}\mathbf{X}+\mathbf{D}_o=\mathbf{O} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (224)$$

If the number r is still too large for handling by the digital computer we solve the problem in two steps. Eqs. (224) are first put in the partitioned form

$$\begin{bmatrix} \mathbf{D}_I & \mathbf{D}_{III}' \\ \mathbf{D}_{III} & \mathbf{D}_{II} \end{bmatrix}\begin{bmatrix} \mathbf{X}_I \\ \mathbf{X}_{II} \end{bmatrix}+\begin{bmatrix} \mathbf{D}_{oI} \\ \mathbf{D}_{oII} \end{bmatrix}=\begin{bmatrix} \mathbf{O} \\ \mathbf{O} \end{bmatrix} \quad \ldots\ldots \quad (240)$$

where the number of rows in the matrices with suffices I and II is ρ and $r-\rho$ respectively and $\mathbf{D}_I$, $\mathbf{D}_{II}$ are square matrices. Eq. (240) gives

$$\left.\begin{aligned} \mathbf{D}_I\mathbf{X}_I+\mathbf{D}_{III}'\mathbf{X}_{II}+\mathbf{D}_{oI}=\mathbf{O} \\ \mathbf{D}_{III}\mathbf{X}_I+\mathbf{D}_{II}\mathbf{X}_{II}+\mathbf{D}_{oII}=\mathbf{O} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (240a)$$

We split next the column matrix $\mathbf{X}_{II}$ into matrices

$$\mathbf{X}_{II}=\mathbf{x}+\mathbf{y} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (241)$$

where $\mathbf{x}$ satisfies the equation

$$\mathbf{D}_{II}\mathbf{x}+\mathbf{D}_{0II}=\mathbf{O} \text{ or } \mathbf{x}=-\mathbf{D}_{II}^{-1}\mathbf{D}_{0II}\ldots\ldots\ldots\ldots \quad (242)$$

Hence

$$\mathbf{D}_{III}\mathbf{X}_I+\mathbf{D}_{II}\mathbf{y}=\mathbf{O} \text{ or } \mathbf{y}=-\mathbf{D}_{II}^{-1}\mathbf{D}_{III}\mathbf{X}_I \quad \ldots\ldots \quad (242a)$$

Substituting for $\mathbf{x}$ and $\mathbf{y}$ into the first of Eqs. (240a) we find

$$\mathbf{D}_I\mathbf{X}_I+\mathbf{D}_{III}'[-\mathbf{D}_{II}^{-1}\mathbf{D}_{0II}-\mathbf{D}_{II}^{-1}\mathbf{D}_{III}\mathbf{X}_I]+\mathbf{D}_{0I}=\mathbf{O}$$

or

$$\mathbf{X}_I=-(\mathbf{D}_I-\mathbf{D}_{III}'\mathbf{D}_{II}^{-1}\mathbf{D}_{III})^{-1}(\mathbf{D}_{oI}-\mathbf{D}_{III}'\mathbf{D}_{II}^{-1}\mathbf{D}_{oII})\ldots\ldots(243)$$

from which we deduce $\mathbf{y}$ and hence $\mathbf{X}$. Eqs. (243) are identical with the elastic compatibility equation for a basic system with $n-\rho$ redundancies.

The matrix form (224) of the equation of compatibility is particularly suitable to illustrate the transformation (see p. 29),

$$\mathbf{X}=\mathbf{B}\mathbf{Y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (195)$$

when the equations are ill-conditioned. Thus, by substitution of (195) into (224) we find

$$\bar{\mathbf{b}}_1'\mathbf{f}\mathbf{b}_1\mathbf{B}\mathbf{Y}+\bar{\mathbf{b}}_1'\mathbf{f}\mathbf{b}_0\mathbf{R}=\mathbf{O}$$

and premultiplying by $\mathbf{B}'$

$$\mathbf{B}'\bar{\mathbf{b}}_1'\mathbf{f}\mathbf{b}_1\mathbf{B}\mathbf{Y}+\mathbf{B}'\bar{\mathbf{b}}_1'\mathbf{f}\mathbf{b}_0\mathbf{R}=\mathbf{O}$$

or

$$\bar{\mathbf{b}}_2'\mathbf{f}\mathbf{b}_2\mathbf{Y}+\bar{\mathbf{b}}_2'\mathbf{f}\mathbf{b}_0\mathbf{R}=\mathbf{O} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (244)$$

where (see also Eqs. (196) and (197)),

$$\mathbf{b}_2=\mathbf{b}_1\mathbf{B} \text{ and } \bar{\mathbf{b}}_2=\bar{\mathbf{b}}_1\mathbf{B} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (245)$$

are merely the matrices for the true and statically equivalent stress systems in the basic system due to

$$\mathbf{Y}=\{1 \quad 1 \ldots\ldots\ldots\ldots\ldots 1\}$$

The form (244) of the equations of compatibility may, of course, be written down directly when starting *ab initio* with the group unknowns $\mathbf{Y}$.

Application to a typical aircraft structure

We present now a detailed investigation of a type of system characteristic of aircraft wings. Consider to this purpose the structure shown in FIG. 37 which consists essentially of an orthogonal or nearly orthogonal grid of spars and ribs covered with sheet material. Longitudinal and transverse flanges may be placed at the intersections of spar and rib webs with the covers. In addition the covers may be stiffened with further longitudinal and/or transverse flanges. The cross-section is assumed arbitrary and the spars may taper differently in plan view and elevation but the angle of taper 2θ is taken to be so small that $\cos 2\theta \approx 1$ and $\sin 2\theta \approx 2\theta$. The analysis is not restricted to structures with continuous rib and spar-webs, covers and flanges and includes hence any kind of minor or major cut-out.

The geometry considered excludes swept-back wings with ribs parallel to the line of flight. On the other hand swept back wings with ribs perpendicular to the main wing axis can be analysed by the present method as long as we are given the necessary information for the triangular root-section. Delta wings may also be investigated by our theory as long as the grid of ribs and spars conforms to the geometry stipulated here. Naturally, many of the restrictions imposed limit the applicability of the method. Indeed we intend our analysis only as an exploratory and tentative first attack on the more general problem. We hope to return to this and similar points in later publications.

The problem of finding the stress distribution in the shell type of structure considered is strictly infinitely redundant. Hence it is necessary to introduce for practical calculations considerable simplifications. First we adopt the standard assumption in wing stress analysis of a membrane state of stress, i.e. we exclude any bending of covers and flanges normal to the surface of the wing. For the very thin wings now coming into prominence this idealization is open to grave doubts and will no doubt have to be reconsidered in future. An essential characteristic of our theory is the assumption that the longitudinal and direct stresses vary linearly between the nodal points of a three-dimensional grid of lines traced on the wing cover. This system of lines should, in general, be at least as fine as the grid of spars and ribs whose intersection with the covers forms the best minimum set of grid lines. The latter grid will often be sufficiently close if we are dealing with a multi-web structure and ribs at not too great a distance. However, many instances occur where it is necessary to select additional nodal points between which the direct stress is taken to vary linearly. For example, we may choose points intermediate between spar webs on the rib stations if the spacing of the spars and the sheet thickness of the cover are large. Similarly, if the structure has few ribs we may have to introduce new transverse stations in order to reduce the spacing of the grid in the longitudinal direction. In either case there need not be an actual longitudinal or transverse reinforcement along the new grid lines. We call the surface enclosed between two adjoining grid lines in the z- and s-direction a field, and denote by 'bay' a part of the wing structure which lies between two cross-sections taken through adjoining grid lines running in the s-direction (see FIG. 37). The assumption of a linear direct stress distribution along the edges of an orthogonal and flat field yields from overall equilibrium conditions a parabolic shear flow distribution along the edges. Naturally, neither the linear direct stress nor the parabolic shear flow variation are, in general, exact and violate the internal and boundary compatibility conditions of the field. This is not serious as long as we keep the spacing of the grid lines reasonably close. Moreover, we simplify further the problem by neglecting the quadratic and linear terms in the shear flow and considering it to be constant within each field. We note that for non-orthogonal grid lines (tapered structure) the uniform shear flow offends against the equilibrium conditions even if the direct stress is constant between adjoining nodal points. The errors introduced by the assumption of uniform shear flow are, however, practically insignificant for the geometry of structure considered here when the nodal point distances are small.

If the direct stresses along the grid lines were known we could calculate the fraction of sheet area to be added to the reinforcements to form the equivalent or effective flanges. This applies to the cover, spar-webs and rib-webs and yields an idealized structure in which the fields are only shear carrying and the direct stress carrying ability is concentrated in flanges; an assumption widely used in aircraft practice. Neglecting the Poisson's ratio effect and assuming the same material for flanges and sheet material cover, the fraction of sheet cross-sectional area to be added to the flanges varies between 1/6 and 1/2 if the fields are flat; the former value applies when the field is in pure bending in its own plane and the latter when it is under uniform stress. Since the stress distribution is unknown we can at best only estimate the effective areas of the flanges but may use an iteration process if the first guess proves inadequate. However, the latter procedure is really clumsy and lengthy and a direct method, obviating the guessing of flange areas would evidently be useful, in particular at the root or other

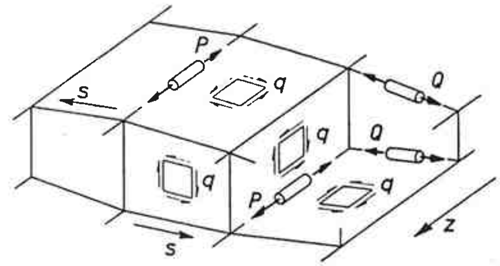


Fig. 38.—Sign convention for flange loads and shear flows

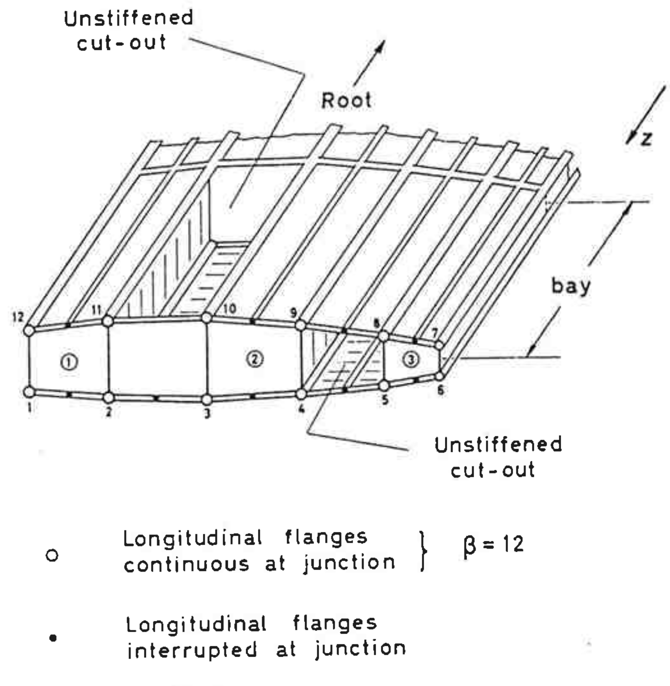


Fig. 39.—Geometry of typical bay for determination of number of redundancies

structural or loading discontinuities where the stress distribution is more difficult to estimate and the Poisson's ratio effect more pronounced. Such a method is given here at the end of this sub-section but at first we develop the theory under the assumption that the effective flanges areas are known and that they are constant between two adjoining nodal points.* For the webs, when considering torque and lift loads, it is always sufficient to add 1/6 of the web cross sectional area to the longitudinal and transverse flanges at the intersection of the spars and ribs with the cover.

We summarize now the main assumptions underlying the idealized structure selected for analysis. Thus, our system consists of an orthogonal or nearly orthogonal grid of spars and ribs with top and bottom covers. Effective flanges of constant area between adjoining nodal points and carrying only direct stresses are assumed placed along the grid lines in longitudinal and transverse directions. For the time being we assume that the flange areas are known. The direct stresses and hence also the flange loads are taken to vary linearly between nodal points. All sheet material for covers, spars and webs is assigned a purely shear carrying role and a constant thickness within each field. The angles of taper of spars in plan view and elevation are assumed to be small. The shear deformability of covers, ribs and spars is included *ab initio* in the analysis. For the stresses and loads in the various elements we adopt the sign convention illustrated in FIG. 38. Naturally, the idealizations and simplifications introduced are strictly only necessary for the calculation of the redundancies. The basic or statically equivalent stress system may and should preferably be determined in the (cut) actual structure.

Degree of redundancy of idealized structure

We proceed next to the enumeration of the redundancies in our idealized structure. In addition to the simplifications introduced previously we ignore here the bending stiffness of the flanges for displacements tangential to the wing surface. This is, no doubt, sufficiently accurate for the present exploratory analysis. The wing structure supported at the root and free at

* Strictly, the latter assumption is only necessary when finding the flexibilities of the flanges for the calculation of the redundancies.

the tip is assumed stiffened by ribs at least at the root and the tip. These ribs need not necessarily consist of a web with flanges but may take the form of a stiff-jointed frame or ring. However, independently of the design of the ribs we may always substitute an equivalent shear web with flanges. The wing structure is subdivided into a number of bays of which we show a typical intermediate one in FIG. (39). The cross-section at the junction nearer to the tip may be stiffened by a rib carried across some or all cells. FIG. (39) indicates also those longitudinal flanges which are continuous across the same junction. It should be noted that if there is a change of transverse slope of the cover at a longitudinal flange the latter must be connected to a spar web.

We use the following notation:

β=number of longitudinal effective flanges which are *continuous across the junction*, i.e. are not interrupted there.

N=number of *closed* cells *stiffened by ribs* at the tip end of the bay.

Then the number of redundancies arising from the geometry of the bay is

$$\beta-3+N-1$$

Hence, in a tubular structure of the type shown in FIG. 37, free at the tip and either fully built-in at the root or with prescribed displacements there at all longitudinal flanges, the total number of redundancies is

$$n=\sum_{\text{bays}}[(\beta-3)+(N-1)] \quad \ldots\ldots\ldots\ldots \quad (247)$$

If certain of the flanges are not held at the root section the number of redundancies reduces accordingly. For example, if the root-section is at the aircraft centre line and the wing is subjected to anti-symmetrical loading the number of unknowns reduces by β_r-3, β_r being the number of longitudinal flanges at the root. The number in the square brackets in (247) can, of course, vary from bay to bay since effective flanges may be interrupted at such stations. Also the number N of stiffened cells may be made different in each bay by the addition or removal of spar webs. However, when β and N are the same in all bays and all the flanges are held at the root formula (247) becomes simply

$$n=\alpha[\beta+N-4] \quad \ldots\ldots\ldots\ldots \quad (247a)$$

where α=number of bays. If the sheet cover is missing between two adjoining longitudinal flanges in a bay and the cut-out is not provided with a stiff-jointed closed frame to restore partially the lost shear stiffness of the sheet then the corresponding cell is open in this bay and by definition is not included in N. Similarly, if there is no rib or equivalent frame in a cell at the section considered this cell is excluded from the numbering for N. Note that spar webs need not be continuous throughout the length of the wing and may be discontinued at any junction. Formula (247) still remains valid.

If the cross-section is singly symmetrical the n redundancies of Eq. (247) split into two groups:

$$\left.\begin{aligned} n_1&=\sum_{\text{bays}}\left[\left(\frac{\beta}{2}-1\right)+(N-1)\right] \\ n_2&=\sum_{\text{bays}}\left(\frac{\beta}{2}-2\right) \end{aligned}\right\} \quad (248)$$

of which n_1 applies for the lift and torque loads and n_2 for the drag loads.

If all cells are closed, with the same number N in all bays and effective flanges are only placed at the corners of the cells, then

$$\beta=2(N+1)$$

and from (247a) the total number of redundancies is

$$n=\alpha(3N-2) \quad \ldots\ldots\ldots\ldots \quad (249)$$

which formula again assumes that all the flanges are held at the root.

Of considerable importance in modern aircraft structural practice are the multispar systems with few, often only two, end ribs. A typical wing of the latter type is shown in FIG. (40). To analyse this structure we subdivide it into a number of bays whose length should not exceed say five times the spar pitch. Effective flanges will by virtue of our idealization process be acting at the junction of these bays although no ribs are provided there. For such a system the number n of redundancies when there are no cut-outs in the sheet, when all spars are continuous for the full length of the wing and all flanges are held at the root, is given by

$$n=(N-1)+1+\alpha(\beta-4)=N+\alpha(\beta-4) \quad \ldots\ldots\ldots\ldots \quad (250)$$

The last system to be considered is a flat panel which is of special importance for diffusion investigations (see FIG. 41). It is assumed built-in at $z=0$ or held with prescribed displacements and free at the other three edges. Here the number n of redundancies when there are no unstiffened cut-outs is simply

$$n=\sum_{\text{bays}}(\beta-2) \quad \ldots\ldots\ldots\ldots \quad (251)$$

where β is defined as in the case of the wing. When M fields are removed without being replaced by stiff-jointed frames the number of redundancies reduces by M.

The next step in our investigations is the discussion of suitable self-equilibrating systems which may be chosen as redundancies. Consider first the simple case of a rectangular flat panel shown in FIG 41. For the redundancies we may select n systems of the type $X=1$ illustrated in the

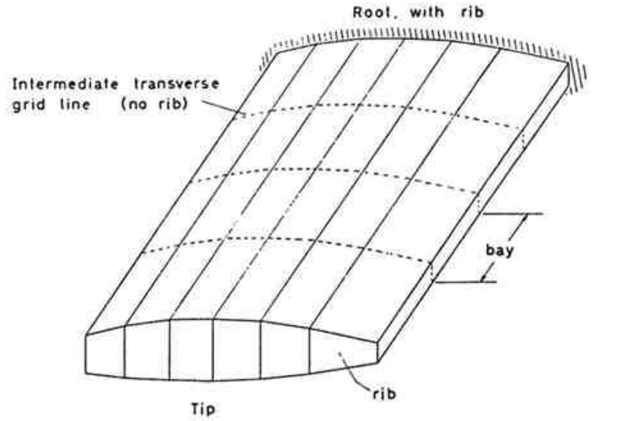


Fig. 40.—Multi-web wing without intermediate ribs

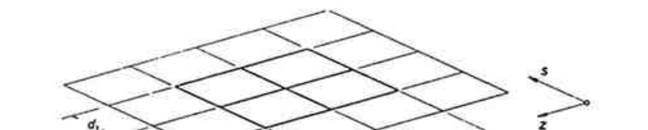


Fig. 41.—Rectangular stiffened panel. Unit self-equilibrating stress system of type $X=1$

figure. All information as to flange loads and shear flows is given there. The corresponding equations (182) or (224) for the unknown X_i are easily seen to be reasonably well conditioned. Naturally, we can further improve the conditioning by introducing group loads

$$\mathbf{X}=\mathbf{B}\overline{\mathbf{X}} \quad \ldots\ldots\ldots\ldots \quad (252)$$

where **B** is a suitable square matrix. We do not enter at this stage into the choice of **B** but hope to discuss these points in Part III. When the panel is symmetrical about its middle line it is preferable to combine the X-systems into symmetrical and antisymmetrical groups.

In a wing structure of the type investigated previously we can describe three simple types of self-equilibrating internal systems. They are shown in FIGS. 42, 43 and 44 and denoted by

$$X=1, \quad Y=1, \quad Z=1$$

respectively. The first is the generalization of the X-system used in the flat panel and the second and third may be considered as slightly modified four boom load systems taken in the longitudinal and transverse directions respectively. The longitudinal four-boom load systems are applied extensively in standard wing analysis.* The three figures are self-explanatory and give all flange loads and shear flows associated with the unit systems. Note, however, that the effect of taper is neglected except that we introduce the true local dimensions in the evaluation of the self-equilibrating

* See J. H. Argyris and P. C. Dunne, 'The General Theory, etc.,' *J.R.Ae.S.*, Vol. LI February, September, November 1947

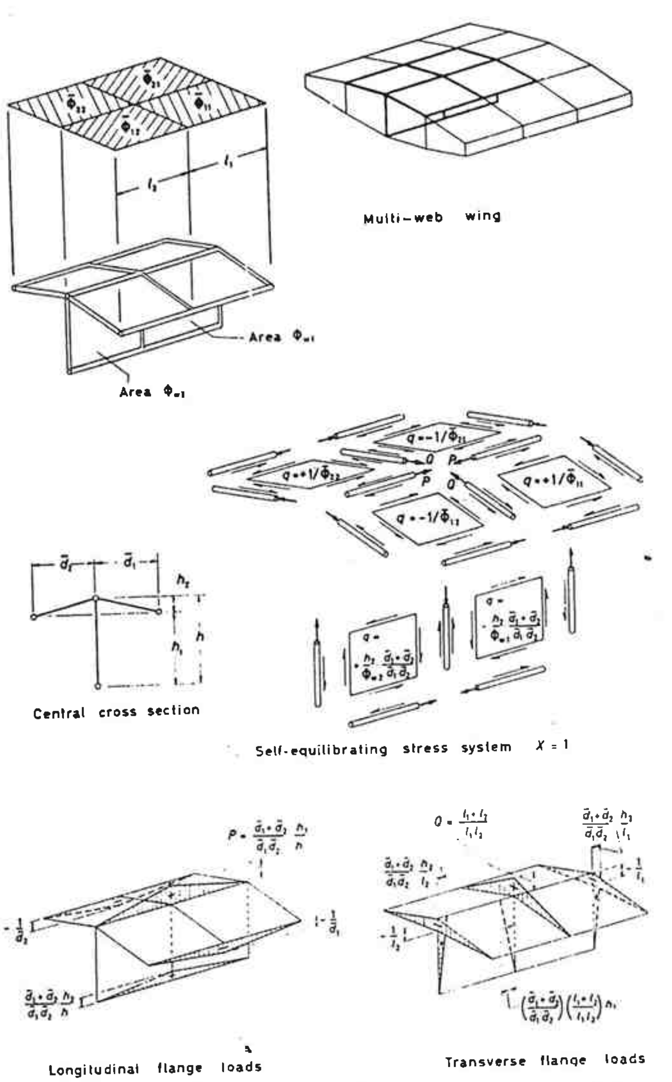


Fig. 42.—Multi-web wing. Unit self-equilibrating stress system of type $X=1$

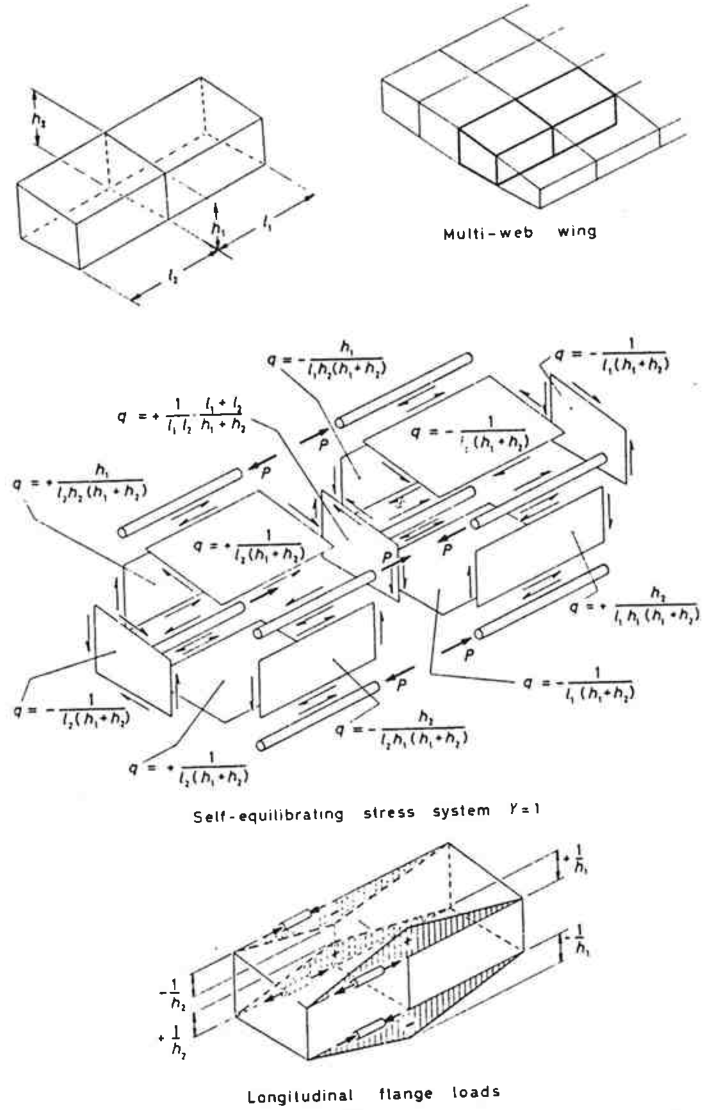


Fig. 43.—Multi-web wing. Unit self-equilibrating stress system of type $Y=1$. (Longitudinal four-boom tube)

systems. This ignoring of the influence of taper gives rise only to small errors as long as the lengths l of the bays are reasonably small and the angles of taper restricted to the order of magnitude mentioned initially. The expressions for the shear flows in the ribs for the Y- and Z-systems are also approximate, being derived for an equivalent rectangular rib. Again the error introduced by this assumption is for practical purposes insignificant. The conditions of equilibrium for the X- and Z-systems yield a load in an 'effective' vertical flange at the intersection of the ribs and webs. This flange load may always be neglected.

