ENERGY THEOREMS AND STRUCTURAL ANALYSIS

A Generalised Discourse with Applications on
Energy Principles of Structural Analysis
Including the Effects of Temperature and
Non-Linear Stress-Strain Relations.

by

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THE PRESENT WORK was originally published as a series of articles in Aircraft Engineering between October 1954 and May 1955. The purpose of these papers was two-fold. Firstly to generalize and extend but at the same time also to unify the fundamental energy principles of analysis of elastic structures. Although much of the corresponding theory has been available for a number of years, to the best of the author's knowledge it has not been given before in such generality. As an example, whilst keeping within the small deflection theory the arguments have been developed ab initio to include non-linear elasticity and arbitrary initial strains e.g. thermal strains. The first assumption introduces naturally the twin concepts of work and complementary work first put forward by Engesser. The author has attempted in this connexion to refer to all relevant and historically important papers. Since the appearance of the present articles, a few papers have been published which touch upon the same subject but suffer, unfortunately, from a rather incomplete list of references.

Secondly, the writer developed in considerable detail practical methods of analysis of complex structures—in particular for aeronautical engineering applications. The most important contributions are the matrix methods of analysis. Since they are only cursorily referred to in the Introduction, it may be appropriate here to describe their use and origin in greater detail. The matrix formulation besides providing an elegant and concise expression of the theory of such structures, is ideally suited for modern automatic computation because of the systematic ordering of numerical operation which the matrix calculus affords. The necessary programming for the digital computer is simplified since it can be preprogrammed to carry out matrix operations with only simple orders as to location and size of the matrix concerned and the operation to be performed. The specific programming for a particular problem may therefore be written comparatively quickly and easily and, moreover, follows closely the algebraic analysis.

As developed here, the matrix methods of analysis follow from particular forms of the two fundamental energy principles applicable to structures made up as an assembly of discrete elements. The one principle leads to an analysis in terms if displacements as unknowns (displacement method), while the second leads to an analysis in terms of forces (force method). Besides revealing more clearly the duality of the two methods, this derivation shows also the close connexion between the approximate methods (like the Rayleigh-Ritz method) for continuous systems and the matrix methods for finite assemblies. This is particularly valuable in providing suitable techniques for establishing the basic properties—stiffness and flexibility—of the individual elements of a complex structure where these elements have to be assigned simplified stress or strain patterns.

But in stressing the advantages of a unified approach to these diverse problems, a word of caution is necessary against carrying over into the modern methods too many ideas associated with practical calculations by the established or classical methods. The ability to tackle successfully problems in which the number of unknowns is measured in hundreds carries with it the necessity of rethinking one's practical approach if maximum advantage is to be gained from modern computational techniques. In the force method of analysis the choice of basic system and of the redundant forces must be governed primarily by the requirements of simplicity and standardization, in order to reduce the manual preparation of data to a minimum, and reduce the probability of errors.

At the time of publication of the original articles it was intended to reprint them as a single volume and to follow up the Parts I and II, contained here, with further parts dealing specifically with the practical application of the matrix methods. Unfortunately it was not possible, for a number of reasons, to complete this plan and the articles have for some time been unavailable. Since there appears to be a persistent interest in them the present reprint has been produced to meet the deficiency. Grateful thanks and acknowledgment are due to the Editor of Aircraft Engineering for permission to reprint the articles in this form. The method of reproduction has not permitted complete rearrangement of the text into book form, so that the divisions into monthly installments are still marked by blank spaces. However, errors in the text have been corrected as far as possible, and the pages have been renumbered consecutively to make for easier reference. Grateful thanks are due to Miss J. A. Berg for her care and skill in effecting these changes. The author would also like to thank here those correspondents who have written to point out textual errors and misprints.

A list of references to further work is also appended. These are all concerned with the matrix methods of analysis whose basic theory is developed here. In particular, Ref. 6 is an expanded and developed form of part of the work which was initially planned for the original series.

FURTHER REFERENCES TO RECENT WORK

I. INTRODUCTION

The increasing complexity of aircraft structures and the many exact or approximate methods available for their analysis demand an integrated view of the whole subject, not only in order to simplify their formulations but also to discover some more general truths and methods. There are also other reasons demanding a more comprehensive discussion of the basic theory. We mention only the increasing attention paid to temperature stresses and the realization of the importance of nonlinear effects. When viewed from all these aspects the ideas of a presenting a unified analysis appears more than necessary.

With this present paper we set out to develop a comprehensive system for the determination of stresses and deformations in elastic structures based on two fundamental energy principles. Although much of the theory has naturally been known for many years we believe that the present approach and the generality of the results are new. The loading systems considered are of an arbitrary nature and include ab initio the effect of temperature or other initial strains. Neither do we restrict ourselves to elastic bodies obeying Hooke's law but take a more general approach to the problem.

The purpose is to investigate, within the small-deflection theory, the stresses and deformations in elastic bodies not necessarily obeying a linear stress-strain law and under any load and temperature distribution. Dynamic effects are initially not considered and hence it is assumed for the present that the loads and temperature are of the quasi-static type. When investigating thermal strain effects we sought strictly to base the analysis on thermodynamic considerations. These are, however, only slightly touched upon here.

As in all theoretical work, we start by discussing the exact implications and equations derived from the initial assumptions, but we do not restrict ourselves here to this aspect. On the contrary, we pay close attention to approximate methods of analysis based on the physical concepts of work and strain energy. In particular we attempt to give upper and lower bounds to overall properties of the structure such as its stiffness. No attempt is made to estimate the error of stress and deformations at any particular point.

This series of papers originally arose12,13 from lectures given by the author since 1949-50 at the Imperial College, University of London. Naturally, the scope of the present work has grown beyond the narrower concept of undergraduate teaching, but the basis of the analysis dates back to that time. It is appropriate here to point out that certain of the basic ideas originate with Engesser2 who unfortunately does not seem to have followed them up. We refer, of course, to the two complementary concepts of work and complementary work. If we consider an ordinary load displacement diagram, then even if we restrict ourselves to small-displacements, this may be curvilinear, if the material follows a non-linear stress-strain law. Work is the area between the displacement axis and the curve, while complementary work is that included between the force axis and the curve. Thus, the two areas complement each other in the rectangular area (force)×(displacement) which would be the work if the ultimate force was acting with its full intensity from the beginning of the displacement. Naturally, in the case of a body following Hooke's law, the two complementary areas are equal, but it is still useful for the purpose of analysis to keep them apart. Since writing a previous paper14 on the subject the author has had the opportunity of consulting the most interesting book of Stephen Timoshenko. There a reference is made to the work of Westergaard,11 who indeed has developed further the basic ideas of Engesser, but not on quite such a general basis as here. Since approximate methods figure prominently in this paper reference ought to be made to the work of Prager and Syngue. They too set out to develop systematically the determination of stress and strain energy, restricting themselves, however, to Hooke's law and excluding temperature effects. Moreover, it appears that although many of their results are identical with existing ideas they clothed them in a language not too familiar to engineers. The discussion of past authors' work brings us to a few points which are preferably stated now. In much of present day structural analysis there seems to be an unfortunate tendency to overemphasize certain methods of analysing redundant structures and to neglect more useful ideas readily available for many years. This refers particularly to Castigliano's principles which are so often set out as the basis of all considerations, not only in theory, but also in the actual methods of calculation. This is, in our opinion, unfortunate, even though all methods naturally lead to the same results if based on the same assumptions. For example, if we select forces as redundancies then much the best means of obtaining the basic equations for their determination is the long established δ_i method of Mueller-Breslau based on the Unit Load idea. We do not need, in fact, even the concept of strain energy for this purpose. All we require is the idea of work and kinematics as used in rigid-body mechanics. From such ideas we can write down immediately our equations in the unknowns without bothering about strain energy. These methods have been in use by civil engineers for the past sixty years and it is surely time that we accepted them in the aeronautical world as standard analytical equipment. Actually, the basic principles go much farther back than Mueller-Breslau and were, in fact, developed independently by Maxwell1 and Mohr2 nearly a hundred years ago. The first systematic application of the δ_i method to stressed skin structures was given in the classical investigations of Ehlers.4 Regrettably enough this lucid work was occasionally referred to in the past as obscure, a lack of comprehension, no doubt, at least partly due to the too narrow understanding of redundant structures arising from a concentration on Castigliano's methods. However, the limitations of Castigliano's formulation of the problem are being at last increasingly recognized in aeronautical circles due to the demands of calculations for highly redundant systems. Naturally, most of the contemporary methods suggested are really nothing more than a transcription of the Mueller-Breslau and Ehlers technique.

We start our investigation in Section 3 with a discussion on work and complementary work in the presence of temperature effects and for non-linear stress-strain laws. With this basic knowledge we then proceed to the standard principle of virtual displacements or virtual work in Section 4. This is very similar to the currently used principle in rigid-body mechanics. Thus, we consider a state of equilibrium, apply virtual displacements to it and develop hence the classical principle of virtual work which substitutes, of course, for the equations of equilibrium. Since virtual displacements are kinematically possible ones this theorem starts from the assumption of inherent compatibility to find the necessary and sufficient condition for equilibrium. It is of course, well known that the theorem applies also to large displacements but this aspect is ignored here. However, temperature effects and an arbitrary law of elasticity are considered as long as the latter is monotonically increasing. Having established this principle we deduce easily some important theorems and applications.

GENERAL REFERENCES


Additional references are given as footnotes.

Firstly the principle of virtual displacements may always be used to derive, for any particular structural problem, the governing differential equations and the appropriate static boundary conditions in terms of the displacements. This method, however, is not recommended in general as a substitute for the derivation from consideration of equilibrium and elastic compatibility.

Next the principle of virtual work is used to derive Castigliano's theorem Part I, generalized for thermal effects. As is well known, this principle applies not only for non-linear stress-strain laws but also for large displacements. Our line of argument leads us then naturally to the principle of minimum strain energy for a fixed set of displacements and a given temperature distribution. This theorem applies also for non-linear stress-strain laws and is of great interest for approximate calculations in terms of assumed forms of displacements. It shows us that, while the strain energy is for a given set of displacements a minimum when the compatible state is also a solution of equilibrium, it is on the other hand a maximum for a given set of forces under the same conditions. These theorems were first developed for linearly elastic bodies by Lord Rayleigh more than seventy-five years ago. They are shown to apply also in the presence of thermal strain and for non-linear elasticity. In the remainder of the chapter we investigate in more detail approximate methods of analysis using the Rayleigh-Ritz Method, which is then applied in such applications that the principle of virtual displacements shows its greatest power. The particular form of the Rayleigh-Ritz procedure as known in the Galerkin method is also discussed. It is of importance when the assumed deformations satisfy all boundary conditions. The methods indicated apply again in the presence of thermal strains and non-linear stress-strain laws. The next, Section 5, gives simple illustrations to the method of virtual displacements.

The second fundamental principle is developed in Section 6. We call it the principle of virtual forces or complementary virtual work. Here we consider a state of equilibrium, apply a statically consistent and infinitely small virtual force and stress system and find, by using the idea of complementary work, the second principle. This is a necessary and sufficient condition that the position of equilibrium is also one of elastic compatibility. Again this theorem may be used to derive the differential equations of any particular problem, this time in terms of stresses or stress resultants. However, our comments on the parallel method in the case of the virtual displacements are equally applicable here. It should never be used as substitute for more physical and geometric reasoning.

Next, we derive what is essentially a generalization of Castigliano's Part II theorem. Contrary to what is generally believed this theorem does apply for non-linear stress-strain laws as long as we replace strain energy by complementary strain energy, which is defined in the same way as complementary work. It is extended to include temperature effects. We proceed then with the generalization of Castigliano's principle of minimum strain energy (or least work) for non-linear stress-strain laws and thermal strains. Some interesting developments derive from this and are given in the form of maximum and minimum theorems complementary to those developed under the virtual displacement method. They do not seem to have been given previously in this form and provide a useful background to approximate methods. They show us that any assumed statically equivalent stress distribution must always underestimate the stiffness. This is most valuable for practical purposes and is exactly opposite to the effect of assumed displacement distributions which always overestimate the stiffness. The two in conjunction give us hence lower and upper bounds to overall characteristics of the structure such as its stiffness. In this section we discuss also the Unit Load Method, which as mentioned previously, provides the basis for one of the more convenient methods for the calculation of displacements and of redundant forces. It is shown to be applicable to structures with non-linear stress-strain laws. Section 7 presents some simple illustrations of the principle of virtual forces.

In the last section we develop a slightly more generalized version of the $\delta u$ method of Mueller-Breslau. These equations lend themselves readily to presentation in matrix form. Next we obtain the corresponding equations when displacements and net forces are introduced as the unknowns.

A Note on the Mathematics
The mathematics used in this paper is, in general, elementary and should be familiar to any university graduate. We have avoided the more formal application of the calculus of variations which can be singularly unattractive to those more physically inclined. Chapter 3 and parts of Chapters 4 and 5 may prove, at first, rather difficult for a student. However, it is always possible to gain an understanding of the basic ideas by substituting simple examples (e.g. frame problems) for the necessarily general proofs given here.

The later parts of this series of papers will present a number of applications of the basic methods developed here.

2. BASIC EQUATIONS AND NOTATION

\begin{align*}
\text{Body forces (e.g. gravity forces) per unit volume} & \quad \omega_x, \omega_y, \omega_z \\
\text{Surface forces per unit surface} & \quad \phi_x, \phi_y, \phi_z \\
\text{Direction cosines of external normal to surface} & \quad l, m, n
\end{align*}

\begin{align*}
\text{Displacements} & \quad u, v, w
\end{align*}

\begin{align*}
\text{Parallel to a Cartesian co-ordinate system} & \quad O_x, O_y, O_z \\
\text{(see Figs. 1 and 2)} & \quad \Omega_x, \Omega_y, \Omega_z
\end{align*}

\begin{align*}
\text{Forced motion} & \quad f_x, f_y, f_z
\end{align*}

\begin{align*}
\text{Displacement field} & \quad \xi_x, \xi_y, \xi_z \\
\text{Strain field} & \quad \epsilon_x, \epsilon_y, \epsilon_z \\
\text{Stress field} & \quad \sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}
\end{align*}

\begin{align*}
\text{Strains} & \quad \epsilon_x, \epsilon_y, \epsilon_z \\
\text{Stresses} & \quad \sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}
\end{align*}

\begin{align*}
\text{Body Forces} & \quad \text{Stresses}
\end{align*}

\begin{align*}
\text{Normal} & \quad (l, m, n)
\end{align*}

\begin{align*}
\text{Fig. 1.- Stresses and body forces}
\end{align*}

\begin{align*}
\text{Fig. 2.- Stresses and surface forces}
\end{align*}

\begin{align*}
\sigma_{xx}, \sigma_{yy}, \sigma_{zz} & \quad \text{Direct stresses} \\
\sigma_{xy}, \sigma_{xz}, \sigma_{yz} & \quad \text{Shear stresses}
\end{align*}

\begin{align*}
\gamma_{xx} = \frac{\partial \sigma_{xx}}{\partial x}, \quad \gamma_{yy} = \frac{\partial \sigma_{yy}}{\partial y}, \quad \gamma_{zz} = \frac{\partial \sigma_{zz}}{\partial z} & \quad \text{Total direct strains} \\
\gamma_{xy} = \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{yx}}{\partial x}, \quad \gamma_{xz} = \frac{\partial \sigma_{xz}}{\partial z} + \frac{\partial \sigma_{zx}}{\partial x}, \quad \gamma_{yz} = \frac{\partial \sigma_{yz}}{\partial z} + \frac{\partial \sigma_{zy}}{\partial y} & \quad \text{Total shear strains}
\end{align*}

\begin{align*}
\epsilon_{xx} = \gamma_{xx} & \quad \text{Initial direct strains (e.g. thermal strains)} \\
\epsilon_{yy} = \gamma_{yy} & \quad \text{Initial shear strains} \\
\epsilon_{xy} = \gamma_{xy} & \quad \text{Elastic direct strains} \\
\epsilon_{xz} = \gamma_{xz} & \quad \text{Elastic shear strains}
\end{align*}

\begin{align*}
\sigma_{xy} = \sigma_{yx} + \sigma_{xx} \eta_{xx} + \sigma_{yy} \eta_{yy} + \sigma_{xz} \eta_{xz} + \sigma_{yz} \eta_{yz} + \sigma_{xy} \eta_{xy} & \quad \text{Element of volume} \\
\sigma_{xy} = \sigma_{yx} + \sigma_{xx} \eta_{xx} + \sigma_{yy} \eta_{yy} + \sigma_{xz} \eta_{xz} + \sigma_{yz} \eta_{yz} & \quad \text{Element of surface}
\end{align*}

\begin{align*}
\alpha & \quad \text{Linear coefficient of thermal expansion (may vary with $\Theta$)} \\
\Theta & \quad \text{Rise of temperature} \\
E & \quad \text{Young's modulus} \\
G & \quad \text{Shear modulus} \\
P & \quad \text{Poisson's ratio}
\end{align*}

\begin{align*}
\sigma_{xy} = \sigma_{yx} + \sigma_{xx} \eta_{xx} + \sigma_{yy} \eta_{yy} + \sigma_{xz} \eta_{xz} + \sigma_{yz} \eta_{yz} & \quad \text{Element of volume} \\
\sigma_{xy} = \sigma_{yx} + \sigma_{xx} \eta_{xx} + \sigma_{yy} \eta_{yy} + \sigma_{xz} \eta_{xz} + \sigma_{yz} \eta_{yz} & \quad \text{Element of surface}
\end{align*}

\begin{align*}
W & \quad \text{Work of external forces} \\
U_{e} & \quad W + \text{const.} \quad \text{Potential energy of external forces} \\
U_{s} & \quad \text{Strain energy (or potential energy of elastic deformation)} \\
W^* & \quad \text{Complementary work, complementary potential of external forces and complementary potential energy of elastic deformation} \\
U^* & \quad \text{Complementary potential energy of total deformation}
\end{align*}

From a consideration of equilibrium on an element $dV = dx dy dz$, illustrated for the $x$-direction in Fig. 3.

\begin{align*}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \omega_x = 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \omega_y = 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \omega_z = 0
\end{align*}
From a consideration of equilibrium on the surface (see Fig. 2)

\[
\begin{align*}
\sigma_{xx} + m\sigma_{xy} + n\sigma_{yy} &= \phi_x \\
-\sigma_{yy} + m\sigma_{xy} + n\sigma_{xx} &= \phi_y \\
n\sigma_{xx} + m\sigma_{xy} + n\sigma_{yy} &= \phi_n
\end{align*}
\]

(5)

Over part of the surface the boundary conditions may be expressed in terms of stresses or forces (static boundary conditions) and over the remainder in terms of displacements or strains (kinematic or geometric boundary conditions). Naturally, the boundary conditions may be of both types over the same part of the surface. Consider, for example, the tube shown in Fig. 4. It is assumed fully built in at the root \(z = 0\) and free at the tip \(z = l\). Ribs rigid in their own plane but freely flexible to deflexions out of their plane are assumed at \(z = 0\) and \(z = l\). The boundary conditions are: at \(z = 0\), \(u = v = w = 0\), i.e. pure kinematic conditions; at \(z = l\), \(\sigma_{zz} = 0\), \(\frac{\partial v}{\partial x} = 0\) for the vertical walls and \(\frac{\partial w}{\partial x} = 0\) for the horizontal walls, i.e. both static and kinematic conditions.

To denote infinitesimal elements of geometric properties of the structure (e.g. co-ordinates, area, volume) we use the standard symbol \(d\).

To denote infinitesimal increments of forces, stresses, displacements, strains and work we use the symbol \(\delta\).

Thus, \(dV = \text{infinitesimal element of volume} = dx dy dz\), \(\delta P = \text{infinitesimal increment of force} P\).

The symbols

\[f(\ldots)\,dV, \quad f(\ldots)\,dS\]

denote integrations over a volume and surface respectively.

The formal mathematical proof of some of the basic theorems in this paper is shortened by using Green's theorem.* Let \(\phi\) and \(\psi\) be two continuous functions and let also the first partial derivatives of \(\phi\) and the first and second partial derivatives of \(\psi\) be also continuous. Green's theorem states:

\[
\int_V \left[ \nabla \cdot \begin{bmatrix} \phi_x & \phi_y \\ \phi_y & \phi_z \end{bmatrix} \right] dV = -\int_S \phi \Delta \psi dS + \int_S \left[ \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial z} \right] dS
\]

(6)

* See Courant, Differential and Integral Calculus, translated by J. E. McShane, Blackie and Son Ltd., London and Glasgow, 1949, Vol. II.

\[
\Delta \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}
\]

and \(l, m, n\) are the direction cosines on the surface.

This theorem can be proved by integration by parts.

Examples of application:

Take \(\phi = \sigma_{xx} + \frac{\partial \psi}{\partial x} = \delta u, \quad \frac{\partial \psi}{\partial x} = 0\)

Then

\[
\int_S \sigma_{xx} \delta u \, dS - \int_S \frac{\partial \sigma_{xx}}{\partial x} \delta y_{xx} \, dV + \int_S \sigma_{xx} \delta u \, dS
\]

where \(\delta y_{xx} = \frac{\partial u}{\partial x} \delta \phi = \frac{\partial u}{\partial x} \delta \psi\) is the increment of the strain \(\gamma_{xx}\) corresponding to \(\delta u\).

Similarly take \(\phi = \sigma_{xy} + \frac{\partial \psi}{\partial y} = \delta v, \quad \frac{\partial \psi}{\partial y} = 0\)

Then

\[
\int_S \sigma_{xy} \delta v \, dS - \int_S \frac{\partial \sigma_{xy}}{\partial y} \delta y_{xy} \, dV + \int_S \sigma_{xy} \delta v \, dS
\]

where

\[
\delta y_{xy} = \frac{\partial u}{\partial y} \delta \phi + \frac{\partial v}{\partial x} \delta \phi = \frac{\partial u}{\partial y} \delta \psi + \frac{\partial v}{\partial x} \delta \psi = \delta u + \delta v
\]

Note that although Green's theorem is helpful for the mathematical understanding of the present theory it is not really necessary for the physical understanding; a reader unfamiliar with these aspects of the integral calculus may omit the relevant parts.

3. WORK AND COMPLEMENTARY WORK—STRAIN ENERGY AND COMPLEMENTARY STRAIN ENERGY

The analysis of the present paper is restricted to small strains which can be expressed by the linear formulae given in the notation. Such displacements and corresponding strains are obviously additive (algebraically). Thus, if \(\sigma_1, \gamma_1\), and \(\sigma_2, \gamma_2\) are displacements and strains in a deformed state 1 and 2 respectively, then \(\sigma_1 + \sigma_2, \gamma_1 + \gamma_2\) represent also a compatible state of deformation of the body. Our assumption does not impose, however, a linear stress-strain relationship; hence if \(P_1, \sigma_1\), and \(P_2, \sigma_2\) are the forces and stresses corresponding to the above two states of deformation of the body, the forces and stresses corresponding to the deformed state \(\sigma_1 + \sigma_2\) are not \(P_1 + P_2, \sigma_1 + \sigma_2\) except in the case of a linearly elastic body. In all cases, however, the stress-strain law is assumed to increase monotonically as shown in Fig. 5. In conclusion we can state that the law of superposition is assumed to hold for strains and displacements but not necessarily for the stresses.