We enumerate now the number of X, Y, Z systems, independent within their own group, which can possibly be applied. We find easily, with the notation of Eq. (247) that we can use for each bay

$(\beta-4)$ X-systems, N Y-systems and $(N-1)$ Z-systems

Thus there are more systems than we require, the difference from the number n of redundancies being obviously linearly dependent systems. Evidently the

$(N-1)$ Z-systems

are independent of the X- and Y-systems and hence must all be chosen as redundancies. To complete the number n of unknowns we may use

$(\beta-4)$ X- and one Y-system

However, it is preferable to apply more longitudinal four-boom (Y) systems since they are better conditioned, and to reduce accordingly the number of X-systems. Thus, if we introduce all

N Y-systems

we have to adopt

$(\beta-3)-N$ X-systems

The last number reduces to $N-1$ when the effective longitudinal flanges are only placed at the corners of the N-cells.

It is often advisable to improve the conditioning of the **D**-matrix by the introduction of group loads $\overline{\mathbf{X}}$ and $\overline{\mathbf{Y}}$ where

$$\mathbf{X}=\mathbf{B}_1\overline{\mathbf{X}} \qquad \mathbf{Y}=\mathbf{B}_2\overline{\mathbf{Y}} \qquad \text{(252a)}$$

When the cross-section is singly symmetrical the number of redundancies for lift and torque loads reduces to n_1 given by the first of Eqs. (248). The $N-1$ Z-systems must still be included in the analysis for such loading cases. If, in addition, we use all N Y-systems the necessary number of X-systems becomes

$$\frac{\beta}{2}-1-N$$

and is zero when the effective flanges are arranged merely at the corners of the N-cells.

For the multispar wing of FIG. (40) with ribs only at the root and tip the n redundancies of Eq. (250) may be selected as

$(N-1)$ Z-systems at the root

one Y-system at the root

and

$(\beta-4)$ X-systems at each junction of bays and at the root.

The Y-system may involve a considerable length of the tube and if the latter is tapered a more accurate estimate of the longitudinal variation of the flange loads may become necessary.

Having selected a suitable system X, Y, Z of redundancies we can write down the $\mathbf{b}_1$ matrix with the information given in FIGS. 42, 43, 44. To obtain the $\mathbf{b}_0$ matrix we may use any suitable statically equivalent stress system in the actual or idealized structure, but preferably the former. It was mentioned on page 32 that it is advantageous to select a basic stress system which, while being simple, approximates as closely as possible to the true stress system and reference was made to the method of example (*a*) of Section 9. Nevertheless, if the work in finding such a $\mathbf{b}_0$ matrix proves excessive it may be preferable—since the choice of $\mathbf{b}_0$ does not affect the conditioning of the **D** matrix—to sacrifice the closeness to the true stress system and to select a $\mathbf{b}_0$ as simple as possible. Thus, we can calculate a $\mathbf{b}_0$ matrix for a basic system in which the spars act independently; a choice differing, in general, widely from the final **b** matrix.

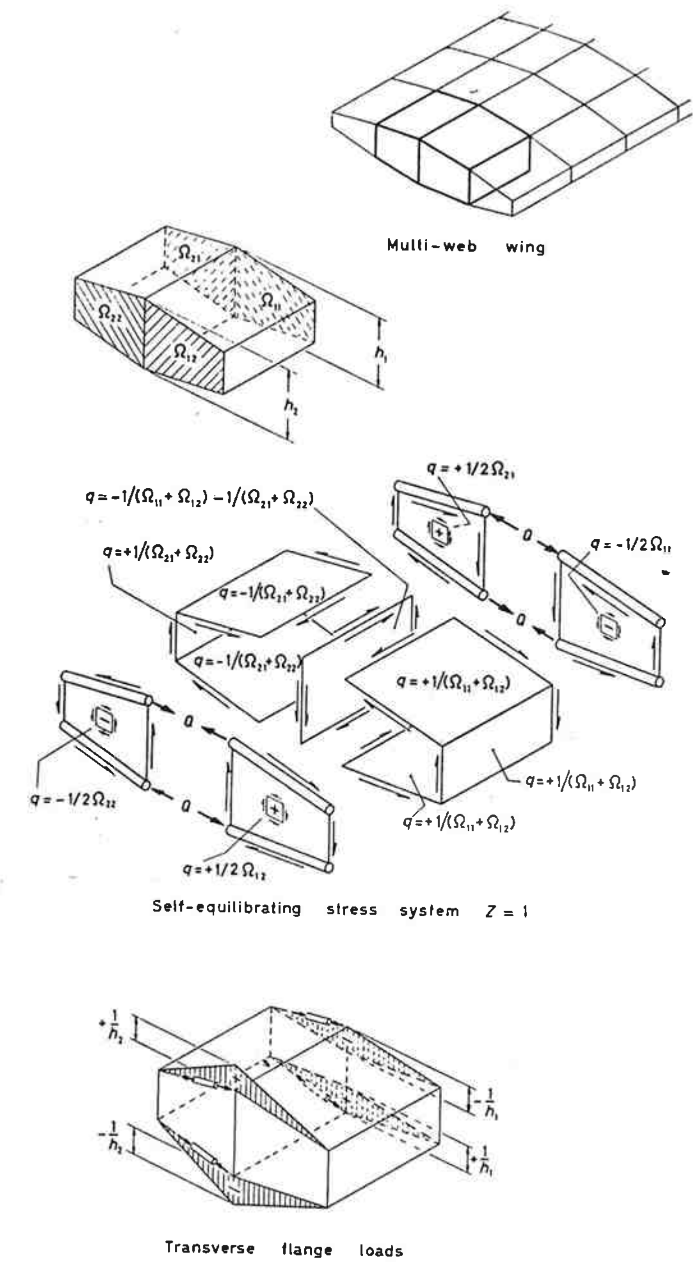


Fig. 44:—Multi-web wing. Unit self-equilibrating stress system of type $Z=1$. (Transverse four-boom tube)

We write now the $\mathbf{b}_0$ and $\mathbf{b}_1$ matrices in the partitioned form:

$$\mathbf{b}_0=\begin{bmatrix}\mathbf{b}_{0l}\\ \mathbf{b}_{0t}\\ \mathbf{b}_{0s}\\ \mathbf{b}_{0w}\\ \mathbf{b}_{0r}\end{bmatrix} \qquad \mathbf{b}_1=\begin{bmatrix}\mathbf{b}_{1l}\\ \mathbf{b}_{1t}\\ \mathbf{b}_{1s}\\ \mathbf{b}_{1w}\\ \mathbf{b}_{1r}\end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (253)$$

where the suffices l, t, s, w and r denote matrices for the longitudinal flange loads, transverse flange loads, shear flows in the fields of the cover, shear flows in the webs and shear flows in the ribs respectively.

Since the flange loads are assumed to vary linearly between nodal points we need at least two entries in the **b** matrices to describe the loads in each flange. As such we use the loads at the ends (nodal-points) of each flange element and denote them by the suffices 1 and 2 where 1 is the end first met when we proceed along the $+z$ or $+s$ direction. These two associated loads are entered in the assigned column of the appropriate sub-matrix of $\mathbf{b}_0$ or $\mathbf{b}_1$ in two consecutive rows, the first of which always corresponds to the end 1. Since the shear flow is assumed constant in each field only one entry appears for a field. In the $\mathbf{b}_1$ matrix we arrange the columns in three groups, the first referring to the X, the second to the Y- and the third to the Z-systems. It is of the utmost importance to organize *ab initio* a rigid and consistent system for the setting up of the $\mathbf{b}_0$ and $\mathbf{b}_1$ matrices.

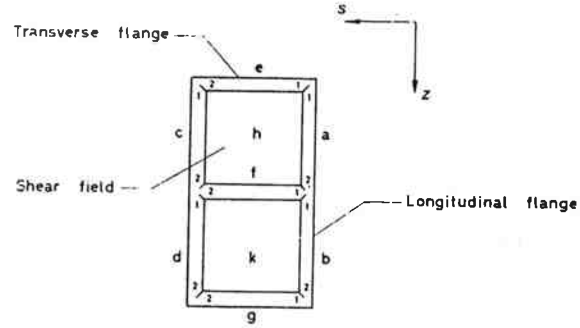


Fig. 45.—Simple panel to illustrate $\mathbf{b}_0$ and $\mathbf{b}_1$ matrices

Consider the simple grid shown in FIG. 45 and denote by $P_0\,(P_i)$, $Q_0\,(Q_i)$, $q_0\,(q_i)$ the longitudinal flange loads, transverse loads and shear flows corresponding to some $R=1$ $(X_i=1)$. The corresponding columns in the $\mathbf{b}_0$ and $\mathbf{b}_1$ matrices are

$$\left.\begin{array}{l}\{P_{0a1}P_{0a2}P_{0b1}P_{0b2}P_{0c1}P_{0c2}P_{0d1}P_{0d2}Q_{0e1}Q_{0e2}Q_{0f1}Q_{0f2}Q_{0g1}Q_{0g2}q_{0h}q_{0k}\}\\ \text{and}\\ \{P_{ia1}P_{ia2}P_{ib1}P_{ib2}P_{ic1}P_{ic2}P_{id1}P_{id2}Q_{ie1}Q_{ie2}Q_{if1}Q_{if2}Q_{ig1}Q_{ig2}q_{ih}q_{ik}\}\end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (254)$$

respectively.

To find the **D** and $\mathbf{D}_0$ matrices it only remains to give the flexibility matrix **f** of the elements. We write it in the partitioned form associated with the $\mathbf{b}_0$ and $\mathbf{b}_1$ matrices of Eqs. (253),

$$\mathbf{f}=\begin{bmatrix}\mathbf{f}_l & 0 & 0 & 0 & 0\\ 0 & \mathbf{f}_t & 0 & 0 & 0\\ 0 & 0 & \mathbf{f}_s & 0 & 0\\ 0 & 0 & 0 & \mathbf{f}_w & 0\\ 0 & 0 & 0 & 0 & \mathbf{f}_r\end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (255)$$

where the suffices have the same meaning as in Eqs. (253). The matrices $\mathbf{f}_l$ and $\mathbf{f}_t$ are themselves partitioned diagonal matrices, the sub-matrices being the flexibility matrices of the longitudinal and transverse flange elements respectively. Since the flange loads vary linearly and the effective flange area of each element is assumed constant within each element the flexibility of the flange elements is that given on p. 22. Thus, for the grid of FIG. 45 the $\mathbf{f}_l$ and $\mathbf{f}_t$ are,

$$\mathbf{f}_l=\begin{bmatrix}\mathbf{f}_a & 0 & 0 & 0\\ 0 & \mathbf{f}_b & 0 & 0\\ 0 & 0 & \mathbf{f}_c & 0\\ 0 & 0 & 0 & \mathbf{f}_d\end{bmatrix}, \quad \mathbf{f}_t=\begin{bmatrix}\mathbf{f}_e & 0 & 0\\ 0 & \mathbf{f}_d & 0\\ 0 & 0 & \mathbf{f}_f\end{bmatrix} \qquad \ldots\ldots\ldots \quad (256)$$

typical sub-matrices being

$$\mathbf{f}_b=\begin{bmatrix}\dfrac{l_b}{3EB_{eb}} & \dfrac{l_b}{6EB_{eb}}\\[2ex] \dfrac{l_b}{6EB_{eb}} & \dfrac{l_b}{3EB_{eb}}\end{bmatrix}, \quad \mathbf{f}_e=\begin{bmatrix}\dfrac{d_e}{3EC_{ee}} & \dfrac{d_e}{6EC_{ee}}\\[2ex] \dfrac{d_e}{6EC_{ee}} & \dfrac{d_e}{3EC_{ee}}\end{bmatrix} \quad \ldots\ldots \quad (257)$$

The flexibility matrices $\mathbf{f}_s$, $\mathbf{f}_w$, $\mathbf{f}_r$ are diagonal matrices with elements Φ/Gt, Φ_w/Gt_w and Ω/Gt_r respectively.

$$\mathbf{f}_s=\begin{bmatrix}\cdot 0 & \cdots & & \cdots & 0\\ 0\cdot & \ddots & & & \\ & & & & \\ 0 & \cdots 0 & \dfrac{\Phi}{Gt} & 0 \cdots & 0\\ & & & \ddots & \\ & & & & \cdot 0\\ 0 & \cdots & & \cdots & 0\cdot\end{bmatrix}, \quad \mathbf{f}_w=\begin{bmatrix}\cdot 0 & \cdots & & \cdots & 0\\ 0\cdot & \ddots & & & \\ & & & & \\ 0 & \cdots 0 & \dfrac{\Phi_w}{Gt_w} & 0 \cdots & 0\\ & & & \ddots & \\ & & & & \cdot 0\\ 0 & \cdots & & \cdots & 0\cdot\end{bmatrix}$$

$$f_r = \begin{bmatrix} \cdot 0 \cdots\cdots\cdots\cdots 0 \\ 0 \cdot \\ \quad \cdot \\ \quad \cdot \\ 0 \cdots 0 \dfrac{\Omega}{Gt_r} 0 \cdots 0 \\ \quad \cdot \\ \quad \cdot \\ \quad \cdot \\ \cdot 0 \\ 0 \cdots\cdots\cdots\cdots 0 \cdot \end{bmatrix} \qquad (258)$$

where Φ, Φ_w, Ω are the areas of the shear fields in the wing surface, webs and ribs respectively and t, t_w, t_r are the corresponding thicknesses.

We have now all the information to form the matrices

$$\mathbf{D}=\mathbf{b}_1'\mathbf{f}\mathbf{b}_1 \text{ and } \mathbf{D}_0=\mathbf{b}_1'\mathbf{f}\mathbf{b}_0\mathbf{R}$$

and can hence solve the system of equations

$$\mathbf{b}_1'\mathbf{f}\mathbf{b}_1\{\mathbf{X}\ \mathbf{Y}\ \mathbf{Z}\}+\mathbf{b}_1'\mathbf{f}\mathbf{b}_0\mathbf{R}=\mathbf{O} \qquad (259)$$

for the unknowns $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$.

Finally, we find the true flange loads and shear flows of the idealized structure from

$$\mathbf{S}=\mathbf{b}_0\mathbf{R}+\mathbf{b}_1\{\mathbf{X}\ \mathbf{Y}\ \mathbf{Z}\} \qquad (260)$$

It will then be necessary to translate the results (260) into stresses of the actual structure. Finally the flexibility $\mathbf{F}$ of the structure at the points and directions of the loads $\mathbf{R}$ may be determined from Eq. (229).

When the number of equations (259) is too large to be dealt with by our digital computer we may proceed in two (or more) steps by the method given on page 36. Essentially this introduces into our analysis a redundant basic system.

If initial strains $\mathbf{H}$ are imposed on the structure in addition to the loads $\mathbf{R}$ we have merely to add the column matrix

$$\mathbf{b}_1'\mathbf{H}$$

on the left hand side of (259). Thus the analysis includes *inter alia* the complete calculation of wings under thermal loading.

A new approach to the problem of cut-outs

We emphasize that our above analysis is valid in the presence of any kind of cut-out stiffened or unstiffened by closed frames as long as the overall geometry and idealization conforms with the initial assumptions. Nevertheless, when we have a structure which is essentially continuous with only minor unstiffened cut-outs it may be worthwhile to apply an artifice which avoids the lack of uniformity in the pattern of the equations inevitably associated with cut-outs. Moreover, it is the ideal method of finding the alteration in the stresses due to the subsequent introduction of cut-outs in our system without having to repeat all the computations *ab initio*.

The method is as follows. To preserve the pattern of equations disturbed by missing shear panels or flanges we eliminate the cut-outs by introducing fictitious shear panels or flanges with an arbitrary thickness or area. Naturally, it is usually preferable to select for the latter dimensions those of the surrounding structure. To obtain nevertheless the same flange loads and shear flows in our altered structure as in the original system initial strains are imposed on the additional elements of such a magnitude that their stresses become zero. The effect of the fictitious elements is thus nullified whilst the uniform pattern of our equations is retained.

Let the column matrix of the unknown initial strains, in the additional elements only, be

$$\mathbf{H}$$

In the new structure (i.e. without the cut-outs) we determine the flexibility matrix $\mathbf{f}$ and the matrices $\mathbf{b}_0$ and $\mathbf{b}_1$ which we write in the partitioned form

$$\mathbf{b}_0=\begin{bmatrix}\mathbf{b}_{0g}\\ \mathbf{b}_{0h}\end{bmatrix}, \quad \mathbf{b}_1=\begin{bmatrix}\mathbf{b}_{1g}\\ \mathbf{b}_{1h}\end{bmatrix} \qquad (261)$$

where the sub-matrices with the suffices g and h refer to the forces in the elements of the original structure and the fictitious new elements respectively.

Denoting the column matrix $\{\mathbf{X}\ \mathbf{Y}\ \mathbf{Z}\}$ simply by $\mathbf{X}$ and taking the initial strains in the original structure as zero the Eqs. (259) in the unknown $\mathbf{X}$ become,

$$\mathbf{b}_1'\mathbf{f}\mathbf{b}_1\mathbf{X}+\mathbf{b}_1'\mathbf{f}\mathbf{b}_0\mathbf{R}+\mathbf{b}_1'\begin{bmatrix}\mathbf{O}\\ \mathbf{H}\end{bmatrix}=\mathbf{O}$$

and hence using the second equation of (261)

$$\mathbf{X}=-\mathbf{D}^{-1}\mathbf{b}_1'\mathbf{f}\mathbf{b}_0\mathbf{R}-\mathbf{D}^{-1}\mathbf{b}_{1h}'\mathbf{H} \qquad (262)$$

where

$$\mathbf{D}=\mathbf{b}_1'\mathbf{f}\mathbf{b}_1$$

The stress matrix $\mathbf{S}$ follows as,

$$\mathbf{S}=[\mathbf{b}_0-\mathbf{b}_1\mathbf{D}^{-1}\mathbf{b}_1'\mathbf{f}\mathbf{b}_0]\mathbf{R}-\mathbf{b}_1\mathbf{D}^{-1}\mathbf{b}_{1h}'\mathbf{H} \qquad (263)$$

The expression in the square bracket is the matrix $\mathbf{b}$ which we write in the partitioned form

$$\mathbf{b}=\begin{bmatrix}\mathbf{b}_g\\ \mathbf{b}_h\end{bmatrix} \qquad (264)$$

To find now the column matrix $\mathbf{H}$ we put the stresses in the additional elements to zero. Thus, the matrix $\mathbf{S}$ must be

$$\mathbf{S}=\begin{bmatrix}\mathbf{S}_g\\ \mathbf{O}\end{bmatrix} \qquad (265)$$

where $\mathbf{S}_g$ are the true stresses (forces) in the original structure. Applying Eqs. (261), (264) and (265) in (263) we find

$$\begin{bmatrix}\mathbf{S}_g\\ \mathbf{O}\end{bmatrix}=\begin{bmatrix}\mathbf{b}_g\\ \mathbf{b}_h\end{bmatrix}\mathbf{R}-\begin{bmatrix}\mathbf{b}_{1g}\\ \mathbf{b}_{1h}\end{bmatrix}\mathbf{D}^{-1}\mathbf{b}_{1h}'\mathbf{H} \qquad (256)$$

Hence

$$\mathbf{O}=\mathbf{b}_h\mathbf{R}-\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}'\mathbf{H}$$

or

$$\mathbf{H}=(\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}')^{-1}\mathbf{b}_h\mathbf{R} \qquad (267)$$

The true stresses in our actual structure are thus

$$\mathbf{S}_g=[\mathbf{b}_g-\mathbf{b}_{1g}\mathbf{D}^{-1}\mathbf{b}_{1h}'(\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}')^{-1}\mathbf{b}_h]\mathbf{R} \qquad (268)$$

which solves our problem completely. As mentioned already the method is ideally suited for finding the alteration of the stresses in a structure through a subsequent introduction of cut-outs, such as access doors which usually seem to materialize at a late stage of design. Another particularly useful application of the new approach may be found in the analysis of fuselages with window-openings. Naturally, the degree of redundancy is increased by the 'filling-in' of the cut-outs but this is of no importance for the automatic computations envisaged here.

A more refined wing stress analysis

The above general method of wing stress analysis suffers from the serious defect mentioned initially that the effective flange areas have first to be guessed since the stress distribution on which they depend is unknown. It is certainly feasible to apply an iteration technique but this is not only necessarily lengthy but also rather uninspiring.

To obviate these difficulties we develop a method which eliminates the determination of the effective flange areas and works directly with effective flange loads. The method has the further virtue that it takes full account of the Poisson's ratio effect which may be important at the root and at other structural and loading discontinuities. The addition of 1/6 of the web area to the flanges is always sufficiently accurate for lift and torque loads and is retained here. Hence, our problem is restricted to the wing surface alone.

The previously introduced assumptions that the loads are carried in the idealized structure by a grid system of effective flange loads and fields purely shear-carrying form also the basis of the new method. We assume also that both the direct stress distribution and the effective flange loads vary linearly between consecutive nodal points. However, our analysis does not presume that the so-called effective flange areas—which do not enter into our developments—are constant between nodal points. The shear flow is again taken to be constant within each field. When replacing the linearly varying direct stresses across a grid line by effective flange loads at the nodal points we introduce the additional assumption that the actual flange areas and thicknesses do not vary across this grid line. Since in wing structures plate thicknesses and possibly flange areas may vary just there it is suggested to take for this particular calculation the mean values of areas and thicknesses on either side of the grid line; on the other hand when there is a cut-out on one side of the grid line or the flange is interrupted the corresponding values should be taken as zero. These simplifications are not necessary for the purpose of the analysis but ease the problem of notation; moreover they do not affect seriously the accuracy of the computations. Contrary to our previous practice of numbering the flange elements with letters we number here only the nodal points with numerals.

We derive now the equation connecting the effective flange loads at nodal points in the z and s directions of the idealized structure and the direct stress distribution in the plate material. It is more convenient to fix a particular point and for this purpose we select the point 9 in the grid-system shown in FIG. 46.

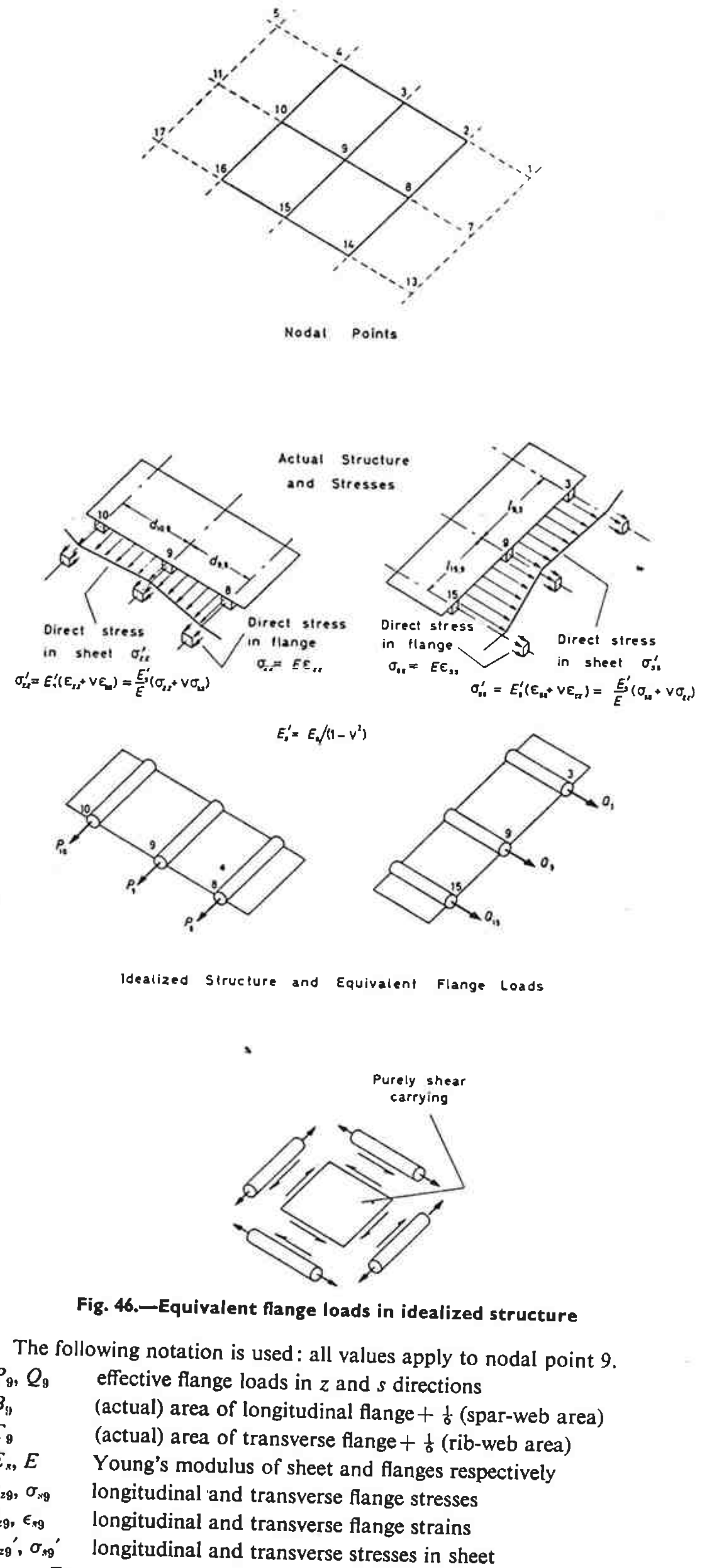


Fig. 46.—Equivalent flange loads in idealized structure

The following notation is used: all values apply to nodal point 9.

P_9, Q_9 effective flange loads in z and s directions

B_9 (actual) area of longitudinal flange + $\frac{1}{6}$ (spar-web area)

C_9 (actual) area of transverse flange + $\frac{1}{6}$ (rib-web area)

E_s, E Young's modulus of sheet and flanges respectively

σ_{z9}, σ_{s9} longitudinal and transverse flange stresses

ϵ_{z9}, ϵ_{s9} longitudinal and transverse flange strains

σ_{z9}', σ_{s9}' longitudinal and transverse stresses in sheet

$E_s'=\dfrac{E_s}{1-\nu^2}$ effective elastic modulus of sheet

$A_{3,9}=\frac{1}{6}\frac{E_s'}{E}l_{3,9}t_{3,9}$, $A_{8,9}=\frac{1}{6}\frac{E_s'}{E}d_{8,9}t_{8,9}$ effective flange areas due to plate in pure bending

Note that the material of flanges and sheet is assumed to be different. We have,

$$\left.\begin{aligned}\sigma_{z9}'&=E_s'(\epsilon_{z9}+\nu\epsilon_{s9})=\frac{E_s'}{E}(\sigma_{z9}+\nu\sigma_{s9})\\ &\text{and}\\ \sigma_{s9}'&=E_s'(\epsilon_{s9}+\nu\epsilon_{z9})=\frac{E_s'}{E}(\sigma_{s9}+\nu\sigma_{z9})\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\quad(269)$$

The conditions of equilibrium for the actual and idealized systems yield in conjunction with Eq. (269)

$$\left.\begin{aligned}P_9=&\sigma_{z9}[B_9+2(A_{8,9}+A_{9,10})]+\sigma_{z8}A_{8,9}+\sigma_{z,10}A_{9,10}\\&+\sigma_{s9}2\nu(A_{8,9}+A_{9,10})+\sigma_{s8}\nu A_{8,9}+\sigma_{s10}\nu A_{9,10}\end{aligned}\right\}\quad\cdots\cdots\quad(270)$$

$$\left.\begin{aligned}Q_9=&\sigma_{s9}[C_9+2(A_{3,9}+A_{9,15})]+\sigma_{s3}A_{3,9}+\sigma_{s15}A_{9,15}\\&+\sigma_{z9}2\nu(A_{3,9}+A_{9,15})+\sigma_{z3}\nu A_{3,9}+\sigma_{z15}\nu A_{9,15}\end{aligned}\right\}\quad\cdots\cdots\quad(271)$$

These equations are expressed more concisely in the form,

$$\begin{bmatrix}P_9\\Q_9\end{bmatrix}=\mathbf{L}_{9,9}\begin{bmatrix}\sigma_{z9}\\\sigma_{s9}\end{bmatrix}+\mathbf{L}_{9,8}\begin{bmatrix}\sigma_{z8}\\\sigma_{s8}\end{bmatrix}+\mathbf{L}_{9,10}\begin{bmatrix}\sigma_{z10}\\\sigma_{s10}\end{bmatrix}\quad\cdots\cdots\cdots\cdots\quad(272)$$

$$+\mathbf{L}_{9,3}\begin{bmatrix}\sigma_{z3}\\\sigma_{s3}\end{bmatrix}+\mathbf{L}_{9,15}\begin{bmatrix}\sigma_{z15}\\\sigma_{s15}\end{bmatrix}$$

where the matrices $\mathbf{L}$ are as follows:

$$\mathbf{L}_{9,9}=\begin{bmatrix}B_9+2(A_{8,9}+A_{9,10}) & 2\nu(A_{8,9}+A_{9,10})\\ 2\nu(A_{3,9}+A_{9,15}) & C_9+2(A_{3,9}+A_{9,15})\end{bmatrix}\quad(273)$$

$$\mathbf{L}_{9,8}=\begin{bmatrix}A_{8,9} & \nu A_{8,9}\\ 0 & 0\end{bmatrix}\qquad \mathbf{L}_{9,3}=\begin{bmatrix}0 & 0\\ \nu A_{3,9} & A_{3,9}\end{bmatrix}\quad\cdots\cdots(273a)$$

The matrix $\mathbf{L}_{9,10}$ ($\mathbf{L}_{9,15}$) is obtained from $\mathbf{L}_{9,8}$ ($\mathbf{L}_{9,3}$) by substituting 10(15) for 8(3). Equations corresponding to (272) may be written down for any other nodal point. We see immediately that

$$\mathbf{L}_{rs}=\mathbf{L}_{sr}\quad\cdots\cdots\cdots\cdots\cdots\cdots\quad(274)$$

and that,

$$\mathbf{L}_{rs}=\mathbf{O}\quad\cdots\cdots\cdots\cdots\cdots\cdots\quad(274a)$$

when r and s are not adjoining nodal points of the grid. We deduce also that Eqs. (273a) are the general formulae of the $\mathbf{L}$-matrices for adjoining nodal points in the z- and s- directions respectively.

Consider now the column matrices for the flange loads and stresses at all p nodal points

$$\left.\begin{aligned}\mathbf{S}&=\{P_1Q_1P_2Q_2\cdots\cdots P_9Q_9\cdots\cdots P_pQ_p\}\\ \mathbf{s}&=\{\sigma_{z1}\sigma_{s1}\sigma_{z2}\sigma_{s2}\cdots\cdots\sigma_{z9}\sigma_{s9}\cdots\cdots\sigma_{zp}\sigma_{sp}\}\end{aligned}\right\}\quad\cdots\quad(275)$$

From the set of equations of the type (272) we find,

$$\mathbf{S}=\mathbf{L}\mathbf{s}\quad\cdots\cdots\cdots\cdots\cdots\cdots\quad(276)$$

where $\mathbf{L}$ is the symmetrical partitioned matrix

$$\mathbf{L}=\begin{bmatrix}\mathbf{L}_{11} & \mathbf{L}_{12}\cdots\cdots & \mathbf{L}_{19}\cdots\cdots & \mathbf{L}_{1p}\\ \cdots & \cdots & \cdots & \cdots\\ \mathbf{L}_{91} & \mathbf{L}_{92}\cdots\cdots & \mathbf{L}_{99}\cdots\cdots & \mathbf{L}_{9p}\\ \cdots & \cdots & \cdots & \cdots\\ \mathbf{L}_{p1} & \mathbf{L}_{p2}\cdots\cdots & \mathbf{L}_{p9}\cdots\cdots & \mathbf{L}_{pp}\end{bmatrix}\quad\cdots\cdots\cdots\cdots\quad(277)$$

From (274a) the submatrices with suffices referring to non-adjoining nodal points are zero. Solving Eq. (276) for $\mathbf{s}$ we find

$$\mathbf{s}=\mathbf{L}^{-1}\mathbf{S}\quad\cdots\cdots\cdots\cdots\cdots\cdots\quad(278)$$

and hence also

$$\mathbf{e}=\frac{\mathbf{s}}{E}=\frac{1}{E}\mathbf{L}^{-1}\mathbf{S}\quad\cdots\cdots\cdots\cdots\cdots\cdots\quad(279)$$

where $\mathbf{e}$ is the column matrix of the flange strains at the nodal points, i.e.

$$\mathbf{e}=\{\epsilon_{z1}\epsilon_{s1}\cdots\cdots\cdots\epsilon_{z9}\epsilon_{s9}\cdots\cdots\cdots\epsilon_{zp}\epsilon_{sp}\}\quad\cdots\cdots\cdots\cdots\quad(279a)$$

Thus, once we have determined the effective flange loads the flange stresses and strains follow from Eqs. (279) and the direct stresses in the sheet from Eqs. (269). No guessing of effective flange areas is involved in this procedure but we have on the other hand to invert the matrix $\mathbf{L}$ with $2p$ rows and columns. It is apparent that if we knew the effective flange areas B_e, C_e at the nodal points we could immediately write down the inverted matrix as a diagonal matrix whose elements are the unit flange flexibilities at the nodal points. In fact, then

$$\frac{\mathbf{L}^{-1}}{E}=\begin{bmatrix} \frac{1}{EB_{e1}} & 0 & \cdots\cdots & 0 \\ 0 & \frac{1}{EC_{e1}} & 0 \cdots\cdots & 0 \\ \hline 0\cdots\cdots 0 & \frac{1}{EB_{e9}} & 0\cdots\cdots & 0 \\ 0\cdots\cdots 0 & \frac{1}{EC_{e9}} & 0\cdots\cdots & 0 \\ \hline 0\cdots\cdots & 0 & \frac{1}{EB_{ep}} & 0 \\ 0\cdots\cdots & & 0 & \frac{1}{EC_{ep}} \end{bmatrix} \quad \cdots\cdots \quad (277a)$$

Consider now the $\mathbf{b}_0$ and $\mathbf{b}_1$ matrices of the basic system. We emphasize that the $\mathbf{b}_0$ system can in the present method only be derived from pure static considerations since the effective flange areas are unknown. Thus, $\mathbf{b}_0$ cannot be derived for a multicell wing from an Engineers' theory of bending cum Bredt-Batho analysis unless we assign arbitrary areas to the flanges. To find, in fact, the $\mathbf{b}_0$ system in this example it is probably best to select a set of longitudinal flange loads in equilibrium with the applied loads and derive a statically consistent set of shear flows and transverse flange loads in equilibrium with the applied shear forces and torque. Contrary to the system adopted in the previous method we write here the statically determinate $\mathbf{b}_0$ and $\mathbf{b}_1$ matrices in the partitioned form

$$\mathbf{b}_0=\begin{bmatrix}\mathbf{b}_{0f}\\ \mathbf{b}_{0s}\\ \mathbf{b}_{0w}\\ \mathbf{b}_{0r}\end{bmatrix} \qquad \mathbf{b}_1=\begin{bmatrix}\mathbf{b}_{1f}\\ \mathbf{b}_{1s}\\ \mathbf{b}_{1w}\\ \mathbf{b}_{1r}\end{bmatrix} \quad \cdots\cdots\cdots\cdots \quad (280)$$

where the submatrices $\mathbf{b}_{0f}$, $\mathbf{b}_{1f}$ give the flange loads at the nodal points. The rows in $\mathbf{b}_{0f}$, $\mathbf{b}_{1f}$ are arranged in p pairs corresponding to the p nodal points; the first and second row in each pair refers to longitudinal and transverse flange loads respectively.

Returning now to Eq. (279) we apply it in the basic system for the external loads $\mathbf{R}$ and the redundancies $\{\mathbf{X}\ \mathbf{Y}\ \mathbf{Z}\}$ respectively. We find with the notation of Eq. (280) the flange strains

$$\mathbf{e}_0=\frac{1}{E}\mathbf{L}^{-1}\mathbf{b}_{0f}\mathbf{R} \quad \text{and} \quad \mathbf{e}_1=\frac{1}{E}\mathbf{L}^{-1}\mathbf{b}_{1f}\{\mathbf{X}\ \mathbf{Y}\ \mathbf{Z}\} \quad \cdots\cdots \quad (281)$$

We seek next the contribution of the flange strains $\mathbf{e}_1$ and $\mathbf{e}_0$ to the matrices $\mathbf{D}$ and $\mathbf{D}_0$ of the relative displacements δ_{ik} and δ_{i0}. For this purpose we apply a self-equilibrating unit load system $X_i=1$ (which may be also a Y- or Z-system) and denote by P_{i3} and P_{i9} its longitudinal effective flange loads at the points 3 and 9 and by Q_{i8} and Q_{i9} its transverse effective flange loads at the points 8 and 9. Let δ_i be the relative displacement at the points and directions of $X_i=1$ due to some given flange strains ϵ_z and ϵ_s. The contribution to δ_i of the straining of the flanges (3, 9) and (8, 9) is then

$$\delta_i=\cdots\cdots+\int_3^9 P_i\epsilon_z dz+\int_8^9 Q_i\epsilon_s ds+ \quad \cdots\cdots$$

and since we assume that both flange loads and strains vary linearly between nodal points we find,

$$\delta_i=\cdots+[P_{i3}P_{i9}]\cdot\begin{bmatrix} l/3 & l/6\\ l/6 & l/3\end{bmatrix}_{3,9}\begin{bmatrix}\epsilon_{z3}\\ \epsilon_{z9}\end{bmatrix}+[Q_{i8}Q_{i9}]\cdot\begin{bmatrix} d/3 & d/6\\ d/6 & d/3\end{bmatrix}_{8,9}\begin{bmatrix}\epsilon_{s8}\\ \epsilon_{s9}\end{bmatrix}+\cdots \quad \cdots\cdots \quad (282)$$

The complete expression of δ_i is arranged by pairing the terms involving the longitudinal and transverse strains at the same nodal point. Thus, showing only the typical terms involving $\{\epsilon_{z9}\epsilon_{s9}\}$,

$$\delta_i=\cdots+\{[P_{i3}Q_{i3}]\mathbf{l}_{9,3}+[P_{i8}Q_{i8}]\mathbf{l}_{9,8}+[P_{i9}Q_{i9}]\mathbf{l}_{9,9}+$$
$$+[P_{i10}Q_{i10}]\mathbf{l}_{9,10}+[P_{i15}Q_{i15}]\mathbf{l}_{9,15}\}\begin{bmatrix}\epsilon_{z9}\\ \epsilon_{s9}\end{bmatrix}+ \quad \cdots\cdots \quad (282a)$$

where the $\mathbf{l}$ matrices are,

$$\mathbf{l}_{9,9}=\begin{bmatrix} l_{3,15}/3 & 0\\ 0 & d_{8,10}/3\end{bmatrix} \quad \cdots\cdots\cdots\cdots \quad (283)$$

$$\mathbf{l}_{9,8}=\begin{bmatrix} 0 & 0\\ 0 & d_{8,9}/6\end{bmatrix},\ \mathbf{l}_{9,3}=\begin{bmatrix} l_{9,3}/6 & 0\\ 0 & 0\end{bmatrix} \quad \cdots\cdots \quad (283a)$$

The matrix $\mathbf{l}_{9,10}(\mathbf{l}_{9,15})$ is obtained from $\mathbf{l}_{9,8}(\mathbf{l}_{9,3})$ by substituting 10(15) for 8(3). It is simple now to write down in Eq. (282) the terms for any other pairs of strains $(\epsilon_z\epsilon_s)$. We deduce immediately that,

$$\mathbf{l}_{rs}=\mathbf{l}_{sr} \quad \cdots\cdots\cdots\cdots \quad (284)$$

and that all $\mathbf{l}_{rs}$ matrices are zero when they do not refer to adjoining points. Moreover, matrices (283a) are typical for adjoining nodal points in the z and s directions respectively.

Introducing the matrix

$$\mathbf{l}=\begin{bmatrix} \mathbf{l}_{1,1}\cdots\cdots\mathbf{l}_{1,9}\cdots\cdots\mathbf{l}_{1,p}\\ \hline \mathbf{l}_{9,1}\cdots\cdots\mathbf{l}_{9,9}\cdots\cdots\mathbf{l}_{9,p}\\ \hline \mathbf{l}_{p1}\cdots\cdots\mathbf{l}_{p9}\cdots\cdots\mathbf{l}_{pp}\end{bmatrix} \quad \cdots\cdots\cdots\cdots \quad (285)$$

we can express Eq. (282) concisely as

$$\delta_i=[P_{i1}Q_{i1}\cdots\cdots P_{i9}Q_{i9}\cdots\cdots P_{ip}Q_{ip}]\,\mathbf{le} \quad \cdots\cdots \quad (286)$$

Observing that the matrix

$$[P_{i1}Q_{i1}\cdots\cdots P_{i9}Q_{i9}\cdots\cdots P_{ip}Q_{ip}]$$

is the ith row of the $\mathbf{b}_{1f}$ matrix and noting Eqs. (281) we find that the contributions of the flange strains to the $\mathbf{D}$ and $\mathbf{D}_0$ matrices are

$$\mathbf{D}_f=\mathbf{b}_{1f}'\mathbf{l}\frac{\mathbf{L}^{-1}}{E}\mathbf{b}_{1f} \qquad \mathbf{D}_{0f}=\mathbf{b}_{1f}'\mathbf{l}\frac{\mathbf{L}^{-1}}{E}\mathbf{b}_{0f}\mathbf{R} \quad \cdots\cdots\cdots\cdots \quad (287)$$

We conclude that the flexibility of the flanges at the nodal points is given by

$$\mathbf{f}_f=\mathbf{l}\frac{\mathbf{L}^{-1}}{E} \quad \cdots\cdots\cdots\cdots \quad (288)$$

Note the structural similarity between the $\mathbf{l}$ and $\mathbf{L}$ matrices. It is particularly pronounced when $\nu=0$. The total flexibility of the elements of the structure is now

$$\mathbf{f}=\begin{bmatrix}\mathbf{f}_f & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{f}_s & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{f}_w & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{f}_r\end{bmatrix} \quad \cdots\cdots\cdots\cdots \quad (289)$$

where the flexibilities of the cover, webs and ribs are as before.

We find for the matrices $\mathbf{D}$ and $\mathbf{D}_0$.

$$\mathbf{D}=\mathbf{b}_1'\mathbf{f}\mathbf{b}_1 \text{ and } \mathbf{D}_0=\mathbf{b}_1'\mathbf{f}\mathbf{b}_0\mathbf{R} \quad \cdots\cdots\cdots\cdots \quad (225b)$$

where $\mathbf{b}_1$ and $\mathbf{b}_0$ are arranged as in Eqs. (280).

Using Eqs. (225b) in (259) we solve the problem completely.

The refinement introduced by the $\mathbf{L}$ matrix need not of course extend over the complete wing but may be restricted to the root and other marked changes of structure and loading. For the rest of the structure it may still be sufficient to estimate the effective flange areas and to use the simple form (277a).

D. The Analysis of Structures by the Displacement Method

The analogy between the developments for the flexibilities and stiffnesses given under A and B and summarized in TABLE I shows clearly that parallel to the analysis of structures with forces as unknowns there must be a corresponding theory with deformations as unknowns. As mentioned in the introduction to this section Ostenfeld* when investigating frameworks was the first to draw attention to such an analogy. In fact, his equations are the exact counterpart of the classic δ_{ik} equations given by Mueller-Breslau for forces as unknowns. In more recent times Southwell† and his pupils have used his relaxation technique to solve the elasticity equations in the finite difference form with displacements as unknowns for a great number of problems. Hoff‡ has applied the latter method to diffusion and related problems in aircraft structures and has solved also the corresponding equations directly. Lately Williams|| has outlined an analysis of wing-structures of the standard or solid type by introducing the deflexions at a finite grid of points as unknowns; his technique, which is intended for use in combination with the automatic digital computer, neglects however the shear deflexions, which may have an important influence.

* loc. cit. p. 43.

† loc. cit. p. 43.

‡ N. J. Hoff and Paul A. Libby, 'Recommendations for numerical solution of reinforced-panel and fuselage-ring problem. *N.A.C.A. Rep.* 934 (1949).

|| loc. cit. p. 43.

Naturally, a theory using displacements as unknowns would only be of value if it could show some concrete advantages. It is clear that such an advantage may possibly arise when the stiffnesses are simpler to calculate than the flexibilities, which is, as we have seen previously, very often the case. In particular in the egg-box structure, characteristic of aircraft wings, the stiffnesses k_{jh} are much easier to find than the influence or flexibility coefficients δ_{jh}. Another obvious advantage arises when the number of unknowns is smaller for the displacement analysis. This may occur in frameworks, especially the stiff-jointed type with few degrees of freedom at the joints. The equations in the displacements for stiff-jointed frameworks are almost invariably well conditioned; a further point in their favour, not only for iteration techniques but also for the direct solution. On the other hand in continuous structures, like wings and fuselages, this is not the case. Here, in fact, the equations in the displacements are nearly always ill-conditioned and it then becomes necessary to introduce generalized or group displacements as unknowns in order to improve the conditioning. This is a pronounced drawback of the displacement method when applied to aircraft structures. Furthermore, in such continuous systems the displacement method will usually involve a considerably greater number of unknowns than the force analysis in order to achieve a comparable degree of accuracy. It is apparent then that the choice between the two parallel techniques must be made on an *ad hoc* basis after careful consideration of the possible advantages and disadvantages of each method for a particular problem. It would, however, appear that at least with the present types of construction the force method is to be preferred for aircraft structures.

Before proceeding to the general development of the displacement analysis we introduce first a simple example to familiarize ourselves with the ideas. Consider the framework shown in FIG. 47, symmetrical both in structure and loading. The number of unknown forces or moments when the engineers' theory of bending is assumed to hold is evidently six. On the other hand, if we neglect the deformations due to shear and end load, two deformations alone, the rotations r_1 and r_2 at the stiff joints, suffice to specify completely the deformation of the system. The analysis may proceed as follows (moments and rotations are taken as positive if in the anticlockwise sense). We freeze first the joints, i.e. put

$$r_1 = r_2 = 0$$

Then, due to the loading on the upper member, moments M_0 are applied at the joints and are, with the notation of FIG. 47

$$\left.\begin{aligned} M_{120} &= -\frac{p_1 l_1^2}{12}, \quad & M_{210} &= +\frac{p_1 l_1^2}{12} \\ M_{230} &= -\frac{p_2 l_2^2}{12}, \quad & M_{320} &= +\frac{p_2 l_2^2}{12} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (290)$$

The out-of-balance moments on the joints are then

$$M_1 = -\frac{p_1 l_1^2}{12}, \quad M_2 = -\frac{p_2 l_2^2}{12} + \frac{p_1 l_1^2}{12}, \quad M_3 = 0 \quad \ldots\ldots\ldots \quad (291)$$

Consider next the system with free joints and no transverse loading subjected to the loading by the joint-moments

$$R_1 = -M_1 = +\frac{p_1 l_1^2}{12}, \quad R_2 = -M_2 = \frac{p_2 l_2^2}{12} - \frac{p_1 l_1^2}{12} \quad \ldots\ldots\ldots \quad (291a)$$

The superposition of this and the previous case yields the true solution of the given system under the transverse loading. To analyse the second problem we apply Eqs. (138) which take the form

$$\left.\begin{aligned} k_{11} r_1 + k_{12} r_2 &= R_1 \\ k_{21} r_1 + k_{22} r_2 &= R_2 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (292)$$

The stiffnesses k_{jh} are easily found as (see also Eqs. (144))

$$\left.\begin{aligned} k_{11} &= \frac{4EI_h}{h} + \frac{4EI_l}{l_1}, \quad k_{12} = \frac{2EI_l}{l_1} = k_{21} \\ k_{22} &= \frac{4EI_h}{h} + \frac{4EI_l}{l_1} + \frac{4EI_l}{l_2} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (293)$$

which assumes that the horizontal beams and supporting struts have the constant bending stiffness EI_l and EI_h respectively.

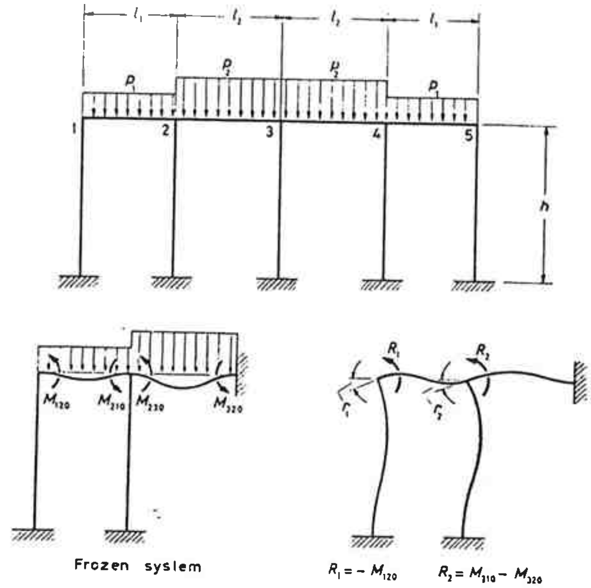


Fig. 47.—Displacement analysis of stiff-jointed frame

We obtain from Eqs. (292)

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \mathbf{K}^{-1} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \quad \ldots\ldots\ldots \quad (292a)$$

where

$$\mathbf{K}^{-1} = \frac{1}{k_{11}k_{22} - k_{12}^2} \begin{bmatrix} k_{22} & -k_{21} \\ -k_{12} & k_{11} \end{bmatrix} \quad \ldots\ldots\ldots \quad (293a)$$

Having the rotations r_1 and r_2 and using the stiffnesses of the individual elements contained in (293) we easily derive the actual moments in the structure. Thus, introducing again the usual sign convention giving positive bending moment when upper fibres are in compression, we find for the bending moment M_{12} at the junction (1) of element (1, 2)

$$M_{12} = -\frac{p_1 l_1^2}{12} - \frac{4EI_l}{l_1} r_1 - \frac{2EI_l}{l_1} r_2 \quad \ldots\ldots\ldots \quad (294)$$

This method forms also the basis of the Hardy-Cross or the more general Southwell relaxation technique in stiff-jointed frameworks.

We develop next the general theory of the displacement method. We introduce immediately the matrix notation and assume that the structure consists of a finite number s of elements whose stiffnesses k, due to relative displacements at the ends or boundaries of each element, are known. In order to show most clearly and concisely the striking analogy between the force and displacement methods we present them side by side in the following TABLE II; most of the information with respect to forces has been given previously under C. The complete duality between the two theories is, of course, a direct consequence of the twin principles of virtual displacements and virtual forces from which they derive most naturally. We believe that the analysis has not been given previously in this generality. It includes *ab initio* any effects of initial strains like temperature, lack of fit and 'give' at the foundations. The great advantage obtained in deeper insight and new theorems and applications by developing the theory on the most general lines is too apparent to need stressing.