In general, we assume also that the displacements are so small that the equilibrium conditions can be written down for the undeformed body. It follows then that the question of stability or instability of equilibrium does not enter in the analysis of this paper and there is a unique solution to every problem.

Consider a three-dimensional deformable body (not necessarily elastic) in equilibrium subjected to a self-equilibrating system of body forces \(\omega\), etc., surface forces \(\phi\), etc., and a temperature \(\Theta\). These forces and temperature may vary with time but the variations are assumed so slow that the dynamic effects are negligible. Let, in a time interval \(\delta t\), the forces increase by \(\delta \omega_x, \delta \omega_y, \delta \omega_z\) etc. and the temperature by \(\delta \Theta\). The displacements increase at the same time by \(\delta u\) etc. There arises hence an increment of work (see Fig. 6).

\[
\delta W = \int_0^1 \delta \omega_x \delta u_x + \int_0^1 \delta \omega_y \delta u_y + \int_0^1 \delta \omega_z \delta u_z \, dV
\]

\[+ \int_S \left[ \frac{\partial \phi_x}{\partial x} \delta u_x + \frac{\partial \phi_y}{\partial y} \delta u_y + \frac{\partial \phi_z}{\partial z} \delta u_z \right] \, dS
\]

+ terms of higher order.
The terms of higher order involve expressions \( \frac{1}{2} \delta \omega \cdot \delta u + \frac{1}{2} \delta \phi \cdot \delta u \) etc., and may be neglected to the first order of magnitude considered here. Thus,

\[
\delta W = \int \left[ \omega \delta u + \alpha \delta v + \alpha \delta w \right] dV + \int \left( \delta \phi \delta u + \phi \delta v + \phi \delta w \right) dS.
\]  

(7)

Note that Eq. (7) does not presume any specific force-displacement law, but be it elastic or non-elastic.

It is simple to derive an alternative expression by considering the additive effect of the work done by the stress resultants on each volume element \( dV \). A perusal of Fig. 7 shows that the deformation \( \delta u, \delta v, \delta w \) gives rise to an increment of work for an element \( dV \)

\[
(\delta \phi \delta u + \delta \phi \delta v + \delta \phi \delta w) dV = \delta \phi \delta w dV
\]

again neglecting terms of higher order.

The incremental (infinitesimal) strains \( \delta \gamma \) etc. are those due to the displacements \( \delta u, \delta v, \delta w \) etc. Thus,

\[
\delta \gamma = \frac{\delta u}{\delta x} \frac{\delta u}{\delta y}, \quad \delta \gamma = \frac{\delta v}{\delta y} \frac{\delta v}{\delta x}, \quad \delta \gamma = \frac{\delta w}{\delta x} \frac{\delta w}{\delta y},
\]

(8)

It follows that the increment of work \( \delta W \) may also be expressed as

\[
\delta W = \int \left[ \delta \phi \delta w \right] dV
\]

(9)

The formal equivalence of Eqs. (7) and (9) may be proved without difficulty by using Green's theorem. To this effect multiply each of the internal equilibrium conditions (4) by \( \delta u \cdot dV, \delta v \cdot dV, \delta w \cdot dV \) respectively, and integrate over the body. We obtain,

\[
\int \left[ \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \omega \right) \delta u + \left( \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + \alpha \right) \delta v + \left( \frac{\partial \tau_{xz}}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \phi \right) \delta w \right] dV = 0.
\]

Applying now Green's theorem to the terms involving the stresses, as shown in the previous section, we find, using the surface equilibrium conditions, Eqs. (5),

\[
\int \delta \phi \delta w dV = \int \left[ \frac{\partial \tau_{xx}}{\partial x} \delta u + \frac{\partial \tau_{yy}}{\partial y} \delta v + \frac{\partial \tau_{zz}}{\partial z} \delta w \right] dV + \int \left[ \frac{\partial \tau_{xy}}{\partial y} \delta v + \frac{\partial \tau_{yx}}{\partial x} \delta w \right] dS = \delta W
\]

where the \( \delta \phi \) 's satisfy Eqs. (8) and are hence compatible strains. Note that where \( \delta u \) is unknown but the displacement (say \( u \)) fixed, the corresponding \( \delta u \) is zero and hence the relevant terms in the last relation vanish. Integrating Eq. (9) from the initial unstrained state \( O \) to a final state \( I \) we find the total work

\[
W = \int_{O}^{I} \left[ \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \omega \right) \delta u + \left( \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + \alpha \right) \delta v + \left( \frac{\partial \tau_{xz}}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \phi \right) \delta w \right] dV
\]

(10)

where

\[
W = \int_{O}^{I} \left( \sigma_{xx} \delta u + \sigma_{yy} \delta v + \sigma_{zz} \delta w \right) dV + \left[ \frac{\partial \tau_{xy}}{\partial y} \delta v + \frac{\partial \tau_{yx}}{\partial x} \delta w \right] dS
\]

(11)

Note that, in general, the work \( W \) done to reach a state \( I \) starting from a state \( O \) depends on the path chosen due to say plasticity, viscous effects, etc. In such cases \( \delta W \) is not the total differential of the right-hand side of Eq. (10).

In what follows we assume that the body is fully elastic and isotropic.

\* We say that \( \delta W = dW \) is a total differential of \( W \) if \( dW = 0 \), where the integration is taken around a closed curve; if this applies \( W \) is obviously independent of the path of deformation taken between states \( O \) and \( I \).

\* The stress used only be assumed at each point; the properties of the body may vary from point to point.
\[ W = U + \frac{E\varepsilon_i}{2(1-\nu)} \]  

(23)

where

\[ U_i = \begin{cases} 0 \frac{1}{1-(1-\nu)(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2 - (\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2))} \end{cases} \]  

(26)

is the strain energy as a function of the elastic strains.

The following relations hold only for linear elasticity,

\[ \frac{\partial U}{\partial \varepsilon_{xx}} = -\frac{\partial U}{\partial \varepsilon_{yy}} = \varepsilon_{xx} \]  

(27)

\[ \frac{\partial U}{\partial \varepsilon_{yy}} = -\frac{\partial U}{\partial \varepsilon_{zz}} = \varepsilon_{yy} \]  

(28)

For two-dimensional stress distributions substitute the factor \(1/(1-\nu)\) for \((1-(1-\nu)^2)\) in Eqs. (24), (25) and (26). The corresponding strain energy function \( U_i \) is

\[ U_i \propto \frac{1}{(1-\nu)}(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 - 2\varepsilon_{xx}\varepsilon_{yy}) \]  

(26a)

Parallel to the concepts of work and strain energy two further ideas essentially due to Engesser, are of particular importance to our investigations. Consider to that effect a one-dimensional force-displacement and a stress-strain elastic stress diagram (Fig. 6). The vertically shaded areas are obviously those of work and strain energy respectively. It is natural to inquire if the horizontally shaded areas complementing the previous areas in the rectangular areas \( P \) and \( \sigma \) are of any importance (see Fig. 6). In fact, as is shown further on, the introduction of these new concepts is proved a particularly happy one when generalizing some theorems, currently assumed to be valid only for linear elasticity, to bodies with non-linear stress-strain relations. Although the complementary areas are equal for linear elasticity it is still useful in such cases to differentiate between them.

It is interesting to note that in thermodynamics two similar complementary functions are used: the free energy function \( H \) of von Helmholtz and the function \( G \) of Gibbs. In what follows we call the horizontally shaded areas complementary work and complementary strain energy and denote them by \( W^* \) and \( U^* \) respectively.

We generalize next our new concepts by considering the three-dimensional case. Let the actual displacements in a body subjected to body forces \( \mathbf{u} \), surface forces \( \mathbf{f} \) and temperature \( \Theta \) be \( u, v, w \). The increment of the complementary work as these displacements increase to \( u + \delta u, v + \delta v, w + \delta w \), due to load increments \( \delta \varepsilon_{xx}, \delta \varepsilon_{yy}, \delta \varepsilon_{zz} \) etc. is given by

\[ \delta W^* = \int \left[ \sum_{\delta \varepsilon_{xx}} + \sum_{\delta \varepsilon_{yy}} + \sum_{\delta \varepsilon_{zz}} \right] dV \]  

(28)

or

\[ \delta W^* = \int \left[ \sum_{\delta \varepsilon_{xx}} + \sum_{\delta \varepsilon_{yy}} + \sum_{\delta \varepsilon_{zz}} \right] dV \]  

(28a)

since terms of higher order like \( \delta \phi \cdot \delta \varepsilon \) or \( \delta \varepsilon \cdot \delta \phi \) can be neglected. It is simple, as in the case of work, to derive an alternative expression to Eq. (28) in terms of stresses and total strains. To find it, note that the increments of the stresses must be in equilibrium with the corresponding increments of the body forces \( \delta \varepsilon_{xx}, \delta \varepsilon_{yy}, \delta \varepsilon_{zz} \) etc. and surface forces \( \delta \phi \). There are thus six relations of the type of Eqs. (4) and (5). We write here only the two for equilibrium in the \( x \)-direction

\[ \partial (\sigma_{xx}) + \partial (\sigma_{yx}) + \partial (\sigma_{zy}) = \partial \dot{u} \]  

(29)

and

\[ \partial (\sigma_{xx}) + \partial (\sigma_{yx}) + \partial (\sigma_{zy}) = \partial \dot{v} \]  

(30)

Multiplying now each of the first set of equations by the displacements \( u, v, w \) respectively, summing and integrating over the body we obtain by applying Green's Theorem similarly to when we derived Eq. (9), and using Eq. (30)

\[ \int_{\Omega} \left[ \sum_{\delta \varepsilon_{xx}} + \sum_{\delta \varepsilon_{yy}} + \sum_{\delta \varepsilon_{zz}} \right] dV = \int_{\Omega} \left[ \sum_{\delta \varepsilon_{xx}} + \sum_{\delta \varepsilon_{yy}} + \sum_{\delta \varepsilon_{zz}} \right] dV \]  

(31)

where \( \gamma_{\varepsilon_{xx}}, \gamma_{\varepsilon_{yy}}, \gamma_{\varepsilon_{zz}} \) are the total strains. Note that where the forces, for example, \( \phi \), are fixed values \( \phi = 0 \). Thus ultimately

\[ \delta W^* = \int \gamma_{\varepsilon_{xx}} dV \]  

(32)

where

\[ \gamma_{\varepsilon_{xx}} = \gamma_{\varepsilon_{yy}} + \gamma_{\varepsilon_{zz}} \]  

(32a)

Integrating Eq. (32) between the initial unstrained state \( O \) and a final state \( I \) we find the total complementary work

\[ W^* = \int \gamma_{\varepsilon_{xx}} dV \]  

(33)

where

\[ W^* = \int \gamma_{\varepsilon_{xx}} dV \]  

(34)

Note that as in the case of work \( W \) the complementary work depends, in general, on the chosen path of deforming between \( O \) and \( I \).

We assume now the body is fully elastic and isotropic as described previously and find from Eq. (34)

\[ \delta W^* = \delta U^* + \eta \delta \varepsilon \]  

(35)

and

\[ \delta W^* = -\varepsilon_0^* + \eta \delta \varepsilon \]  

(35a)

where

\[ U^* = \int \delta \varepsilon dV \]  

(36)

and

\[ \varepsilon_0^* = \int \delta \varepsilon dV \]  

(37)

(see also Eqs. (18)). Note that the law of elasticity is arbitrary in Eq. (37). If \( U^* \) is given in terms of the stresses Eq. (37) show that this determines uniquely the strain-stress laws and vice versa. It is natural hence to call \( U^* \) the complementary strain energy function.

Integrating over the body we find

\[ W^* = U^* + \int \left[ \frac{1}{2} \dot{\sigma} dV \right] \]  

(38)

where

\[ U^* = \int \delta \varepsilon dV \]  

(38a)

where \( U^* \) is called the complementary strain energy or complementary elastic potential energy. When \( \Theta = 0 \) then

\[ W^* = U^* \]  

(39)

In the case of linear elasticity \( \varepsilon_0^* = W \) and \( U^* = U_0 \) but it is useless to differentiate still between them. For linear stress-strain laws of Eqs. (21) and (22), Eq. (38a) becomes, if temperature and loads are increased together from zero,

\[ W^* = U^* + \frac{1}{2} \]  

(38b)

and

\[ U^* = \int \left[ \frac{1}{2} \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \right] dV \]  

(38b)

(40)

which for two-dimensional stress distributions reduces to

\[ U^* = \int \left[ \frac{1}{2} \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \right] dV \]  

(40a)

The above considerations on strain energy and complementary strain energy may be used to derive all Castigliano theorems as generalized for non-linear stress-strain relations. We postpone, however, these investigations to subsequent chapters.

When the body obeys linear stress-strain relations the principle of superposition applies also to forces and corresponding stresses. Some important theorems derive from this property.

Thus, if the forces on a body are increased from zero to their final value then

\[ W = W^* + \int \left[ \frac{1}{2} \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \right] dV \]  

(41)

where all symbols in the brackets refer to final values. Eqs. (41) are known as Clapeyron's theorem. Another very useful theorem is due to Bertrami. Assume that the body is subjected to two force systems \( P_1 \) and \( P_2 \). Let the deflections due to system \( P_1 \) alone be \( u_i \) and due to \( P_2 \) alone \( u_j \). By applying first system \( P_1 \) and subsequently \( P_2 \) and then reversing the process we prove easily—nor that the final state is in each case the same—that

\[ W_{12} = W_{21} \]  

(42)

where

\[ W_{12} = \sum P_1 u_i \]  

(42a)

\[ W_{21} = \sum P_2 u_i \]  

(42b)

* See Nau, C. (2), Vols. 7, 8. 1872.
are the work done by the system of forces \( P_i \) over the displacements \( u_3(u_1) \) respectively. Relation (42) is known as the generalized reciprocal theorem of Betti.

A special form of Eq. (42) is Maxwell’s reciprocal theorem. Thus, if systems 1 and 2 consist each of on one force (or moment) only then

\[
\begin{align*}
   \mu_2 & = 1, \quad \delta u_1 \\
   \mu_3 & = 1, \quad \delta u_2
\end{align*}
\]

(43)

where \( \mu_2(\delta u_1) \) is the displacement or rotation in the direction of force or moment \( P_1(\delta u_1) \) due to force or moment \( P_2(\delta u_2) \).

### 4. THE PRINCIPLE OF VIRTUAL DISPLACEMENTS OR VIRTUAL WORK

We assumed in the previous section when discussing work and strain energy that the displacements \( \delta u \), \( \delta v \), \( \delta w \) arise from an actual variation of the applied forces and/or temperature distribution. However, this is an unnecessary restriction. We need only remember that to the first order of magnitude considered \( \delta W \) and \( \delta U \) are independent of the \( \delta F \)'s and corresponding \( \delta u \)'s since we ignore terms of the order \( \delta F \cdot \delta u \). Also for the purpose in hand no variation in the temperature is called for. Hence, when finding \( \delta W \) and \( \delta U \) we can assume that forces, stresses and temperature remain constant while the displacements are varied to \( u + \delta u, \quad v + \delta v, \quad w + \delta w \).

It is only necessary that the \( \delta u, \delta v, \delta w \)'s are compatible infinitesimal displacements (see Section 3, p. 4) thus they must be piecewise continuous in the interior and satisfy the kinematic boundary conditions. For example, if the \( u, v, w \) are prescribed on part of the boundary then the selected variations \( \delta u, \delta v, \delta w \) must be zero there too. Similar arguments apply if the derivatives of any of the displacements are fixed. However, where the forces and stresses are prescribed the variation of the \( \delta F \)'s is necessarily free. Note that there are cases when it is useful to relax even the kinematic boundary conditions when selecting the \( \delta u \)'s.

Such geometrically possible infinitesimal displacements are used extensively in rigid body mechanics and are called virtual displacements. Noting that the temperature and hence thermal strains remain constant we can restate now Eqs. (15), (19) and (19a) more generally as follows:

An elastic body is in equilibrium under a given system of loads and temperature distribution if for any virtual displacements \( \delta u, \delta v, \delta w \) from a compatible state of deformation considered

\[
\delta W = \delta U
\]

which is the standard principle of virtual displacements or virtual work. As stated here it is valid for an elastic body subjected both to loads and temperature effects.

Note that Eq. (44) is also valid for large displacements but the formulation of \( U \) is in such cases more complicated since the strain expressions are not any longer linear in the displacements and the equilibrium conditions have to be considered on the deformed element.

The point made above that the virtual displacements are arbitrary as long as they are infinitesimal and satisfy the internal compatibility conditions and kinematic boundary conditions is worth emphasizing. To fix ideas consider the statically determinate framework shown in Fig. (8) subjected to a transverse force \( P \). We apply to the system a virtual displacement \( \delta u \) in the form shown in the figure which allows only an elastic deformation of the upper flange 1, 2. This virtual displacement satisfies the kinematic boundary conditions

\[
\delta u = 0 \quad \text{at A and B}
\]

but bears obviously no relation to the actual displacements of the framework due to the force \( P \).

We find easily:

- force in member 1, 2: \( N_2 = - \frac{P h}{h} \)
- virtual displacement of force \( P \): \( \delta u_v = \delta u/\mu_b \)
- virtual elongation of member 1, 2: \( \Delta l_2 = \frac{\delta u}{h} \)

Thus,

\[
\delta W = - \delta U
\]

and

\[
U = \int \left[ \left( \omega_2 \omega_u + \omega_3 \omega_v + \omega_4 \omega_w \right) - \left( \phi_u \phi_u + \phi_v \phi_v + \phi_w \phi_w \right) \right] dS
\]

Hence

\[
\delta W = - \delta U
\]

as indicated by the principle of virtual displacements or virtual work.

In order to realize best some of the implications of the principle of virtual displacements let us consider again the derivation of Eq. (44) which applies when \( \eta = 0 \). In accordance with the analysis of p. 3, if we multiply the internal equilibrium Eqs. (4) with the virtual displacements \( \delta u, \delta v, \delta w \) sum the three expressions, integrate over the body, apply Green’s theorem (Eq. 6), and note the boundary conditions (5) we obtain Eq. (44). Again if we start from Eq. (44) we can apply Green’s theorem in the opposite direction and are led to

\[
\begin{align*}
   \int \left[ \left( \frac{\partial \delta F_2}{\partial x} + \frac{\partial \delta F_3}{\partial y} + \frac{\partial \delta F_4}{\partial z} \right) \delta u + \left( \frac{\partial \delta F_4}{\partial x} + \frac{\partial \delta F_1}{\partial y} + \frac{\partial \delta F_2}{\partial z} \right) \delta v + \left( \frac{\partial \delta F_1}{\partial x} + \frac{\partial \delta F_2}{\partial y} + \frac{\partial \delta F_3}{\partial z} \right) \delta w \right] dV \\
   = \int \left[ \left( \omega_2 \omega_u + \omega_3 \omega_v + \omega_4 \omega_w \right) - \left( \phi_u \phi_u + \phi_v \phi_v + \phi_w \phi_w \right) \right] dS
\end{align*}
\]

For arbitrary virtual displacements \( \delta u, \delta v, \delta w \) this relation can only be true if each of the brackets vanishes separately; thus, starting from the principle of virtual work we have re-established the conditions of equilibrium. An exception occurs, of course, where a displacement (say \( u \)) is fixed on the surface and \( \delta u \) is automatically zero there.

We conclude that the principle of virtual displacements Eq. (44) is a necessary and sufficient condition for the existence of equilibrium of an elastic body. Or otherwise we can state that by using virtual, i.e. kinematically possible compatible displacements, we substitute Eq. (44) for the internal and external equilibrium conditions. Note that the idea of strain energy is not necessary to the establishment of the principle of virtual work.

Since the forces are assumed to be applied on the undiscreted system and to remain constant during the virtual displacements we can regard \( \delta W \) as the variation of a potential \( -U \). Thus

\[
\delta W = - \delta U
\]

(45)

and

\[
U = - \int \left( \omega_2 \omega_u + \omega_3 \omega_v + \omega_4 \omega_w \right) dV - \int \left( \phi_u \phi_u + \phi_v \phi_v + \phi_w \phi_w \right) dS
\]

(46)

Thus

\[
\omega_2 = \frac{\delta u}{\delta u}
\]

(47)

which is the usual definition of a potential of forces. Note that \( \delta U \) is a total differential of the elastic displacement increments \( \delta u, \delta w \) denoted as potential of external forces.

We can write now Eq. (44) in the concise form

\[
\delta U = 0
\]

(47a)

where the suffix \( \epsilon \) indicates that only elastic strains and displacements are varied.

\[
U = U_1 + U_1 + \text{const.}
\]

(47b)

where \( U \) is the total potential energy of the system. Eq. (47) states that a position of elastic compatibility of an elastic body is also one of equilibrium (i.e. the body is at the true position of equilibrium) if any virtual variation of the displacements and strains whilst forces, stresses and temperature remain constant does not give rise to any first order variation of the total potential energy. The particular form (47) is known as the principle of a stationary value of total potential energy, if the latter is expressed in terms of displacements. Note that \( U \) itself may be calculated from formula (19b).

Actually the stationary value of \( U \) in our case always a minimum and this confirms our previous assertion that with the assumptions of our analysis all systems are stable. The mathematical proof that \( U \) is a minimum at the true position of equilibrium is straightforward*; the point is discussed in greater detail under (C) below.

We have assumed until now that the initial strains \( \eta \) are due to a temperature variation. However, this is an unnecessary restriction and there may be stresses arising from any source of self-straining. For example, in a framework they may be due to manufacturing errors in the lengths of the bars. In the more general cases of self-straining not only may the \( \eta \), \( \eta_1 \), \( \eta_2 \), \( \eta_3 \) be different, but there may also arise initial shear strains \( \eta_1 \), \( \eta_2 \), \( \eta_3 \). In such problems substitute

\[
\omega_2 = \omega_2 + \omega_2 \omega_3 + \omega_3 \omega_2 + \omega_3 \omega_3 + \omega_3 \omega_1 + \omega_1 \omega_3
\]

(48)

for \( \omega_2 \) in Eqs. (15) and (19) and in the other related expressions.

Equations (44) or (47) may be used to derive the results which follow.

(A) The differential equations of the theory of elasticity for arbitrary loading and temperature distribution or to particular structural problems in terms of the displacements; the appropriate static boundary conditions

* See Biezeno and Grammel (1), p. 74.
in terms of the displacements follow also from this analysis. It is important to note that in all applications it is best to form directly
\[
\delta U = \int \sigma e \, dV
\]
and not to evaluate first \( U \) and then to take its increment \( \delta \).

(B) Castigliano's theorem Part I generalized for thermal effects

\[
\left[ \frac{dU}{dP} \right]_n = P_n \tag{49}
\]

where \( P_n \) is the force (moment) applied in the direction of the deflection (rotation) \( u \). This relation may be obtained immediately if we apply a virtual elastic displacement \( \delta u \), solely to one external load \( P_n \). Note that Eq. (49) applies also for non-linear stress-strain laws and may also be generalized for large displacements.

(C) The Principle of Minimum Stain Energy when \( U \) is expressed in terms of the displacements and the temperature is not varied.