TABLE II

A COMPARATIVE PRESENTATION OF STRUCTURAL ANALYSIS BY THE FORCE AND DISPLACEMENT METHODS

METHOD OF FORCES		METHOD OF DISPLACEMENTS	
External forces	$\mathbf{R}$	Joint displacements	$\mathbf{r}$
Flexibility	$\mathbf{F}$	Stiffness	$\mathbf{K}$
Displacements	$\mathbf{r} = \mathbf{FR}$	Forces	$\mathbf{R} = \mathbf{Kr}$
	$\mathbf{FK} = \mathbf{I}$		$\mathbf{KF} = \mathbf{I}$
See also TABLE I		See also TABLE I	

TABLE 11 (continued)

Unit Load Method	Unit Displacement Method

Unit Load Method

Given the true strains ϵ in a structure the kinematically related displacement r at a given point and direction can be calculated from

(295a) .... $1 \cdot r = \int_V \bar{\sigma} \epsilon \, dV$

where $\bar{\sigma}$ is a virtual or otherwise statically equivalent stress system due to unit load in given direction. Statically equivalent stresses ignore compatibility conditions. (See also Section 6 and FIG. 15).

Cf. Eq. (84b)

Unit Displacement Method

Given the true stresses σ in a structure the equilibrating force R at a given point and direction can be calculated from

$1 . R = \int_V \bar{\epsilon} \sigma \, dV$ .... (295b)

where $\bar{\epsilon}$ is a virtual strain system due to unit displacement in given direction. In what follows we denote virtual strains as kinematically equivalent strains. Kinematically equivalent strains ignore equilibrium conditions. (See also Section 4 and FIG. 8.)

Cf. Eq. (146)

Statically Determinate System

Internal forces (stresses) on elements determined from

(296a) .... $\mathbf{S} = \mathbf{bR}$

where matrix $\mathbf{b}$ is obtained by static reasoning alone.

Flexibility of individual (unassembled) elements $\mathbf{f}$

Internal strains,

(297a)........................ $\mathbf{v} = \mathbf{fS} = \mathbf{fbR}$

External displacements,

(298a) .... $\mathbf{r} = \mathbf{b'v} = \mathbf{b'fbR}$

Flexibility,

(299a) .... $\mathbf{F}_0 = \mathbf{b'fb}$

Cf. Eqs. (121), (122), (125), (126)

Kinematically Determinate System

Internal relative displacements (strains) of elements determined from

$\mathbf{v} = \mathbf{ar}$ .... (296b)

where matrix $\mathbf{a}$ is obtained by kinematic reasoning alone by displacing one joint at a time whilst keeping all others fixed.

Stiffness of individual (unassembled) elements $\mathbf{k}$

Internal stresses,

$\mathbf{S} = \mathbf{kv} = \mathbf{kar}$ .... (297b)

External forces,

$\mathbf{R} = \mathbf{a'S} = \mathbf{a'kar}$ .... (298b)

Stiffness,

$\mathbf{K}_0 = \mathbf{a'ka}$ .... (299b)

Cf. Eqs. (153), (154), (159), (160)

Statically Indeterminate System

In the relation

(296a) .... $\mathbf{S} = \mathbf{bR}$

$\mathbf{b}$ cannot be determined by statics alone.

Flexibility of structure needs to be considered, entering as compatibility conditions.

On the other hand if the internal strains $\mathbf{v}$ are known the kinematically related external displacements may be derived from

(300a) .... $\mathbf{r} = \bar{\mathbf{b}}'\mathbf{v}$

where $\bar{\mathbf{b}}$ is merely a *statically equivalent* (virtual) matrix due to unit R's.

Hence flexibility

(301a) .... $\mathbf{F} = \bar{\mathbf{b}}'\mathbf{fb}$

Eq. (300a) is a special form of the Unit Load method (Principle of Virtual Forces).

FIG. 48a illustrates the matrices $\mathbf{b}$ and $\bar{\mathbf{b}}$ on a particularly simple example of a singly redundant system.

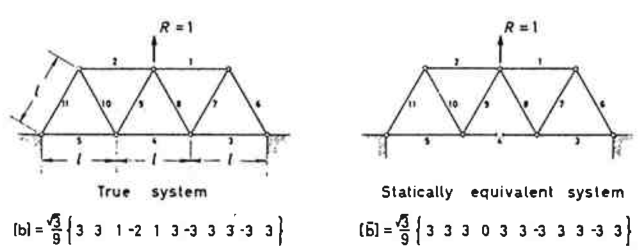


Fig. 48(a).—True and statically equivalent stress systems in singly redundant, pin-jointed framework

Cf. Eqs. (125a), (126)

Kinematically Indeterminate System

In the relation

$\mathbf{v} = \mathbf{ar}$ .... (296b)

$\mathbf{a}$ cannot be determined by kinematics alone.

Stiffness of structure needs to be considered, entering as equilibrium conditions.

On the other hand if internal stresses $\mathbf{S}$ are known the equilibrating external forces may be derived from

$\mathbf{R} = \bar{\mathbf{a}}'\mathbf{S}$ .... (300b)

where $\bar{\mathbf{a}}$ is merely a *kinematically equivalent* (virtual) matrix due to unit r's.

Hence stiffness

$\mathbf{K} = \bar{\mathbf{a}}'\mathbf{ka}$ .... (301b)

Eq. (300b) is a special form of the Unit Displacement method (Principle of Virtual Displacements).

FIG. (48b) illustrates the matrices $\mathbf{a}$ and $\bar{\mathbf{a}}$ on the same example as in FIG. 48a.

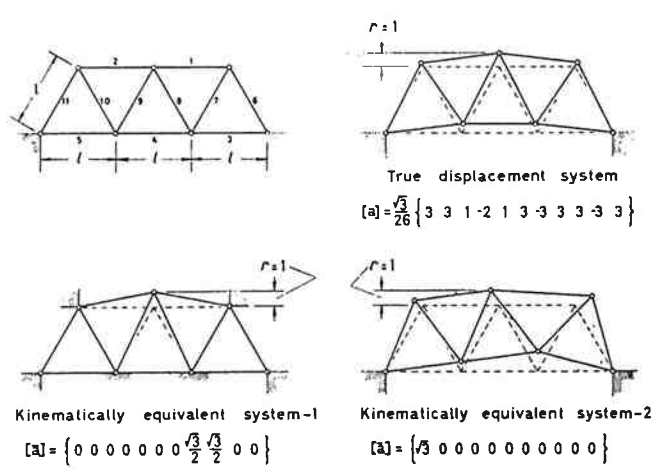


Fig. 48(b).—True and kinematically equivalent displacement systems in pin-jointed framework

Cf. Eqs. (159a), (160b)

Problem a

Given a set of forces $\mathbf{R}$, determine a set of statically indeterminate forces $\mathbf{X}$ necessary to satisfy the compatibility conditions. Find also the displacements $\mathbf{r}$ in the directions of $\mathbf{R}$.

Complete force matrix

$$\{\mathbf{R}\ \ \mathbf{X}\} \tag{302a}$$

By putting $\mathbf{X}=\mathbf{O}$ we obtain the so-called basic system which is statically determinate within limits of idealization.

Stresses in basic system

$$\mathbf{S}_0=\mathbf{b}_0\mathbf{R} \tag{303a}$$

Stresses due to $\mathbf{X}$ (with $\mathbf{R}=\mathbf{O}$)

$$\mathbf{S}_1=\mathbf{b}_1\mathbf{X} \tag{304a}$$

where $\mathbf{b}_0$ and $\mathbf{b}_1$ are obtained from statics alone.

True stresses in actual structure

$$\mathbf{S}=\mathbf{S}_0+\mathbf{S}_1=\mathbf{b}_0\mathbf{R}+\mathbf{b}_1\mathbf{X} \tag{305a}$$

Strains of elements

$$\mathbf{v}=\mathbf{f}\mathbf{S}=\mathbf{f}\mathbf{b}_0\mathbf{R}+\mathbf{f}\mathbf{b}_1\mathbf{X} \tag{306a}$$

Compatibility condition in actual system at points of application of forces $\mathbf{X}$

$$\mathbf{b}_1'\mathbf{v}=\mathbf{b}_1'\mathbf{f}\mathbf{b}_0\mathbf{R}+\mathbf{b}_1'\mathbf{f}\mathbf{b}_1\mathbf{X}=\mathbf{O} \tag{307a}$$

or

$$\mathbf{D}\mathbf{X}+\mathbf{D}_0=\mathbf{O} \tag{308a}$$

where

$$\mathbf{D}=\mathbf{b}_1'\mathbf{f}\mathbf{b}_1\ , \quad \mathbf{D}_0=\mathbf{b}_1'\mathbf{f}\mathbf{b}_0\mathbf{R} \tag{309a}$$

Hence

$$\mathbf{X}=-\mathbf{D}^{-1}\mathbf{D}_0=-(\mathbf{b}_1'\mathbf{f}\mathbf{b}_1)^{-1}\mathbf{b}_1'\mathbf{f}\mathbf{b}_0\mathbf{R} \tag{310a}$$

True stresses

$$\mathbf{S}=\mathbf{b}\mathbf{R} \tag{296a}$$

where

$$\mathbf{b}=\mathbf{b}_0-\mathbf{b}_1(\mathbf{b}_1'\mathbf{f}\mathbf{b}_1)^{-1}\mathbf{b}_1'\mathbf{f}\mathbf{b}_0 \tag{311a}$$

True strains

$$\mathbf{v}=\mathbf{f}\mathbf{S}=\mathbf{f}\mathbf{b}\mathbf{R} \tag{297a}$$

Displacements $\mathbf{r}$ due to $\mathbf{R}$

$$\mathbf{r}=\bar{\mathbf{b}}'\mathbf{v}=\bar{\mathbf{b}}'\mathbf{f}\mathbf{b}\mathbf{R}=\mathbf{F}\mathbf{R} \tag{300c}$$

where

$$\mathbf{F}=\mathbf{F}_0-\mathbf{b}_0'\mathbf{f}\mathbf{b}_1(\mathbf{b}_1'\mathbf{f}\mathbf{b}_1)^{-1}\mathbf{b}_1'\mathbf{f}\mathbf{b}_0 \tag{312a}$$

and

$\mathbf{F}_0=\mathbf{b}_0'\mathbf{f}\mathbf{b}_0$ is the flexibility of the basic system since we may choose

$$\bar{\mathbf{b}}=\mathbf{b}_0 \tag{313a}$$

Cf. Eqs. (217), (223), (222a), (224), (225), (226), (227a), (228), (229), (230)

Problem a

Given a set of joint displacement $\mathbf{r}$, determine the set of kinematically indeterminate joint displacements $\mathbf{U}$ necessary to satisfy the equilibrium conditions. Find also the forces $\mathbf{R}$ in the directions of $\mathbf{r}$.

Complete displacement matrix

$$\{\mathbf{r}\ \ \mathbf{U}\} \tag{302b}$$

By putting $\mathbf{U}=\mathbf{O}$ we obtain the so-called basic system which is kinematically determinate within limits of idealization.

Strains in basic system

$$\mathbf{v}_0=\mathbf{a}_0\mathbf{r} \tag{303b}$$

Strains due to $\mathbf{U}$ (with $\mathbf{r}=\mathbf{O}$)

$$\mathbf{v}_1=\mathbf{a}_1\mathbf{U} \tag{304b}$$

where $\mathbf{a}_0$ and $\mathbf{a}_1$ are obtained by kinematics alone.

True strains in actual structure

$$\mathbf{v}=\mathbf{v}_0+\mathbf{v}_1=\mathbf{a}_0\mathbf{r}+\mathbf{a}_1\mathbf{U} \tag{305b}$$

Forces on elements

$$\mathbf{S}=\mathbf{k}\mathbf{v}=\mathbf{k}\mathbf{a}_0\mathbf{r}+\mathbf{k}\mathbf{a}_1\mathbf{U} \tag{306b}$$

Equilibrium condition in actual system at non-prescribed displacements $\mathbf{U}$

$$\mathbf{a}_1'\mathbf{S}=\mathbf{a}_1'\mathbf{k}\mathbf{a}_0\mathbf{r}+\mathbf{a}_1'\mathbf{k}\mathbf{a}_1\mathbf{U}=\mathbf{O} \tag{307b}$$

or

$$\mathbf{C}\mathbf{U}+\mathbf{C}_0=\mathbf{O} \tag{308b}$$

where

$$\mathbf{C}=\mathbf{a}_1'\mathbf{k}\mathbf{a}_1\ , \quad \mathbf{C}_0=\mathbf{a}_1'\mathbf{k}\mathbf{a}_0\mathbf{r} \tag{309b}$$

Hence

$$\mathbf{U}=-\mathbf{C}^{-1}\mathbf{C}_0=-(\mathbf{a}_1'\mathbf{k}\mathbf{a}_1)^{-1}\mathbf{a}_1'\mathbf{k}\mathbf{a}_0\mathbf{r} \tag{310b}$$

True strains

$$\mathbf{v}=\mathbf{a}\mathbf{r} \tag{296b}$$

where

$$\mathbf{a}=\mathbf{a}_0-\mathbf{a}_1(\mathbf{a}_1'\mathbf{k}\mathbf{a}_1)^{-1}\mathbf{a}_1'\mathbf{k}\mathbf{a}_0 \tag{311b}$$

True stresses

$$\mathbf{S}=\mathbf{k}\mathbf{v}=\mathbf{k}\mathbf{a}\mathbf{r} \tag{297b}$$

Forces $\mathbf{R}$ due to $\mathbf{r}$

$$\mathbf{R}=\bar{\mathbf{a}}'\mathbf{S}=\bar{\mathbf{a}}'\mathbf{k}\mathbf{a}\mathbf{r}=\mathbf{K}\mathbf{r} \tag{300d}$$

where

$$\mathbf{K}=\mathbf{K}_0-\mathbf{a}_0'\mathbf{k}\mathbf{a}_1(\mathbf{a}_1'\mathbf{k}\mathbf{a}_1)^{-1}\mathbf{a}_1'\mathbf{k}\mathbf{a}_0 \tag{312b}$$

and

$\mathbf{K}_0=\mathbf{a}_0'\mathbf{k}\mathbf{a}_0$ is the stiffness of the basic system since we may choose

$$\bar{\mathbf{a}}=\mathbf{a}_0 \tag{313b}$$

Problem b

Given a set of displacements $\mathbf{r}$ find forces $\mathbf{R}$, stresses $\mathbf{S}$ and strains $\mathbf{v}$

From Eqs. (300c)

$$\mathbf{r}=\mathbf{F}\mathbf{R} \tag{314a}$$

Hence

$$\mathbf{R}=\mathbf{F}^{-1}\mathbf{r} \tag{315a}$$

$$\mathbf{S}=\mathbf{b}\mathbf{R}=\mathbf{b}\mathbf{F}^{-1}\mathbf{r} \tag{316a}$$

$$\mathbf{v}=\mathbf{f}\mathbf{S}=\mathbf{f}\mathbf{b}\mathbf{F}^{-1}\mathbf{r} \tag{317a}$$

Once $\mathbf{F}$ is known the question of statical determinacy or indeterminacy is irrelevent in this problem.

Problem b

Given a set of forces $\mathbf{R}$ find joint displacements $\mathbf{r}$, strains $\mathbf{v}$ and stresses $\mathbf{S}$

From Eqs. (300d)

$$\mathbf{R}=\mathbf{K}\mathbf{r} \tag{314b}$$

Hence

$$\mathbf{r}=\mathbf{K}^{-1}\mathbf{R} \tag{315b}$$

$$\mathbf{v}=\mathbf{a}\mathbf{r}=\mathbf{a}\mathbf{K}^{-1}\mathbf{R} \tag{316b}$$

$$\mathbf{S}=\mathbf{k}\mathbf{v}=\mathbf{k}\mathbf{a}\mathbf{K}^{-1}\mathbf{R} \tag{317b}$$

Once $\mathbf{K}$ is known the question of kinematical determinacy or indeterminacy is irrelevent in this problem.

Problem c

Given a set of initial strains $\mathbf{H}$ imposed on free unassembled elements due to temperature, lack of fit, 'give' at foundations, find stresses $\mathbf{S}$ and total strains $\mathbf{v}$ when forces $\mathbf{R}=\mathbf{O}$.

Total strains of elements

$$\mathbf{v}=\mathbf{f}\mathbf{b}_1\mathbf{X}+\mathbf{H} \tag{318a}$$

Compatibility condition in actual system at points of application of forces $\mathbf{X}$

$$\mathbf{b}_1'\mathbf{v}=\mathbf{b}_1'\mathbf{f}\mathbf{b}_1\mathbf{X}+\mathbf{b}_1'\mathbf{H}=\mathbf{O} \tag{319a}$$

Hence,

$$\mathbf{X}=-(\mathbf{b}_1'\mathbf{f}\mathbf{b}_1)^{-1}\mathbf{b}_1'\mathbf{H} \tag{320a}$$

and

$$\mathbf{S}=-\mathbf{b}_1(\mathbf{b}_1'\mathbf{f}\mathbf{b}_1)^{-1}\mathbf{b}_1'\mathbf{H} \tag{321a}$$

$$\mathbf{v}=-\mathbf{f}\mathbf{b}_1(\mathbf{b}_1'\mathbf{f}\mathbf{b}_1)^{-1}\mathbf{b}_1'\mathbf{H}+\mathbf{H} \tag{322a}$$

Note,

$$\mathbf{H}=-\mathbf{f}\mathbf{J}$$

Cf. Eq. (236)

Problem c

Given a set of initial stresses $\mathbf{J}$ imposed on elements with frozen joints (i.e. all joint displacements zero) due to temperature, lack of fit, 'give' at foundations, find strains $\mathbf{v}$ and stresses $\mathbf{S}$ when displacements $\mathbf{r}=\mathbf{O}$.

Total stresses on elements

$$\mathbf{S}=\mathbf{k}\mathbf{a}_1\mathbf{U}+\mathbf{J} \tag{318b}$$

Note that the column matrix $\mathbf{U}$ must here include all unknown joint displacements.

Equilibrium condition in actual system in the directions of $\mathbf{U}$

$$\mathbf{a}_1'\mathbf{S}=\mathbf{a}_1'\mathbf{k}\mathbf{a}_1\mathbf{U}+\mathbf{a}_1'\mathbf{J}=\mathbf{O} \tag{319b}$$

Hence,

$$\mathbf{U}=-(\mathbf{a}_1'\mathbf{k}\mathbf{a}_1)^{-1}\mathbf{a}_1'\mathbf{J} \tag{320b}$$

and

$$\mathbf{v}=-\mathbf{a}_1(\mathbf{a}_1'\mathbf{k}\mathbf{a}_1)^{-1}\mathbf{a}_1'\mathbf{J} \tag{321b}$$

$$\mathbf{S}=-\mathbf{k}\mathbf{a}_1(\mathbf{a}_1'\mathbf{k}\mathbf{a}_1)^{-1}\mathbf{a}_1'\mathbf{J}+\mathbf{J} \tag{322b}$$

Note,

$$\mathbf{J}=-\mathbf{k}\mathbf{H}$$

Problem d

Assume that we write the total set of forces (including the statically indeterminate forces) in the partitioned form

(323a) $$\begin{bmatrix} \mathbf{R} \\ \mathbf{X} \\ \mathbf{Z} \end{bmatrix}$$

in which $\mathbf{Z}$ is known in terms of $\mathbf{R}$ and $\mathbf{X}$. We set now the modified problem (a): Given the set of forces $\mathbf{R}$ determine the set of forces $\mathbf{X}$ necessary to satisfy the compatibility conditions.

Here in the basic system obtained by putting $\mathbf{X}=\mathbf{O}$ the stresses

(303a) $$\mathbf{S}_0=\mathbf{b}_0\mathbf{R}$$

are completely known although the system is statically indeterminate.

Similarly we know the stresses

(304a) $$\mathbf{S}_1=\mathbf{b}_1\mathbf{X}$$

when $\mathbf{R}=\mathbf{O}$

True stresses in actual structure,

(305a) $$\mathbf{S}=\mathbf{S}_0+\mathbf{S}_1=\mathbf{b}_0\mathbf{R}+\mathbf{b}_1\mathbf{X}$$

strains in elements

(306a) $$\mathbf{v}=\mathbf{f}\mathbf{S}=\mathbf{f}\mathbf{b}_0\mathbf{R}+\mathbf{f}\mathbf{b}_1\mathbf{X}$$

Compatibility condition in actual system at points of application of $\mathbf{X}$

(324a) $$\bar{\mathbf{b}}_1'\mathbf{v}=\bar{\mathbf{b}}_1'\mathbf{f}\mathbf{b}_0\mathbf{R}+\bar{\mathbf{b}}_1'\mathbf{f}\mathbf{b}_1\mathbf{X}=\mathbf{O}$$

where $\bar{\mathbf{b}}_1$ is a set of stresses statically equivalent to unit X's (and $\mathbf{R}=\mathbf{O}$) preferably found for $\mathbf{Z}=\mathbf{O}$. In the latter case the rows of $\bar{\mathbf{b}}_1$ are the same as the corresponding rows of $\mathbf{b}_1$ of Problem (a).

Thus,

(325a) $$\mathbf{X}=-(\bar{\mathbf{b}}_1'\mathbf{f}\mathbf{b}_1)^{-1}\bar{\mathbf{b}}_1'\mathbf{f}\mathbf{b}_0\mathbf{R}$$

True stresses and strains

(296a) $$\mathbf{S}=\mathbf{b}\mathbf{R}\ , \quad \mathbf{v}=\mathbf{f}\mathbf{b}\mathbf{R}$$

where

(326a) $$\mathbf{b}=\mathbf{b}_0-\mathbf{b}_1(\bar{\mathbf{b}}_1'\mathbf{f}\mathbf{b}_1)^{-1}\bar{\mathbf{b}}_1'\mathbf{f}\mathbf{b}_0$$

Displacements $\mathbf{r}$ due to $\mathbf{R}$ (see Eq. (300a))

(327a) $$\mathbf{r}=\bar{\mathbf{b}}_0'\mathbf{v}=\mathbf{F}\mathbf{R}$$

where

(328a) $$\mathbf{F}=\mathbf{F}_0-\bar{\mathbf{b}}_0'\mathbf{f}\mathbf{b}_1(\bar{\mathbf{b}}_1'\mathbf{f}\mathbf{b}_1)^{-1}\bar{\mathbf{b}}_1'\mathbf{f}\mathbf{b}_0$$

and

(329a) $$\mathbf{F}_0=\bar{\mathbf{b}}_0'\mathbf{f}\mathbf{b}_0$$

is the flexibility of the basic system.

The matrix $\bar{\mathbf{b}}_0$ is a set of stresses statically equivalent to unit R's preferably found for $\mathbf{Z}=\mathbf{O}$ and $\mathbf{X}=\mathbf{O}$. In the latter case $\bar{\mathbf{b}}_0$ is identical with $\mathbf{b}_0$ of problem (a).

Problem d

Assume that we write the total set of joint displacements in the partitioned form

$$\begin{bmatrix} \mathbf{r} \\ \mathbf{U} \\ \mathbf{W} \end{bmatrix}$$ (323b)

in which $\mathbf{W}$ is known in terms of $\mathbf{r}$ and $\mathbf{U}$. We set now the modified problem (a): Given the set of displacements $\mathbf{r}$ determine the set of displacements $\mathbf{U}$ necessary to satisfy the equilibrium conditions.

Here in the basic system obtained by putting $\mathbf{U}=\mathbf{O}$ the strains

$$\mathbf{v}_0=\mathbf{a}_0\mathbf{r}$$ (303b)

are completely known although the system is kinematically indeterminate.

Similarly we know the strains

$$\mathbf{v}_1=\mathbf{a}_1\mathbf{U}$$ (304b)

when $\mathbf{r}=\mathbf{O}$

True strains in actual structure,

$$\mathbf{v}=\mathbf{v}_0+\mathbf{v}_1=\mathbf{a}_0\mathbf{r}+\mathbf{a}_1\mathbf{U}$$ (305a)

forces on elements

$$\mathbf{S}=\mathbf{k}\mathbf{v}=\mathbf{k}\mathbf{a}_0\mathbf{r}+\mathbf{k}\mathbf{a}_1\mathbf{U}$$ (306a)

Equilibrium condition in actual system at displacements $\mathbf{U}$

$$\bar{\mathbf{a}}_1'\mathbf{S}=\bar{\mathbf{a}}_1'\mathbf{k}\mathbf{a}_0\mathbf{r}+\bar{\mathbf{a}}_1'\mathbf{k}\mathbf{a}_1\mathbf{U}=\mathbf{O}$$ (324b)

where $\bar{\mathbf{a}}_1$ is a set of strains kinematically equivalent to unit U's (and $\mathbf{r}=\mathbf{O}$) preferably found for $\mathbf{W}=\mathbf{O}$. In the latter case the rows of $\bar{\mathbf{a}}_1$ are the same as the corresponding rows of $\mathbf{a}_1$ of Problem (a).

Thus,

$$\mathbf{U}=-(\bar{\mathbf{a}}_1'\mathbf{k}\mathbf{a}_1)^{-1}\bar{\mathbf{a}}_1'\mathbf{k}\mathbf{a}_0\mathbf{r}$$ (325b)

True strains and stresses

$$\mathbf{v}=\mathbf{a}\mathbf{r}\ , \quad \mathbf{S}=\mathbf{k}\mathbf{a}\mathbf{r}$$ (296b)

where

$$\mathbf{a}=\mathbf{a}_0-\mathbf{a}_1(\bar{\mathbf{a}}_1'\mathbf{k}\mathbf{a}_1)^{-1}\bar{\mathbf{a}}_1'\mathbf{k}\mathbf{a}_0$$ (326b)

Forces $\mathbf{R}$ due to $\mathbf{r}$ (see Eq. (300b))

$$\mathbf{R}=\bar{\mathbf{a}}_0'\mathbf{S}=\mathbf{K}\mathbf{r}$$ (327b)

where

$$\mathbf{K}=\mathbf{K}_0-\mathbf{a}_0'\mathbf{k}\mathbf{a}_1(\bar{\mathbf{a}}_1'\mathbf{k}\mathbf{a}_1)^{-1}\bar{\mathbf{a}}_1'\mathbf{k}\mathbf{a}_0$$ (328b)

and

$$\mathbf{K}_0=\bar{\mathbf{a}}_0'\mathbf{k}\mathbf{a}_0$$ (329b)

is the stiffness of the basic system.

The matrix $\bar{\mathbf{a}}_0$ is a set of strains kinematically equivalent to unit r's preferably found for $\mathbf{W}=\mathbf{O}$ and $\mathbf{U}=\mathbf{O}$. In the latter case $\bar{\mathbf{a}}_0$ is identical with $\mathbf{a}_0$ of problem (a).

Cf. Eqs. (226), (227a), (229), (230).

Condensation of flexibility matrix

The calculation of the flexibility matrix $\mathbf{F}$ given under problem (a) can be developed concisely as a condensation of the complete flexibility matrix for the forces $\mathbf{R}$ and $\mathbf{X}$.

This matrix may be written as

$$\begin{bmatrix} \mathbf{F}_I & \mathbf{F}_{III}' \\ \mathbf{F}_{III} & \mathbf{F}_{II} \end{bmatrix}$$

where I(II) is for forces $\mathbf{R}$ ($\mathbf{X}$) only and was denoted by $\mathbf{F}_0$ ($\mathbf{D}$) in problem (a). Evidently $\mathbf{F}_{III}\mathbf{R}=\mathbf{D}_0$.

The flexibility matrix $\mathbf{F}$ of the actual structure under the forces $\mathbf{R}$ is then

(331a) $$\mathbf{F}=\mathbf{F}_I-\mathbf{F}_{III}'\mathbf{F}_{II}^{-1}\mathbf{F}_{III}$$

Naturally this condensation may be performed in two or more stages and is then equivalent to the method of problem (d).

Condensation of stiffness matrix

The calculation of the stiffness matrix $\mathbf{K}$ given under problem (a) can be developed concisely as a condensation of the complete stiffness matrix for the displacements $\mathbf{r}$ and $\mathbf{U}$.

This stiffness may be written as

$$\begin{bmatrix} \mathbf{K}_I & \mathbf{K}_{III}' \\ \mathbf{K}_{III} & \mathbf{K}_{II} \end{bmatrix}$$

where I(II) is for displacements $\mathbf{r}$ ($\mathbf{U}$) only and was denoted by $\mathbf{K}_0$ ($\mathbf{C}$) in problem (a). Evidently $\mathbf{K}_{III}\mathbf{r}=\mathbf{C}_0$.

The stiffness matrix $\mathbf{K}$ of the actual structure for the displacements $\mathbf{r}$ is then

$$\mathbf{K}=\mathbf{K}_I-\mathbf{K}_{III}'\mathbf{K}_{II}^{-1}\mathbf{K}_{III}$$ (331b)

Naturally this condensation may be performed in two or more stages and is then equivalent to the method of problem (d).

Cf. Eq. (175)

Elimination and rigidification of structural elements

Assume a set of initial strains, written as column matrix **H**, in the structural elements to be removed, of such magnitude as to give zero stress in resultant system.
Write the **b** and $\mathbf{b}_1$ matrices of the complete structure in the partitioned form

(332a) .... $\mathbf{b}=\begin{bmatrix}\mathbf{b}_g\\ \mathbf{b}_h\end{bmatrix}, \quad \mathbf{b}_1=\begin{bmatrix}\mathbf{b}_{1g}\\ \mathbf{b}_{1h}\end{bmatrix}$

where the suffix h refers to those elements that are to be removed.

We find

(333a) .... $\mathbf{H}=(\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}')^{-1}\mathbf{b}_h\mathbf{R}$

and hence forces in the new structure

(334a) .... $\mathbf{S}_g=\{\mathbf{b}_g-\mathbf{b}_{1g}\mathbf{D}^{-1}\mathbf{b}_{1h}'(\mathbf{b}_{1h}\mathbf{D}^{-1}\mathbf{b}_{1h}')^{-1}\mathbf{b}_h\}\mathbf{R}$

In this process the number of statically indeterminate forces **X** has been reduced to a degree depending on the number of elements removed.

In the inverse process of making infinitely rigid certain of the structural elements we have merely to put $\mathbf{f}_h=\mathbf{o}$ for the affected elements. The number of statically indeterminate forces remains the same.

Cf. Eqs. (264), (267), (268)

Rigidification and elimination of structural elements

Assume a set of initial stresses, written as column matrix **J**, in the structural elements to be made infinitely rigid, of such magnitude as to give zero strain in resultant system.
Write the **a** and $\mathbf{a}_1$ matrices of the complete structure in the partitioned form

$\mathbf{a}=\begin{bmatrix}\mathbf{a}_g\\ \mathbf{a}_h\end{bmatrix}, \quad \mathbf{a}_1=\begin{bmatrix}\mathbf{a}_{1g}\\ \mathbf{a}_{1h}\end{bmatrix}$ .... (332b)

where the suffix h refers to those elements that are to be made infinitely rigid.
We find

$\mathbf{J}=(\mathbf{a}_{1h}\mathbf{C}^{-1}\mathbf{a}_{1h}')^{-1}\mathbf{a}_h\mathbf{r}$ .... (333b)

and hence strains in the new structure

$\mathbf{v}_g=\{\mathbf{a}_g-\mathbf{a}_{1g}\mathbf{C}^{-1}\mathbf{a}_{1h}'(\mathbf{a}_{1h}\mathbf{C}^{-1}\mathbf{a}_{1h}')^{-1}\mathbf{a}_h\}\mathbf{r}$ .... (334b)

In the process of making elements infinitely rigid (stiff) we introduce kinematic relations between displacements and hence reduce the number of unknown displacements **U** accordingly.

In the inverse process of eliminating certain of the structural elements we have merely to put $\mathbf{k}_h=\mathbf{o}$ for the affected elements. The number of kinematically indeterminate displacements remains the same.

Generalized Forces

Generalized forces given by

(335a) .... $\begin{bmatrix}\mathbf{R}\\ \mathbf{X}\end{bmatrix}=\begin{bmatrix}\mathbf{B}_0 & \mathbf{O}\\ \mathbf{O} & \mathbf{B}_1\end{bmatrix}\begin{bmatrix}\bar{\mathbf{R}}\\ \bar{\mathbf{X}}\end{bmatrix}$

The equation for the unknown $\bar{\mathbf{X}}$ is

(336a) .... $\bar{\mathbf{D}}\bar{\mathbf{X}}+\bar{\mathbf{D}}_0=\mathbf{O}$

where

(337a) .... $\bar{\mathbf{D}}=\mathbf{B}_1'\mathbf{D}\mathbf{B}_1$ and $\bar{\mathbf{D}}_0=\mathbf{B}_1'\mathbf{D}_0\mathbf{B}_0$

Then

(338a) .... $\mathbf{S}=\bar{\mathbf{b}}\bar{\mathbf{R}}$

where

(339a) .... $\bar{\mathbf{b}}=\mathbf{b}_0\mathbf{B}_0-\mathbf{b}_1\mathbf{B}_1(\mathbf{B}_1'\mathbf{D}\mathbf{B}_1)^{-1}\mathbf{B}_1'\mathbf{b}_1'\mathbf{f}\mathbf{b}_0\mathbf{B}_0$

and the flexibility of the actual structure for the forces $\bar{\mathbf{R}}$ is

(340a) $\bar{\mathbf{F}}=\bar{\mathbf{F}}_0-\mathbf{B}_0'\mathbf{b}_0'\mathbf{f}\mathbf{b}_1\mathbf{B}_1(\mathbf{B}_1'\mathbf{D}\mathbf{B}_1)^{-1}\mathbf{B}_1'\mathbf{b}_1'\mathbf{f}\mathbf{b}_0\mathbf{B}_0$

where

(340c) .... $\bar{\mathbf{F}}_0=\mathbf{B}_0'\mathbf{b}_0'\mathbf{f}\mathbf{b}_0\mathbf{B}_0$

is the flexibility of the basic system under the forces $\bar{\mathbf{R}}$

Generalized Displacements

Generalized displacements given by

$\begin{bmatrix}\mathbf{r}\\ \mathbf{U}\end{bmatrix}=\begin{bmatrix}\mathbf{A}_0 & \mathbf{O}\\ \mathbf{O} & \mathbf{A}_1\end{bmatrix}\begin{bmatrix}\bar{\mathbf{r}}\\ \bar{\mathbf{U}}\end{bmatrix}$ .... (335b)

The equation for the unknown $\bar{\mathbf{U}}$ is

$\bar{\mathbf{C}}\bar{\mathbf{U}}+\bar{\mathbf{C}}_0=\mathbf{O}$ .... (336b)

where

$\bar{\mathbf{C}}=\mathbf{A}_1'\mathbf{C}\mathbf{A}_1$ and $\bar{\mathbf{C}}_0=\mathbf{A}_1'\mathbf{C}\mathbf{A}_0$ .... (337b)

Then

$\mathbf{v}=\bar{\mathbf{a}}\bar{\mathbf{r}}$ .... (338b)

where

$\bar{\mathbf{a}}=\mathbf{a}_0\mathbf{A}_0-\mathbf{a}_1\mathbf{A}_1(\mathbf{A}_1'\mathbf{C}\mathbf{A}_1)^{-1}\mathbf{A}_1'\mathbf{a}_1'\mathbf{k}\mathbf{a}_0\mathbf{A}_0$ .... (339b)

and the stiffness of the actual structure for the displacements $\bar{\mathbf{r}}$ is

$\bar{\mathbf{K}}=\bar{\mathbf{K}}_0-\mathbf{A}_0'\mathbf{a}_0'\mathbf{k}\mathbf{a}_1\mathbf{A}_1(\mathbf{A}_1'\mathbf{C}\mathbf{A}_1)^{-1}\mathbf{A}_1'\mathbf{a}_1'\mathbf{k}\mathbf{a}_0\mathbf{A}_0$ ...... (340b)

where,

$\bar{\mathbf{K}}_0=\mathbf{A}_0'\mathbf{a}_0'\mathbf{k}\mathbf{a}_0\mathbf{A}_0$ .... (340d)

is the stiffness of the basic structure for the displacements $\bar{\mathbf{r}}$

THE application of the general theory with displacements as unknowns to frameworks—both of the pin-jointed and stiff-jointed type—is straightforward. For the stiff-jointed system the method is particularly simple when direct and shear deformations are ignored. In fact, for all frameworks the determination of the matrices $\mathbf{C}$ and $\mathbf{C}_0$ is trivial once we consider all possible degrees of freedom of the joints. See for example, the systems of FIGS. (23), (24) and (48) investigated on pp. 23, and 45, which show clearly how elementary the matrices $\mathbf{a}$ and stiffness $\mathbf{k}$ are when we break up the structure into its simplest constituent components. We need not therefore concern ourselves any more here with frameworks, and we turn our attention to the membrane type of system characteristic of aircraft applications. Essentially, a major aircraft structure like a wing consists of an assembly of plates (fields) stiffened by flanges along their edges. The field may be a curved and/or tapered surface but we ignore here both these effects and consider only rectangular flat elements of constant thickness. For convenience the element formed by the plate (sheet) and its fouredge members is denoted by the term unit panel. It is assumed that the flange areas are constant along each edge.

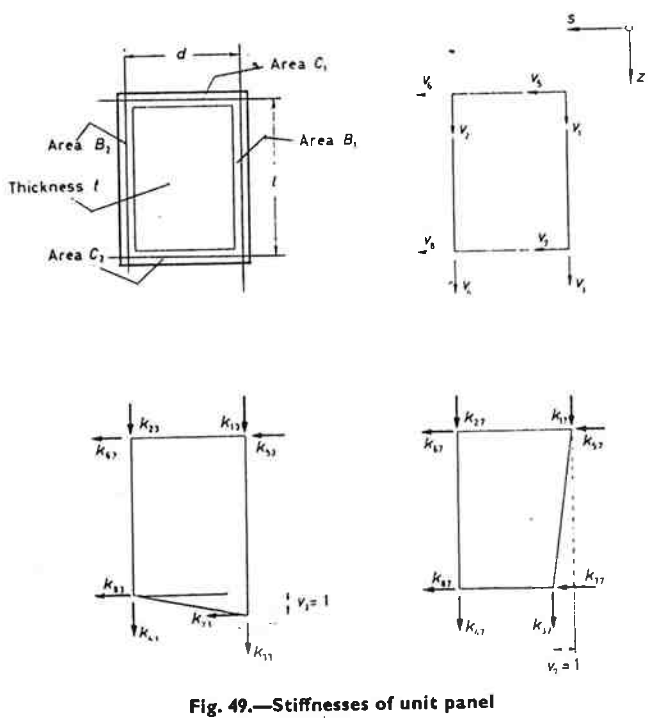


Fig. 49.—Stiffnesses of unit panel

We determine first the stiffnesses k_{jh} of the unit panel shown in FIG. 49 for unit displacements in the z- and s-directions at the four corners or nodal points of the idealized system. The stiffness of our element is hence an 8×8 matrix. As in the case of the force method it is necessary for the practical evaluation of the k_{jh} to introduce simplifying assumptions which are, naturally, concerned here with the state of deformations. Thus, we assume that the displacements vary linearly between the nodal points. Although this idealization offends against the equilibrium conditions its effect upon the stiffness is not pronounced as long as we keep the unit panels reasonably small. Nevertheless, it is inevitable that the stress distribution derived from an approximate deformation analysis should, in general, be less accurate than the one obtained from the approximate force method in the same grid system.

We denote for the purpose of the analysis of the unit panel the displacements parallel to the s and z axes by u and w respectively and introduce also the local coordinate system ξ, ζ. Consider now the state of strain and stress arising from a unit displacement

$$v_3=1 \qquad \text{(341)}$$

Following our assumption the internal displacements are given by

$$u_3=0 \qquad w_3=\frac{\zeta}{l}\left(1-\frac{\xi}{d}\right) \qquad \text{(341a)}$$

where the suffix 3 indicates that these displacements are due to $v_3=1$.

The strains and stresses in the sheet are*:

$$\left.\begin{aligned} \epsilon_{zs3}&=-\frac{\zeta}{ld}, & \sigma_{zs3}&=-G\frac{\zeta}{ld}\\ \epsilon_{zz3}&=\frac{1}{l}\left(1-\frac{\xi}{d}\right), & \epsilon_{ss3}&=0\\ \sigma_{zz3}&=\frac{E'}{l}\left(1-\frac{\xi}{d}\right), & \sigma_{ss3}&=\frac{\nu E'}{l}\left(1-\frac{\xi}{d}\right) \end{aligned}\right\} \qquad \text{(342)}$$

where $E'=E/(1-\nu^2)$

Strain ϵ_1 and load P_1 in flange B_1

$$\epsilon_{1,3}=\frac{1}{l}, \qquad P_{1,3}=B_1\frac{1}{l} \qquad \text{(342a)}$$

all other flange strains and loads are zero.

Similar formulae are obtained for the strains and stresses due to any other $v_j=1$. To derive the stiffnesses we apply the unit displacement method Eq. (295b), which takes here the form,*

$$1.k_{jh}=\int_0^d\int_0^l \sigma_j\epsilon_h\,d\xi\,d\zeta$$

where the integral extends over sheet and flange. For example, for the stiffnesses associated with $v_3=1$ we obtain,

$$\left.\begin{aligned} k_{13}&=-\frac{E'dt}{3l}-\frac{EB_1}{l}+\frac{Gtl}{6d}\\ k_{23}&=-\frac{E'dt}{6l}+0-\frac{Gtl}{6d}\\ k_{33}&=+\frac{E'dt}{3l}+\frac{EB_1}{l}+\frac{Gtl}{3d}\\ k_{43}&=+\frac{E'dt}{6l}+0-\frac{Gtl}{3d} \end{aligned}\right\} \qquad \text{(343)}$$

and

$$\left.\begin{aligned} k_{53}&=-\frac{\nu E't}{4}+\frac{Gt}{4}, & k_{63}&=+\frac{\nu E't}{4}+\frac{Gt}{4}\\ k_{73}&=-\frac{\nu E't}{4}-\frac{Gt}{4}, & k_{83}&=+\frac{\nu E't}{4}-\frac{Gt}{4} \end{aligned}\right\} \qquad \text{(343a)}$$

It is simple now to write down the stiffnesses corresponding to any other unit displacement. For convenience we express the total stiffness matrix in the form

$$\mathbf{k}=\mathbf{k}_s+\mathbf{k}_d+\mathbf{k}_f \qquad \text{(344)}$$

* Contrary to our usual notation subscripts are used here to denote stresses and strains due to unit displacements.

where the suffices s, d, f indicate the partial stiffnesses for shear strains and direct strains in sheet, and direct strains in flanges, We find

$$\mathbf{k}_s=\begin{bmatrix} \frac{Gtl}{3d} & -\frac{Gtl}{3d} & \frac{Gtl}{6d} & -\frac{Gtl}{6d} & \frac{Gt}{4} & \frac{Gt}{4} & -\frac{Gt}{4} & -\frac{Gt}{4} \\ -\frac{Gtl}{3d} & \frac{Gtl}{3d} & -\frac{Gtl}{6d} & \frac{Gtl}{6d} & -\frac{Gt}{4} & -\frac{Gt}{4} & \frac{Gt}{4} & \frac{Gt}{4} \\ \frac{Gtl}{6d} & -\frac{Gtl}{6d} & \frac{Gtl}{3d} & -\frac{Gtl}{3d} & \frac{Gt}{4} & \frac{Gt}{4} & -\frac{Gt}{4} & -\frac{Gt}{4} \\ -\frac{Gtl}{6d} & \frac{Gtl}{6d} & -\frac{Gtl}{3d} & \frac{Gtl}{3d} & -\frac{Gt}{4} & -\frac{Gt}{4} & \frac{Gt}{4} & \frac{Gt}{4} \\ \frac{Gt}{4} & -\frac{Gt}{4} & \frac{Gt}{4} & -\frac{Gt}{4} & \frac{Gtd}{3l} & \frac{Gtd}{6l} & -\frac{Gtd}{3l} & -\frac{Gtd}{6l} \\ \frac{Gt}{4} & -\frac{Gt}{4} & \frac{Gt}{4} & -\frac{Gt}{4} & \frac{Gtd}{6l} & \frac{Gtd}{3l} & -\frac{Gtd}{6l} & -\frac{Gtd}{3l} \\ -\frac{Gt}{4} & \frac{Gt}{4} & -\frac{Gt}{4} & \frac{Gt}{4} & -\frac{Gtd}{3l} & -\frac{Gtd}{6l} & \frac{Gtd}{3l} & \frac{Gtd}{6l} \\ -\frac{Gt}{4} & \frac{Gt}{4} & -\frac{Gt}{4} & \frac{Gt}{4} & -\frac{Gtd}{6l} & -\frac{Gtd}{3l} & \frac{Gtd}{6l} & \frac{Gtd}{3l} \end{bmatrix} \quad (345)$$

$$\mathbf{k}_d=\begin{bmatrix} \frac{E'dt}{3l} & \frac{E'dt}{6l} & -\frac{E'dt}{3l} & -\frac{E'dt}{6l} & \frac{\nu E't}{4} & -\frac{\nu E't}{4} & \frac{\nu E't}{4} & -\frac{\nu E't}{4} \\ \frac{E'dt}{6l} & \frac{E'dt}{3l} & -\frac{E'dt}{6l} & -\frac{E'dt}{3l} & \frac{\nu E't}{4} & -\frac{\nu E't}{4} & \frac{\nu E't}{4} & -\frac{\nu E't}{4} \\ -\frac{E'dt}{3l} & -\frac{E'dt}{6l} & \frac{E'dt}{3l} & \frac{E'dt}{6l} & -\frac{\nu E't}{4} & \frac{\nu E't}{4} & -\frac{\nu E't}{4} & \frac{\nu E't}{4} \\ -\frac{E'dt}{6l} & -\frac{E'dt}{3l} & \frac{E'dt}{6l} & \frac{E'dt}{3l} & -\frac{\nu E't}{4} & \frac{\nu E't}{4} & -\frac{\nu E't}{4} & \frac{\nu E't}{4} \\ \frac{\nu E't}{4} & \frac{\nu E't}{4} & -\frac{\nu E't}{4} & -\frac{\nu E't}{4} & \frac{E'lt}{3d} & -\frac{E'lt}{3d} & \frac{E'lt}{6d} & -\frac{E'lt}{6d} \\ -\frac{\nu E't}{4} & -\frac{\nu E't}{4} & \frac{\nu E't}{4} & \frac{\nu E't}{4} & -\frac{E'lt}{3d} & \frac{E'lt}{3d} & -\frac{E'lt}{6d} & \frac{E'lt}{6d} \\ \frac{\nu E't}{4} & \frac{\nu E't}{4} & -\frac{\nu E't}{4} & -\frac{\nu E't}{4} & \frac{E'lt}{6d} & -\frac{E'lt}{6d} & \frac{E'lt}{3d} & -\frac{E'lt}{3d} \\ -\frac{\nu E't}{4} & -\frac{\nu E't}{4} & \frac{\nu E't}{4} & \frac{\nu E't}{4} & -\frac{E'lt}{6d} & \frac{E'lt}{6d} & -\frac{E'lt}{3d} & \frac{E'lt}{3d} \end{bmatrix} \quad (345a)$$

$$\mathbf{k}_f=\begin{bmatrix} \frac{EB_1}{l} & 0 & -\frac{EB_1}{l} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{EB_2}{l} & 0 & -\frac{EB_2}{l} & 0 & 0 & 0 & 0 \\ -\frac{EB_1}{l} & 0 & \frac{EB_1}{l} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{EB_2}{l} & 0 & \frac{EB_2}{l} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{EC_1}{d} & -\frac{EC_1}{d} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{EC_1}{d} & \frac{EC_1}{d} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{EC_2}{d} & -\frac{EC_2}{d} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{EC_2}{d} & \frac{EC_2}{d} \end{bmatrix} \quad (345b)$$

In assembling the panels of FIG. 50 to form a wing structure the stiffnesses (345) and (345a) may be simplified considerably when applied to rib and spar webs. Thus, for these cases we can always neglect the expansion in the direction of the height of the web. The corresponding formulae may be obtained by putting $v_5=v_6$ and $v_7=v_8$ which apply when C_1 and C_2 are infinite.

The stiffnesses $\mathbf{k}_s$ and $\mathbf{k}_d$ contract to 6×6 matrices and are

$$\mathbf{k}_s=\begin{bmatrix} \frac{Gtl}{3d} & \frac{Gtl}{3d} & \frac{Gtl}{6d} & -\frac{Gtl}{6d} & \frac{Gt}{2} & -\frac{Gt}{2} \\ -\frac{Gtl}{3d} & \frac{Gtl}{3d} & \frac{Gtl}{6d} & \frac{Gtl}{6d} & -\frac{Gt}{2} & \frac{Gt}{2} \\ \frac{Gtl}{6d} & \frac{Gtl}{6d} & \frac{Gtl}{3d} & \frac{Gtl}{3d} & \frac{Gt}{2} & \frac{Gt}{2} \\ -\frac{Gtl}{6d} & \frac{Gtl}{6d} & \frac{Gtl}{3d} & \frac{Gtl}{3d} & -\frac{Gt}{2} & \frac{Gt}{2} \\ \frac{Gt}{2} & -\frac{Gt}{2} & \frac{Gt}{2} & \frac{Gt}{2} & \frac{Gtd}{l} & -\frac{Gtd}{l} \\ -\frac{Gt}{2} & \frac{Gt}{2} & \frac{Gt}{2} & \frac{Gt}{2} & \frac{Gtd}{l} & \frac{Gtd}{l} \end{bmatrix}. \quad \ldots (346)$$

$$\mathbf{k}_d=\begin{bmatrix} \frac{E'dt}{3l} & \frac{E'dt}{6l} & \frac{E'dt}{3l} & \frac{E'dt}{6l} & 0 & 0 \\ \frac{E'dt}{6l} & \frac{E'dt}{3l} & \frac{E'dt}{6l} & \frac{E'dt}{3l} & 0 & 0 \\ \frac{E'dt}{3l} & \frac{E'dt}{6l} & \frac{E'dt}{3l} & \frac{E'dt}{6l} & 0 & 0 \\ \frac{E'dt}{6l} & \frac{E'dt}{3l} & \frac{E'dt}{6l} & \frac{E'dt}{3l} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \ldots. (346a)$$

No contribution of the flanges is called for when evaluating the stiffnesses of the webs since $\mathbf{k}$ is best included in the top and bottom panels.

Further simplification of the stiffness matrices for the webs is possible when the top and bottom panels of our wing structure are identical. Then for vertical loads alone the horizontal displacements in the two covers are antisymmetrical and the stiffness matrices (346) and (346a) may be contracted to 4×4 matrices.

We illustrate now the application of the unit panel stiffnesses to the diffusion problem shown in FIG. (50). The plate is reinforced longitudinally and laterally by stiffeners of area B and C respectively, and edge members of area B_f. Displacements in the x and z directions are defined at all nodes of the grid formed by lateral and longitudinal stiffening. Naturally the grid does not have to be restricted to this definition and we can always choose a finer one if the stiffeners are widely spaced so that the assumption of linear variation between adjacent nodal points can represent adequately the displacement pattern. Using the stiffness matrix of the unit panel already derived, the setting up of the complete stiffness matrix follows quite simply. It is only necessary to identify quickly and easily the displacements defined for the unit panels separately with those defined for the assembled panel. The complete stiffness matrix is obtained as (Eq. 299b).

$$\mathbf{K}=\mathbf{a}'\mathbf{k}\mathbf{a}$$

where $\mathbf{k}$ is the stiffness matrix of the unassembled unit panels and may be written in the diagonal partitioned form

$$\mathbf{k}=\begin{bmatrix} \mathbf{k}_u & \mathbf{o} & \ldots\ldots\ldots\ldots & \mathbf{o} \\ \mathbf{o} & \mathbf{k}_u & \mathbf{o} \ldots\ldots\ldots & \mathbf{o} \\ & & \ddots & \\ \mathbf{o} \ldots\ldots \mathbf{o} & & \mathbf{k}_u \quad \mathbf{o} & \ldots \mathbf{o} \\ & & & \ddots \\ \mathbf{o} & \ldots\ldots\ldots\ldots & \mathbf{o} & \mathbf{k}_u \end{bmatrix} \quad \ldots\ldots\ldots\ldots (347)$$

$\mathbf{k}_g$ is the (8×8) stiffness matrix for the unit panel g as derived previously, see Eq. (344).* To preserve the symmetry of formulation, the stiffnesses of the reinforcement elements are included with the panels. Thus each of the areas B or C is split in two and $B/2$ or $C/2$ associated with the panel on each side. For the boundary member, of course, the whole area must be included with the panel it bounds.