We arrive immediately at this theorem if we select only such virtual displacements \( \delta u \) which are zero at the applied forces. Then
\[
\delta W = 0
\]
and we conclude from Eq. (44a) that
\[
\delta u = 0 \quad \text{and} \quad U = \min.
\]

at true position of equilibrium if only such virtual deformations are allowed that do not cause any additional strains.

Hence, if we compare all possible compatible states of deformation of a body associated with a given set of displacements (not sufficient by themselves to fix completely the deformed shape of the body) then the true position of equilibrium has the minimum strain energy. This is still true if the body is subjected to temperature loading.

The point is of sufficient interest to warrant some elaboration. First it may be helpful to point out that when we state that at the position of equilibrium the total potential energy has a stationary value, this is a minimum, we do not compare physically possible adjoining states. For, in stating that the potential energy has a stationary value, i.e., \( U = 0 \), we compare the true position \( u \) with a position \( u + \delta u \) assuming in both cases that forces and stresses are the same. This can obviously not be true for the second position since for given forces \( u \) there is a unique position of equilibrium. In fact, we mentioned that this arises due to our legitimate neglect of the higher order terms in \( \delta u \). Also, when we go a step further and state that the stationary value is a minimum we prove this by considering the influence of terms \( u + \delta u \) \( \delta u \) in \( U \), arising from the variation \( \delta u \) associated with \( \delta u_n \), but we still keep the forces constant—although this cannot, in general, be true.

Having pointed out these aspects of the virtual displacements approach we shall, in what follows, discuss the question of the extremum of \( U \), from a more physical point of view. Again we prescribe certain displacements on the body and do not allow any forces \( P \) other than those arising due to and in the direction of the given displacements. The structure takes up its natural position of equilibrium, from which we can deduce the value of the forces \( P \). If we want to move the body to a position \( u + \delta u \), \( \delta u \) while keeping the set of prescribed displacements constant we must apply certain additional body and surface forces to push the system away from its initial configuration. The work done by these constraint forces \( \int \sigma e \, dV \) obviously positive), produces by reason of equilibrium in the new position an equal increase in the strain energy stored. Thus, the strain energy in any neighboring compatible configuration is greater than that for the unconstrained original position and hence the strain energy there is a minimum.

An alternative way of producing a state of equilibrium different from the natural one in an elastic body under a prescribed set of displacements is the introduction (prior to the imposition of the displacements) of internal or external constraints that do not work. For example, in a shell or plate analysis, we may assume that the middle surface is inextensible and the transverse shear strains are zero; thus, in this case we impose infinite values for \( E \) and \( G \) in the middle surface and an infinite value for \( E \) in the transverse direction. Another type of constraint may be achieved by the introduction of a rigid support. Also the stress-strain data may be taken locally to be II instead of I (Fig. 9). In all such cases the arguments of the previous paragraph show that the strain energy of the constrained body for the given prescribed set of displacements is greater than for the unconstrained body. Conversely, if we relax any existing constraint whilst again keeping constant the prescribed displacements the strain energy is decreased. The relaxation of constraints may take the form of a stress-strain curve as indicated in Fig. III instead of II. Alternatively we may introduce a hinge in the structure. Another example, is the case of a shell where we ignore the bending stiffness and admit only a membrane state of stress; a current procedure in wing stressing.

Thus we conclude: The strain energy of an elastic body for a given set of displacements is increased (reduced) by the imposition (relaxation) of constraints that do no work. An exception occurs if the effect of imposition or removal of a constraint is nil. For example, in an infinitely long, thin, circular shell under internal pressure it is immaterial whether we take account of or ignore the bending stiffness of the wall.

Also since the constraints do no work it follows that the increase (reduction) of the strain energy can only be produced by the forces \( P \). Thus: the force system \( P \) which is set up at the and in the direction of the prescribed displacements and hence to the stress of the structure is increased (reduced) by the imposition (relaxation) of constraints that do no work.

If we consider now the case of an elastic body under a given set of forces instead of displacements then we conclude immediately from the last theorem: the displacements and hence also the strain energy in a body under a given set of forces are reduced (increased) by the imposition (relaxation) of constraints that do no work.

Although the last two theorems illustrate two complementary effects of the action of constraints on the stiffness of a structure, the last theorem may also be expressed as follows: the strain energy of an elastic body under a given set of forces is a maximum when it is subjected to the least number of constraints that do no work.

The immediate application of the above considerations is, of course, to the effect of actual constraints on elastic structures as illustrated in the examples mentioned. A more important application appears in connexion with approximate analyses of deformations. Thus, if we reduce the freedom of deformation as we do in the Rayleigh-Ritz and related methods we always over-estimate the stiffness of the structure. Hence for a given set of forces (displacements) we under-estimate (over-estimate) the corresponding displacements and strain energy (forces and strain energy).

The above theorems on the effects of constraints on strain energy and stiffness appear to have been given first by Rayleigh in 1875 for linear elastic bodies and no temperature effects. Our arguments indicate, however, that they apply also to elastic bodies with non-linear stress-strain relationship and under thermal loading. The original principles are occasionally referred to as the static analogues of Bertrand's and Kelvin's theorems in dynamics.

(D) The Unit Displacement method. This method will be developed in Section 8.

(E) Approximate methods of displacement analysis, using the Rayleigh-Ritz procedure. In this method we assume for the displacements approximate functions or series of functions satisfying the geometric but not necessarily the static boundary conditions. For example, in a three-dimensional elastic continuum we may express the total displacements \( u, v, w \) in a finite series as follows:

\[
\begin{align*}
\delta u &= \alpha u_n + \alpha v_n + \alpha w_n \\
\delta v &= \beta u_n + \beta v_n + \beta w_n \\
\delta w &= \gamma u_n + \gamma v_n + \gamma w_n
\end{align*}
\]

where \( u_n, v_n, w_n \) satisfy the kinematic conditions where these are prescribed and \( u_n, v_n, w_n \) are linearly independent functions which vanish there. \( u_n, v_n, w_n \) are unknown constants to be determined by the Rayleigh-Ritz procedure. The elastic strains corresponding to (50) are (see Eqs. (1) and (2))

\[
\begin{align*}
\epsilon_{xx} &= \frac{\partial u_n}{\partial x} \\
\epsilon_{yy} &= \frac{\partial v_n}{\partial y} \\
\epsilon_{zz} &= \frac{\partial w_n}{\partial z} \\
\gamma_{xy} &= \frac{\partial u_n}{\partial y} + \frac{\partial v_n}{\partial x} \\
\gamma_{yz} &= \frac{\partial v_n}{\partial z} + \frac{\partial w_n}{\partial y} \\
\gamma_{zx} &= \frac{\partial w_n}{\partial x} + \frac{\partial u_n}{\partial z}
\end{align*}
\]

The chosen series satisfy the displacement boundary conditions and the infinitesimal displacements

\[
\begin{align*}
\delta u &= \alpha u_n + \beta v_n + \gamma w_n \\
\delta v &= \delta u_n + \beta v_n + \gamma w_n \\
\delta w &= \delta u_n + \beta v_n + \gamma w_n
\end{align*}
\]

* See A. Castigliano, "Théorie de l'équilibre des systèmes élastiques," Turin 1879.

** See Ref. 3, Vol. II, p. 94, and also "General Theorems Relating to Equilibrium and Stability of Motion," Phil. Mag., March 1875. They have been discussed more recently by D. Williams in Phil. Mag., Sec. 2, Vol. 26, 1939, p. 631.

obtained by variation of \(a, b, c\), while the other coefficients and the temperature are kept constant are hence virtual elastic displacements. Note again that the chosen \(u, v, w\) functions need not satisfy the given static boundary conditions. Naturally, the accuracy of the analysis is enhanced if the latter are also satisfied.

To determine the coefficients \(a, b, c\), we use the condition of the stationary value of the total potential energy in the forms

\[
\delta U = 0 \quad \text{or} \quad \delta U_i = \delta W
\]

There are also cases when the theorem of minimum strain-energy, \(U_i = \text{min}\), is useful, see for example Part II of this series example 4. The application of \(\delta U = 0\) ensures the average of the equilibrium conditions.

If we evaluate \(U\) in terms of (50) then for a linearly elastic body we obtain a quadratic function in \(a, b, c\). Condition \(\delta U = 0\) which in this case becomes

\[
\begin{align*}
\delta U & = 0 \\
\delta U_i & = 0 \\
\delta U_j & = 0
\end{align*}
\]

leads to a set of 3n linear equations in the 3n unknowns \(a, b, c\). It is, however, superfluous to evaluate first \(U\) and then to differentiate with respect to the unknown coefficients. We can obtain directly the final equations by forming the 3n expressions

\[
\delta U_i (U_i + U_j) = 0, \quad \delta U_i (U_i + U_j) = 0, \quad \delta U_i (U_i + U_j) = 0
\]

(54)

where the suffixes \(a, b, c\) indicate that the virtual displacements are chosen respectively as in Eqs. (52).

Using the first of Eqs. (52) in the first of Eqs. (54) we obtain the more explicit form

\[
\int_0^l d\psi \delta a_1 = 0
\]

or since \(\delta a_1\) is arbitrary

\[
\int_0^l d\psi \delta P = 0
\]

(55)

where \(e\) are the elastic strains due to \(u, v, w\) the stress due to the elastic strains given by Eqs. (51), \(P\) the applied forces and \(\delta s\) the displacements in their directions to the point \(s\). There are in all 3n equations in \(a, b, c\), which are non-linear if the stress-strain relations are non-linear.

By a judicious choice of the \(u, v, w\) functions it is possible to obtain very good approximations to the deformations of the body. The number of necessary functions for a good estimate depends on the problem and on the choice of these component functions. The proof of the convergence to the exact solution with increasing \(n\) is a difficult question which cannot be considered here, see Trefftz (1907), p. 130. We note only, referring to the previous paragraph, that the Rayleigh-Ritz method always over-estimates the stiffness.

While the Rayleigh-Ritz method can provide a good approximation to the deformations, the accuracy of the associated stresses is, in general, not as good. This is obvious if we remember that the accuracy of any approximate function is decreased with every differentiation.

In two- or three-dimensional problems it is possible to improve upon the original Rayleigh-Ritz procedure by adopting a mixed technique of (A) and (B). Thus, in a two-dimensional problem it is often possible to guess accurately the variation of the displacements parallel to one axis, say the \(x\)-axis, while it is much more difficult to make an intelligent assumption about the variation parallel to the other direction. We may then write the displacements \(u, v\) in the form

\[
u = u(x) \phi(y) \quad \psi = v(x) \phi(y)
\]

(56)

where \(u, v, \psi\) are assumed crosswise distributions of the \(u, v,\) displacements and \(\phi, \psi\) are unknown non-dimensional functions of \(y\). Substituting Eqs. (56) into (51) one obtains after some transformations the differential equations in \(\phi, \psi\) together with the necessary boundary conditions. Such an analysis can yield a very accurate result. It will be illustrated on a number of examples of some complexity in Part II of this report.

Note that the Rayleigh-Ritz procedure as presented here is also valid for non-linear stress-strain relations.

(F) Galerkin's method of approximate stress analysis. Consider the internal equilibrium Eqs. (4) which we write here in the by now familiar form

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial z^2} & = 0 \\
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} & = 0 \\
\end{align*}
\]

(57)

where the \(\delta u, \delta v, \delta w\) are virtual, i.e. kinematically possible infinitesimal displacements.

Eqs. (57) lead to the principle of virtual work Eq. (44) if the \(u, v,\) functions satisfy not only the internal but also the boundary equilibrium conditions Eqs. (5).

Assume now that the displacements \(u, v, w\) are written in the approximate form of Eqs. (50) where the \(a, b, c, u, v, w\) are unknown coefficients. However, contrary to what we assumed in paragraph (4), series (50) are taken here to satisfy not only the kinematic conditions where prescribed on the surface but also by substitution into the stress-strain relations the equilibrium conditions where prescribed.

Expressing now the stresses in the brackets of (57) in terms of the displacements and temperature \(T\) and selecting as virtual elastic displacements one of the 3n possibilities (52) we obtain the \(a, b, c, u, v, w\) equations in the 3n unknowns \(a, b, c, u, v, w\).

\[
\begin{align*}
\left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial z^2} \right] u_{\delta} + v_{\delta} & = 0 \\
\left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right] v_{\delta} & = 0 \\
\left[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right] w_{\delta} & = 0
\end{align*}
\]

(58)

This is the Galerkin's procedure usually only given for linear elasticity. To fix ideas take the case of linear elasticity and write \(c, \psi\) etc. in terms of \(u, v, w\). We find easily three equations, the first of which is

\[
\int_0^l \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial z^2} \right] u_{\delta} \psi = 0
\]

(59)

where

\[
\psi = \frac{2u}{1 - 2\nu} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - \frac{\partial u}{\partial z} + \frac{\partial u}{\partial z}
\]

(60)

The bracket in Eq. (59) is, of course, the equilibrium equation in the \(x\)-direction expressed in terms of displacements.

The procedure in any particular structural problem is to form the equilibrium conditions in the stresses or stress resultants and express each in terms of the displacements. Next we multiply each by the corresponding \(u, v, w\) which may be a deflection, slope, twist) and then integrate over the body. Thus, for a beam subjected to a distributed loading \(p\) in the \(x\)-direction, the equilibrium equation in terms of the deflection \(v\) is, assuming engineers' theory of bending to hold

\[
\frac{d^2}{dx^2} \left( \frac{d^2 v}{dz^2} \right) - p = 0
\]

(61)

and the Galerkin form of the virtual work equation is

\[
\int_0^l \left[ d^2 \left( \frac{d^2 v}{dz^2} \right) - p \right] d\psi dz = 0
\]

(62)

It is easy to see that for displacement functions (50) satisfying all boundary conditions the Galerkin and Rayleigh-Ritz methods must yield the same equations for \(a, b, c\) and hence also the same deformations. We need only realize that in this case Eqs. (57) are indeed equivalent to the principle of virtual work. Hence substitution of \(u, v, w\) in Eqs. (57) must give the same result as substitution into

\[
\delta U = 0
\]

The advantage of Galerkin's method lies in a more direct derivation of the equations in \(a, b, c\). However, contrary to what is usually assumed, this advantage is small if we calculate \(\delta U\) directly. We note also that Galerkin's method allows only such approximate functions as satisfy all boundary conditions, while the Rayleigh-Ritz procedure requires only the satisfaction of the kinematic boundary conditions.


Fig. 10.—Cantilever beam
Consider the cantilever under transverse load shown in Fig. 10. To obtain an approximate expression for the deflections $v$ by the Rayleigh-Ritz procedure we need only select a function or sequence of functions giving
\[ v = \frac{dv}{dz} = 0 \text{ at } z = 0 \]
However, when applying the Galerkin method Eq. (61a) must satisfy also the static conditions at $z = l$, i.e.
\[ \text{Shear force} = \text{Bending moment} = 0 \]
\[ (\frac{d^2P}{dz^2}) = 0 \text{ at } \frac{d^2}{dz^2} = 0 \text{ at } z = l \]

5. ILLUSTRATIONS OF THE METHOD OF VIRTUAL DISPLACEMENTS

In this section we present a number of applications of the principle of virtual work. These are not meant to give the shortest possible solutions to the problems considered but merely to illustrate the way in which the method can be applied in some simple cases. More complicated cases are investigated in Part II

(a) Continuous Beam with Non-Linear Spring Support

The uniform beam of bending stiffness $EI$ shown in Fig. 11 carries a uniformly distributed load $P$ and is simply supported at the ends. At the centre an additional support consists of a spring with the load placement law
\[ P = kV \left[ 1 - \frac{\alpha}{1 - \alpha} \right] \]
Thus $k$ is the initial spring stiffness and $V$ is the displacement at which the spring becomes solid.

With the assumption of the Engineers' Theory of Bending, establish by means of the Principle of Virtual Displacements the differential equation for the deflexion $v$ of the beam and the boundary conditions. Find also the deflexion $V$ at the spring.

Fig. 11.—Virtual displacements: Example (a) Simply supported beam with non-linear spring.

The first part of the problem is, of course, trivial and the result known to any undergraduate, but we want to show here how the Virtual Displacements method can be applied in such a case.

We consider virtual displacements consisting of small arbitrary additional deflexions $(\delta v)$ of the beam from its equilibrium position. The Principle of Virtual Displacements then expresses the equilibrium condition in the form
\[ \delta U - 2\delta W = 0 \]
\[ \delta v = 0 \text{ for } z = 0 \text{ and } z = l \]
and since both structure and loading are symmetrical we need only consider symmetrical virtual displacements. Hence
\[ (\delta v)dz = (\delta v) = 0 \text{ for } z = l \]
Strains and stresses in beam due to bending:
\[ e = e_m = -V' \text{, } a = \sigma_m = -4E\nu'' \]
and therefore the virtual strain due to $\delta v$ is
\[ \delta e = -2V'\delta v = -\nu'' \delta v \]
The increment of strain energy in the beam due to bending is thus
\[ 2\int_a^l E\delta e \cdot dA \cdot dz = 2E \int_a^l (f^2 + d\lambda) v'' \delta v' \delta v'' \delta v' \delta v dz \]
which becomes, on integrating twice by parts,
\[ 2E \int_a^l (v'' \delta v')' - v'' \delta v' \delta v \delta v' \delta v dz \]
The increment of strain energy in the spring is
\[ P\delta V \]
and the increment of work done by the distributed load is
\[ \delta W = 2P \delta v \delta v' \delta v'' \delta v' \delta v dz \]
Hence the complete virtual work Eq. (a2) becomes
\[ 2\int_a^l (Ei'v'' - P)\delta v' \delta v'' \delta v' \delta v'' + 2EI'v''(\delta v')' - 2EIv''(\delta v')' + P\delta V = 0 \] (a11)
But
\[ (\delta v')' = 0 \text{, } (\delta v')' = 0 \text{ and } (\delta v) = -\delta V \]
Since otherwise $\delta V$ is arbitrary we conclude that to satisfy Eq. (a11) we must have
\[ (Ei'v'')_o = 0 \text{, } 2(Ei'v'')_o + P = 0 \] (a12)
and $\delta v$ must satisfy the differential equation
\[ Ei'v'' - P = 0 \] (a13)
Eqs. (a12) and (a13) together with the kinematical conditions
\[ (\delta v')' = 0 \text{ and } (\delta v')' = 0 \] (a14)
give all the necessary information for the determination of $\delta v$.

Integrating Eq. (a13) and using the boundary conditions (a12) and (a14) we find finally for $\delta V$ the quadratic equation
\[ (1 + \psi(V/V_o)^2) - (1 + \psi(V/V_o)^2) = 0 \] (a15)
where
\[ \psi = kP/6EI, \quad V_o = 5p^4/24 \]

(b) Plane Redundant Framework

A plane framework consisting of a single joint connected by a number of hinged bars to a rigid foundation is loaded by forces $X$ and $Y$ along the axes $Ox$ and $Oy$ respectively (Fig. 12). In addition, the bars are heated to arbitrary temperatures and have also initial strains due to errors in manufacture. Find by application of the Principle of Virtual Displacements the forces in the bars.

Let $u, v$ be the displacements of the loaded hinge, measured from the position for which all bars have the correct length and are at zero temperature. Then the total direct stress in the $r$th bar for these displacements is
\[ \sigma_r = u \cos \theta_r + v \sin \theta_r \] (b1)
The total strain is made up of the elastic strain $\epsilon_r$ together with the thermal strain $\eta_r = \alpha \theta_r$ and the initial strain $\eta_{ir} = \Delta \ell_r / \ell_r$ (b2)
$\Delta \ell_r$ being the additional length of bar (in excess of the correct length) due to manufacturing or other causes. Hence the elastic strain due to $u, v$ is
\[ \epsilon_r = \eta_r + \eta_{ir} \] (b3)
and the direct stress in the bar is
\[ N_r = A_r E_r = E \left[ u \cos \theta_r + v \sin \theta_r - (\eta_r + \eta_{ir}) \right] \] (b4)
If we now impose on the joint the virtual displacements $\delta u, \delta v$ there arises an increment of strain energy $\delta U_r$ and an increment of work $\delta W$ of the applied forces, where

Fig. 12.—Virtual displacements: Example (b) Redundant system of bars.
and the work done by the distributed torque and the end torques
\[ \delta W = \int_0^l m_1 \delta \theta dz + T_1 \delta \theta \]  

(c7)

Applying the Principle of Virtual Work
\[ \delta U_i - \delta W = 0 \]

(c8)

Integrating by parts the first term twice and the second once, we finally obtain
\[ \int_0^l \left[ \delta \theta \right] + \int_0^l \left[ \frac{m_1 \theta}{J} + \frac{m_2}{J} \theta' \right] \right] = 0 \]

(c9)

For the integral to vanish, since \( \delta \theta \) is arbitrary, \( \theta \) must satisfy the differential equation
\[ \frac{\delta \theta}{J} - \frac{m_2}{J} \theta' = 0 \]

(c10)

which is recognized as the usual Wagner equation differentiated with respect to \( z \).

If the twist at both ends of the tube is specified then \( \delta \theta \) must therefore be zero and the first bracketed term in (c9) vanishes also. If in addition the warping (and hence \( \delta \theta \)) is specified (e.g. built-in end) then \( \delta \theta' \) is also zero at \( z = 0 \) and \( z = l \) and the remaining term also vanishes.

If, however, the end \( z = l \) is free to twist, \( \delta \theta' \) and \( \delta \theta \) are arbitrary and we have as further conditions from (c9)
\[ \frac{m_2}{J} \theta' - \frac{m_1}{J} \theta = 0 \]  

(c11)

which are the necessary static boundary conditions at the free end. The first is of course the condition for equality of external and internal torque and the second the condition for zero direct stress.

For the series solution, we represent the twisted shape by the Fourier series
\[ \theta = \frac{1}{n} \sum_{n=1}^{\infty} a_n \sin \left( \frac{nnz}{L} \right) \]

(c12)

and take for virtual displacements the increments of twist produced by a small variation \( \delta a_n \) of the coefficient \( a_n \).

Using Eq. (c8) we find
\[ \delta U_i - \delta W = \delta a_n \left[ \frac{GJ}{l} \sum_{n=1}^{\infty} \frac{a_n nnz}{L^2} \cos \frac{nnz}{L} \cos \frac{nnz}{L} \right] 
+ \left( \frac{m_2}{J} \right) \delta \theta' \]

(c13)

which gives on integrating
\[ a_n = \frac{m_2}{J} \int \frac{m_2}{J} \sin \frac{nnz}{L} \frac{nnz}{L} \frac{nnz}{L} \frac{nnz}{L} \]

(c14)

If \( m_1 \) is constant
\[ a_n = \frac{4m_1}{J} \int \frac{m_2}{J} \sin \left( \frac{nnz}{L} \right) \]

(c15)

and
\[ a_n = 0 \]

for \( m \) even

which gives the twist distribution
\[ \theta = \frac{4m_1}{J} \int \frac{m_2}{J} \sin \left( \frac{nnz}{L} \right) \frac{nnz}{L} \frac{nnz}{L} \frac{nnz}{L} \frac{nnz}{L} \]

(c16)

In this case, since the assumed form of solution satisfies also the static boundary conditions \( \theta' = 0 \) for \( z = 0 \) and \( l \) we can alternatively use the Galerkin form of the Virtual Work equation, which is in this case
\[ \int \left( \delta \theta \right) = 0 \]

(c17)

and is given immediately from Eq. (c9).

If we approximate our solution for \( \theta \) by retaining only the first term in (c16) we underestimate the average angle of twist. This ties up with our statements on p. 81; for, by putting \( \theta = a_n \sin \left( \frac{nnz}{L} \right) \) we apply constraints on the tube and hence overestimate the stiffness.

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*See Angstria, 'The Open Tube', AIRCRAFT ENGINEERING, Vol. XXVI, No. 301, April 1954, p. 102 et seq.
6. THE PRINCIPLE OF VIRTUAL FORCES OR COMPLEMENTARY VIRTUAL WORK

It is natural in reviewing the developments of Sections 3 and 4 to inquire if it is possible to enlarge upon the conception of complementary work and strain energy by the introduction of virtual displacements. In fact, if we consider Eqs. (28) and (36) we realize immediately that the functions \( \delta W^* \) and \( \delta U^* \) are independent of the variations \( \delta u \) and \( \delta \varepsilon \) associated with the force and stress increments, just as \( \delta \lambda \) and \( \delta \sigma \) are independent of the variations \( \delta \lambda \) and \( \delta \sigma \) associated with the chosen \( \delta u \) and \( \delta \varepsilon \)'s. Hence, when finding \( \delta W^* \) and \( \delta U^* \), we may assume that displacements and strains remain constant. Also, the infinitesimal increments \( \delta \varepsilon \) and \( \delta \sigma \) are arbitrary as long as they satisfy the equilibrium conditions in the interior and, where such are prescribed, on the surface. Thus, if we fix that the surface forces are not to be varied over part of the boundary we must have there \( \delta \sigma = 0 \); however, where kinematic conditions are prescribed on the boundary the \( \delta \sigma \) variation cannot be assigned. It is apparent that our incremental stress system need not even be an elastically compatible one. It is only restricted by the condition that it must be statically equivalent to the load increments \( \delta \sigma \) and \( \delta \phi \). While these increments are applied it is assumed as in Section 4 that the temperature remains constant.

Such infinitesimal variations of forces and stresses which are arbitrary as long as they satisfy the prescribed equilibrium conditions we call virtual forces and stresses.

Before restating theorem (35) for the more general conceptions introduced here let us consider again its derivation in the light of our new ideas. Thus, if we multiply the true displacements \( u, v, w \) by the internal equilibrium conditions (29) which the virtual stresses \( \delta \sigma \) and \( \delta \lambda \) must satisfy, sum, integrate over the body, apply Green's Theorem and note the boundary conditions (30) we obtain Eq. (35) where \( y_{x, y} \) etc. are the total work strain associated with the displacements \( u, v, w \). Next let us apply Green's Theorem in the opposite direction by starting from the right-hand side of Eq. (35). We find that this function can only be equal to the left-hand side if the terms \( y_{x, y} \) etc. are indeed expressions for the strains (3) and (12) and satisfy the kinematic boundary conditions.

Thus, we conclude that an elastic body is in an elastically compatible state under a given system of forces and temperature distribution if for any virtual increments of forces and stresses from a position of equilibrium

\[
\delta W^* = \delta U^* + \int \delta \lambda d\sigma
\]

where

\[
\delta U^* = \int \delta \varepsilon dV
\]

See also Eqs. (25) and (38a).

Eq. (62) is, in fact, a necessary and sufficient condition for the elastic compatibility of the equilibrium.

Theorem (62) we call the principle of virtual forces or virtual complementary work for elastic bodies subjected to loads and temperature distribution. Note that (62) applies for non-linear stress-strain laws.

The above discussion indicates that there is a close parallel between the principles of virtual displacements and virtual forces. Thus, by substituting virtual forces (stresses) for virtual elastic displacements (strains), actual total displacements (strains) for forces (stresses), and invariant state of straining for invariant state of forces we obtain Eq. (62) from Eq. (44). However, this duality is only complete for continuous structures which are infinitely redundant. If, on the other hand, we consider a statically determinate structure we find that while it is still possible to describe an infinite set of virtual displacements \( \delta u \) associated with a prescribed set of certain of the displacements, only one stress system can exist for given external forces; hence no \( \delta \sigma \) can be assigned in the latter case and the principle of virtual forces has no application. A more fundamental limitation of the principle of virtual forces we see if we extend our theorems to finite displacements. Here we find that it is, in general, impossible to achieve this for the principle of virtual forces while, as mentioned in Section 4, no basic difficulty arises in the case of the principle of virtual displacements. However, for the usual analyses of redundant systems involving small displacement theory the principle of virtual forces with its many particular forms is the most useful since the standard procedure introduces forces as unknowns. Naturally, there are many cases, especially in multi-redundant structures, where it is advantageous to introduce displacements as unknowns; here the principle of virtual displacements is the indicated method as shown in Example (b) of Section 5.

We return now to Eq. (62) and will illustrate its validity on a very simple example. Consider to that effect the redundant beam of uniform flexural stiffness \( EI \) built-in at \( z = 0 \), simply supported at \( z = l \), and subjected to a uniform load \( p \) (see Fig. 14). Under the assumption that the ordinary engineers' theory of bending holds and that the shear deflections are negligible the deflection \( v \) is given by

\[
v = \frac{pI}{6EI} \left[ \frac{3}{2} - \frac{5}{2} \left( \frac{Z}{l} \right)^2 + \frac{1}{2} \left( \frac{Z}{l} \right)^4 \right]
\]

As a virtual force we select \( \delta P \) at the centre of the beam as shown in Fig. 14(a). However, since we require only a statically equivalent stress system to equilibrate the applied virtual load we may eliminate the moment redundancty and select a statically determine beam. The two alternative choices leading to a simply supported beam and a cantilever are seen in Figs. 14(b) and 14(c). Denoting the true deflexion at the centre \( v \) we have

\[
\delta W^* = \delta P \cdot v = \delta P \cdot \frac{pI}{192EI}
\]

Also since the Engineers' theory of bending applies \((y_{x, y} = 0)\)

\[
\delta U^* = \int \left[ \frac{1}{2} \delta \sigma_{x,y} dV \right] = \int -\frac{dV}{2} d\varepsilon_{x,y} d\sigma = \frac{pI}{192EI} = \delta W^* q.e.d.
\]

where the integral in the square bracket refers to the integration over the cross-section. For the case shown in Fig. 14(c),

\[
\begin{align*}
\delta M &= \delta P \left( \frac{1}{2} - z \right) & \text{for } 0 < z < l/2 \\
\delta M &= 0 & \text{for } l/2 < z < l
\end{align*}
\]

Also since,

\[
\frac{d^2v}{dz^2} = \frac{pI}{8EI} \left[ \frac{1}{2} - \frac{5}{2} \left( \frac{z}{l} \right)^2 + \frac{1}{2} \left( \frac{z}{l} \right)^4 \right]
\]

\[
\delta U^* = \delta P \left( \frac{1}{2} - z \right) \int dV = \delta P \cdot \frac{pI}{192EI} = \delta W^* q.e.d.
\]

If we apply now a force \(-\delta P \) at the free end our virtual system (c) is transformed into (a). No additional \( \delta W^* \) arises since \( v = 0 \) at \( z = l \). The additional bending moment \( \delta M \) produced by \(-\delta P \) is

\[
\frac{1}{16} \left( \frac{1}{2} - z \right) \delta \sigma_{x,y}
\]

and this is easily found not to create an additional \( \delta U^* \). By relaxing the moment restraint at A we may finally prove without difficulty that \( \delta W^* = \delta U^* \) applies also for the virtual system (b).

We return now to Eq. (62) and note that since the displacements are assumed constant when the virtual forces are applied we may regard \( \delta W^* \) as the variation of a potential \(-U^* \), where

\[
U^* = -\int \left[ w_{x,y} + \psi_{x,y} \right] dV - \int \left[ \phi_{x,y} + \psi_{x,y} + \psi_{x,y} \right] d\sigma
\]

Thus, \( \delta W^* = -\delta U^* \) and \( \delta U^* \) may be termed the complementary potential of the external forces. Note, however, that \( W^* \) is not \(-U^* \) since in obtaining \( W^* \) from \( \delta W^* \) we must, naturally, perform the integration for displacements varying with load. In fact, for a linear system and no temperature effects \( W^* = -U^* \), compare also Eqs. (64) and (65).

Also, since the thermal strains are kept constant we may write the right-hand side of (62) as

\[
\delta U^* + \int \delta \lambda d\sigma = \delta (U^* + \int \phi_{x,y} dV) = -\delta U^*
\]

where

\[
\delta U^* = \int \delta \lambda d\sigma
\]

14. Example of an arbitrary virtual force
but
\[ U_* = U_* + f \pi \delta v \cdot \int \left( \eta \sigma_{rr} + \eta \sigma_{\theta \theta} + \eta \sigma_{z z} \right) \, dV \]  
(66)

since \( \Theta = \text{const.} \) we term the complementary potential energy of total deformation. Note that it is always simpler to calculate directly \( \delta U_* \) from Eq. (65). Particular care is necessary in evaluating \( U_* \) for as Eq. (66) shows in the first integral \( \eta \) is taken to vary with \( \sigma \) from the initial to the final state while \( \gamma, \theta, \) and \( \kappa \) in the second integral only refer to the final values. Physically speaking we may consider \( U_* \) as the complementary work necessary to reach the final true state of deformation from an initial state in which we allowed free thermal expansion and destroyed compatibility.

Formulas (65) and (66) may be extended immediately to the case of arbitrary initial straining by substituting
\[ \eta_0 = \eta_{0,rr} + \eta_{0,\theta \theta} + \eta_{0,z z} \]
for \( \eta \).

Eq. (62) can now be written more concisely
\[ \delta U_* = \delta U_* + U_* \]
(67)
where the suffix \( e \) indicates that only forces and stresses are varied and
\[ U_* = U_* + U_* \]
(67a)
is defined as the total complementary potential energy of the system. Eq. (67) states that a state of equilibrium of an elastic body is also one of elastic compatibility (i.e., the body is at the true position of equilibrium) if each virtual variation of the stresses and forces, while displacements remain constant, does not give rise to any (first order) variation of the total complementary potential energy. This theorem we call the principle of a stationary value of total complementary potential energy if the latter is expressed in terms of forces and stresses. Actually the stationary value of \( U_* \) is a minimum as we may prove without difficulty.\(^*\) This point is discussed in more detail later (C) below.

Eqs. (62) or (67) may be used to derive the results which follow.

(A). The differential equations of the theory of elasticity (for arbitrary loading and temperature distribution), or any particular structural problem, in terms of stresses or stress-resultants: the appropriate kinematic conditions in terms of forces and stresses follow also from this analysis. It is important to note that in all applications it is best to form directly
\[ \delta U_* = \frac{\partial \delta U_*}{\partial \delta \Pi} \]
and not to evaluate first \( U_* \) and then to take its increment \( \delta U_* \).

(B). Castigliano’s Theorem Part II generalized for Thermal Effects and non-linear elasticity

\[ \frac{\partial U_*}{\partial \Pi} = 0 \quad \text{const.} \]
(68)
where \( u_t \) is the deflection (rotation) in the direction of the force (moment) \( P_t \). This relation may be obtained immediately if we apply one virtual external force \( \delta \Pi_t \) in the direction of the displacement \( u_t \).

(C). The Principle of Stationary (Minimum) Value of Complementary Potential Energy of total deformation for internally redundant structures

This may be derived from (62) if we do not apply any virtual external forces, i.e.
\[ \delta \Pi_t = \delta \Pi_t - \delta \Pi_t = \delta \Pi_t - \delta \Pi_t = 0 \]
while varying the stresses \( \sigma \).
Then
\[ \delta \Pi_t = 0 \]
(69)
which is our generalization of the standard principle of Castigliano of Minimum Strain Energy to include temperature effects and non-linear stress-strain laws. Note again that Principle (69) necessarily applies only to internally redundant structures, since for given external loads only one stress distribution can exist in statically determinate structures.

Eq. (69) itself only indicates that \( U_* \) has a stationary value in that particular state of equilibrium in which all the elastic and kinematic compatibility conditions are satisfied. Note, as mentioned before, that with the limitations of the present assumptions, i.e., small displacements and monotonically increasing stress-strain diagram, there is only one position possible where both the equilibrium and compatibility conditions are satisfied. We now indicate the necessary extremum of \( U_* \), which naturally requires the consideration of second order terms as in Section 4.

Consider an elastic body under given loads and temperature distribution in its compatible equilibrium position. We make a series of cuts in the body but at the same time apply stresses \( \sigma \) acting across and along the cuts of the same magnitude as in the uncut body; these are obviously the stresses required to maintain the compatibility condition of perfect fit at the cuts. If we impose the virtual stresses \( \delta \sigma \), it is apparent that since these produce corresponding deformations \( \delta \Pi_t \), on the cuts the latter are not any longer compatible. It is important to realize that the \( \delta \sigma \) systems are self-equilibrating since the external loads \( P \) remain constant. Thus, in a framework we may obtain a system \( \delta \Pi_t \) by cutting a redundant bar and applying a virtual \( \delta \Pi_t \) to the true force \( P \) in the bar.

We now investigate the differences in complementary work \( W^* \) and \( U_* \) between the original equilibrium position of the uncut body and the new enforced equilibrium position of the cut body. Comparing Eqs. (38) and (66) we find
\[ W^* = U_* - \eta \int \left[ \sigma_{rr} + \sigma_{\theta \theta} + \sigma_{z z} \right] \, dV \]
(70)
in moving from the uncot (compatible) equilibrium state to the cut one we note first that the integral does not vary since \( \eta \) is constant in this step. Also the first order increment, \( \delta U_* \), of \( U_* \) is zero since this is the condition for compatible equilibrium of the original body and no first order increment \( \delta W^* \) can arise since the loads \( P \) remain constant. We are then left only with second order increments.

For the complementary work this is
\[ \delta W^* = -\frac{\partial \Pi_t}{\partial \Pi} \delta \Pi_t \delta \Pi_t \]
(70a)
where the integral is taken over the cut faces) which is the work of the virtual stresses \( \delta \Pi_t \) over the displacements \( \delta \Pi_t \), they themselves produce. This is clearly positive. The second order increment of \( U_* \) is merely
\[ \delta U_* = \frac{\partial \Pi_t}{\partial \Pi} \delta \Pi_t \delta \Pi_t \]
(70b)
since \( \Pi \) remains constant; \( \delta \Pi_t \) and \( \delta \Pi_t \) are the stresses and strains due to \( \delta \Pi_t \).

Terms (70a) and (70b) are equal and both positive.

We conclude that the complementary potential energy of total deformation \( U_* \) and the complementary work \( W^* \) have for given forces and temperature distribution a minimum at that position of equilibrium of the uncut body at which compatibility is satisfied.

It follows that if \( U_* \) is overestimated by assuming a statically equivalent stress system which does not satisfy all compatibility conditions and we, ignoring the latter fact, equate \( U_* \) to \( W^* \) of the applied loads \( P \) alone we cannot but overestimate the magnitude of the displacement system under the loads \( P \). Conversely to achieve a given displacement system our calculations based on a non-compatible stress system must underestimate the corresponding load system \( P \). Thus, the latter has its maximum for the unique equilibrium position which is also truly compatible. This may be expressed also as follows:

For given displacements and temperature distribution the complementary potential energy of total deformation has a maximum when the state of equilibrium satisfies also the compatibility conditions.

The above theorems may be combined to give:

The stiffness of an elastic body in which the equilibrium conditions are satisfied is a maximum when the elastic compatibility conditions are all met.

Thus we see that the effect of introducing assumed forms of stress distribution for the purpose of approximate solutions is the opposite to that of the method of Virtual Displacements and therefore application of both methods to a given problem yields upper and lower bounds to such aggregate quantities as stiffness. No general conclusion as to bounds can, of course, be drawn for the details of the stress distribution.

The above theorems which apply also in the presence of initial strains other than those due to temperature do not appear to have been given before with this degree of generality.

(D). The Unit Load Method

Assume that we require the deformation (deflection or slope) \( u \) at a given point and direction of an elastic redundant body subjected to given forces and thermal effects. Let the actual total strains in the structure be known and given by
\[ \gamma_{xx} = \gamma_{xx} + \gamma_{xx} + \gamma_{xx} = \gamma_{xx} \]

Applying a load (force or moment), \( P \), in the direction of \( u \), and using Eq. (62) we find
\[ \delta \Pi_t = [\gamma_{xx} + \gamma_{xx} + \gamma_{xx} + \gamma_{xx} + \gamma_{xx} + \gamma_{xx}] \]
(71)
where \( \gamma_{xx} \) are the virtual stresses due to \( \delta \Pi_t \). In a linearly elastic system \( \delta \Pi_t = \delta \Pi_t \), are proportional to \( \delta \Pi_t \), and Eq. (69) can be written
\[ \gamma_{xx} = \gamma_{xx} = \gamma_{xx} = \gamma_{xx} = \gamma_{xx} = \gamma_{xx} \]
(71a)
where \( \gamma_{xx} \) are the stresses due to a unit load. Since \( \gamma_{xx} \) need only satisfy the internal equilibrium conditions and the external one for \( \delta \Pi_t = -1 \) it is obviously advantageous to determine \( \gamma_{xx} \) in the most simple statically determinate basic system.

For a non-linear system Eq. (71a) is still applicable as long as \( \gamma_{xx} \) etc. are calculated in a statically determinate basic system. For only in the

* See Bieszko and Grammer, p. 75.
† A. Castigliano, Théorie de l'Equilibre des systèmes élastiques, Turin 1879.
Unit load method for displacement of redundant framework

The stresses corresponding to a unit load, $\sigma_{x_0} \Delta P = \sigma_{a_0} \Delta P$, be the stresses in the structure with a finite number of redundancies expressed as stresses or stress resultants.

This will be shown in some detail in Section 8.

Example of the application of Eq. (71a).

Consider the plane framework with a redundant support as shown in Fig. 15(a). We seek the deflection $\Delta F$ at joint 2 for the loading case shown. Let the actual elongation or the members due to loads $P_1, \ldots, P_6$, temperature and manufacturing errors be denoted by $\Delta$, Next we apply a unit load at 2 in the direction of $\Delta F$ and find the force $\sigma$ in the bars. Since we need only consider a statically determinate case we ignore the support at C and are left with the simple problem of finding $N$ in the left-hand span only. Application of Eq. (71a) yields the simple formula

$$\Delta F = \sum N_\Delta$$

(71b)

where the summation extends only over the continuously drawn bars. The formula given is due essentially to Maxwell and Mohr, who applied it to statically indeterminate frameworks. Actually Mohr derived this type of equation by using the principle of virtual displacements with the actual elongations taken as virtual ones and the unit load as the actual one. Although such a procedure is in the present case of small displacements permissible, it should nevertheless be avoided since Eq. (71b) follows more naturally from the principle of virtual forces.

Approximate method of stress analysis

Consider an elastic body subjected to external loads (body and surface forces) and a temperature distribution $\Theta$. The boundary conditions are assumed to be of the static and kinematic kind; however, where the latter are prescribed they are taken to be of the rigid kind, e.g. rigidly built-in or sliding in a rigid groove (see Fig. 16). To limit the present analysis we restrict our investigation to a state of plane stress. The solution of such problems is often expedited by the use of the Airy stress function $F$.

Then the stresses are given by

$$\sigma_{x} = \frac{\partial F}{\partial x} = \int \omega_x dx_x, \quad \sigma_{y} = \frac{\partial F}{\partial x} = \int \omega_y dy_x$$

(72)

which satisfy automatically the internal equilibrium conditions (4).

Eliminating the displacements from the strain expressions (1) we obtain the compatibility condition for the strains,

$$\frac{\partial^2 \gamma_{x y}}{\partial x \partial y} = \frac{\partial^2 \gamma_{y x}}{\partial y \partial x}$$

(73)

where $\gamma_{x y}$ etc. are the total strains $\gamma_{x y}$ etc. For a given stress-strain law we can express $\gamma_{x y}$ in terms of the stresses (72) and the temperature $\Theta$. Hence by substituting into (73) we obtain the differential equation in the unknown $F$, which will, in general, be non-linear. However, in the case of bodies obeying Hooke's law we obtain for $\sigma = \text{const.}$ the simple linear result

$$\frac{\partial^2 \gamma_{x y}}{\partial x \partial y} = \frac{\partial^2 \gamma_{y x}}{\partial y \partial x}$$

(74)

$F$ must, of course, also satisfy the given boundary conditions. In fact, where surface forces are prescribed we have (see Eqs. (5) and Fig. 16)

$$\frac{\partial F}{\partial x} = \phi_x$$

(75)

An approximate method may proceed as follows. Assume that the stress function $F$ is expressed in the form of a finite series

$$F = F_0 + \sum b_i F_i$$

(76)

where $F_0, F_1, \ldots, F_n$ are known functions of which $F_0$ satisfies the static boundary conditions (75) where these are prescribed and the functions $F_i$ vanish there. $b_i$ to $b_n$ are constants to be determined by the virtual forces principle. The system (76) satisfies by definition all given equilibrium conditions and the increment

$$\delta F = \delta b_i F_i$$

(77)

may be regarded as a virtual stress system corresponding to zero increments of external loads where the latter are fixed.

Since $\delta W = 0$ (either forces are given or displacements are zero) the principle of minimum complementary potential energy of total deformation is applicable here and the form

$$\delta U = \int \delta b_i \delta \omega_x dx_x + \int \delta b_i \delta \omega_y dy_x + \int \delta b_i \delta \omega_{x y} dA = 0$$

(78)

We use here $(\ldots) dA$ to denote the integration $\int f(\ldots) dx dy$ over the area of the two-dimensional continuum.

Substituting $\delta \omega_x$ etc. in terms of (77) we find

$$\int \left[ \frac{\partial^2 \gamma_{x y}}{\partial x \partial y} + \frac{\partial^2 \gamma_{y x}}{\partial y \partial x} \right] dA = 0$$

(79a)

and since $\delta b_i$ is arbitrary.

$$\int \left[ \frac{\partial^2 \gamma_{x y}}{\partial x \partial y} + \frac{\partial^2 \gamma_{y x}}{\partial y \partial x} \right] dA = 0$$

(79a)

If the total strains are expressed in terms of the stresses (72) and temperature distribution $\Theta$ we obtain from (79a) $n$ equations for the unknowns $\gamma_{x y}$ to $\gamma_{y x}$. These are only linear if the body follows a linear stress-strain law. In the latter case and for constant body forces Eq. (74) shows that the solution must be independent of the Poisson's ratio $\nu$ if all boundary conditions are of the static type. Hence we may take $\nu = 0$ and (79a) becomes

$$\int \left[ \frac{\partial^2 \gamma_{x y}}{\partial x \partial y} + \frac{\partial^2 \gamma_{y x}}{\partial y \partial x} \right] dA = 0$$

(79b)

from which we may obtain without difficulty the $n$ linear equations for $b_i$ to $b_n$.

The case when all prescribed boundary conditions are static is interesting for all surface conditions are then exactly satisfied by (76). This indicates that it should be possible to express (79a) in a form similar to that given as Galerkin's method under the virtual displacement principle. In fact, if we integrate (79a) twice by parts, or better, if we apply Green's Theorem assisted by the following assumptions.