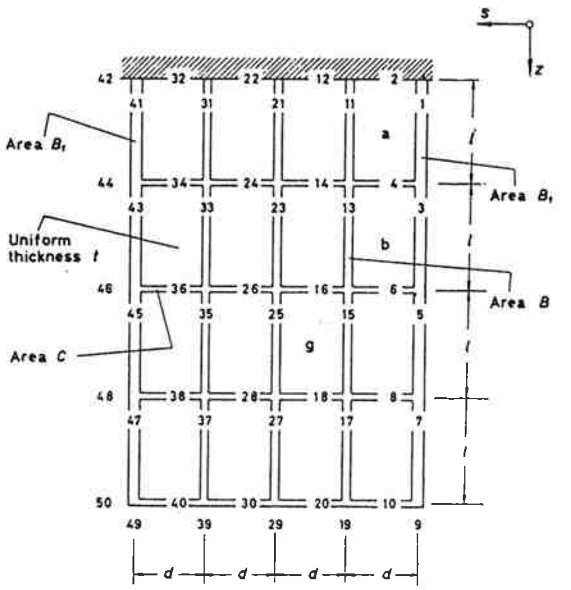


Fig. 50.—Rectangular stiffened panel. Assembly of stiffness matrix from unit panels

Since the terms in the $\mathbf{a}$ matrix are either unity or zero their formation is particularly simple. Writing $\mathbf{a}$ in the partitioned form

$$\mathbf{a}=\{\mathbf{a}_a\mathbf{a}_b\ldots\ldots\mathbf{a}_g\ldots\ldots\mathbf{a}_s\} \quad\ldots\ldots\ldots\ldots \quad (348)$$

$\mathbf{a}_g$ is the sub-matrix of 8 rows and 50 columns relating the displacements defined for the unit panel g (FIG. 49) to the displacements as defined for the complete system of FIG. (50). Superimposing the unit panel on panel g of the complete assembly we find that the directions 1, 2, 3, 4, 5, 6, 7, 8 of the unit panel coincide with 15, 25, 17, 27, 16, 26, 18, 28 respectively and the sub-matrix $\mathbf{a}_g$ is thus

$$\begin{array}{c} \begin{array}{cccccccc} 1\ldots15 & 16 & 17 & 18\ldots25 & 26 & 27 & 28\ldots50 \end{array} \\ \mathbf{a}_g=\begin{bmatrix} 0\ldots1 & 0 & 0 & 0\ldots0 & 0 & 0 & 0\ldots0 \\ 0\ldots0 & 0 & 0 & 0\ldots1 & 0 & 0 & 0\ldots0 \\ 0\ldots0 & 0 & 1 & 0\ldots0 & 0 & 0 & 0\ldots0 \\ 0\ldots0 & 0 & 0 & 0\ldots0 & 0 & 1 & 0\ldots0 \\ 0\ldots0 & 1 & 0 & 0\ldots0 & 0 & 0 & 0\ldots0 \\ 0\ldots0 & 0 & 0 & 0\ldots0 & 1 & 0 & 0\ldots0 \\ 0\ldots0 & 0 & 0 & 1\ldots0 & 0 & 0 & 0\ldots0 \\ 0\ldots0 & 0 & 0 & 0\ldots0 & 0 & 0 & 1\ldots0 \end{bmatrix} \end{array} \quad (349)$$

All remaining columns in $\mathbf{a}_g$ are zero.

As typical terms in the complete stiffness matrix the stiffnesses associated with direction 15 are, for a uniform panel with t, d, l, B and C, the same in each bay:

$$\left.\begin{aligned} k_{15,15}&=\frac{4E'dt}{3l}+\frac{2EB}{l}+\frac{4Glt}{3d} \\ k_{17,15}&=k_{13,15}=-\frac{2E'dt}{3l}-\frac{EB}{l}+\frac{Glt}{3d} \\ k_{25,15}&=k_{5,15}=\frac{E'dt}{3l}-\frac{2Glt}{3d} \\ k_{23,15}&=k_{3,15}=k_{7,15}=k_{27,15}=-\frac{E'dt}{6l}-\frac{Glt}{6d} \\ k_{24,15}&=-k_{4,15}=-k_{28,15}=k_{8,15}=\frac{\nu E't}{4}+\frac{Gt}{4} \end{aligned}\right\} \quad\ldots\quad (350)$$

All the remaining k's associated with 15 are here zero due to symmetry.

If $\mathbf{R}$ is the column matrix (50 rows) of forces applied at the nodes then the displacements $\mathbf{r}$ are given by

$$\mathbf{r}=\mathbf{K}^{-1}\mathbf{R}$$

Naturally, loads may not be applied at all nodes (joints) in which case it may be desirable partially to solve the problem by eliminating the displacements where forces are not applied and to use the condensed matrix. (See TABLE II.)

Finally we apply the unit panel to the assembly and analysis of the egg box type of structure illustrated in FIG. 51 where upper and lower plates are connected together by longitudinal and transverse webs. Any stiffeners on the plates are assumed for the present example to be along the lines of web-plate intersections. The structure is taken to be symmetrical about the horizontal middle surface and we consider the application of vertical loads only. With these assumptions it is only necessary to specify three displacements at each web intersection : the vertical displacement and the two rotations of the web intersection line (FIG. 51). In many cases the webs may be too widely spaced for the assumed linear variation of displacements between them to give satisfactory accuracy. It then becomes necessary to introduce further grid lines intermediate between the actual webs, the displacements being defined at all nodal points formed by grid line intersections. Where such nodal points do not lie on a web then obviously we define there only the two rotations, since vertical displacement does not affect the cover plates. Naturally, further lateral and longitudinal reinforcement of the plates can lie along the extra grid lines.

The analysis of such a structure under vertical loads follows that given under Problems (a) and (b) in TABLE II. Thus, we designate the vertical displacements as $\mathbf{r}$ and take the rotations as the redundant displacements $\mathbf{U}$.

The strains of the elements are here identified as the displacements of the unit panel defined in FIG. 49 and can therefore be written (Eq. 305b).

$$\mathbf{v}=\mathbf{a}_0\mathbf{r}+\mathbf{a}_1\mathbf{U}$$

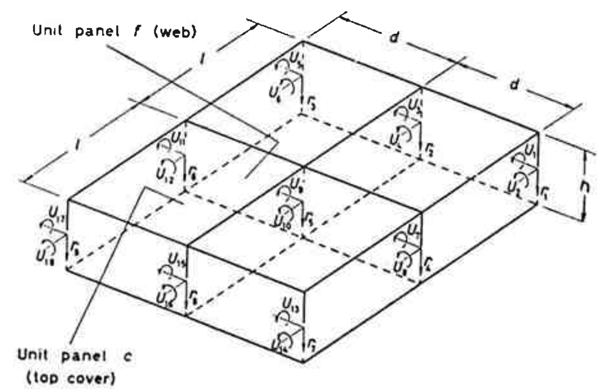


Fig. 51.—Egg-box type of structure. Analysis by displacement method

The equation for the unknowns $\mathbf{U}$, which is here the condition of equilibrium of the moments corresponding to the rotations $\mathbf{U}$ at the joints is (TABLE II, Eqs. (310b)),

$$\mathbf{a}_1'\mathbf{k}\mathbf{a}_1\mathbf{U}+\mathbf{a}_1'\mathbf{k}\mathbf{a}_0\mathbf{r}=\mathbf{O}$$

or

$$\mathbf{U}=-(\mathbf{a}_1'\mathbf{k}\mathbf{a}_1)^{-1}\mathbf{a}_1'\mathbf{k}\mathbf{a}_0\mathbf{r}$$

The stiffness for displacements $\mathbf{r}$ only is then

$$\mathbf{K}=\mathbf{a}_0'\mathbf{k}\mathbf{a}_0-\mathbf{a}_0'\mathbf{k}\mathbf{a}_1(\mathbf{a}_1'\mathbf{k}\mathbf{a}_1)^{-1}\mathbf{a}_1\mathbf{k}\mathbf{a}_0 \quad\ldots\ldots\ldots\ldots \quad (312b)$$

* Suffix s refers here to the number of unit panels and should not be confused with suffix s for stiffness due to shear stresses in (344).

from which the displacements $\mathbf{r}$ under loads $\mathbf{R}$ are found as

$$\mathbf{r}=\mathbf{K}^{-1}\mathbf{R}$$

The total strains of the elements due to $\mathbf{R}$ are then

$$\mathbf{v}=\mathbf{a}\mathbf{r}=\{\mathbf{a}_0-\mathbf{a}_1(\mathbf{a}_1'\mathbf{k}\mathbf{a}_1)^{-1}\mathbf{a}_1'\mathbf{k}\mathbf{a}_0\}\mathbf{K}^{-1}\mathbf{R} \quad \ldots\ldots\ldots\ldots \quad (316b)$$

from which the stresses in the unit panels are calculated.

Due to the simple geometry of the structure the matrices $\mathbf{a}_0$, $\mathbf{a}_1$ are again quite straightforward. Thus writing $\mathbf{a}_0$ in the partitioned form

$$\mathbf{a}_0=\{\mathbf{a}_{0a}\mathbf{a}_{0b}\ldots\ldots\ldots\mathbf{a}_{0g}\ldots\ldots\ldots\mathbf{a}_{0s}\} \quad \ldots\ldots\ldots\ldots \quad (351)$$

it is apparent that for the cover plates the $\mathbf{a}_0$'s are all zero since vertical displacements $\mathbf{r}$ can cause no strain in the plates* (with $\mathbf{U}=\mathbf{O}$).

For the web f, the $\mathbf{a}_{0f}$ matrix is easily seen to be

$$\mathbf{a}_{0f}=\begin{array}{c} \scriptstyle 1\ldots\ldots 5 \quad 6\ldots\ldots 9 \\ \left[\begin{array}{cc} 0\ldots\ldots 0 & 0\ldots\ldots 0 \\ 0\ldots\ldots 0 & 0\ldots\ldots 0 \\ 0\ldots\ldots 0 & 0\ldots\ldots 0 \\ 0\ldots\ldots 0 & 0\ldots\ldots 0 \\ 0\ldots\ldots 1 & 0\ldots\ldots 0 \\ 0\ldots\ldots 1 & 0\ldots\ldots 0 \\ 0\ldots\ldots 0 & 1\ldots\ldots 0 \\ 0\ldots\ldots 0 & 1\ldots\ldots 0 \end{array}\right] \end{array} \quad \ldots\ldots\ldots\ldots \quad (352)$$

Likewise the $\mathbf{a}_1$ matrix for unit panel c of the top cover is

$$\mathbf{a}_{1c}=\begin{array}{c} \scriptstyle \ldots\ldots\ 9 \quad 10 \quad 11 \quad 12\ \ldots\ldots 15 \quad 16 \quad 17 \quad 18\ \ldots\ldots \\ \left[\begin{array}{cccccccccc} \ldots\ldots h/2 & 0 & 0 & 0 & \ldots\ldots & 0 & 0 & 0 & 0 & \ldots\ldots \\ \ldots\ldots 0 & 0 & h/2 & 0 & \ldots\ldots & 0 & 0 & 0 & 0 & \ldots\ldots \\ \ldots\ldots 0 & 0 & 0 & 0 & \ldots\ldots & h/2 & 0 & 0 & 0 & \ldots\ldots \\ \ldots\ldots 0 & 0 & 0 & 0 & \ldots\ldots & 0 & 0 & h/2 & 0 & \ldots\ldots \\ \ldots\ldots 0 & h/2 & 0 & 0 & \ldots\ldots & 0 & 0 & 0 & 0 & \ldots\ldots \\ \ldots\ldots 0 & 0 & 0 & h/2 & \ldots\ldots & 0 & 0 & 0 & 0 & \ldots\ldots \\ \ldots\ldots 0 & 0 & 0 & 0 & \ldots\ldots & 0 & h/2 & 0 & 0 & \ldots\ldots \\ \ldots\ldots 0 & 0 & 0 & 0 & \ldots\ldots & 0 & 0 & 0 & h/2 & \ldots\ldots \end{array}\right] \end{array} \quad \ldots \quad (353)$$

and for the web plate f

$$\mathbf{a}_{1f}=\begin{array}{c} \scriptstyle \ldots\ldots\ 10 \quad 11 \quad 12\ \ldots\ldots \\ \left[\begin{array}{ccccc} \ldots\ldots & h/2 & 0 & 0 & \ldots\ldots \\ \ldots\ldots & -h/2 & 0 & 0 & \ldots\ldots \\ \ldots\ldots & 0 & 0 & h/2 & \ldots\ldots \\ \ldots\ldots & 0 & 0 & -h/2 & \ldots\ldots \\ \ldots\ldots & 0 & 0 & 0 & \ldots\ldots \\ \ldots\ldots & 0 & 0 & 0 & \ldots\ldots \\ \ldots\ldots & 0 & 0 & 0 & \ldots\ldots \\ \ldots\ldots & 0 & 0 & 0 & \ldots\ldots \end{array}\right] \end{array} \quad \ldots\ldots\ldots\ldots \quad (354)$$

All other columns are zero.

The matrices for the web plates may of course be reduced to six or even four rows by using the assumption of zero vertical direct strain and the antisymmetric character of the U displacements (see also Eqs. (346) and (346a)). However, we retain here the full eight displacements of the unit panels to show the simple formation of the $\mathbf{a}$ matrices with a completely standard unit panel.

The stiffness matrix $\mathbf{k}$ of the unassembled unit panels is written again in the diagonal partitioned form:

$$\mathbf{k}=\left[\begin{array}{ccccc} 2\mathbf{k}_a & \mathbf{o} & \cdots & \cdots & \mathbf{o} \\ \mathbf{o} & 2\mathbf{k}_b & \mathbf{o} & \cdots & \mathbf{o} \\ & & \ddots & & \\ \mathbf{o}\cdots\mathbf{o} & & \mathbf{k}_f & & \mathbf{o}\cdots\mathbf{o} \\ & & & \ddots & \\ \mathbf{o} & \cdots & \cdots & \mathbf{o} & \mathbf{k}_s \end{array}\right] \begin{array}{l} \left.\right\} \textit{covers} \\ \\ \ldots\ldots \\ \\ \left.\right\} \textit{webs} \end{array} \quad (355)$$

The factor 2 is introduced for the cover plates to take advantage of the symmetry of the structure by including the unit panels of the lower cover with their opposite numbers in the top cover. The $\mathbf{k}_a$, $\mathbf{k}_b$ etc., are of course the stiffnesses of the unit panels discussed earlier.

The method of formulation of the matrices $\mathbf{a}_1'\mathbf{k}\mathbf{a}_1$, etc., given above is probably the most convenient for use with the automatic digital computer since the various terms in the constituent matrices are reduced to their simplest and most standard forms. However, it is instructive to consider directly the components of $\mathbf{a}_1'\mathbf{k}\mathbf{a}_1$, etc., and gain some physical insight into their formation.

We call

$$\left.\begin{array}{l} \mathbf{a}_0'\mathbf{k}\mathbf{a}_0=\mathbf{K}_0 \\ \mathbf{a}_1'\mathbf{k}\mathbf{a}_1=\mathbf{C} \\ \mathbf{a}_1'\mathbf{k}\mathbf{a}_0=\overline{\mathbf{C}} \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (356)$$

Thus the *complete* stiffness matrix for the displacement column $\{\mathbf{r}\ \ \mathbf{U}\}$ is

$$\left[\begin{array}{cc} \mathbf{K}_0 & \overline{\mathbf{C}}' \\ \overline{\mathbf{C}} & \mathbf{C} \end{array}\right] \quad \ldots\ldots\ldots\ldots \quad (357)$$

$\mathbf{K}_0$ is clearly the set of vertical forces $\mathbf{R}$ which arise due to unit r displacements when $\mathbf{U}$ is zero. Evidently only the webs are involved and we find easily as a typical example the vertical forces at the joints due to $r_5=1$

$$\left.\begin{array}{l} k_{0,5,5}=\dfrac{2Ght_w}{l}+\dfrac{2Ght_r}{d} \\[2ex] k_{0,6,5}=k_{0,4,5}=-\dfrac{Ght_r}{d} \\[2ex] k_{0,8,5}=k_{0,2,5}=-\dfrac{Ght_w}{l} \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (358)$$

Similarly $\mathbf{C}$ is the set of moments arising at the joints due to unit displacements (rotations) U. By using the stiffnesses of the unit panel (or by carrying out the matrix multiplication $\mathbf{a}_1'\mathbf{k}\mathbf{a}_1$) we find for the moments due to $U_9=1$

$$\left.\begin{array}{l} c_{9,9}=\dfrac{2E'h^2dt}{3l}+\dfrac{EBh^2}{l}+\dfrac{E'h^3t_w}{6l}+\dfrac{2Gh^2lt}{3d}+\dfrac{Ghlt_w}{3} \\[2ex] c_{3,9}=c_{15,9}=-\dfrac{E'h^2dt}{3l}-\dfrac{EBh^2}{2l}-\dfrac{E'h^3t_w}{12l}+\dfrac{Gh^2lt}{6d}+\dfrac{Ghlt_w}{6} \\[2ex] c_{11,9}=c_{7,9}=\dfrac{E'h^2dt}{6l}-\dfrac{Gh^2lt}{3d} \\[2ex] c_{17,9}=c_{13,9}=c_{5,9}=c_{1,9}=-\dfrac{E'h^2dt}{12l}-\dfrac{Gh^2lt}{12d} \\[2ex] -c_{18,9}=c_{14,9}=-c_{2,9}=c_{6,9}=\dfrac{\nu E'h^2t}{8}+\dfrac{Gh^2t}{8} \end{array}\right\} \quad \ldots \quad (359)$$

Finally $\overline{\mathbf{C}}$ is the set of moments arising at the joints due to the vertical displacements. Again only the webs are involved, and we obtain easily

$$\left.\begin{array}{l} \bar{c}_{12,5}=-\bar{c}_{8,5}=\dfrac{Ght_r}{2} \\[2ex] \bar{c}_{15,5}=-\bar{c}_{3,5}=\dfrac{Ght_w}{2} \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (360)$$

---

* This, of course, is no longer true if the webs are tapered in depth, when an appreciable part of the shear force carried here by the web is equilibrated by the vertical component of the direct stresses in the covers.

In some problems it may be possible to neglect the shear deformations of the webs. Quite obviously this introduces kinematic relations between the rotations and the vertical displacements. However, the above breakdown of the structure into the simple unit panels is then not the most suitable. A suggested method for the setting up of the **K** matrix for this case, based on the theory of bending of plates, has been given by Williams.*

9. ILLUSTRATIONS TO THE ANALYSIS OF REDUNDANT STRUCTURES BY THE FORCE METHOD

In this section we present two very simple applications of the force method developed in Section 8C. The first example shows how to determine the statically equivalent stress system in an N cell tube typical of a wing structure.† With the direct stresses distributed according to E.T.B. we find using the δ_{ik} method the corresponding shear flow distribution for the multi-cell cross-section with the assumption that the ribs are rigid in their plane. Naturally, the axial constraint stresses and the effect of rib deformation remain to be investigated but the statically equivalent stress system derived here is particularly useful, being, in general, a reasonable first approximation when the structural design has to be based merely on a statically equivalent stress system. This, of necessity, has been the approach in most cases up to the present owing to the difficulty of computing highly redundant systems.

Although the problem may, with some justice, be described as trivial in relation to the powerful analytical techniques of Section 8 it is astonishing to see what an unfortunate treatment it often receives—even today.

The second example analyses—first by the δ_{ik} method—the axial constraint stresses in a four flange tube with shear carrying walls and deformable ribs under arbitrary loading at the rib stations. The solution of the same problem is also obtained by the matrix method and the effect of a cut-out is investigated by the **H** matrix device of p. 41. The 'exact' flexibility of the structure is derived and compared with that given by E.T.B. and Bredt-Batho. A thermal loading is also investigated by the matrix method.

These two problems are only meant as preliminary illustrations of the force method. Thermal applications of the δ_{ik} method are investigated in Part II. More complicated structures, particularly suitable for showing the power of the matrix formulation of the theory, will be analysed in a later publication.

(a) Shear Flow Distribution in a Multi-cell Tube Due to E.T.B. Direct Stresses

Consider the uniform cylindrical and multi-cell tube of the type shown in FIG. 52 subjected to shear forces S_y, S_x through and a torque T_o about the point O. Find the corresponding distribution of shear flow if the direct stresses are given by the engineers' theory of bending; thus axial constraint stresses due to restrained warping are ignored. Instead of referring the loads to the arbitrary point O we may alternatively give the point D through which the resultant of all transverse forces is acting, i.e. $T_D=0$.

The direct stress σ due to bending moments M_x, M_y about the axes $\overline{Gx}$, $\overline{Gy}$ through the section centroid $\overline{G}$ and parallel to Ox, Oy, is

$$\sigma=\overline{M}_x\frac{\overline{y}}{I_x}+\overline{M}_y\frac{\overline{x}}{I_y} \quad \text{(a1)}$$

where

$$\overline{M}_x=\frac{M_x-M_y(I_{xy}/I_y)}{1-[(I_{xy})^2/I_xI_y]},\ \overline{M}_y=\frac{M_y-M_x(I_{xy}/I_x)}{1-[(I_{xy})^2/I_xI_y]} \quad \text{(a2)}$$

are the effective bending moments for the chosen axes which are, in general, not the principal axes of the cross-section. Physically $\overline{M}_x$, $\overline{M}_y$ are the combinations of M_x, M_y which give rise to pure bending strains about $\overline{Gx}$, $\overline{Gy}$ respectively. We could alternatively restrict ourselves to principal axes of the cross-section but in practice, unless these are obvious, it is preferable to use Eqs. (a1), (a2). They are not only more convenient from the computational point of view but permit also the retention of parallel axes Ox, Oy at all cross-sections of a wing regardless of the change of directions of the principal axes.

The condition of equilibrium in the z-direction of an element $dsdz$ of a wall gives,

$$\frac{\partial q}{\partial s}+t_s\frac{\partial\sigma}{\partial z}=0 \quad \text{(a3)}$$

where

$q=t\sigma_{zs}$=shear flow in the wall.

t_s=effective direct stress carrying thickness of the wall (i.e. including an allowance for the stringers).

Similarly, we find from the equilibrium of an element dz of a typical flange g placed at a web-cover intersection (see FIG. 52),

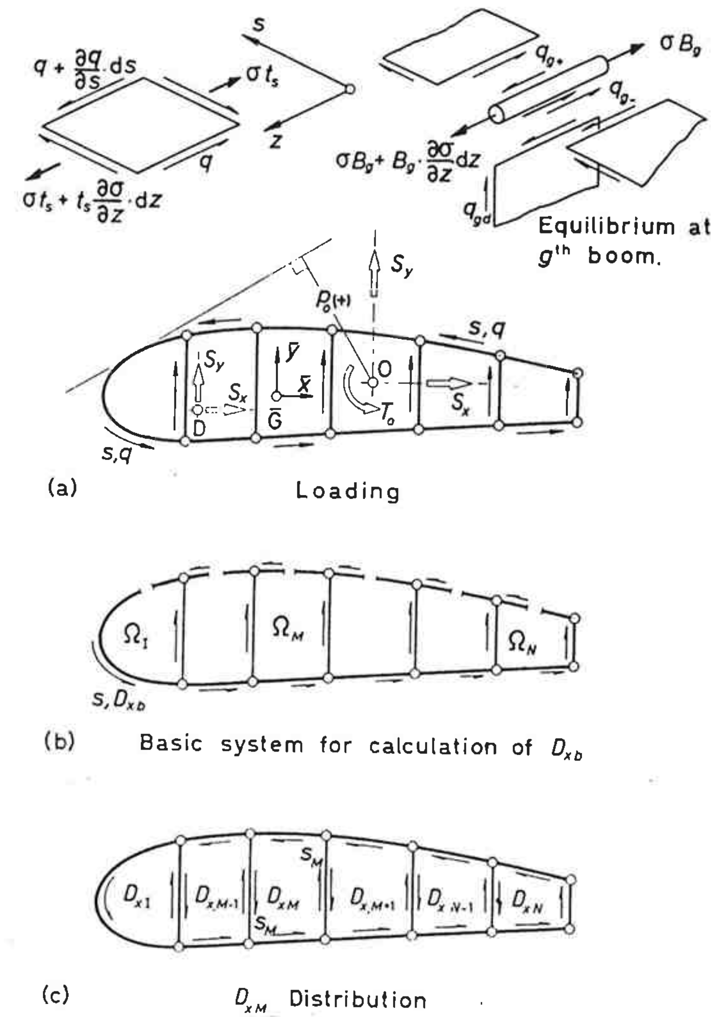


Fig. 52.—Shear flow distribution in multi-cell tube. Sign conventions and equilibrium conditions

$$B_g\frac{\partial\sigma_g}{\partial z}-q_{g-}-q_{gd}+q_{g+}=0 \quad \text{(a4)}$$

Eqs. (a3) and (a4) yield, except for a constant of integration in each of the N cells, a shear flow distribution q whose resultants in the y and x directions are S_y and S_x respectively. Since there remains one further equilibrium condition,

$$T_0=\int qp_o ds \quad \text{(a5)}$$

for torque about O, the degree of redundancy is $(N-1)$; (the integration in (a5) extends over all walls and the normal p_o is taken positive (negative) if movement along s leads to an anticlockwise (clockwise) rotation about O). It follows that the shear flow distribution in a single-cell tube under prescribed transverse loading is statically determinate once we stipulate that the direct stresses are distributed as per E.T.B.