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* J. C. Maxwell, Phil. Mag., vol. 37, p. 294, 1864
* The presentation is restricted to simply connected domains.
17. — Virtual Forces: Example (a) Diffusion problem

we find

\[ I = \int_{A} \left[ \frac{\partial F}{\partial y} \cdot y - \frac{\partial F}{\partial x} \cdot x \right] \, dy \\
+ \int_{\partial A} \left[ \frac{\partial F}{\partial y} \, dy \right]. \]

But, on the boundary \( F = 0 \) and also

\[ \frac{\partial F}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F}{\partial y} = 0 \]

(see Eqs. (75)).

Hence Eq. (79) reduces to the slightly simpler form,

\[ \int_{A} \frac{\partial F}{\partial x} \, dx + \int_{\partial A} \frac{\partial F}{\partial y} \, dy = 0 \]

which shows clearly how the method of virtual forces satisfies in the average the compatibility condition (73). When the body is linearly elastic Eq. (80) may be written as

\[ \int_{A} \left[ \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \right] \, dx \, dy = 0 \]

for \( r = 1 \) to \( n \) \hspace{1cm} (80a)

the expression in the bracket being of course Eq. (74). Note again the independence of the solution from \( \nu \) when body forces are constant.

The above application of the principle of virtual forces is a generalization of a method developed by Timoshenko, p. 167. A thermal stress example of the above analysis is given in Part II.

Naturally, the method can be extended to three-dimensional cases.

W. Ritz proposed as early as 1908 a similar procedure for the solution of St. Venant's torsion problem; this method is illustrated on an example of considerable complexity in Part II.

A slightly more refined approach than that shown above may be adopted when it is possible to estimate accurately the variation \( f \) of \( F \) parallel to one co-ordinate say \( y \) while the distribution parallel to the other co-ordinate is more difficult to guess. Then, we may set

\[ F = f(y) \cdot \phi(x) \] \hspace{1cm} (81)

where \( \phi(x) \) is an unknown function of \( x \). It is, of course, possible to formulate the analysis in any other suitable co-ordinate system. Substituting (81), with \( \delta F = \delta F \phi \) in place of \( F \), into (79) or (80) (or related expressions), we obtain after some simple integrations the differential equation in \( \phi \); when there are also kinematic boundary conditions the corresponding boundary expressions for \( \phi \) follow also from (79).

Consider, for example, the case of linear elasticity, zero body forces and pure static boundary conditions. Eq. (80a) takes here the simple form

\[ \int_{A} (\Delta F + \phi \delta F) \, dA = 0 \] \hspace{1cm} (80b)

Using (81) in (80b), integrating with respect to \( y \) and noting that (80b) must be true for any virtual variation \( \delta \phi \), we obtain the differential equation in \( \phi \).

\[ \frac{d^2 \phi}{dx^2} + \frac{2}{4} \frac{d^2 \phi}{dx^2} + \phi \frac{d^2 \phi}{dx^2} + \phi \frac{d^2 \phi}{dx^2} + \phi \frac{d^2 \phi}{dx^2} = 0 \] \hspace{1cm} (80c)

where \( \gamma_1 \) and \( \gamma_2 \) are the extreme (boundary) values of \( y \) corresponding to the same \( x \) (see Fig. 16). Thus, the coefficients of the homogeneous part of (80c) are only constants in the case of a rectangular field.

This method can yield very accurate results and is actually the one adopted in Part II.

7. ILLUSTRATIONS OF THE PRINCIPLE OF VIRTUAL FORCES

In this section we present a number of applications of the principle of virtual forces to quite simple problems. Again, it is not necessarily suggested that the method is the most suitable one for the problems considered. It is only intended to show how it can be applied in these simple cases. In subsequent parts of the paper somewhat rather more complex problems will be dealt with. In all the examples of this section linear elasticity is assumed.

(a) Diffusion Problem

The panel shown in Fig. 17 is subjected to loads \( P \) applied at the free ends of the edge members. Assuming that the sheet carries only shear stress which is constant across the width \( b \) of each half (usual diffusion assumption) obtain by application of the principle of virtual forces the differential equation for the load \( P \) in the central stringer. Find also the displacement \( w \) of the free end of the stringer.

From the equilibrium of an element of the stringer, we find for the shear flow \( q \cdot \sigma_{y} \) in the sheet

\[ q = -\frac{1}{2} \frac{dP}{dc} \]

and from the equilibrium of the free end of the panel we find for the load \( P \) in the edge members

\[ P_{n} = 2 \frac{P}{b} \] \hspace{1cm} (a2)

For the virtual forces we consider a variation \( \delta P \) in the stringer load. The applied forces \( P \) are maintained constant and hence to satisfy the equilibrium conditions on the free end \( (P_{n} = 0) \) we must take \( \delta P \), to be zero there

i.e. \( (\delta P)_{n} = 0 \) \hspace{1cm} (a3)

Otherwise the variation \( \delta P \), is arbitrary.

The virtual shear flow in the sheet is thus

\[ \delta q = -\frac{1}{2} \frac{dP}{dc} \]

(a4) and the virtual load in the edge members

\[ \delta P_{n} = -2 \frac{P}{b} \] \hspace{1cm} (a5)

Since the applied forces are not varied, the virtual forces principle (Eq. (62) for \( \Theta = 0 \) reduces to

\[ \delta U_{*} = 0 \]

The virtual complementary energy due to \( \delta P \), is

\[ \delta U_{*} = \left[ \frac{P_{n} \delta P_{n}}{2EB} + \frac{2P \delta P_{n}}{2EB} + \frac{2}{G_{1}} \delta \phi \right] \] \hspace{1cm} (a6)

Substituting for \( P_{n} \), \( \delta P_{n} \), and \( \delta \phi \) in terms of \( P \), \( \delta \phi \), and integrating the last term by parts we find

\[ \delta U_{*} = \left[ \frac{P_{n} \delta P_{n}}{2EB} \left( \frac{1}{A + 2B} \right) - \frac{b \delta P_{n}}{2G_{1}} \right] + \frac{\delta \phi}{2G_{1}} \frac{d^2 \phi}{dx^2} \frac{d^2 \phi}{dx^2} = 0 \] \hspace{1cm} (a7)

and therefore, since \( \delta P \), is arbitrary, we must have

\[ \frac{d^2 \phi}{dx^2} = -2 \frac{G_{1}}{2} \left( \frac{1}{A + 2B} \right) \] \hspace{1cm} (a8)

or

\[ \frac{d^2 \phi}{dx^2} = -\mu^2 \phi \] \hspace{1cm} (a9)

where

\[ \mu = \frac{G_{1}}{2} \left( \frac{1}{A + 2B} \right) \]

which is the required differential equation in \( \phi \). Since \( \delta P_{n} \) is zero for \( z = l \), the remaining term in Eq. (a7) vanishes for the upper limit \( z = l \). At the lower limit \( z = 0 \), however, \( \delta P_{n} \) is arbitrary and hence.
18. Virtual Forces: Example (b) Unit load method for twist of multicell tube

\[ \frac{dP}{dz} = 0 \quad \text{for } z = 0 \]  
\[ P_0 = 0 \quad \text{for } z = 1 \]  
\[ \frac{dP}{dz} = \frac{\cosh (1-z)}{\cosh \mu (1-z)} \]  

To determine the displacement \( w \) of the free end of the stringer we apply a unit force and since we need only consider a statically determinate system, we assume the stringer alone loaded by the unit force. We find then (see Eq. (71))

\[ 1 w = \int_0^1 A_2 \sigma_2 \sin \theta dz \]  

and since

\[ A_2 = \text{unit load} = 1, \quad \sigma_2 = P_0 / EA \]  

we have

\[ w = \frac{P_0}{EA} \]  

which is, of course, merely the extension of the stringer under the varying end load \( P_0 \).

(b) Rate of Twist of Multicell Tube

In the uniform thin-walled tube whose cross-section is shown in Fig. 18(d), \( q_2 \) is the known shear flow distribution in the walls of the tube due to a given loading. Using the unit load method, find the rate of twist \( dB/dz \) of the tube.

We consider unit length of the tube and apply a unit torque \( T = 1 \). Since we only need consider a statically determinate system for unit load stresses we select the single cell (1, 2, 3, 4) shown by full lines in Fig. 18(b). The unit torque gives then merely a constant shear flow

\[ q = \frac{1}{2A_n} \]  
\[ \frac{d\theta}{dz} = \frac{q_2}{c G_1} \]  

around the single cell and the rate of twist is given immediately as

\[ \frac{d\theta}{dz} = \frac{c}{A_n} \]  

where the integral is obviously only taken around the single cell (1, 2, 3, 4).

(c) Plan. Stress-Strain Relations for Oblique Co-ordinates

In a uniform isotropic plate, the stresses \( \sigma_{xx}, \sigma_{yy}, \sigma_{xy} \) are referred to the oblique co-ordinates \( Oa, Ob \) (Fig. 19). Using the principle of virtual forces and assuming the stress-strain relations for rectilinear stresses, find expressions for the strains \( \epsilon_{xx}, \epsilon_{yy}, \epsilon_{xy} \) in terms of the stresses.

The oblique strains \( \epsilon_{xx}, \epsilon_{yy}, \epsilon_{xy} \) are defined as the elongations in the directions \( Oa \) and \( Ob \) and the decrease in the angle \( \theta \) respectively of the unit parallelogram (see Fig. 19).

For the stresses \( \sigma_{xx}, \sigma_{yy}, \sigma_{xy} \) equivalent to the oblique stresses, we find easily from statics

\[ \sigma_{xx} = \sigma_{xx} \sin \theta \]  
\[ \sigma_{yy} = \sigma_{yy} \cos \theta \]  
\[ \sigma_{xy} = \sigma_{xy} \cos \theta + 2 \sigma_{xy} \cos \theta \sin \theta \]  

The rectilinear strains are

\[ \epsilon_{xx} = \frac{1}{2} (\sigma_{xx} - \sigma_{yy}) \]  
\[ \epsilon_{yy} = \frac{1}{2} (\sigma_{xx} - \sigma_{yy}) \]  
\[ \epsilon_{xy} = \frac{v(\sigma_{xx} + \sigma_{yy})}{E} \]  

and hence the virtual complementary energy per unit thickness of the element \( dxdy \) is

\[ \delta U_{cr} = dxdy \left[ \epsilon_{xx} \sigma_{xx} + \epsilon_{yy} \sigma_{yy} + \epsilon_{xy} \sigma_{xy} \right] \]  

Substituting for \( \sigma_{xx}, \sigma_{yy}, \sigma_{xy} \) from (c) we can now express \( \delta U_{cr} \) in terms of the oblique stresses and virtual stresses. Thus for the virtual stress \( \sigma_{ba} \) we find for the virtual complementary energy

\[ \delta U_{cr}^* = \frac{dxdy}{\sin \theta} \left[ \epsilon_{xx} \sigma_{ba} - \lambda \sigma_{ba} + 2 \sigma_{ba} \cos \theta \right] \]  

where

\[ \lambda = v \sin^2 \theta - \cos^2 \theta \]  

From Fig. 19 the virtual complementary work of \( \delta \sigma_{ba} \) is seen to be

\[ \delta W^* = \frac{dxdy}{\sin \theta} \epsilon_{xx} \delta \sigma_{ba} \]  

and therefore from the Virtual Forces principle

\[ \delta W^* = \delta U_{cr}^* \]  

we find, using Eqs. (48) and (60) in (70)

\[ \epsilon_{xx} = \frac{1}{2} (\sigma_{xx} - \lambda \sigma_{yy} + 2 \sigma_{xy} \cos \theta) \]  

Applying a virtual stress \( \delta \sigma_{ba} \) in the same way we obtain for the strain \( \epsilon_{xx} \) the corresponding expression

\[ \epsilon_{xx} = \frac{1}{2} (\sigma_{xx} - \lambda \sigma_{yy} + 2 \sigma_{xy} \cos \theta) \]  

Consider now the virtual shear stress \( \delta \sigma_{ba} \). From Eq. (3) we find for the virtual complementary strain energy due to \( \delta \sigma_{ba} \)

\[ \delta U_{cr}^* = \frac{dxdy}{\sin \theta} \left[ 2(1+v) \sin^2 \theta + 2 \cos \theta \right] \]  

Calculating the complementary work of the virtual shear stress \( \sigma_{ba} \) we find (see Fig. 19)

\[ \delta W^* = \delta \sigma_{ba} \epsilon_{xx} \cos \theta + \epsilon_{yy} \sin \theta \delta \sigma_{ba} \]  

Thus the virtual shear stress \( \delta \sigma_{ba} \) does work not only due to the shear strain \( \epsilon_{xy} \), but also due to the strains \( \epsilon_{xx} \) and \( \epsilon_{yy} \).

Substituting from Eqs. (87) and (90) for the strains \( \epsilon_{xx}, \epsilon_{yy}, \epsilon_{xy} \) and equating \( \delta U_{cr}^* \) and \( \delta W^* \) of Eqs. (100) and (101) respectively we finally obtain for the shear strain:

\[ \epsilon_{xx} = \frac{2(1+v)}{E} \left[ \sigma_{xx} + 2 \sin \theta \sigma_{yy} + \sigma_{xy} \right] \]  

Note that with the strains defined as above the increment of complementary energy is

\[ \delta U_{cr}^* = \epsilon_{xx} \sigma_{xx} + \epsilon_{yy} \sigma_{yy} + (\epsilon_{xx} + \epsilon_{yy}) \cos \theta + \epsilon_{xy} \sin \theta \delta \sigma_{ba} \]  

as compared with the simple result for rectilinear axes in Eq. (3).
8. METHODS OF ANALYSIS OF STRUCTURES WITH A FINITE NUMBER OF REDUNDANCIES

The general theorems given in Sections 4 and 6 include, from the fundamental point of view, all that is required for the analysis of redundant structures. However, to facilitate practical calculations it is helpful to develop more explicit methods and formulae. To find these is the purpose of this Section.

A structure is by common definition redundant if there are not sufficient conditions of equilibrium to obtain all internal forces (stresses or stress-resultsants) and reactions; the number of redundancies is the difference between the number of unknown forces (or stresses) and the number of independent equilibrium conditions. Strictly all actual structures are infinitely redundant but for practical purposes it is, in general, necessary and justified to simplify and idealize the structure and/or stress distribution in order to obtain a system with a finite (or even zero) number of redundancies. Such typical processes of simplification are, for example, the assumption of pin-joints in frameworks and the assumption of the engineers' theory of bending in the analysis of beams. Note, moreover, that the Rayleigh-Ritz procedure discussed in Section 6F amounts also, in fact, to the substitution of a finitely redundant structure for the actual elastic body.

All our considerations in this Section are restricted to linearly elastic bodies but Example 2 in Para II shows how the present methods may be extended to the analysis of non-linear redundant structures.

It is curious to note that, while the solution of problems in the theory of elasticity is derived very often from the differential equations in the displacements, the stress-deformation analysis of engineering structures was, until a few years ago, generally based on the concept of force-redundancies. Interestingly enough, Navier, * who was the first to evolve a general method for the analysis of redundant systems, when investigating problem (b) in Section 5 used also the displacement method. The analysis of indeterminate structures on the basis of redundant forces goes back to Clerk Maxwell and Otto Mohr and was ultimately developed by Mueller-Breslau.† This technique is, as mentioned in the introduction, more concise and physically more illuminating than the Castigliano approach; it derives most naturally from the unit load method (see Section 6D, Eq. (71a)). Mueller-Breslau's technique is generalized here and presented also in matrix form. The effect of temperature or other initial strains is included ab initio.

Parallel to the rapid development of the force-redundant theory occasional practical problems were solved by selecting deformations as unknowns. Fundamentally this method is equivalent to the virtual displacement analysis given in Section 4. Mohr‡ was probably the first to use such an approach in engineering structures when finding the secondary bending stresses in frameworks of the type usually assumed to be pin-jointed.

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† See also H. Mueller-Breslau, Die geographische Statistik der Baukonstruktionen, 2 ed., Koenner, Leipzig, 1894.


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Additional Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$R$, $Q$, $P$</td>
<td>single, generalized, orthogonal force (moment)</td>
</tr>
<tr>
<td>$r$, $q$, $p$</td>
<td>single, generalized, orthogonal displacement (rotation)</td>
</tr>
<tr>
<td>$s_i$, $t_i$</td>
<td>corresponding column matrices</td>
</tr>
<tr>
<td>$e_i$</td>
<td>strain due to unit load at $i$</td>
</tr>
<tr>
<td>$f_{ij}$, $f_{jk}$</td>
<td>direct and cross-flexibility</td>
</tr>
<tr>
<td>$F_{ij}$, $F_{jk}$</td>
<td>matrix of flexibilities $f_{ij}$, $f_{jk}$</td>
</tr>
<tr>
<td>$B$</td>
<td>transformation matrix for forces</td>
</tr>
<tr>
<td>$F_{ij}$, $F_{jk}$</td>
<td>generalized and orthogonal flexibility matrices</td>
</tr>
<tr>
<td>$S$</td>
<td>column matrix of internal forces (stresses)</td>
</tr>
<tr>
<td>$v_i$</td>
<td>column matrix of strains</td>
</tr>
<tr>
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</tr>
<tr>
<td>$g_i$</td>
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<tr>
<td>$e_i^t$, $e^{-t}$</td>
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<tr>
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<tr>
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<tr>
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<td>$D_e$</td>
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</tr>
<tr>
<td>$M$</td>
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</tr>
<tr>
<td>$\Delta$</td>
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</tr>
<tr>
<td>$C_i$</td>
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<tr>
<td>$A^T$, $A^{-1}$</td>
<td>transpose and inverted (reciprocal) matrix of $A$</td>
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</tbody>
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Following Mohr's analysis his ideas were applied to stiff-jointed frameworks, the first systematic work being that of the Danish engineer Axel Bendixen. However, the great potentialities of the method were only discovered by Ostenfeld, a colleague of Bendixen. He was the first to point out the duality of the force and displacement approach. In fact, his equations for the unknown displacements in a structure complement Mueller-Breslau's equations for the redundant forces. It is regrettable that Timoshenko in his fascinating History does not mention Ostenfeld's classical book. We give here a considerable generalization of Ostenfeld's ideas to include any structures under any load and temperature distribution.

The 'slope-deflexion' equations of Bendixen form the basis of the method of successive approximation due to Cramer and developed by Hardy-Cross as the well-known moment distribution method. The technique used is essentially a particular example of the relaxation method of Southwell which has been successfully applied to a wide range of problems. In its application to elasticity and structural problems this latter method is particularly representative of the modern tendency in making practical the numerical solution of highly redundant systems and has been used in conjunction with both forces or stresses and displacements as unknowns. Further discussion of this method is beyond the scope of the present work which is not concerned with iteration methods but the reader is referred to the original literature on the subject.

In this Section we make use, where appropriate, of the matrix notation. Although the complete analysis could be developed ab initio in this form it is thought preferable to give first most of the basic principles in the more familiar 'long-hand' notation. Only the most elementary properties of the matrix algebra like matrix partition, multiplication, transposition and inversion are necessary for the understanding of our theory. The reader may consult the classical work of Frazer, Duncan and Collar for these and more advanced matrix operations. Another modern and readable account is given in the recent book of Zurnuemling. The most comprehensive work to date on the formulation of aircraft structural analysis in matrix notation is Blandany on this side of the Atlantic, that of B. Langenflor's D. Williams recently presented an interesting account of some aspects of matrix operations in static and dynamic elastic problems.

Before proceeding to a discussion of the general methods for the analysis of redundant structures we introduce some concepts helpful to the understanding of the following theories and their subsequent matrix formulation.

### A. Flexibilities

Consider a cantilever beam with a plane of symmetry xy consisting of three connected segments a, b, and c with bending stiffnesses for deflections in the xy-plane (EI), (EI), and (EI) respectively (see Fig. 20). Let the corresponding shears and moments be (M), (M), and (M), respectively.

Transverse forces (F), (F), and (F) are applied in the xy plane at the joints B, C, and D. Since the system is assumed to be linear the principle of superposition holds and we can express the deflections in B, C, and D in terms of the loads as follows:

\[ r_1 = f_1, \quad r_2 = f_2, \quad r_3 = f_3 \]

where \( f_1, f_2, f_3 \) are, of course, the well-known influence coefficients. In fact, \( f_n \) is the displacement in the y-direction due to a unit force \( \Delta y \) in the \( n \)-direction. We call also \( f_{xy} \) and \( f_{xz} \) the direct and cross-flexibilities respectively and deduce immediately from Maxwell's reciprocity theorem (Eq. 43, Section 3) that

\[ f_{xy} = f_{yx}, \quad f_{xz} = f_{zx} \]

(83)

To find the flexibilities \( f_{ij} \) in any linearly elastic body we may use the unit load method developed in Section 6D. Thus, from Eq. (71a),

\[ f_{ij} = \frac{1}{\sigma} \frac{dJ}{dV} \]

(84)

where \( \sigma_{ij} \) is the stress and \( J \) is the moment of the unit load by the flexibilities. For example, with an axial load \( P \) at \( y \) and a transverse load \( M \) at \( x \),

\[ f_{xy} = \frac{1}{P} \frac{d}{dy} \left( \frac{P}{y} \right) = \frac{1}{P} \frac{P}{y} = \frac{1}{y} \]

(84b)

It is, of course, possible to substitute in the above formulae true stresses and virtual strains for true stresses and virtual strains but for reasons of logical consistency this is best avoided.

### B. Analysis

In the case of the beam shown in Fig. 20 that the Engineers' theory of bending stresses is true we find, noting that the system is statically determinate and hence \( \bar{\sigma} = \sigma \),

\[ f_{xy} = \int \frac{M_s M_s}{E I} + \frac{S_s S_s}{G A} \left( \frac{d^2}{dz} + 1 \right) \]

(85)

\[ \sigma_{ij} = \sigma_{ij} + \sigma_{ij} \]

(84c)

where \( \sigma_{ij} \) are the strains corresponding to a unit load at \( j \) and in the direction of \( k \). Under load we understand either force or moment. Similarly the flexibilities may represent either displacements or rotations. Naturally, formulae (84) yield also linear (angular) flexibilities due to moments (forces) respectively. Note that while \( \sigma_{ij} \) must be the true strains due to unit loads at \( j \) and \( k \), respectively, \( \sigma_{ij} \) need only be virtual, i.e., spatially equivalent, stresses due to the same loads.

It is of great importance in redundant structures. Thus, denoting by \( \bar{\sigma}, \sigma \) any set of statically equivalent stress system due to unit loads at \( j \) and \( k \), respectively in a redundant structure we can write Eqs. (84) also in the form:

\[ f_{xy} = \int \frac{M_s M_s}{E I} \]

(84d)

It is, of course, possible to substitute in the above formulae true stresses and virtual strains for true stresses and virtual strains but for reasons of logical consistency this is best avoided.