For the analysis of the general case of an N cell tube we find it more convenient to use a slightly different approach. Thus, for the moment, we prescribe instead of the torque equilibrium condition, the rate of twist

$$\phi=\theta'=\frac{d\theta}{dz} \quad \text{(a6)}$$

in all cells. The prescribed ϕ may be considered as an initial 'give' experienced by the ribs maintaining the shape of the cross-section and is subsequently determined from the torque equilibrium equation. Our modified problem has now N redundancies. The basic system is obtained by cutting the wall in each of the N cells and the unknowns

$$X_I,\ X_{II},\ \ldots\ldots\ldots,\ X_N$$

are then the shear flows at the cuts. They are determined from the compatibility equations,

$$\sum_{r=I}^{N}\delta_{Mr}X_r+\delta_{Mo}=0 \quad \text{(a7)}$$

which express the conditions of zero relative warping δ at the cuts.*

* loc. cit. p. 17.

† See J. H. Argyris and P. C. Dunne, 'The General Theory, etc.', Part V, *J.R. Aer. Soc.*, Vol. LI, September 1947, p. 770, and J. H. Argyris and P. C. Dunne, *Structural Analysis*, Part II of *Handbook of Aeronautics*, Pitman 1952.

* We denote here the warping by the unconventional symbol δ to apply directly Eqs. (282) in their original notation.

The shear flow distribution q_b in the open tube forming the basic system is obtained from Eqs. (a3) and (a4). Integrating (a3) with respect to s and using Eq. (a2) we find

$$q_b=\overline{S}_y\frac{D_{xb}}{I_x}+\overline{S}_x\frac{D_{yb}}{I_y} \quad \text{(a8)}$$

where

$$\overline{S}_y=\frac{S_y-S_x(I_{xy}/I_y)}{1-[(I_{xy})^2/I_xI_y]},\ \overline{S}_x=\frac{S_x-S_y(I_{xy}/I_x)}{1-[(I_{xy})^2/I_xI_y]} \quad \text{(a9)}$$

and

$$D_{xb}=-\int_0^s\overline{y}t_sds,\ D_{yb}=-\int_0^s\overline{x}t_sds \quad \text{(a10)}$$

To determine completely the D_b distributions from Eqs. (a10) we require also the equilibrium conditions of the type (a4) at each joint of spar web and cover. Using (a1) and (a8) in (a4) we have

$$\left.\begin{aligned}(D_{x+}-D_{xd}-D_{x-}+B\overline{y})_{bg}=0\\(D_{y+}-D_{yd}-D_{y-}+B\overline{x})_{bg}=0\end{aligned}\right\} \quad \text{(a11)}$$

The positive directions of q and s are indicated on FIG. 52.

Choice of Basic System

The reduction of the multi-cell tube to an open section can, of course, be achieved in a variety of ways. For example, we may cut the upper or lower cover in each cell or we may cut N of the vertical walls (see FIG. 52). However, consideration of the form of the δ's shows that the compatibility equations (a7) are very much simpler for the former choice. We confirm this immediately by applying the unit load method for the calculation of the δ's which measure the relative warping at the cuts. Thus, if we apply unit shear flows at each of the cuts of the open section in FIG. 52 we produce merely constant shear flow around each of the individual cells.

The Redundant Shear Flows

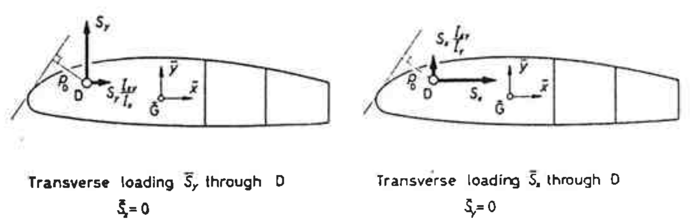


Fig. 53.—Effective shear forces for bending about non-principal axes

If we consider now the total transverse loading S_y and S_x as acting through D and split it into the two component loads $\overline{S}_y$ and $\overline{S}_x$ (see FIG. 53) we can express the statically indeterminate shear flows in the form

$$X_M=q_M=\overline{S}_y\frac{D_{xM}}{I_x}+\overline{S}_x\frac{D_{yM}}{I_y} \quad \text{(a12)}$$

where the D_{xM} and D_{yM} are unknown. For the basic system of FIG. 52 the shear flows in the M cell of the actual system can then be written as:

$$\left.\begin{aligned}&\text{external walls} && q=q_b+q_M\\&\text{web between } M-1 \text{ and } M \text{ cells} && q=q_b+q_{M-1}-q_M\\&\text{web between } M \text{ and } M+1 \text{ cells} && q=q_b+q_M-q_{M+1}\end{aligned}\right\} \quad \text{(a13)}$$

We may always put the total shear flow in the form

$$q=\overline{S}_y\frac{D_x}{I_x}+\overline{S}_x\frac{D_y}{I_y} \quad \text{(a14)}$$

and Eqs. (a12), (a13), (a14) yield for the cross-sectional function D_x in the M cell

$$\left.\begin{aligned}&\text{external walls} && D_x=D_{xb}+D_{xM}\\&\text{web between } M-1 \text{ and } M \text{ cells} && D_x=D_{xb}+D_{x,M-1}-D_{xM}\\&\text{web between } M \text{ and } M+1 \text{ cells} && D_x=D_{xb}+D_{xM}-D_{x,M+1}\end{aligned}\right\} \quad \text{(a15)}$$

Similar equations may be written down for D_y.

The δ Coefficients for the Basic System of FIG. 52

The δ_{Mo} coefficients consist of two parts δ_{Mob} and $\delta_{Mo\varphi}$ corresponding to the shear flow q_b in the basic system and the initial 'give' ϕ (see also Eq. (177)). We have:

$$\delta_{Mo}=\delta_{Mob}+\delta_{Mo\varphi}$$

Application of the unit load method yields immediately for δ_{Mob} -

$$\delta_{Mob}=\int_M\frac{q_b}{Gt}ds_M \quad \text{(a17)}$$

where s_M is the circumferential anticlockwise co-ordinate in the M cell and the integral

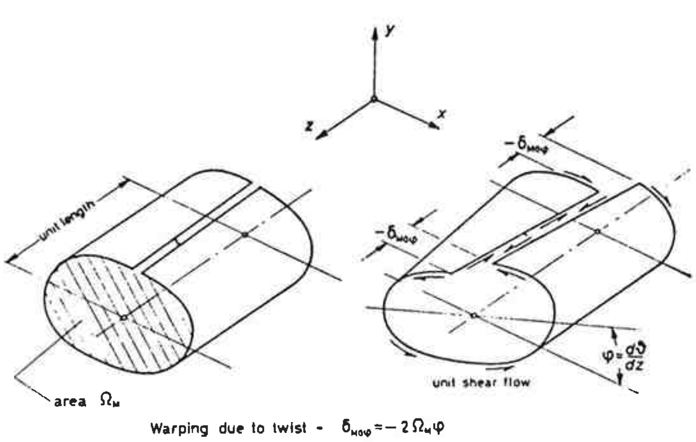


Fig. 54.—Warping in Mth cell of basic system due to rate of twist ϕ

$$\int_M(\ldots)ds_M$$

denotes integration over the M cell. It must be emphasized that the sign convention used for q_b and s in the basic (open) system is in opposition to s_M in the left-hand wall of the M cell and hence in evaluating the above term the sign of q_b must be reversed over this wall.

Similarly, we obtain for the relative warping $\delta_{Mo\varphi}$ due to ϕ (see FIG. 54).

$$\delta_{Mo\varphi}=-2\Omega_M\phi \quad \text{(a18)}$$

The standard derivation of Eqs. (a17) and (a18) is by kinematics. Thus, Eq. (a17) is obtained by integration of the shear strain expression corresponding to zero rate of twist. Also, we find (a18) from the condition of zero shear strain along the middle line of the wall.*

The unit load method also yields directly the coefficients of the unknown X's. Thus, the relative warping δ_{MM} due to unit shear flow in the M cell is

$$\delta_{MM}=\int\frac{ds_M}{Gt}=\frac{\beta_M}{G} \quad \text{(a19)}$$

where

$$\beta_M=\int_M\frac{ds_M}{t} \quad \text{(a20)}$$

All cross-terms but $\delta_{M,M-1}$ and $\delta_{M,M+1}$ vanish since only unit shear flows in adjoining cells act over a common wall. We find

$$\left.\begin{aligned}\delta_{M,M-1}=\delta_{M-1,M}=-\frac{\beta_{M-1,M}}{G}\\\delta_{M,M+1}=\delta_{M+1,M}=-\frac{\beta_{M,M+1}}{G}\end{aligned}\right\} \quad \text{(a21)}$$

where

$$\beta_{M-1,M}=\int_{M-1,M}\frac{ds}{t},\quad \beta_{M,M+1}=\int_{M,M+1}\frac{ds}{t} \quad \text{(a22)}$$

are the integrals extending over the common walls (spar webs) of cells $M-1$, M and M, $M+1$ respectively. The minus sign arises since the shear flows due to unit redundancies have opposite signs in the common walls.

Determination of Redundant Shear Flows

We denote the unknown rates of twist associated with $\overline{S}_y$ and $\overline{S}_x$ by ϕ_y and ϕ_x respectively. For the loading due to $\overline{S}_y$ the M'th compatibility Eq. (a7), which expresses the condition of zero relative warping at the M'th cut is obviously

$$\frac{\overline{S}_y}{I_x}\left\{\int_M\frac{D_{xb}}{t}ds_M-2\Omega_MG\phi_y\frac{I_x}{\overline{S}_y}-\beta_{M-1,M}D_{x,M-1}+\beta_MD_{xM}-\beta_{M,M+1}D_{x,M+1}\right\}=0$$

We obtain hence the set of N equations in the N unknowns D_{xM}

$$\beta_ID_{xI}-\beta_{I,II}D_{xII}=-\int_I\frac{D_{xb}}{t}ds_I+2\Omega_IG\phi_y\frac{I_x}{\overline{S}_y}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-\beta_{M-1,M}D_{x,M-1}+\beta_MD_{xM}-\beta_{M,M+1}D_{x,M+1}=-\int_M\frac{D_{xb}}{t}ds_M+2\Omega_MG\phi_y\frac{I_x}{\overline{S}_y}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-\beta_{N-1,N}D_{x,N-1}+\beta_ND_{xN}=-\int_N\frac{D_{xb}}{t}ds_N+2\Omega_NG\phi_y\frac{I_x}{\overline{S}_y} \quad \text{(a23)}$$

* See e.g. J. H. Argyris, 'The Open Tube', AIRCRAFT ENGINEERING, Vol. XXVI, No. 302, April 1954, p. 102.

These formulae are usually derived more simply from the condition of equal rate of twist*

$$\phi=\frac{d\theta}{dz}=\frac{1}{2\Omega_M G}\oint_M \frac{q}{t}ds_M \quad \text{(a24)}$$

in all N cells in the actual system. In this approach we specify initially zero relative warping at all cuts and express the compatibility condition by the equality of ϕ in all cells. On the other hand in our present method we specify initially the same rate of twist ϕ in all cells and express the compatibility by the condition of zero relative warping at the cuts.

The solution of Eqs. (a23) and the corresponding ones for D_{yM} is straightforward and may be put in the form

$$\left.\begin{aligned} D_{xM}&=d_{xM}+\alpha_M G\phi_y\frac{I_x}{\bar{S}_y}\\ D_{yM}&=d_{yM}+\alpha_M G\phi_x\frac{I_y}{\bar{S}_x}\end{aligned}\right\} \quad \text{(a25)}$$

d_{xM}, d_{yM} are the values of the redundancies corresponding to zero rate of twist. They yield hence the shear flow distribution q_E—commonly known as the engineers' theory of bending shear flows—due to transverse shear forces acting through the shear centre E_s · q_E may be written

$$q_E=\bar{S}_y\frac{d_x}{I_x}+\bar{S}_x\frac{d_y}{I_y} \quad \text{(a26)}$$

where the cross-sectional functions d_x (d_y) are obtained from Eqs. (a15) with d_{xM} (d_{yM}) in place of D_{xM} (D_{yM}). The co-ordinates x_E, y_E of E_s can be determined from a consideration of the two loading cases $\bar{S}_y$ and $\bar{S}_x$ through E_s shown in FIG. 55. We find

$$\left.\begin{aligned} I_x x_E-I_{xy}y_E&=\int d_x p_o ds=\int D_{xb}p_o ds+2\sum_I^N d_{xM}\Omega_M\\ -I_{xy}y_E+I_y y_E&=-\int d_y p_o ds=-\int D_{yb}p_o ds-2\sum_I^N d_{yM}\Omega_M\end{aligned}\right\} \quad \text{(a27)}$$

where the integrals extend over all walls and the normal p_o is taken positive (negative) if movement along the positive s direction produces anticlockwise (clockwise) rotation about O.

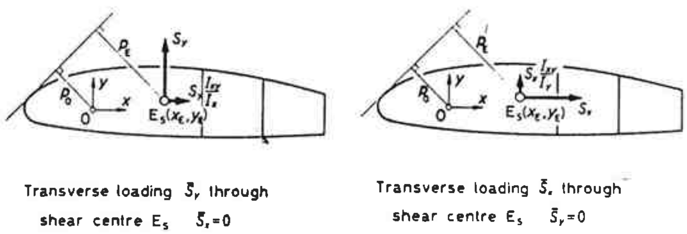


Fig. 55.—Effective shear forces applied through the shear centre E_s

The shear centre allows us to define the loading alternatively by the shear forces through and the torque

$$T_E=-S_y(x_E-x_D)+S_x(y_E-y_D)=T_o-S_y x_E+S_x y_E \quad \text{(a28)}$$

about E_s.

To obtain the total redundancies D_{xM} and D_{yM} we have still to determine the rates of twist ϕ_y and ϕ_x. Observing that the distribution coefficients α_M are the same in the two equations (a25)—a direct result of the rigid diaphragm assumption which allows us to displace the transverse forces anywhere along their lines of action—we can combine the indeterminate shear flows due to $\bar{S}_y$ and $\bar{S}_x$ into a single set

$$q_{BM}=\frac{\bar{S}_y}{I_x}\alpha_M G\phi_y\frac{I_x}{\bar{S}_y}+\frac{\bar{S}_x}{I_y}\alpha_M G\phi_x\frac{I_y}{\bar{S}_x}=\alpha_M G(\phi_x+\phi_y)=\alpha_M G\phi \quad \text{(a29)}$$

where

$$\phi=\frac{d\theta}{dz}=\phi_x+\phi_y \quad \text{(a30)}$$

is the total rate of twist due to the given loading.

To derive the rate(s) of twist we may apply the equilibrium condition about D (see FIG. 53)

$$\int q p_D ds=0 \quad \text{(a31)}$$

where the sign of p_D is defined as for p_o. Applying Eq. (a31) to the $\bar{S}_y$ loading we obtain

$$\int D_{xb}p_D ds+2\sum d_{xM}\Omega_M+2\sum\alpha_M\Omega_M\cdot G\phi_y\frac{I_x}{\bar{S}_y}=0$$

or

$$\left.\begin{aligned} 2\sum_I^N\alpha_M\Omega_M\cdot G\phi_y\frac{I_x}{\bar{S}_y}&=-\int D_{xb}p_D ds-2\sum_I^N d_{xM}\Omega_M\\ \text{Similarly}\qquad&\\ 2\sum_I^N\alpha_M\Omega_M\cdot G\phi_x\frac{I_y}{\bar{S}_x}&=-\int D_{yb}p_D ds-2\sum_I^N d_{yM}\Omega_M\end{aligned}\right\} \quad \text{(a32)}$$

where the integrals extend over all walls.

Having derived the complete set of redundancies we can calculate using formulae (a14) and (a15) the shear flow distribution in the actual system under any given transverse loads.

If we have found the shear centre E_s and the torque T_E it is more convenient to proceed as follows. Thus, using the equilibrium condition (FIG. 55)

$$T_E=\int q p_E ds \quad \text{(a33)}$$

and remembering that the engineers' theory of bending shear flows have no torque contribution about E_s, we obtain with Eq. (a29)

$$T_E=G\phi 2\sum_I^N\alpha_M\Omega_M \quad \text{(a34)}$$

or

$$G\phi=\frac{T_E}{2\sum_I^N\alpha_M\Omega_M} \quad \text{(a34a)}$$

Hence the redundant q_{BM} due to torque T_E are given by:

$$q_{BM}=\alpha_M\frac{T_E}{2\sum_I^N\alpha_M\Omega_M} \quad \text{(a35)}$$

The shear flow is, of course, constant in each wall between two consecutive joints. Thus, we have in the M'th cell

external walls	$q_B=q_{BM}$	
web between $M-1$ and M cells	$q_B=q_{B,M-1}-q_{BM}$	(a36)
web between M and $M+1$ cells	$q_B=q_{BM}-q_{B,M+1}$	

The distribution q_B is known as the Bredt-Batho shear flows in a multi-cell tube. The special case of Eqs. (a23) corresponding to pure torque is now written more conveniently in the form

$$\begin{aligned} \beta_I q_{BI}-\beta_{I,II}q_{BII}&=2\Omega_I G\phi\\ &\cdots\cdots\cdots\\ -\beta_{M-1,M}q_{B,M-1}+\beta_M q_{BM}-\beta_{M,M+1}q_{B,M+1}&=2\Omega_M G\phi\\ &\cdots\cdots\cdots\\ -\beta_{N-1,N}q_{B,N-1}+\beta_N q_{BN}&=2\Omega_N G\phi \quad \text{(a37)}\end{aligned}$$

which are most simply derived from the rate of twist formula Eq. (a24). Expressing the solution as in Eq. (a29) we find the unknown $G\phi$ from Eq. (a34a).

Note that the total shear flow q of Eq. (a14) may be written as

$$q=q_E+q_B \quad \text{(a14a)}$$

where the engineers' theory shear flows q_E are given by Eq. (a26).

With the chosen basic system Eqs. (a23) and (a37) are particularly well conditioned since the diagonal coefficients are predominant and at the most only three unknowns are involved in a given equation. Direct solution by elimination is quite easy and may be performed by slide rule even for a high degree of redundancy; application of the relaxation technique is superfluous. An additional virtue of the basic system of FIG. 52 is that the D_{xb} distribution is quite close to the final d_x distribution and the values of the redundancies are small.

In finding on the other hand the d_{yM} redundancies there is a conflict in the choice of the basic system. For when we cut the external wall in each cell the D_{yb} distribution is vastly different from the final one although the equations are as well conditioned as for the D_{xM} since the matrix **D** is the same in both cases. Naturally, the criterion of good conditioning of the equations is always the most important one. The alternative basic system in which the webs are cut gives a D_{yb} distribution close to the actual d_y but the equations are not so easy to solve since each equation involves all the unknowns. To satisfy both the above requirements it is necessary to make composite cuts, i.e. to cut both upper and lower external walls and instead of the condition $D_{yb}=0$ at a cut to take say equal and opposite values of D_{yb} at the two cuts of each cell. In any case the d_y distribution is, in general, of small importance and a more approximate solution is acceptable.

Generalizations of the Above Analysis

The method given above for the determination of the statically equivalent stress system in uniform cylindrical tubes may be generalized and applied to tubes with conical or non-conical taper and with sheet thicknesses and boom areas varying lengthwise. The angle of taper 2θ is taken, however, to be so small that $\cos 2\theta\approx 1$ and $\sin 2\theta\approx 2\theta$.

* See *loc. cit.* p. 129, 'The General Theory'. See also derivation of Eq. (a24) by the Unit Load method in Section 7.

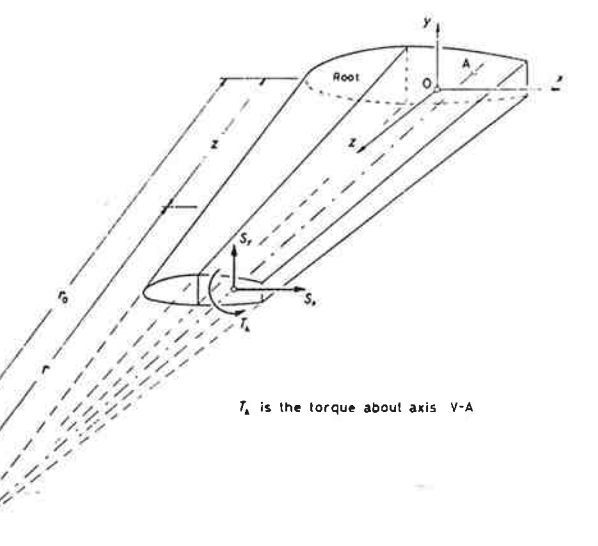


Fig. 56.—Geometry and loading for conical tube

1. *Conical and Cylindrical Tubes with Similar Distribution of Material at All Cross-Sections*

Consider the conical tube shown in FIG. 56. At the root the sheet thicknesses for direct and shear stresses are t_s and t respectively and the (effective) boom areas B. At any intermediate section the corresponding thicknesses and areas are given by

$$\left.\begin{aligned} t_s' &= \psi_s t_s \\ t' &= \psi t \\ B' &= \rho\psi_s B \end{aligned}\right\} \quad \ldots\ldots \quad \text{(a38)}$$

where

$$\rho = r/r_0 \quad \ldots\ldots \quad \text{(a39)}$$

r and r_0 being the distances from the apex V of the current and root cross-sections respectively; for cylindrical tubes $\rho = 1$. ψ and ψ_s are non-dimensional functions of z or ρ.

In what follows under the present heading (1) all cross-sectional dimensions, areas and functions refer to the geometry of the root-section.

As in the previous analysis we assume that the direct stresses are given by the engineers' theory of bending and write them in the form

$$\sigma = \frac{\overline{M}_x}{\rho^2\psi_s}\frac{\bar{y}}{I_x} + \frac{\overline{M}_y}{\rho^2\psi_s}\frac{\bar{x}}{I_y} \quad \ldots\ldots \quad \text{(a40)}$$

where $\overline{M}_x$, $\overline{M}_y$ are given by Eqs. (a2). Note that $\bar{y}$, I_x are based on the root geometry. Since the stresses (a40) act along the generators they give rise to shear resultants at the apex V in the y and x directions which are easily found to be

$\dfrac{M_x}{r}$ and $\dfrac{M_y}{r}$ respectively.

Hence the shear forces resisted solely by shear flows q are

$$Q_y = S_y + \frac{M_x}{r}, \; Q_x = S_x + \frac{M_y}{r} \quad \ldots\ldots \quad \text{(a41)}$$

If now the transverse forces are applied through a line VD where D is a point at the root the shear flow q at any cross-section may be expressed as (see also Eqs. (a14))

$$q = \frac{\overline{Q}_y}{\rho}\frac{D_x}{I_x} + \frac{\overline{Q}_x}{\rho}\frac{D_y}{I_y} \quad \ldots\ldots \quad \text{(a42)}$$

where $\overline{Q}_y$, $\overline{Q}_x$ are determined from Eqs. (a9) with Q_y, Q_x in place of S_y, S_x. Thus,

$$\overline{Q}_y = \frac{Q_y - Q_x(I_{xy}/I_y)}{1-[(I_{xy})^2/I_xI_y]}, \; \overline{Q}_x = \frac{Q_x - Q_y(I_{xy}/I_x)}{1-[(I_{xy})^2/I_xI_y]} \quad \ldots\ldots \quad \text{(a43)}$$

The cross-sectional distribution functions D_x, D_y are obtained at the root cross-section by the method of the previous analysis but with

$$\rho^2\psi\phi_y\frac{I_x}{\overline{Q}_y} \quad \left(\rho^2\psi\phi_x\frac{I_y}{\overline{Q}_x}\right) \quad \text{in place of } \phi_y\frac{I_x}{\overline{S}_y} \quad \left(\phi_x\frac{I_y}{\overline{S}_x}\right) \quad \ldots \quad \text{(a44)}$$

in Eqs. (a23), (a24), (a25), (a29), (a32). Note that $\rho^2\psi\phi_yI_x/\overline{Q}_y$, etc., are constants for the type of loading considered as the transformed Eqs. (a32) show.

The flexural axis is in the present case a straight line VE_s through the apex and the shear centres at all cross-sections. We define this axis by the shear centre E_s at the root. The co-ordinates x_E, y_E are found as before from Eqs. (a27). The engineers' theory of bending shear flows for transverse forces through the flexural axis are now

$$q_E = \frac{\overline{Q}_y}{\rho}\frac{d_x}{I_x} + \frac{\overline{Q}_x}{\rho}\frac{d_y}{I_y} \quad \ldots\ldots \quad \text{(a45)}$$

where d_x, d_y are obtained for the root dimensions from Eqs. (a23) for $\phi_y = \phi_x = 0$.