### C. References


**R. V. Southwell, Relaxation methods in engineering science, Oxford University Press, London, 1946.**


**These shear stiffnesses in bending are commonly based on the assumption of the Engineers' theory of bending stresses. See Argyn and Done, *Structural Analysis (Handbook of aeronautics, Vol. I)*, Pitman 1952, for a derivation of the area A'.**

**The influence coefficients were discovered independently by E. Wiggles, Mitt. Arch. d. Ing. Ver., Berlin 1884, 6 and O. Mohr, Zeit. Arch. d. Ing. Ver., Hannover, 1886, 19.**
where $M_0, S_i (M_0, N_0, S_i)$ are the moments and shear forces corresponding to $R_0 = 1, R_i = 1$. Eqs. (85) yield easily the following set of influence coefficients,

$$f_{ab} = \frac{a}{3(\Delta t)^2} - \frac{b}{3(\Delta t)^3} + \frac{c}{3(\Delta t)^4} + \frac{d}{3(\Delta t)^5}$$

$$+ \frac{a}{(\Delta)^4} + \frac{b}{(\Delta)^5} + \frac{c}{(\Delta)^6}$$

$$J_0 = \frac{1}{(\Delta t)^4} \left[ \begin{array}{c} f_{11} \\ f_{12} \\ f_{13} \\ f_{14} \\ f_{15} \\ f_{16} \\ f_{17} \\ f_{18} \\ f_{19} \end{array} \right]$$

$$+ \frac{a}{(\Delta)^4} + \frac{b}{(\Delta)^5} + \frac{c}{(\Delta)^6}$$

$$\begin{split} f_{ah} &= \frac{f_{ah}}{(\Delta)^4} \left[ \begin{array}{c} f_{11} \\ f_{12} \\ f_{13} \\ f_{14} \\ f_{15} \\ f_{16} \\ f_{17} \\ f_{18} \\ f_{19} \end{array} \right] \\ f_{ah} &= \frac{f_{ah}}{(\Delta)^5} \left[ \begin{array}{c} f_{11} \\ f_{12} \\ f_{13} \\ f_{14} \\ f_{15} \\ f_{16} \\ f_{17} \\ f_{18} \\ f_{19} \end{array} \right] \end{split}$$

To obtain $f_{ab}$ and $f_{ah}$ from the expression for $f_{ah}$ omit in the latter the terms $a, b, c$ respectively. Also to find $f_{ab}$ omit the terms involving $c$ in the last of Eqs. (85). Naturally, we can also derive the influence coefficients (86) by direct integration of the differential equation for the deflected beam when shear deformations are included. A systematic method for deriving the flexibility coefficients for compound engineering structures is given later.

Influence or flexibility coefficients are of great importance in the static and dynamic analysis of linearly elastic engineering structures. In this connection it is most appropriate to make use of the matrix notation not only for conciseness of presentation but also for the systematic programming of the considerable computational work usually involved in practical problems. The matrix algebra is, in fact, ideally suited for the automatic digital computers now available.

The matrix form of Eqs. (82) is

$$r = FR$$

where $r$ and $R$ are the column matrices of the displacements and forces

$$R = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \end{bmatrix}, \quad R = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \end{bmatrix}$$

and

$$F = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

is the so-called flexibility matrix; note that $F$ is a symmetrical square matrix. The relation (87) is, of course, valid for any number $n$ of displacements and rotations in any linear elastic body. To each displacement or rotation there corresponds a force or moment $R$. Thus, in such a matrix the entries are

$$r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \end{bmatrix}, \quad R = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \end{bmatrix}$$

and

$$F = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

where the $f_{ab}$ can be calculated always from Eqs. (84). The flexibilities in (90) need not necessarily refer to $m$ different points. For example, we may choose three directions $x, y, z$ at a particular point of a three dimensional body and define six flexibilities

$$f_{ax}, f_{ay}, f_{az},$$

$$f_{ax}, f_{ay}, f_{az},$$

corresponding to the three forces $R_1 = 1, R_2 = 1, R_3 = 1$ at the same point. Similarly for a beam in which we assume that the engineers' theory of bending is true we may require the slope and the deflexion at a cross-section under transverse force and moment applied there. Three flexibilities are required for this information; note, however, that if shear deformations are included we must specify that the bending moment is applied as engineers' theory direct stresses at the particular cross-section. A characteristic property of the influence coefficients is that any $f_{ab}$ in a given elastic body depends only on the points and directions $b$ and but not on any other directions selected for the calculation of a flexibility matrix (91).

A special of the flexibilities of Eqs. (86) shows that it is possible to split the complete flexibility matrix $F$ into two additive and distinctly different matrices. Thus,

$$F = F_1 + F_2$$

where $F_1$ and $F_2$ are the flexibilities corresponding to pure bending and shear deformations respectively. The first contains only terms involving $EI$ and the second only terms involving $GA$. For example,

$$f_{11} = \frac{a}{3(\Delta t)^2} - \frac{b}{3(\Delta t)^3} + \frac{c}{3(\Delta t)^4} + \frac{d}{3(\Delta t)^5}$$

$$+ \frac{a}{(\Delta)^4} + \frac{b}{(\Delta)^5} + \frac{c}{(\Delta)^6}$$

$$J_0 = \frac{1}{(\Delta t)^4} \left[ \begin{array}{c} f_{11} \\ f_{12} \\ f_{13} \\ f_{14} \\ f_{15} \\ f_{16} \\ f_{17} \\ f_{18} \\ f_{19} \end{array} \right]$$

$$+ \frac{a}{(\Delta)^4} + \frac{b}{(\Delta)^5} + \frac{c}{(\Delta)^6}$$

$$\begin{split} f_{ah} &= \frac{f_{ah}}{(\Delta)^4} \left[ \begin{array}{c} f_{11} \\ f_{12} \\ f_{13} \\ f_{14} \\ f_{15} \\ f_{16} \\ f_{17} \\ f_{18} \\ f_{19} \end{array} \right] \\ f_{ah} &= \frac{f_{ah}}{(\Delta)^5} \left[ \begin{array}{c} f_{11} \\ f_{12} \\ f_{13} \\ f_{14} \\ f_{15} \\ f_{16} \\ f_{17} \\ f_{18} \\ f_{19} \end{array} \right] \end{split}$$

where $M_i, S_i (M_0, N_0, S_i)$ are the bending moment, normal and shear forces due to a unit load at $j (h)$.

It is often convenient not to operate in single loads (or moments) but in groups of loads (or moments), which are known as generalized forces. To fix ideas, consider that in the example of Fig. 20 we select as applied generalized forces the three loads $Q_1, Q_2, Q_3$ given by

$$Q_1 = G_1 R_1 + G_2 R_2 + G_3 R_3$$

$$Q_2 = G_1 R_1 + G_2 R_2 + G_3 R_3$$

$$Q_3 = G_1 R_1 + G_2 R_2 + G_3 R_3$$

or in matrix form

$$Q = GQ$$

where $G$ is the square matrix, in general not symmetrical, of the coefficients $G_{ij}$. We call $G$ a load transformation matrix and assume that it is non-singular, i.e. that the determinant $|G|$ of the coefficients is different from zero. We may solve Eq. (96) for $R$ by premultiplying with $G^{-1}$ and obtain

$$R = BQ$$

where $B = G^{-1}$

is the so-called reciprocal or inverse matrix of $G$. Its determination is equivalent to solving Eqs. (95) and therefore involves considerable numerical labour if the number of equations is large. In such cases approximate methods may have to be used. However, with the advent of the automatic digital computers this difficulty is no longer insuperable. We give later in this Section a systematic procedure suitable for punch-card machines for computing the reciprocal matrix but hope to return in greater detail to this and similar questions in Part III. Next we have to determine the generalized displacements $q$ corresponding to the generalized forces $Q$. By definition $q$ are obtained from the equality of the two expressions for work in the two sets of variables $r, q$ and $q, q$.

$$W = \frac{1}{2} rR = \frac{1}{2} qQ$$

where $r$ and $q$ are the transposed matrices of the column matrices $r$ and $q$ and $r$ and hence the row matrices

$$q = rB$$

or

$$q = q'$$

where $B$ is the transpose of the matrix $B$, i.e.

$$B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

Substituting (87) into (100) we find,

$$q = BFR + BFBQ = FQ$$

Eq. (102) shows that

$$F = BFB$$

is the flexibility matrix corresponding to the generalized forces and displacements $Q, q$.

We illustrate now the application of generalized forces on a simple
and therefore
\[ r = \begin{bmatrix}
-0.6a + 1.5a - 0.4a \\
-2.4a + 1.0a + 0.4a \\
-3.6a - 0.4a - 0.4a
\end{bmatrix} = \begin{bmatrix}
a/2 \\
-a \\
-5a
\end{bmatrix}
\]
in agreement with the previously given values of \( r \). Each of the three columns of the intermediate expressions represents obviously the \( r \)-components of the corresponding \( q \)-coordinate. Fig. (21) illustrates in detail the three \( q \)-modes.

Naturally, Eqs. (97), (100) and (103) are valid for any linearly elastic body and any number \( m \) of forces (moment) \( R \) and displacements (rotations) \( r \). The load transformation matrix takes then the form
\[
B = G^{-1} = \begin{bmatrix}
B_{11} & \cdots & B_{1m} \\
B_{21} & \cdots & B_{2m} \\
\vdots & \ddots & \vdots \\
B_{m1} & \cdots & B_{mm}
\end{bmatrix}
\]  
(108)

and is not, in general, symmetrical. However, the transformation (103), called a congruent transformation, ensures that the flexibility matrix \( F_q \) is still symmetrical.

Attention is drawn to the dual relationship (97) and (100). Thus, if we transform a load system \( R \) by the transformation
\[
R = BQ
\]
the corresponding displacements \( r \) are transformed as
\[
q = B^{-1} r
\]  
(109)

It is often required to find the set of forces and displacements \( p, p \) for which all cross-flexibilities \( f_{pp} \) (when \( f_{pp} = 0 \)) are zero. These are, of course, the elastic eigenmodes corresponding to the set of displacements \( r_1 \) to \( r_m \). The load-displacements law is then given by
\[
p = F_p P
\]
where \( F_p \) is the diagonal matrix,
\[
F_p = \begin{bmatrix}
f_{p1} & 0 & \cdots & 0 \\
0 & f_{p2} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & f_{pm}
\end{bmatrix}
\]
(110)

Thus,
\[
p_1 = f_{p1} p_1, \ldots, p_i = f_{pi} p_i, \ldots, p_m = f_{pm} p_m \quad (111)
\]

It is always possible to find the unique load transformation matrix \( B \), which transforms our system \( R, r \) into the orthogonal system \( F, p \). We do not enter here into its detailed derivation since the reader can consult a number of textbooks on this subject.*

Our above considerations and in particular Eqs. (84) and (84b) show that the flexibilities are particularly simple to derive for a statically determinate structure, e.g. the beam of Fig. 20. For a redundant structure, we must first find the forces or stresses in the redundant members before we can obtain the true strains \( e_j \) for the unit loads. The necessary analysis is developed later but it is helpful to give here a formal matrix derivation of the flexibility \( F \) of an engineering structure, the stress distribution of which is known. To this effect we use again the unit load method given by Eq. (71a). We denote by \( e_x, e_y, e_z \) two column matrices for the true strains and stresses respectively, corresponding to a unit load \( R_k = 1 \) at the point and direction \( k \), thus,
\[
e_x = (e_{x1}, e_{y1}, e_{z1}, e_{x2}, e_{y2}, e_{z2}, e_{x3}, e_{y3}, e_{z3})
\]
\[
e_y = (e_{x1}, e_{y1}, e_{z1}, e_{x2}, e_{y2}, e_{z2}, e_{x3}, e_{y3}, e_{z3})
\]
\[
e_z = (e_{x1}, e_{y1}, e_{z1}, e_{x2}, e_{y2}, e_{z2}, e_{x3}, e_{y3}, e_{z3})
\]
(112)

where the elements of these matrices may, of course, vary with \( x, y, z \).

It is always possible to write
\[
e_x = f_x e_x, \quad e_y = f_y e_y, \quad e_z = f_z e_z
\]
(113)

where \( f_x, f_y, f_z \) is the flexibility matrix of a unit cube at the point \( x, y, z \). Thus for an isotropic body

* See Fraser, Duncan, Collar, loc. cit. Zurnahl, loc. cit.
The matrix $b$ has submatrices with $m$ columns, $f$ is a partitioned diagonal matrix whose elements are the flexibility matrices $f_i$.

We denote now by $b$ a matrix whose $m$ columns are loading systems on the $s$ members statically equivalent to the external loads $R_1 = 1, R_2 = 1, \ldots, R_s = 1$ respectively. If we choose these systems to be also elastically compatible then, of course,

$$\mathbf{b} = \mathbf{b} \quad (124)$$

Applying now the unit load method and using Eq. (122) and the transpose matrix $b'$ we find by an argument similar to that leading to Eq. (117) that the deflections $r$ at the points of application and in the directions of the loads $R$ are given by

$$r = b' \mathbf{fR} \quad (125)$$

Therefore the flexibility matrix $F$ for the prescribed $m$ directions in the complete structure is

$$F = b' \mathbf{f} \quad (126)$$

The matrix operations in (126) are again congruent and thus $F$ is indeed symmetrical. Eqs. (123) show that Eq. (126) can also be written in the form

$$F = \mathbf{S} \mathbf{b} \quad (126)$$

Eq. (126) is the general expression for obtaining the flexibility of a complete structure from the flexibilities of the constituent elements. The configuration of the elements is said to be in series since the assembly condition is expressed by the matrix $b$ which derives from conditions of equilibrium. Thus, Eq. (126) may be regarded as the most general formulation of the flexibility matrix of a structure consisting of elastic elements in series.

It is also clear why Eq. (103) for the flexibility matrix of generalized forces has the same form as Eq. (126). In the first case we derive generalized forces from single forces and in the second internal forces from external forces but in both cases this entails a linear transformation matrix $B$ or $b$. Note also that $F$ is in the first instance the flexibility matrix of the complete structure for the single forces and $f$ in the second instance the flexibility matrix of the individual elements. It is seen, however, that whereas $\mathbf{B}$ is always a square matrix $b$ is not in general, rectangular.

Before illustrating the application of Eq. (126) we draw attention to an interesting dual relationship (see also p. 20). Thus, Eqs. (121) and (125) prove that if the internal loads $S$ are derived from the external load system $R$ with the relationship

$$S = R \quad (121)$$

the deflections $r$ at the points of application of the $R$-loads are found from the internal relative displacements (strains) $v$ from the relationship

$$r = b'v = b'v \quad (125a)$$

Naturally, Eq. (125a) merely restates the unit load theorem. We stress again the fact that $\mathbf{B}$ need only be the matrix of statically equivalent stress systems.

Illustration of Eq. (126).

We observe first that Eq. (126) includes as a particular case Eq. (92) for the splitting of a flexibility matrix. This may be seen as follows. Splitting the flexibility matrix is equivalent to considering the combined effect of two or more geometrically identical structures (elements) each of which is assigned only part of the complete flexibility of the structure (e.g., flexibility in bending or shear, or normal force). Thus, the constituent elements are in this case geometrically identical and hence the load trans- ference matrices $b$, etc., are merely unit matrices $I$. If then, flexibilities of each of the elements are written as $f_i, f_j, \ldots, f_n$, etc., we find

$$F = \begin{bmatrix} f_1 & 0 & \cdots & 0 \\ f_2 & f_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_n & f_{n-1} & \cdots & f_1 \end{bmatrix} = b' \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = b'v \quad (127)$$

The order of the unit matrices and the $f$ matrices is $m$, the number of assigned directions in the structure.

Consider next a beam built-up by two uniform component beams $a$ and $b$ as shown in Fig. 22. We seek the flexibility matrix for the transverse forces $R_x, R_y$ and moments $R_m, R_m$ under the assumption that the E.T.B. holds and shear deflections are negligible. We analyse first each beam separately as a cantilever built-in at the L.H.S. and subjected to transverse force and bending moment at the tip, the signs of which are taken to be those of the
where
\[ \phi_{a,b} = \frac{1}{EI} \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix} \]  

(132)
gives the flexibility per unit length of the cantilevers. The negative sign in the cross-flexibilities arises from the sign convention. The total flexibility follows as
\[ f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \]  

(131a)

Applying now Eq. (126) the flexibility \( F \) of the complete structure is
\[ F = b^T f b = b^1_{s1} f_1 b_1 + b^1_{s2} f_2 b_2 \]  

(133)

In the present case \( b = b \) since the system is statically determinate. The deformations of the structure may finally be obtained from
\[ r = Fr \]  

(87)
The expressions for the deflections \( r_1 \) and \( r_2 \) derived from Eq. (133) agree with those of Eq. (86) when the shear deformations are neglected.

An alternative approach to the problem is to express the loading on the component beam by the end moments (see Fig. 22b). The internal loading matrix is now,
\[ S = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} \begin{bmatrix} M_{A} \\ M_{B} \\ M_{C} \end{bmatrix} = b R \]  

(134)

where
\[ b = \begin{bmatrix} -1 \quad -1 \quad -1 \\ 0 \quad -1 \quad -1 \\ 0 \quad 0 \quad 0 \end{bmatrix} \]  

(135)

The internal flexibilities \( f_1 \) and \( f_2 \) derived in this case solely from the end bending moments and to find them we have to consider only the end slopes of the simply supported beams shown in Fig. 22b; thus, taking end slopes positive in the direction of positive moments we find
\[ \phi_{a,b} = \begin{bmatrix} \frac{1}{EI} & \frac{1}{EI} \\ \frac{1}{EI} & \frac{1}{EI} \end{bmatrix} \]  

(136)

Hence,
\[ F = b^T f b_1 \]  

(137)

where,
\[ f_1 = \begin{bmatrix} f_1 \\ 0 \\ 0 \end{bmatrix} \]  

(136a)

It is easily seen that the flexibility matrix \( F \) of Eq. (137) is identical with that of (133). We observe that the cantilever position of the constituent elements is derived merely by a rigid body rotation from the simply supported beams shown in Fig. 22b. No additional total virtual work is associated with such a linear transformation and the flexibility \( F \) is, hence the same.

The form (136) of the flexibility of the element is important for it applies to a linear variation of any deforming stress, force or moment as long as we substitute for \( \phi \) the appropriate unit flexibility. Thus, for a normal force varying linearly in a beam with constant direct stiffness \( EA \), we can use Eq. (136) with \( \phi = 1/EI \).

B. Stiffnesses

We return now to Eqs. (87) and solve them for \( R_1 \) to \( R_m \) to find equations of the type
\[ R_i = k_i r_i + k_{i1} r_1 + \ldots + k_{im} r_m + \ldots + k_{i1} r_m + \ldots + k_{mm} r_m \]  

(138)

The coefficients \( k_{ij} \) and \( k_{ib} \) are known as direct- and cross-stiffnesses in the directions of the selected \( m \) displacements. In fact, it follows directly from Eqs. (138) that the general stiffness \( k_{ab} \) is the force (or moment) applied in the direction \( j \) if we displace the body by \( r_a = 1 \) whilst keeping the remaining \((m-1)\) \( r \)’s at zero. Using matrix notation the solution (138) of Eqs. (87) may be written as
\[ R = F^{-1} \tau = Kr \]  

(139)

Fig. 22.—Break-down of cantilever for calculation of flexibility matrix
where $R$ and $r$ are the column matrices defined by Eqs. (90) and $K$ is the $m \times m$ stiffness matrix

$$K = \begin{bmatrix} k_{11} & \cdots & k_{1m} \\ \vdots & \ddots & \vdots \\ k_{m1} & \cdots & k_{mm} \end{bmatrix}$$

(140)

The stiffness matrix may be determined either directly or from the identity

$$K = F^{-1}$$

(140a)

by inversion of the flexibility matrix $F$. Eq. (140a) shows that $K$ is symmetrical, i.e.

$$k_{ij} = k_{ji}$$

(141)

This may be seen also as follows:

Let $\sigma'$, $\epsilon'$ and $\sigma^a$, $\epsilon^a$ be the stresses and strains corresponding to unit displacements $R_j = 1$ and $r_k = 1$ respectively while all other $r$ displacements are kept at zero. Applying now the principle of virtual work or displacements to the true state $j(h)$ and virtual state $h(j)$ we obtain

$$\int_0^l \sigma^a \epsilon'dV = \int_0^l \sigma' \epsilon' dV - \int_0^l k_{ij}$$

(142)

Where $\sigma' \epsilon'$ etc. stands for an expression analogous to Eq. (84a). This application of the principle of virtual displacements, by an obvious analogy with Eq. (84), is called the 'unit displacement method'.

We remarked on page 19 that the direct or cross-flexibilities depend for a given structure only on the points and directions to which they refer. This is not so for the stiffnesses which by definition depend on the correct set of points and directions selected to describe the stiffness of the body. Thus, if we choose an additional direction $m+1$ to augment our description of the elastic behaviour of the structure all the original $k_{ij}$ will in general change whilst the $f_j$ remain unaffected.

Consider again now the example of Fig. 20. A study of Eqs. (89) and (140a) shows that the stiffnesses $\sigma^a$ corresponding to unit deflexions at $B$, $C$ and $D$ are considerably more complicated than the expressions for the flexibilities. However, this is not always the case. Naturally, we can calculate the stiffnesses directly. For example, the $k_{ij}$ may be obtained by analysing a continuous beam built-in at $A$ and simply supported at $B$, $C$ and $D$ at which last support there is a fixed 'give' of unity. We may solve this threes redundant problem either with the three-moment equation or by the slope-deflection method.

Assume now that not only transverse forces but also moments are applied at the junctions of the component beams and at the tips (Fig. 23).

To simplify the argument we ignore, moreover, the effect of shear deformability. The modes and stiffnesses corresponding to unit deflexions in the directions, 1, 2, 3, 4 can now be determined very easily. For example, for the modes $r_1 = 1$ and $r_2 = 1$ shown in Fig. 23 we find respectively,

$$k_{11} = 12 \left( \frac{E}{\beta} \right)_a + 12 \left( \frac{E}{\gamma} \right)_b \quad , \quad k_{33} = -12 \left( \frac{E}{\gamma} \right)_b$$

$$k_{21} = -6 \left( \frac{E}{\gamma} \right)_a + 6 \left( \frac{E}{\gamma} \right)_b \quad , \quad k_{11} = 6 \left( \frac{E}{\gamma} \right)_b$$

(143)

$$k_{14} = 0 \left( \frac{E}{\gamma} \right)_a \quad , \quad k_{14} = -6 \left( \frac{E}{\gamma} \right)_b$$

(144)

The important point about this example is that it shows how easy the determination of the stiffnesses can be when we consider all possible modes of deformation of joints connecting simple component elements of a structure. Another example will help to clarify the argument further. Consider the symmetrical framework of Fig. 24 and assume that we seek the flexibility or stiffness at the central point 2 for vertical deflexions. We first case must solve a threes redundant problem and in the second a four times redundant problem with a central unit "give". If, on the other hand, we select the complete set of stiffnesses corresponding to vertical and horizontal deflexions at all movable joints then the calculations are most simple. In fact, for the typical cases shown in Fig. 24 we find by inspection,

$$k_{2a} = k_{2a} + 2k_{a} \left( \frac{a}{h} \right)^2$$

$$k_{2b} = k_{2b} + k_{a} \left( \frac{a}{h} \right)^2$$

$$k_{4a} = k_{4a} - k_{a} \left( \frac{a}{h} \right)^2$$

$$k_{4b} = k_{4b} - k_{a} \left( \frac{a}{h} \right)^2$$

(145)

and,

$$k_{12} = 2k_{2a} + 2k_{a} \left( \frac{a}{h} \right)^2$$

$$k_{13} = k_{2b} - k_{a} \left( \frac{a}{h} \right)^2$$

$$k_{14} = k_{4a} - k_{a} \left( \frac{a}{h} \right)^2$$

(145a)

where

$$\kappa = E\alpha$$

(145a)

**Fig. 23.—Stiffnesses of a cantilever**

The stiffnesses per unit length of a bar.

The stiffnesses at a point associated with the unit displacements at the same point are, in fact, already derived by the method of virtual displacements in Example 5b where they appear at the coefficients to the displacements $u$, $v$. Naturally, the problem of deriving the single stiffness $k_{ij}$ at the point 2 from the set (145) still remains. A general method for solving

**Fig. 24.—Stiffnesses of pin-jointed framework**
this and related problems is given further below. It is characteristic that
the direct calculation of the flexibilities corresponding to (145) is as com-
plicated as that for the single flexibility at 2.
In both examples we see that the stiffnesses are determined most straight-
forwardly, in fact, practically by inspection, once we find the set of unit
deformations for which it is simple to calculate the strains and hence
stresses and forces. The advantage of the second is that the stiffnesses may
be particularly marked in highly redundant structures but it requires, as
the example of Fig. 24 shows, the consideration of many degrees of freedom
which may also have its disadvantages. On the other hand flexibilities are
always easier to calculate if the stresses corresponding to unit forces can be
found without difficulty as in statically determinate structures.
We gave in Eq. (142) a general formula for the determination of the
stiffnesses. Let us consider again in more detail. Observing first that,
while $\alpha^i$ and $\phi^i$ must be the true stresses corresponding to $r_{i}=1$ and $r_{i}=1$
respectively, the strains $\epsilon^1$ and $\epsilon^2$ need only be virtual strains $\epsilon^0$ and $\epsilon$
(i.e., compatible but not necessarily statically consistent strains*), correspond-
ing to $r_{i}=1$ and $r_{i}=1$ respectively: this may contribute to a con-
siderable simplification of the calculations. However, the actual practical
use of Eqs. (142) rewritten here in the form

$$k_{ij} = \frac{\partial^2 v}{\partial^2 y}$$

$$k_{ij} = \int \frac{\partial^2 v}{\partial^2 y} dy$$

is somewhat limited. This follows from the previous discussion which
shows that stiffnesses are best found either by considering all possible
degrees of freedom at the joints, in which case the determination of the
$k$'s is usually performed by inspection, or by inverting the flexibility
matrix $F$. Moreover, even if we calculate the stiffnesses $k$ for a restricted
total number of degrees of freedom at the joints (e.g., example of Fig. 20)
Eq. (146) are really superfluous. Thus, in the example of Fig. 20 we have
to find the true stress $s$ for a times redundant structure, the analysis of
which includes the derivation of the forces $k_{p}$ and the use of Eq. (146)
is hence unnecessary. Nevertheless, Eq. (146) is of considerable value when
the elements into which the structure is broken down are characterized,
not by simple loading systems (e.g. beam elements or bars) for which the
$k$'s are determinate by inspection, but by simple (assumed) dis-
placement patterns. An example of this application is given in D of this
section. Also given later is the matrix formulation of Eq. (142), which is
most useful in practical cases.
We now find a generalization of the concept of stiffness corresponding
to the generalized flexibility given on page 19. Thus, following the argu-
ment there we introduce the generalized displacements $q$ and forces $Q$
developed by

$$r \rightarrow Aq \quad Q = AR$$

where

$$A = G(B^{-1})'$$

(147)

see also Eqs. (96), (97) and (100). Substituting the expressions for $r$ and $R$
in Eq. (139) we find immediately

$$Q = Kq$$

(148)

where

$$K = KA'K = (B^{-1})^{-1}$$

(149)

is the generalized stiffness corresponding to the $m$ generalized displace-
ments $q$. Eq. (149) may naturally also be derived by inversion of Eq. (103), i.e.

$$K = (F')^{-1}$$

(149a)

The particular linear transformation $B$ (or the corresponding matrix $A$)
which reduces the cross-flexibilities $f_{ab}$ to zero nullifies also the corre-
sponding cross-stiffness $k_{ab}$. In fact, we obtain from Eq. (109)

$$P = Kp$$

(150)

where $K_{p}$ is the diagonal matrix

$$K_{p} = \begin{bmatrix}
    k_{p11} & 0 & \cdots & 0 \\
    0 & k_{p22} & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & \cdots & 0 & k_{pmm}
\end{bmatrix}$$

and

$$k_{p11} = \int k_{11} \, dy, \quad k_{p22} = \int k_{22} \, dy, \quad k_{p11} = \int k_{11} \, dy, \quad k_{p22} = \int k_{11} \, dy.$$  

(151)

Fig. 25.—Generalized displacement and forces

We illustrate now in Fig. (25) the application of formulae (147)-(149)
on the simple example of Fig. (21) and for the same load transformation
matrix $B$ of Eq. (105), but seek here the components of forces correspond-
ing to a generalized displacement

$$q_{p} = 1$$

The displacement transformation matrix $A$ of Eq. (147a) is given by Eq. (107)

$$A = G^{-1}(B^{-1})'$$

(147a)

Hence

$$r \rightarrow Aq \rightarrow 0 \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$  

(107a)

The generalized stiffness matrix $K$, is best obtained by inversion of $F$, in
Eq. (106) and is

$$K = \begin{bmatrix}
    0.0169 & -0.00615 & -0.17 \\
    -0.00615 & 0.548 & 0.406 \\
    -0.17 & 0.406 & 0.218
\end{bmatrix}$$

(106a)

where $a$ is given by Eq. (104a), Hence,

$$Q = Kq = Kq_{0}, \begin{bmatrix} 1 \end{bmatrix}$$

or

$$Q = \begin{bmatrix}
    0 & -0.00615 & 0.548 & 0.406
\end{bmatrix}$$

We analyse finally the generalized forces $Q$ in their $R$-components. From
Eq. (97) or (147)

$$R = BQ$$

and carrying out the multiplication,

$$R = \begin{bmatrix}
    1 & 0 & -0.548 & -0.406 \\
    -0.006 & 0.274 & -0.406 \\
    -0.006 & 0.274 + 0.203
\end{bmatrix} = \begin{bmatrix}
    -0.954/a \\
    -0.138/a \\
    -0.077/a
\end{bmatrix}$$

Each of the three columns of the intermediate expression represents ob-
viously the $R$ components of $Q_{1}, Q_{2}, Q_{3}$ respectively. We can check now
the previously given result for $r$ from

$$r = FR$$

where $F$ is given in Eq. (104).

Next we derive a general formula for the stiffness $K$ of a structure con-
sisting of a finite number of simple elements. The expression given is the
matrix formulation of Eqs. (146) and corresponds to the flexibility matrix

---

* See Section 4.
F of Eq. (126). Since the analysis follows closely the arguments on page 21 we need present here only the outlines of the proof.

Consider again an assembly of $s$ structural elements joined together at their ends or boundaries. $m$ displacements $r$ are selected to describe the stiffness $K$ of the complete structure. Let $k_g$ be the stiffness matrix of the $g$ element due to the characteristic strains of the element arising from the displacements $v_2$ at the boundaries. Naturally, there are usually several different but equivalent possible ways of expressing the strain or the stiffness of the element. Let $a_j$ be the matrix, in general rectangular, which transforms the displacements $r$ into the true strains $v_2$ of the element. I.e.

$$v_2 = a_j r \quad \quad \quad (153)$$

Then

$$s_j = k_j v_2 = k_j a_r \quad \quad \quad (154)$$

is the matrix for the forces (moments, etc.) applied on the element due to the displacements $r$. The internal force or stress matrix $S$ of the aggregate structure is now given by

$$S = k a r \quad \quad \quad (155)$$

where

$$S = (S_1, S_2, \ldots, S_s) \quad \quad \quad (156)$$

and

$$a = (a_1, a_2, \ldots, a_s) \quad \quad \quad (157)$$

$k$ is the symmetrical diagonal partitioned matrix.

Applying now the principle of virtual work, taking the internal forces $S$ and external forces $R$ as the true state and selecting as virtual state the internal strains corresponding to unit displacements $r_1 = 1, r_2 = 1, \ldots, r_m = 1$ respectively we find

$$R = k a r \quad \quad \quad (159)$$

where $a'$ is the transpose of $a$. Thus, the stiffness matrix $K$ of the compound structure is

$$K = a' k a \quad \quad \quad (160)$$

Eq. (160) may also be written as

$$K = 2 a' k a \quad \quad \quad (160a)$$

Since the virtual strains need only satisfy the compatibility but not necessarily the equilibrium conditions we may select for the virtual states a simpler matrix $a$ which satisfies only the former. Eq. (160) becomes then

$$K = a' k a \quad \quad \quad (160b)$$

However, the application of a possibly simplified matrix $a$ is really not required in practice. As mentioned on page 23 the stiffness matrix $K$ is best calculated for all degrees of freedom at the joints, yielding very simple matrices.

The configuration of the elements of the compound structure is said to be in parallel in Eq. (160) since the assembly condition is expressed by the matrix $a$ which derives from conditions of compatibility. Thus Eq. (160) may be regarded as the most general formulation of the stiffness matrix for a structure with constituent elements in parallel. It is immediately apparent why Eq. (149) which expresses the stiffness matrix for generalized displacements must have the same form as Eq. (160). In the first case we derive generalized displacements from single displacements and in the second, internal strains from external displacements. In both applications this entails a linear transformation matrix which, however, is a square matrix in the former case. Also $K$ is the stiffness of the complete structure for the single displacements while $k$ is the stiffness of the matrix of the individual members.

Eqs. (153) and (159) show that there is a most illuminating parallel development to Eqs. (121) and (125a). Thus, if the internal relative displacements (strains) $v$ derive from the external displacements $r$ with the relationship

$$v = a r \quad \quad \quad (153a)$$

Then the external forces $R$ derive from the internal forces (stresses) $S$ with the relationship

$$R = a s a' r \quad \quad \quad (159a)$$

Eq. (159a) restates, of course, the principle of Virtual Work.

### Table I

#### Duality of Force and Displacement Methods

<table>
<thead>
<tr>
<th>Method of Forces</th>
<th>Method of Displacements</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>$K$</td>
</tr>
<tr>
<td>Flexibility</td>
<td>Stiffness</td>
</tr>
<tr>
<td>Displacement $r$</td>
<td>Force $R$</td>
</tr>
</tbody>
</table>

- **Generalized Force** $Q = BQ$
- **Generalized Displacement** $q = Aq$
- **Generalized Flexibility** $A'B = I = B'A'$
- **Generalized Stiffness** $K_{i} = A'K_{i}A$
- **Generalized Displacement** $q = B'r = F_{i}Q$
- **Generalized Force** $Q = AR = K_{i}q$

<table>
<thead>
<tr>
<th>Generalized Series Assembly</th>
<th>Generalized Parallel Assembly</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stress on elements $S$</td>
<td>Strain of elements $V$</td>
</tr>
<tr>
<td>$S = bR$</td>
<td>$V = ar$</td>
</tr>
<tr>
<td>Strain of elements $s$</td>
<td>Stress on elements $S$</td>
</tr>
<tr>
<td>$s = b'V$</td>
<td>$S = a's$</td>
</tr>
<tr>
<td>Flexibility of elements $f$</td>
<td>Stiffness of elements $k$</td>
</tr>
<tr>
<td>(for stresses $S$)</td>
<td>(for strains $V$)</td>
</tr>
<tr>
<td>$F = b'f b$</td>
<td>$K = a'k a$</td>
</tr>
</tbody>
</table>

**Addition of Flexibilities** (Special series assembly)

- **Addition of Stiffnesses** (Special parallel assembly)

Before illustrating applications of Eq. (160) we draw attention to the by now also apparent complete parallel between the flexibility and stiffness approach in the analysis of structures. We may express this concisely by the tabular arrangement under the two headings: ‘Methods of Forces’ and ‘Methods of Displacements’.

The analogy between the two methods is developed considerably in what follows and is shown in greater detail in Table II.

**Illustrations to Eq. (160).**

Consider the beams $I$ and $II$ of Fig. (26) joined by inextensible bars which connect the set of points $B, C, D$ and $B', C', D'$ respectively. Let $K_I$ and $K_{II}$ be the stiffness matrices of the upper and lower beam respectively defined for vertical displacements $r_1, r_2$ and $r_3$. From the definition of the stiffness it follows immediately that the stiffness $K$ for the displacements $r_1, r_2$ and $r_3$ in the compound structure is given by

$$K = K_I + K_{II} \quad \quad \quad (161)$$

This simple result may also be derived from the general Eq. (160). For in this special case the joint displacements $r$ etc. of the complete structure and the straining displacements $v_I$ and $v_{II}$ of the component beams are the same. Thus,

$$v_I = a_I r, \quad v_{II} = a_{II} r \quad \quad \quad (162)$$

and

$$r = (r_1, r_2, r_3) \quad \quad \quad (162a)$$

where

$$a_I = a_{II} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \quad \quad (163)$$

We conclude,

$$K = \begin{bmatrix} K_I & 0 \\ 0 & K_{II} \end{bmatrix} = K_I + K_{II} \quad \quad \quad (161a)$$
Eq. (161) applies to any compound structure in which the stiffnesses are defined for at least all common degrees of freedom associated with the joining of structures I and II. Thus, in the example of Fig. (26) the common degrees of freedom are the vertical displacement $r_1$, $r_2$ and $r_3$. Formula (161) is, however, still true if we define the stiffnesses of the upper beam and the complete structure for both the vertical displacements $r_1$, $r_2$ and $r_3$ and the slopes $s_1$, $s_2$, $s_3$. Then $K_{II}$ can be calculated by the methods given previously. $K_{II}$ is still only definable for vertical displacement, the corresponding entries associated with $s_1$, $s_2$, and $s_3$ being zero. Naturally, we can define the stiffness matrix $K$ and say $K$, for points not connected to II. Again the corresponding terms of $K$ are zero. Fig. (27) shows the joining of two arbitrary structures to give $K = K_1 + K_2$. Note that at a joint point like (2) we must define the stiffnesses for two displacements, say the $x$ and $y$-directions.

Formula (161) may, of course, also be applied in obtaining the stiffness matrix of the compound cantilever consisting of elements a and b, Fig. (23). Again, we must define the stiffness for all common deflections and slopes at the joints, assuming the E.T. B. to be true and the shear deflections zero. The total stiffness $K$ is then

$$K = K_a + K_s$$

(161b)

where the elements of the split matrices may be found from Eqs. (143) and (144) for the displacements $r_1$ and $r_4$ and similar equations for $r_2$ and $r_5$. Thus,

\[
K_a = \begin{bmatrix}
\frac{EI}{L_3} & 0 & -\frac{EI}{L_3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-\frac{EI}{L_3} & 0 & \frac{EI}{L_3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(164)

\[
K_s = \begin{bmatrix}
\frac{EI}{L_5} & 0 & -\frac{EI}{L_5} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-\frac{EI}{L_5} & 0 & \frac{EI}{L_5} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(164a)

where the columns and rows refer to displacements $r_1$, $s_1$, $r_2$, $s_2$, $r_3$ respectively. Formula (161b) may be generalized for any numbers of component beams, and for any structure in which the joint displacements $r$ express also the straining displacements $v$ of the elements (i.e. $a = 1$). In such cases the stiffness matrix $K$ can be written

$$K = K_a + K_s + \cdots + K_{a'} + \cdots + K_{s'}$$

(165)

Note that the only non-zero coefficients in $K_s$ are the stiffness $k$ corresponding to the displacements at the ends or boundaries of the $g$-element.

The flexibility matrix $F$ corresponding to (165) is

$$F = K^{-1} = (K_a + K_s + \cdots + K_{a'} + \cdots + K_{s'})^{-1} = (F_a^{-1} + F_s^{-1} + \cdots + F_{a'}^{-1} + \cdots + F_{s'}^{-1})^{-1}$$

(166)

The parallel between the displacement and force method is underlined further by comparison of Eqs. (92) and (161). The first shows the case of additive partial flexibilities for series assembly and the second, the case of additive partial stiffnesses for parallel assembly. A very simple application of Eq. (160) is given by the pin-jointed framework shown in Fig. 12 of example 50. Thus, the stiffness $K_e$ of the bar corresponding to unit elongation $\Delta l = 1$ is

$$k_e = \frac{(EA)}{l_e} = \frac{k_e}{l_e}$$

(167)

where $k_e = (EA)$, is the stiffness per unit length of the $e$th component bar.

The transformation matrix $a$, for displacements in the x- and y-directions is

$$a = \begin{bmatrix}
\cos \theta & \sin \theta \\
0 & 0 \\
\end{bmatrix}$$

(168)

Hence

$$a = [a_1 a_2 \cdots a_{n-1} a_n]$$

(168a)

and

$$K = \begin{bmatrix}
\kappa_1/l_1 & 0 & \cdots & 0 & 0 \\
0 & \kappa_2/l_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \kappa_{n-1}/l_{n-1} & 0 \\
0 & 0 & \cdots & 0 & \kappa_n/l_n \\
\end{bmatrix}$$

(169)

in agreement with Eq. (67).

We mentioned previously that the simplest method of calculating stiffnesses is to define them for as many degrees of freedom as are necessary to obtain simple deformation patterns of the elements of the structure. Having calculated such a stiffness matrix it will become necessary to "condense" it—i.e. to reduce it to the smaller number of displacements in which we may be interested. This changes, of course, all the stiffness coefficients, but the necessary analysis is easily arranged in matrix form for automatic computation. Let the original stiffness matrix be of the order $m \times m$ and denoted by $K$. We want to find the matrix $K$ referred to $p$-directions only, where $p < m$. We have

$$R = K_r$$

where

$$R = [r_1 \cdots r_p r_{p+1} \cdots r_n] = [r_1' r_2']$$

(171)

in which we write first the $p$-directions required for the condensed matrix $K$. $K_r$ may be expressed as a partitioned matrix as follows.

* See p. 10.
where \( K_i \) and \( K_{ij} \) are square matrices of order \( p \) and \( m-p \) respectively. Eq. (170) can now be transformed to
\[
R_i = K_i r_i + K_{ij} r_j \tag{173}
\]
Also, by definition, in the structure with stiffness defined in \( p \) directions
\[
R_i = K_i r_i \tag{173a}
\]
Putting \( R_i = 0 \) in Eq. (173) and eliminating \( r_j \), we find
\[
R_i = (K_i - K_{ij} K_{jj}^{-1} K_{ji}) r_i \tag{174}
\]
and hence comparing with Eq. (173a)
\[
K = K_i - K_{ij} K_{jj}^{-1} K_{ji} \tag{175}
\]
Eq. (175) gives a general solution to the particular problem of specifying the reaction at a point. Another example illustrating the application of Eq. (175) is discussed under D of this Section.

Consider a structure subject to arbitrary external loads \( R \), temperature strains \( \alpha \Delta T \) and any other initial strains \( \eta \). We assume that the system has \( n \) internal or external redundancies
\[
X_1, X_2, \ldots, X_n
\]
which may be stresses, forces, moments or linear combinations of such (generalized forces). By including the supporting body—assumed rigid—in our structure we can define all redundancies as internal. The stress distribution in the body varies statically indeterminate until an elastic analysis yields the unknown \( X \)'s. If, irrespective of compatibility, we assume the \( X \)'s to be zero, we obtain the 'basic' (principal or null) system which is statically indeterminate. This procedure of obtaining the basic structure may sometimes be identified with the process of an actual physical cut of redundant members (e.g. of bars in a redundant pin-jointed framework).

However, the simple idea of a cut is not always applicable to continuous structures typical of aircraft. We discuss this point later but for the sake of linguistic simplicity continue to use the expression 'cut redundancy'.

Let the stress-system in the basic system be denoted by \( \sigma_0 \)

It must obviously be in equilibrium with the applied loads. We describe it as a 'statically equivalent stress system', thus drawing attention to the fact that in its determination only statical conditions enter. We find also in the basic structure the stress systems
due to
\[
X_1 = 1, X_2 = 1, \ldots, X_n = 1
\]
respectively. The systems \( \sigma_1, \sigma_2, \ldots, \sigma_n \) are obviously self-equilibrating. Since our structure is by definition linearly elastic the true stresses \( \sigma \) in the eenut original structure can be expressed as
\[
\sigma = \sigma_0 + \sum \sigma_i X_i \tag{176}
\]

Similar equations may be written down for stress resultants (forces or moments). Thus, the problem reduces to the determination of the \( X_i \)'s, which as already mentioned, need not be simple forces or moments but can be linear combinations of such (generalized forces).

The Equations in the unknown \( X_i \)'s. We define the following set of deformations in the basic system.

\( \delta_{1s} \)
Relative movement of ends of cut \( i \)th redundancy due to all external causes, i.e. loads, temperature changes, lack of fit, 'give' at supports, \( i = 1 \) to \( n \).

\( \delta_{ik} \)
Relative movement of cut \( k \)th (and \( k \)th) redundancy due to self-equilibrating load system \( X_k = 1 \). \( i \) and \( k \) take values \( 1 \) to \( n \). The \( \delta \)-coefficients are taken positive if the relative movements are in the positive direction of the \( X_i \)'s.

C. The Calculation of Redundant Structures by the Force-Method

We develop now a generalization of the Mueller-Breslau* technique for the calculation of linearly elastic redundant structures. Following our investigations under (A) we could easily formulate immediately the complete analysis in matrix notation. However, since the basic ideas do not appear to be generally known we think it preferable to develop them first in the more standard form.

The above method is, of course, the basis of the solution of partly homogeneous equations. A parallel relationship exists also in the 'force-method' investigation of structures. Thus, in this case, we have to find the flexibility \( F \) of a redundant structure in which we know the flexibility \( F \), of the basic structure. The analysis is given under C below.

![Image](image)

Fig. 28.—Singly redundant, pin-jointed framework. Contribution to \( \delta_{1s} \) from sinking or 'give' of support and excess lengths of bars.

The \( \delta \)'s are, of course, the influence or flexibility coefficients of the basic structures for the directions of the redundant forces. We use here the symbol \( \delta \) for these flexibilities since it is standard in the literature. To calculate the \( \delta \)'s we apply the unit load method of Section (A).