If the shear centre E_s has been found we may calculate at any cross-section the torque T_E of all applied transverse forces about the flexural axis. It is then preferable to calculate the Bredt-Batho shear flows q_B of the total statically equivalent shear flow

$$q = q_E + q_B \quad \ldots\ldots \quad \text{(a41a)}$$

by a slightly modified version of the method on p. 55. Thus, Eqs. (a37) for the redundant q_{BM} become

$$\begin{aligned} &\beta_I q_{BI} - \beta_{I,II} q_{BII} = 2\Omega_I\,\rho\psi G\phi \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &-\beta_{M-1,M} q_{B,M-1} + \beta_M q_{BM} - \beta_{M,M+1} q_{B,M+1} = 2\Omega_M \rho\psi G\phi \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &-\beta_{N-1,N} q_{B,N-1} + \beta_N q_{BN} = 2\Omega_N \rho\psi G\phi \end{aligned} \quad \ldots\ldots \quad \text{(a46)}$$

The solution of which may be written

$$q_{BM} = \alpha_M G\rho\psi\phi \quad \ldots\ldots \quad \text{(a47)}$$

Next we deduce from the equilibrium condition about the flexural axis

$$G\rho\psi\phi = \frac{T_E}{2\rho^2\sum_I^N \alpha_M\Omega_M} \quad \ldots\ldots \quad \text{(a48)}$$

Hence the shear flows q_{BM} are

$$q_{BM} = \alpha_M\frac{T_E}{2\rho^2\sum_I^N \alpha_M\Omega_M} \quad \ldots\ldots \quad \text{(a49)}$$

Finally, we derive the shear flow q_B in all walls from Eqs. (a36).

If the torque is given initially about an arbitrary axis VA we can calculate T_E with the formula

$$T_E = T_A - \rho Q_y(x_E - x_A) + \rho Q_x(y_E - y_A) \quad \ldots\ldots \quad \text{(a50)}$$

where x_A, y_A are the co-ordinates of the point A at the root.

2. *Conical and Cylindrical Tubes with Arbitrary Variation of Boom Areas and Wall Thicknesses*

Here we investigate conical or cylindrical tubes with an arbitrary lengthwise variation of skin thicknesses and boom areas. The engineers' theory shear flows are not any longer proportional to Q_y and Q_x and Eq. (a42) does not apply. The concept of a flexural axis, either straight or curved, is also not any longer strictly true.

Under this heading all cross-sectional dimensions, areas and functions are based on the current cross-section.

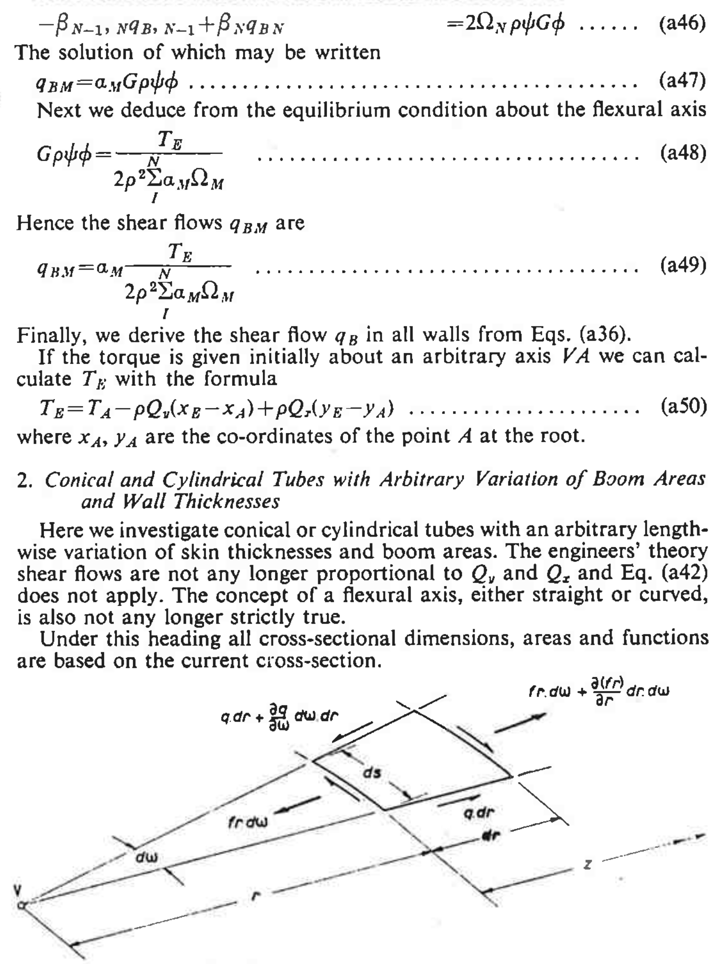


Fig. 57.—Equilibrium condition in element of conical tube

For the subsequent analysis we require the modified forms of the internal equilibrium conditions (a3) and (a4). Thus, we deduce immediately from the geometry of FIG. 57 the equilibrium condition on a conical element on the surface

$$-\frac{\partial(rf)}{\partial r} + \frac{\partial q}{\partial\omega} = 0$$

or

$$-\frac{1}{r}\frac{\partial(rf)}{\partial r} + \frac{\partial q}{\partial s} = 0 \quad \ldots\ldots \quad \text{(a51)}$$

where

$$f = \sigma t_s \quad \ldots\ldots \quad \text{(a52)}$$

Hence

$$\frac{\partial q}{\partial s} = \frac{1}{r}\frac{\partial(rf)}{\partial r} = Df \quad \ldots\ldots \quad \text{(a51a)}$$

where

$$Df = \frac{\partial f}{\partial r} + \frac{f}{r} = -\frac{\partial f}{\partial z} + \frac{f}{r} \quad \ldots\ldots \quad \text{(a53)}$$

In practical calculations we find $\partial f/\partial z$ numerically by finite differences

The equilibrium condition Eq. (a4) for a flange B_g at the intersection of cover and web becomes here

$$\frac{dP_g}{dz}-q_{g-}-q_{gd}+q_{g+}=0 \quad \text{(a54)}$$

where

$$P_g=B_g\sigma_g \quad \text{(a55)}$$

To find the statically equivalent stresses for an arbitrary system of transverse loads defined by the shear forces (and associated bending moments)

$S_y(M_z)$ and $S_z(M_y)$

and the torque

$$T_A$$

about an axis VA we proceed as follows. First we equilibrate the applied bending moments by the E.T.B. direct stresses

$$\sigma=\overline{M}_x\frac{\bar{y}}{I_x}+\overline{M}_y\frac{\bar{x}}{I_y} \quad \text{(a1)}$$

The co-ordinates $\bar{x}$, $\bar{y}$ from the centre of gravity and the corresponding moments of inertia may vary arbitrarily from section to section.

The statically equivalent shear flows are calculated as before in two parts

$$q=q_E+q_B \quad \text{(a56)}$$

where q_E are in equilibrium with $\overline{Q}_y$, $\overline{Q}_z$ with the imposed condition of zero twist throughout the tube and additional shear flow necessary to balance the applied torque.* The solution for an N cell tube follows closely the method given initially and the selection of a basic system derives from similar considerations. The calculations must, of course, be repeated for every section to be analysed.

Calculation of q_E:

The shear flow q_b in the basic system is obtained at all sections we want to analyse by integration from (a51a)

$$q_b=\int_0^s Df_E ds \quad \text{(a57)}$$

where f_E is calculated from Eqs. (a1) and (a52). The necessary constants of integration to compute the shear flow in the webs, etc., are derived from equations of the type (a54). The N unknown shear flows q_{EM} at the cuts of each section are determined by the same method leading to Eq. (a23). We find

$$\beta_I q_{EI}-\beta_{I,II}q_{EII}=-\int_I\frac{q_b}{t}ds_I$$

$$\text{- - - - - - - - - - - - - - - - - - - -}$$

$$-\beta_{M-1,M}q_{E,M-1}+\beta_M q_{EM}-\beta_{M,M+1}q_{E,M+1}=-\int_M\frac{q_b}{t}ds_M$$

$$\text{- - - - - - - - - - - - - - - - - - - -}$$

$$-\beta_{N-1,N}q_{E,N-1}+\beta_N q_{EN}=-\int_N\frac{q_b}{t}ds_N \quad \text{(a58)}$$

Note that all dimensions are based on the current cross-section. Having solved these equations we calculate the shear flow q_E with Eqs. (a13).

Calculation of q_B:

The torque T_Q of the shear flows q_E about the VA axis is

$$T_Q=\int q_E p_A ds=\int q_b p_A ds+2\sum_I^N\Omega_M q_{EM} \quad \text{(a59)}$$

where p_A is the normal from the point A to the tangent at the wall; note the usual sign convention. Hence the torque T_B to be resisted by the shear flows q_B is

$$T_B=T_A-T_Q \quad \text{(a60)}$$

The shear flows q_B at any section may now be determined from Eqs. (a37) and (a29) with

$$G\phi=\frac{T_B}{2\sum_I^N a_M\Omega_M} \quad \text{(a61)}$$

3. *Tubes with Non-Conical Taper*

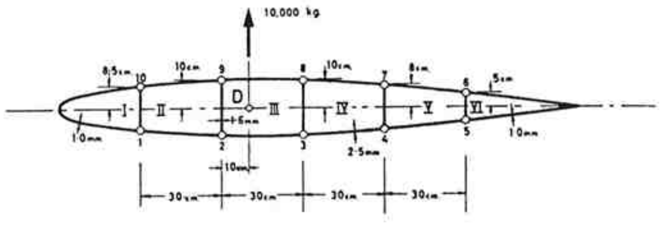
For tubes with non-conical taper we can find the statically equivalent stress system

f_B and q_E+q_B

by the method under (2). However, when finding the torque T_B to be carried by the q_B shear flows we must make allowance for the torque T_P carried by the direct flow f_E and the boom loads. Thus

$$T_B=T_A-T_Q-T_P \quad \text{(a62)}$$

* q_E may be regarded as quasi-engineers' shear flow; see also J. H. Argyris and P. C. Dunne, *loc. cit.*, p. 53 (*Handbook*).



	Top and Bottom Covers				Webs					Nose cell	Rear cell
	1, 2 / 9, 10	2, 3 / 8, 9	3, 4 / 7, 8	4, 5 / 6, 7	1, 10	2, 9	3, 8	4, 7	5, 6		
Length (cm)	30·0	30·0	30·1	30·2	17	20	20	16	10	65·7	84·6
Thickness (mm)	2·5	2·5	2·5	2·5	1·6	1·6	1·6	1·6	1·6	1·0	1·0

Boom	1 / 10	2 / 9	3 / 8	4 / 7	5 / 6	Cell	I	II	III	IV	V	VI
Area (cm²)	8	12	12	12	10	Area (cm²)	350	555	600	540	390	210

Fig. 58.—Geometrical data for six-cell tube

The above analysis of a statically equivalent stress system in tapered or cylindrical multicell tubes could be used to write down the $\mathbf{b}_0$ matrix. Using the information developed in Section 8C for the $\mathbf{b}_1$ and $\mathbf{f}$ matrices we can calculate next the axial constraint stresses in the structures considered here. However, when an 'exact' analysis is to be performed by the matrix method in what is essentially a single continuous operation the choice of the $\mathbf{b}_0$ matrix should be guided entirely by the requirement of simplicity. Hence, in such cases the system of independent spars is greatly to be preferred to that of the multicell tube in the present example: see also Example 9b.

Numerical Example

To illustrate the application of the foregoing analysis we determine now the shear flow distribution in the single symmetrical six-cell section of a uniform cylinder illustrated in FIG. 58. It was found convenient to use the metric system in this example. The direct stress carrying ability of the walls has been replaced by effective flange areas at the web-cover intersections (augmenting the actual flange areas there) so that the walls carry only shear flow which is constant between flanges (t_s=0). This procedure enormously simplifies the work and is quite accurate enough for practical computations although naturally additional effective flanges may be introduced at intermediate stations (see Section 8C, p. 37). It should be noted that since the distribution of direct stress is known (E.T.B.) the appropriate effective flange areas can be calculated explicitly without any need for guesswork. Strictly, different areas have to be calculated for bending about $\overline{Gx}$ and $\overline{Gy}$ but since in practice the horizontal shear force S_x is of much less importance it is usually sufficient to apply throughout those calculated for bending about $\overline{Gx}$.

For a singly symmetrical section under a shear force normal to the axis of symmetry we are obviously only concerned with the D_x (or d_x) distribution. Due to the assumption of t_s=0 the (constant) values of D_{xb} in the upper and lower walls are zero and the values in the webs can be written down by inspection using the equilibrium equation (a11) for the flanges. The D_{xb} distribution is shown diagrammatically in FIG. 59*a*. In FIG. 59*b* we illustrate also the D_{xb} distribution for the alternative basic system mentioned previously in which the multi-cell tube is reduced to an open section by cutting N vertical webs. No further comment should be necessary to point the moral of this illustration.

Table I of Example 9a

Values of $\int_M D_{xb}ds_M/t$ and coefficients β

Cell M	I	II	III	IV	V	VI
area Ω_M (cm.²)	350	555	600	540	390	210
$\beta_M=\int_M\frac{ds_M}{t}$	763	471	490	466	404	908
$\beta_{M,M+1}$	106	125	125	100	62·5	
D_{xb} (cm.³)	+68	+120	+120	+96	+50	
$D_{xb}\beta_{M,M+1}$	7200	15000	15000	9600	3120	
$\int_M D_{xb}\frac{ds_M}{t}$	7200	7800	0	−5400	−6480	−3120

With D_{xb} having (constant) values in the inter-cell webs only the six compatibility equations are formed easily and systematically by means of the arrangement in TABLE I. All values in the table are obtained directly from the dimensional data of FIG. 58 and the D_{xb} distribution of FIG. 59*a*.

The equations for the unknown D_{xM} are therefore (see Eqs. (a23))

$$763D_{x1}-106D_{x2} = -7200+700G\phi\frac{I_x}{S_y}$$

$$-106D_{x1}+471D_{x2}-125D_{x3} = -7800+1110G\phi\frac{I_x}{S_y}$$

$$-125D_{x2}+490D_{x3}-125D_{x4} = 0+1200G\phi\frac{I_x}{S_y}$$

$$-125D_{x3}+466D_{x4}-100\,D_{x5} = +5400+1080G\phi\frac{I_x}{S_y}$$

$$-100D_{x4}+404D_{x5}-62\cdot5D_{x6} = +6480+780G\phi\frac{I_x}{S_y}$$

$$-62\cdot5D_{x5}+908D_{x6} = +3120+420G\phi\frac{I_x}{S_y} \qquad \text{(a63)}$$

Also the equilibrium Eq. (a31) for torque about D is

$$700D_{x1}+1110D_{x2}+1200D_{x3}+1080D_{x4}+780D_{x5}+420D_{x6}+94600=0 \qquad \text{(a64)}$$

Writing the solution of Eqs. (a63) in the form

$$D_{xM}=d_{xM}+\alpha_M G\phi\frac{I_x}{S_y} \qquad \text{(a25)}$$

$G\phi I_x/S_y$ is determined from (a64) and the results are as in TABLE II.

Table II of Example 9a

Values of d_{xM} and D_{xM} (cm.^{3})

M	d_{xM}	α_M	D_{xM}
I	$-12\cdot17$	$1\cdot46$	$-20\cdot13$
II	$-19\cdot62$	$3\cdot90$	$-40\cdot92$
III	$-\ 1\cdot15$	$4\cdot52$	$-25\cdot85$
IV	$+15\cdot79$	$4\cdot20$	$-\ 7\cdot16$
V	$+20\cdot71$	$3\cdot08$	$+\ 3\cdot89$
VI	$+\ 4\cdot88$	$0\cdot68$	$+\ 1\cdot16$
	From Eq. (a64) $G\phi\frac{I_x}{S_y}=-5\cdot46$		

Adding these D_{xM} to the D_{xb} we obtain the total D_x distribution due to S_y through D. Since D_{xb} is zero in the outer walls we have simply (see also Eqs. (a15))

for outer walls $\qquad D_x=D_{xM}$

for web between M and $M+1$ cells $\qquad D_x=D_{xb}+D_{xM}-D_{x,M+1}$

The total actual shear flows under the 10000 kg. load at D are

$$q=S_y\frac{D_x}{I_x}=\frac{10000}{7992}D_x=1\cdot25D_x \qquad \text{(a64)}$$

in kg./cm., the actual values being shown diagrammatically in FIG. 60.

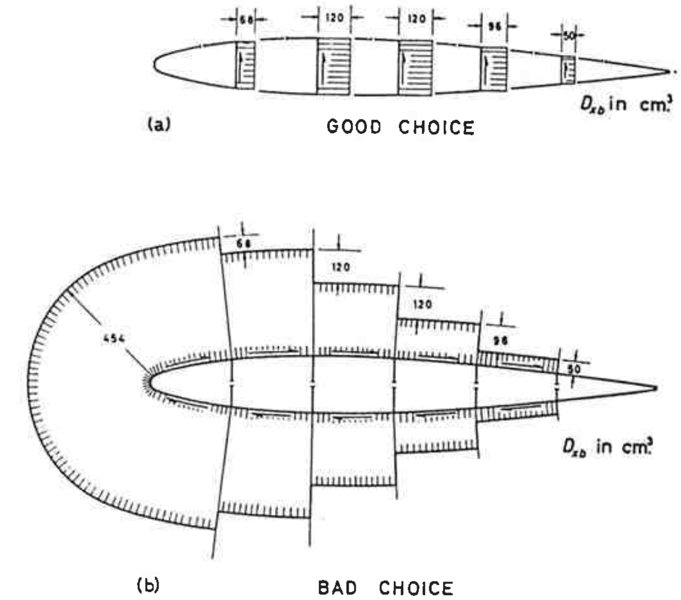


Fig. 59.—D_{xb} distribution in six-cell tube for alternative choices of basic system

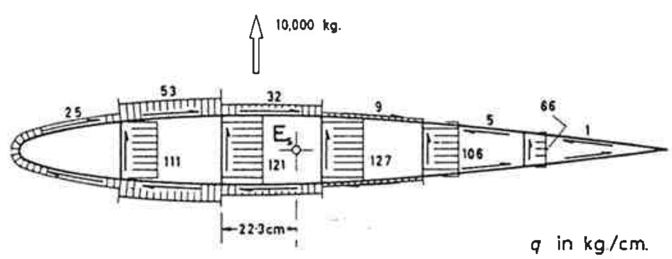


Fig. 60.—Final shear flow distribution in six-cell tube

Shear Centre

Since the present cross-section is singly symmetrical, the shear centre lies on the axis of symmetry and the first of Eqs. (a27) (with $I_{xy}=0$) gives immediately the x_E co-ordinate. Alternatively we note that the shear flows d_x have a resultant $S_y=I_x$ through the shear centre and we can modify slightly Eq. (a64) for torque about D to give

$$700d_{x1}+1110d_{x2}+1200d_{x3}+1080d_{x4}+780d_{x5}+420d_{x6}+94600=I_x(x_E-x_D)$$

Therefore

$$x_E-x_D=12\cdot3 \text{ cm.}$$

(b) The Four-Boom Tube with Deformable Ribs and some more General Structures

In this example we determine, using the force method of analysis, the 'exact' stress distribution in the idealized four-boom tube shown in FIG. 61. The investigation is carried out first by the δ_{ik}-method and illustrated in a numerical example. Subsequently, we analyse the same example by the general matrix method of Section 8C. We show also how the matrix formulation may be used with advantage in the more interesting case of a six-boom tube with or without intermediate spar web. It is hoped that these simple applications of the matrix method will condition the reader to the new ideas and show him their power and basic simplicity.

Consider the cylindrical tube of FIG. 61 with a singly symmetrical trapezoidal cross-section the flange cross-sectional areas B and the wall thicknesses t of which may vary arbitrarily length-wise. Loads are applied only at the rib positions

$$1, 2, \ldots\ldots, i, \ldots\ldots n$$

in the form of shear forces

$$R_{a1}, R_{b1}, \ldots\ldots, R_{ai}, R_{bi}, \ldots\ldots, R_{an}, R_{bn}$$

and moments

$$M_{a1}, M_{b1}, \ldots\ldots, M_{ai}, M_{bi}, \ldots\ldots, M_{an}, M_{bn}$$

at the front (a) and rear (b) spars. Following the general discussion on the idealization of aircraft structures given in Section 8 C (pp. 37and 40) we assume that the walls carry only shear stresses (the direct stress carrying ability is allowed for by suitably increasing the flange areas). The shear flow is hence constant in any field of each bay since changes in the z-direction can only be brought about at the ribs. It follows then that the end loads in the flanges vary linearly between ribs and that a knowledge of the flange loads at the rib positions suffices to determine them everywhere. Having found the flange loads the corresponding shear flows are determined easily from the flange load gradients and the condition of equilibrium with the applied shear force and torque.

In the tube shown in FIG. 61 there are n ribs including those at the free and built-in ends. At each rib position there are four flange loads and only three equilibrium equations are available for their determination. Hence, noting that the flange loads at the tip are zero, the degree of redundancy is $(n-1)$. This trivial result is confirmed by the general Eq. (247a) on p. 38 by substituting $N=1, \beta=4, \alpha=n$.

In selecting the basic system many choices are open to us. We may, for example, make a single cut in one of the flanges at each rib station to reduce the structure to a statically determinate three-flange basic system. Here, however, we calculate the statically equivalent stress system by the E.T.B. and the Bredt-Batho theory of torsion, a more general example of which was investigated in Section 9a. In this choice, instead of making a single cut we have, in fact, cut all the flanges to allow the relative warping consistent with the statically equivalent stresses while at the same time the direct stresses are transmitted across the cuts;* see also the discussion in Section 8C, pp. 32 and 33. The redundancies then consist of self-equilibrating flange load systems at the $(n-1)$ rib stations. A suitable and symmetrical measure of such a system is the boom load function P introduced by Argyris and Dunne.† We prefer to use instead a slight variant of P, the Y-system introduced on p. 39. The $(n-1)$ redundant Y are determined from the compatibility conditions of warping at the $(n-1)$ rib stations.

As a further alternative procedure we could, of course, choose as basic system the very simple structure consisting only of the two spars acting independently, i.e. we cut the top and bottom covers of the tube. However,

* Actually, we may consider this basic system as also derived by a single cut from the given system. Thus, if at the cut flange we apply the corresponding E.T.B. flange load and allow there any out of place movements, the stress distribution in the other structural elements is obviously that of the E.T.B. and Bredt-Batho theories.

† loc. cit. p. 38.

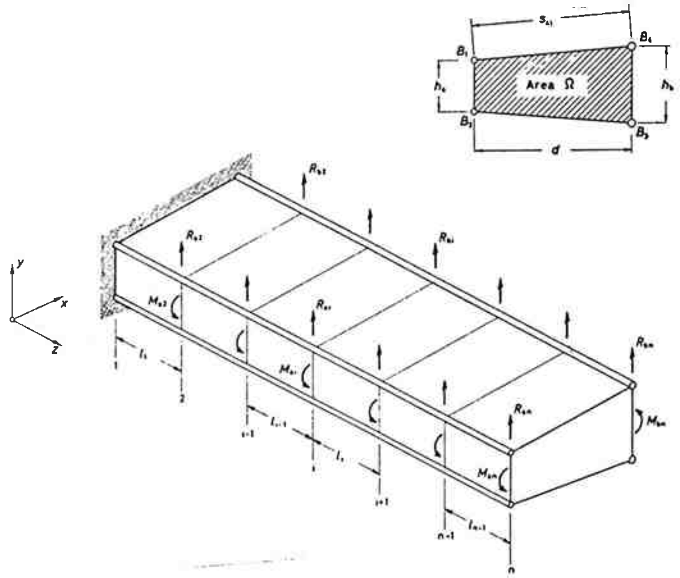


Fig. 61.—Four-flange tube. Geometry and loading.

except for very flexible covers and ribs our previous choice is much closer to the final correct solutions. The latter method is hence to be preferred when the design work has to be checked solely by a good statically equivalent stress system as is, in general, the case when no automatic digital computer is available. Subsequently the exact distribution, if found at all, is obtained only after completion of the design work—usually by a more clumsy version of the δ_{ik} method.

On the other hand when we use the matrix formulation in conjunction with a digital computer and derive the complete stress distribution as a single process it is preferable to select the simplest possible $\mathbf{b}_0$ matrix (see also the discussion on p. 39). Then the basic system formed by the independent spars is obviously indicated. An important advantage of this system is the absence of rib stresses if external loads are only applied in the plane of the spars. In multispar construction the simplicity of the basic system consisting of independent spars in comparison with the multi-cell system of Example (9a) is even more striking. If it is necessary to take into account external loads applied at intermediate points in the ribs and/or if spars are interrupted we may select as basic system for the matrix type of analysis the grid formed by spars and ribs (without covers)—still a very simple structure in which to find the $\mathbf{b}_0$ matrix.

(1) *Analysis by the δ_{ik} Method**

The statically equivalent stress system

Following our previous discussion we select here the E.T.B. (or quasi E.T.B.) and Bredt-Batho stresses as statically equivalent stress system. The flange loads are calculated at the rib stations with the effective areas B_i there. Thus, the m'th flange load at the i'th rib is,

$$P_{omi}=M_{zi}\frac{\bar{y}_{mi}\,B_{mi}}{I_{zi}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(b1)}$$

where M_{zi} is the bending moment at the i rib due to the applied loads R_a

* The particular case of the four-flange single-cell tube under a given loading has also been treated by W. J. Goodey in 'Two-spar Wing Stress Analysis', AIRCRAFT ENGINEERING, Vol. XXI, No. 247, p. 287, September 1949; No. 288, p. 313, October 1949; No. 289, p. 358, November 1949. There the author uses the Castigliano technique to formulate equations analogous to the three- and five-joint equations given here with essentially the same basic idealizations but with the effect of taper included. A comparison of the theoretical results with experimental strain measurements is also given.