Thus, using again the abbreviations
\[
\sigma_0 = \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_n \tag{3}
\]
we find from Eq. (171a),
\[
\delta_{1s} = \int \sigma_0 \cdot (\epsilon_1 + \eta) dV \tag{177}
\]
\[
\delta_{ik} = \int \sigma_0 \cdot \epsilon_k dV = \int \sigma_0 \cdot \epsilon_k dV = \delta_{ik} \tag{178}
\]
where \( \epsilon_1, \epsilon_2, \ldots, \epsilon_n \) are the stresses and strains corresponding to \( X_1 = 1 \) and \( \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_n \) are the stresses and strains due to the applied loads. Eqs. (178) reproduce, of course, merely Eqs. (84) for the flexibility coefficients. The total initial strains \( \eta \) imposed upon the basic system may be separated into thermal and other strains
\[
\eta_{1s} = \eta_{1s} \tag{179}
\]
where \( \eta_{1s}, \eta_{1s}, \ldots, \eta_{1s} \) are initial strains due to say lack of fit, 'give' at the supports. The effect of the latter upon \( \eta_{1s} \) is best considered separately and expressed in terms of the imposed changes of length (rotations) and 'gives'. Consider, for example, the singly redundant framework of Fig. 28 and assume that the manufactured length of the bars exceeds the correct length \( l \) by \( \Delta l \). Let also each bar be subjected to a different thermal straining \( \alpha \). We assume furthermore that the intermediate support gives or sinks by the amount \( \Delta \). As redundancy we select the force \( X_1 \) in the bar (1, 2) and denote by \( N_1 \) and \( N_2 \) the tension forces in the bars of the basic system due to the applied loads and \( X_1 = 1 \) respectively. The leading case \( X_1 = 1 \), with the corresponding force applied to support C by the structure is shown in Fig. 28.

We find immediately
\[
\delta_{1s} = \sum \varepsilon_{1s} \tag{180}
\]

* See also footnote, p. 17.
Applying now the unit load method to the state $X_1 = 1$ and the true total strains $(e_r + \gamma)$ and displacements in the basic system we derive (see Fig. (28) and Eq. (71a)).

$$1 - \gamma = \sum N_i \varepsilon_i \frac{\sigma_l}{\varepsilon_l} + \sum N_i \varepsilon_i \frac{\sigma_l}{\varepsilon_l} + \Delta l \Delta \xi = \frac{h}{\alpha} \left( 1 + \frac{a}{b} \right) \Delta \quad \cdots \quad (181)$$

Naturally, we can alternatively deduce by kinematical reasoning the contribution to $\gamma$ of the initial elongations $\Delta \xi$ and the 'give' $\Delta l$. However, the unit load method yields the results much more conveniently and systematically.

More general formulae for the $\delta$-coefficients are given further below. The condition of consistent deformations at the cut ends of the n redundancies in the actual structure or application of the unit load method yields the following equations in the $n$ unknown $X$: 

$$\delta_{11} X_1 + \delta_{12} X_2 + \cdots + \delta_{1m} X_m + \cdots + \delta_{n1} X_1 + \cdots + \delta_{nm} X_m = 0 \quad \cdots \quad (182)$$

The solution of these equations determines the $X$ and hence also the total stress distribution after substitution into Eqs. (176).

To solve Eqs. (182) by elimination* is particularly simple when they are of the three-moment or five moment type; see for example the tube analysis in Section 9(b). In general, however, all unknowns may appear in each equation and we need a systematic and mnemonic method for the determination of the $X$. The most convenient method for this purpose is the shortened elimination process of Gauss† (known also as Gaussian algorithm). This method is so well known that we need not discuss its formulation in the present 'long-hand' notation but may use immediately the matrix notation. Accordingly, we write the system of Eqs. (182a) in the form

$$b = a X \quad \cdots \quad (186)$$

where $b$ is a square matrix and $Z$ is a column matrix. Thus,

$$a = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad \cdots \quad (187)$$

To eliminate in the first column of $a$ all its elements but $a_{11}$ and to reduce the latter to 1 we premultiply $a$ with the matrix

$$M_1 = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ -a_{21}/a_{11} & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ -a_{n1}/a_{11} & 0 & \cdots & 1 \end{bmatrix} \quad \cdots \quad (188)$$

$$T_0 = \begin{bmatrix} t_{11} & 0 & \cdots & 0 \\ t_{12} & t_{22} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ t_{1n} & t_{2n} & \cdots & t_{nn} \end{bmatrix} \quad \cdots \quad (184)$$

The solution of Eqs. (183) and (184) is straightforward by substitution, starting from the first of Eqs. (183) or in matrix language it is simple to find the inverse $T^{-1}$ of $T$ and write

$$X = T^{-1} T_0 \quad \cdots \quad (185)$$

For the automatic computation techniques now available and, in particular, for the punched card machines it is usually preferable to modify slightly the Gaussian elimination process and obtain directly the inverse of $D$ and hence also the column of $X$. This method is known as the Jordan technique and we restrict here our discussion to this process.

---

* Naturally we may also use iteration techniques but such methods are not discussed in this series of papers.

We obtain

\[
\begin{bmatrix}
1 & 0 & c_{13} & \cdots & c_{1n} \\
0 & 1 & c_{23} & \cdots & c_{2n} \\
& & & \ddots & \vdots \\
& & & & c_{n3}
\end{bmatrix}
\]

\[\epsilon = \begin{bmatrix}
c_{13} \\
c_{23} \\
\vdots \\
c_{n3}
\end{bmatrix}
\]

(1876)

where

\[
c_{3i} = b_{3i}/b_{33} \quad \text{and} \quad c_{1i} = a_{1i} \cdot b_{3i}/b_{33} \quad \text{for} \quad i \neq 2
\]

(189a)

The procedure will now be clear. Thus, at the \((g-1)\)th elimination step we obtain a matrix \(G\) of the form

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 & h_{1n} \\
0 & 1 & \cdots & 0 & h_{2n} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & h_{n-1, n-1} \\
0 & \cdots & \cdots & 0 & 1
\end{bmatrix}
\]

(187c)

where \(I\) is the unit matrix with \(f\) columns and \(O\) is a zero matrix with \(f\) columns and \((n-f)\) rows. \(G\) is a rectangular \(n \times (n-f)\) matrix. For the next step, i.e., to obtain the \(h\) matrix, we premultiply with \(M_s\), which has a \(g\)th column

\[
\begin{bmatrix}
-\frac{b_{31}}{b_{33}} & \cdots & -\frac{b_{3n}}{b_{33}} & \cdots & -\frac{b_{3f}}{b_{33}}
\end{bmatrix}
\]

(1886)

and otherwise unit diagonal elements and zero cross-elements. The resulting \(h\)-matrix is of the form

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 & h_{1n} \\
0 & 1 & \cdots & 0 & h_{2n} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & h_{n-1, n-1} \\
0 & \cdots & \cdots & 0 & 1
\end{bmatrix}
\]

(187d)

where

\[
h_{km} = \frac{g_{km}}{g_{3m}} \quad \text{and} \quad h_{km} = \frac{g_{km} - g_{kn}g_{kn}}{g_{kn}} \quad \text{for} \quad k \neq m
\]

(189b)

and so on until the last premultiplication with

\[
M_s = \begin{bmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{bmatrix}
\]

(190)

From which we find our unknowns

\[
a^1 = M_s^{-1} \cdot a
\]

(191)

In practice it is usually preferable not to determine \(a^1\) explicitly but to perform first the multiplication \(M_s \cdot Z\) and continue then premultiplying to find \(X\) directly.

It is apparent that if in the above procedure any \(h_{ak}\) becomes zero the elimination process cannot be continued. An interchange of rows is indicated but this is obviously inconvenient for automatic computation. Clearly, if

\[
a_{ii} > a_{ik}
\]

no \(h_{ik}\) can vanish. Moreover, the condition

\[
a_{ii} > a_{ik}
\]

(193)

\* As a matter of interest we point out that the actual operations on the digital computer to obtain \(a^1\) do not follow exactly the typical matrix multiplication rules as implied by formula (192).

Fig. 29.—Continuous beam. Good and bad choices of basic system and redundancies

It is necessary to avoid serious accumulation of round-off errors in the most important digits. For if \(a_{ii} < a_{ik}\) the limited number of digits of the machine is not sufficient to ensure reliable computations. Thus, requirement (193) is seen to be essential for well conditioned equations.

Our Eqs. (186) are also ill-conditioned if two or more columns are nearly linearly dependent, e.g.,

\[
(a_1, a_2, \ldots, a_i, \ldots, a_{uk}) \approx (a_1, a_2, \ldots, a_{vk}, \ldots, a_{uk})
\]

(194)

The one diagonal element \(g_{33}\) will inevitably become very small and serious errors will again arise. Naturally, in actual structural problems two columns could never be exactly linearly dependent for otherwise this would indicate that we have underestimated by one our number of redundancies. Nevertheless, a bad choice of redundant forces or moments may give an approximate linear dependence which would yield a result grossly in error. If it is found that our initial choice of redundancies leads to an ill-conditioned set of equations then we can always obtain more suitable equations by introducing as unknowns appropriate linear combinations \(Y\) of the initial unknowns \(X\). Such a transformation may be represented always by

\[
X = BY
\]

(195)

where \(B\) is a non-singular square matrix \(n \times n\). If \(X\) are initially single forces or moments then \(Y\) represents groups of forces or generalized forces. Such groups of forces were first introduced by Mueller-Breslau* guided by pure physical reasoning and this is still the best method of finding them. The transformation (195) may be introduced directly into Eqs. (186a) yielding

\[
\begin{align*}
BDY &= D_0 = 0 \\
DY + D &= 0
\end{align*}
\]

(196)

where \(D\) is determined by matrix multiplication from

\[
D = DB
\]

(196a)

Physically, Eqs. (196) express the compatibility conditions at the cuts of the original unknowns \(X\) in terms of the new unknowns \(Y\). If the transformation matrix \(B\) is unsymmetrical then the resulting matrix will also be unsymmetrical. Although, in general, the simple substitution of (195) into (186) can lead to a slight improvement of ill-conditioned equations the effect is usually small. The next obvious step is to express the compatibility condition (186) in terms of the generalized displacements at the cuts corresponding to the generalized forces \(Y\) of (195). Following our discussion on generalized displacements and flexibilities on p. 19 the generalized compatibility equations are derived by premultiplying Eqs. (196) by \(B^T\) as

\[
BB^T + BD_0 = 0
\]

(197)

where

\[
D_0 = BDB^T
\]

(197a)

It is evident that the column matrix \(D_{0k}\) and the symmetrical matrix \(D_0\) would be obtained directly by selecting \(ab\) initio the generalized forces \(Y\) as redundancies and deriving the corresponding \(b_{ik}\) and \(d_{ik}\) from the standard formulae given.

In many cases it is best, when the equations are ill-conditioned, to select a different basic system and corresponding redundancies. Naturally, the latter are again statically related by a transformation matrix \(B\) with any previous choice \(X\) of redundancies.

We illustrate the above considerations on ill-conditioned equations on a very simple structure. Fig. (29) shows a uniform, continuous beam of three equal spans simply supported at A, B, C, D—a twice redundant system. We discuss four alternative choices for the two redundancies.

(a) \( X_1 \) and \( X_2 \) are taken as the reactions at the supports A and B. The D matrix for this system is

\[
\mathbf{D} = \begin{bmatrix}
\delta_{11} & \delta_{12} \\
\delta_{21} & \delta_{22}
\end{bmatrix} = \frac{a^4 \phi}{6} \begin{bmatrix}
24 & 9 \\
9 & 4
\end{bmatrix}
\]

A remarkably bad choice since \( \delta_{12} \gg \delta_{22} \).

(b) \( X_1, X_2 \) taken as the reactions at the intermediate supports B and C.

Then,

\[
\mathbf{D} = \frac{a^4 \phi}{18} \begin{bmatrix}
8 & 7 \\
7 & 8
\end{bmatrix}
\]

Still a bad choice since all \( \delta \)'s are of the same order of magnitude.

(c) \( Y_1 \) and \( Y_2 \) are generalized redundancies formed from system (b) by the transformation matrix

\[
\mathbf{B} = \begin{bmatrix}
1 & -0.5 \\
-0.5 & 1
\end{bmatrix}
\]

The D matrix may be obtained either directly or from Eq. (197a) and is

\[
\mathbf{D} = \frac{a^4 \phi}{24} \begin{bmatrix}
4 & 1 \\
1 & 4
\end{bmatrix}
\]

The improvement over (b) and (a) is immediately apparent.

(d) \( X_1 \) and \( X_2 \) are the bending moments at supports B and C. This choice of unknowns is statically identical to \( Y_1 \) and \( Y_2 \) of (c), but the basic system is different. The D matrix is

\[
\mathbf{D} = \frac{a^4 \phi}{12} \begin{bmatrix}
4 & 1 \\
1 & 4
\end{bmatrix}
\]

and clearly a scalar multiple of the D matrix of (c).

The final system (d) is recognized as a particular case of the well-known Three-Moment Equation of Clapeyron. Since the basic system approximates more closely to the actual system than that of (c) it is clearly the most suitable choice of all.

The differences between the above four systems become even more pronounced when the number of spans is increased. Moreover, the \( \delta_{ij} \) coefficients of choices (a) and (b) tend, for a large number of spans, to become linearly dependent.

This discussion shows how important the choice of the redundant forces is for the convenient numerical solution of a problem. An extreme case of simplicity is achieved if in all equations only one unknown appears. The particular set of redundancies \( Y_1, Y_2, Y_3, \ldots, Y_n \), for which this condition is satisfied is called orthogonal. Then all but the corresponding direct influence coefficients are zero, i.e.

\[
\delta_{ik} = 0 \quad \text{if} \quad i \neq k
\]

where we introduce the symbol \( \delta \) for \( \delta \) to emphasize the special nature of this system. Eqs. (182) take now the simple form,

\[
\delta_{ij} Y_j + \delta_{ii} Y_i = 0 \quad \text{for} \quad i = 1 \text{ to } n
\]

and hence

\[
Y_i = -\frac{\delta_{ii}}{\delta_{ij}} Y_j \quad \text{for} \quad i = 1 \text{ to } n
\]

This system \( Y \) may always be obtained by a particular linear transformation

\[
\mathbf{X} = \mathbf{B}_r Y
\]

However, the computations involved in determining \( \mathbf{B}_r \) are usually more laborious than the direct solution of Eqs. (182) if these are well conditioned. Nevertheless there are structures in which it is simple and hence advantageous to find the orthogonal set of redundancies. This is particularly so if physical and not mathematical considerations indicate how to find them.

For example, this is so in arches and singly connected rings where, if

we restrict ourselves to bending deformations and assume the E.T.B. to hold, an orthogonal set of redundancies is obtained by referring them to the principal axes \( (C_\theta, C_\phi) \) of the ring neutral axis weighted with the ring flexibility \( \phi = 1/EI \) (see Fig. 30). The origin \( O \) of this system which is, of course, the centroid of the elastically weighted ring is known as the elastic centre. The transformation matrix \( \mathbf{B}_r \) relating the orthogonal set of redundancies \( Y \) to a set \( \mathbf{X} \) consisting of bending moment \( X_1 \), normal force \( X_2 \) and shear force \( X_3 \) at some point is in the notation of Fig. 30,

\[
\mathbf{B}_r = \begin{bmatrix}
\frac{\partial Y_1}{\partial x} & \frac{\partial Y_2}{\partial x} & \frac{\partial Y_3}{\partial x} \\
0 & \cos \theta & -\sin \theta \\
0 & -\sin \theta & -\cos \theta
\end{bmatrix}
\]

In practice it is nearly always worthwhile to find the elastic centre and eliminate two of the cross-flexibilities but determination of the principal axes, unless possible by inspection, is not usually worth the trouble. A further point is that the elastic centre concept is still valid if deformations due to normal and shear forces are included whereas the principal axes requirement becomes more complicated.

Interestingly enough this solution was first given by Mohr\(^*\) more than seventy years ago, but it appears not to be universally known, for otherwise it would not have been necessary to rediscover it so many times. A more recent derivation and application of an orthogonal set of redundancies (in general infinite) is the system of self-equilibrating eigenloads developed by Argyris and Dunne\(^†\) for their general theory of tubes in bending and torsion.

Particular forms of the \( \delta_{kk} \) and \( \delta_{ij} \) coefficients

We return now to Eqs. (179) and (178) for the \( \delta \)-coefficients and give, following our expressions (180) and (181) for a pin-jointed framework, some further explicit formulae for more complex structures.\(^\ddagger\)

**Stiff-jointed plane framework.** We assume the Engineers' theory of bending stresses to be true and introduce the special notation:

\( N_0, S_0, M_0 \), normal force, shear force, bending moment in basic system due to applied loads.

\( N_i, S_i, M_i \), normal force, shear force, bending moment in basic system due to \( X_i = 1 \) where \( i = 1 \) to \( n \).

\( \theta \), coordinate along axis of beam.

\( \Delta \theta/h \), temperature at neutral line of cross-section.

\( \theta \), temperature gradient across depth \( \Delta \theta \) of beam; positive if giving rise to thermal bending strain of the same sign as that due to positive bending moment \( M \).


Fig. 31. $\delta_{s}$, due to initial strains in rigid jointed frame and manufacturing errors and "give" at supports

$\delta_{s} = \frac{1}{E} \left[ N \alpha + M \delta \right] + \left[ \frac{S}{G} \delta \right] + \left[ \frac{M_{s}}{E} \delta \right] \frac{\Delta}{\Delta} \quad \text{(203)}$

Also from Eq. (177) or directly from the unit load method,

$\delta_{s} = \frac{N_{s}}{E} \Delta + \int N_{s} \theta \delta \frac{\Delta}{\Delta} \quad \text{(204)}$

For pin-jointed frameworks we omit the terms involving $M$ and $S$. On the other hand in stiff-jointed frameworks we can, in general, omit the terms involving $S$ and often also the terms in $N$. Fig. (31) illustrates on a twice redundant system how the contribution of the prescribed displacements $\Delta$ to $\delta_{s}$ is calculated. $X_{1}$ and $X_{2}$ are the chosen redundancies and $\Delta_{1}$, $\Delta_{2}$, $\Delta_{3}$, $\Delta_{4}$ are linear or angular imposed displacements arising from errors in the manufacture of the frame.

Two-dimensional stress distribution

We restrict ourselves here to a presentation in cartesian co-ordinates $x, y$ but the formulae are not restricted to stress distributions in flat plates. They are applicable to stress-states in any curved membrane which, by definition, does not allow for any variation of the stress over the thickness. Hence, $x, y$ are, in general, orthogonal curvilinear co-ordinates; for example, in a cylindrical membrane $y$ may be measured along the generator and $x$ along the periphery of the cross-section.

The following special notation applies.

$\Delta_{s}, \Delta_{t}$: prescribed direct (n) and tangential (t) relative displacements either inside the membrane (lack of fit of parts, slip of rivets) or at the boundaries ('give' at supports).

$\xi$: distance along part which experiences $\Delta_{s}$ and/or $\Delta_{t}$.

$\Delta_{s}$, $\Delta_{t}$: prescribed shearing deformation due to incorrect manufacture.

$C_{n}$: shear flow, due to $X_{1}=1$, acting on the element which experiences the $\Delta$, in the direction of this $\Delta$.

$C_{s}$: prescribed shear deflection due to incorrect manufacture.

$C_{n}$: shear flow, due to $X_{1}=1$, acting on the element which experiences the $\Delta$, in the direction of this $\Delta$.

$C_{s}$: prescribed shear deflection due to incorrect manufacture.

$C_{n}$: shear flow, due to $X_{1}=1$, acting on the element which experiences the $\Delta$, in the direction of this $\Delta$.

$C_{s}$: prescribed shear deflection due to incorrect manufacture.

$C_{n}$: shear flow, due to $X_{1}=1$, acting on the element which experiences the $\Delta$, in the direction of this $\Delta$.

$C_{s}$: prescribed shear deflection due to incorrect manufacture.

$C_{n}$: shear flow, due to $X_{1}=1$, acting on the element which experiences the $\Delta$, in the direction of this $\Delta$.

$C_{s}$: prescribed shear deflection due to incorrect manufacture.

$C_{n}$: shear flow, due to $X_{1}=1$, acting on the element which experiences the $\Delta$, in the direction of this $\Delta$.

$C_{s}$: prescribed shear deflection due to incorrect manufacture.

$C_{n}$: shear flow, due to $X_{1}=1$, acting on the element which experiences the $\Delta$, in the direction of this $\Delta$.

$C_{s}$: prescribed shear deflection due to incorrect manufacture.

$C_{n}$: shear flow, due to $X_{1}=1$, acting on the element which experiences the $\Delta$, in the direction of this $\Delta$.

$C_{s}$: prescribed shear deflection due to incorrect manufacture.

$C_{n}$: shear flow, due to $X_{1}=1$, acting on the element which experiences the $\Delta$, in the direction of this $\Delta$.

$C_{s}$: prescribed shear deflection due to incorrect manufacture.

$C_{n}$: shear flow, due to $X_{1}=1$, acting on the element which experiences the $\Delta$, in the direction of this $\Delta$.

$C_{s}$: prescribed shear deflection due to incorrect manufacture.

$C_{n}$: shear flow, due to $X_{1}=1$, acting on the element which experiences the $\Delta$, in the direction of this $\Delta$. 

Thickness.

We deduce from Eq. (178),

$\delta_{i} = \frac{1}{E} \int \left[ \sigma_{xx} \theta_{xx} + \sigma_{yy} \theta_{yy} + \sigma_{xy} \theta_{xy} \right] + 2(1+\nu) \sigma_{xy} \theta_{xy} \right] \ dx \ dy$

Also from Eq. (177),

$\delta_{i} = \frac{1}{E} \int \left[ \sigma_{xx} \theta_{xx} + \sigma_{yy} \theta_{yy} + \sigma_{xy} \theta_{xy} \right] + 2(1+\nu) \sigma_{xy} \theta_{xy} \right] \ dx \ dy$

The immediate application of the above formula to major aircraft components like wings and fuselages. Their matrix formulation is discussed further below.

Fig. (32) illustrates how on a thirteenth redundant beam with shear carrying beam the contribution of a prescribed displacement $\Delta$ to $\delta_{i}$ is calculated. $\Delta$ is in any case an initial shear displacement of a panel due to error of manufacture.

It was assumed in all our above considerations that the basic or cut system is statically determinate. However, nothing in the theory so far given restricts us to such a choice. We can select in a structure with a total number of redundancies $n$ a statically indeterminate basic system with $(n-r)\times 3$ redundancies by cutting only $r$ redundant members. Equations of the type (182) can then be written down for the cut $r$ unknowns, the corresponding $\delta_{s}$-coefficients being still defined as in our previous analysis in the basic system. In fact, to calculate the $\delta_{s}$ we may apply again Eqs. (178). Similarly, for $\delta_{t}$ we may use Eq. (177) if we substitute $\epsilon_{s}$ for $\epsilon_{t}$ where $\epsilon_{s}$ is the true strain in the basic system due to the prescribed initial strains. This modification is necesssary since the basic system is now redundant and the imposed initial strains $\epsilon_{t}$ are not free to develop. However, both formulae for $\delta_{s}$ and $\delta_{t}$ may be simplified considerably if we remember that in the unit load method from which they derive (see Eq. (71a)) only the strains must be the true ones for the system considered—in the present case the redundant basic system. The stresses corresponding to the unit load may be any suitable statically equivalent stress system and may hence
be found in the simplest statically determinate system. Thus, if we introduce
the notation
\[ \delta = \text{stably equivalent stress system in redundant basic system due to } X_0 \]
we may write,
\[ \delta_{ikl} = \int \sigma \varepsilon_{ikl} dV \]  \hspace{2cm} (177a)
\[ \delta_{ijkl} - \int \sigma \varepsilon_{ijkl} dV = \int \sigma \varepsilon_{ikl} dV = \delta_{ik} \]  \hspace{2cm} (178a)

The introduction of \( \delta \) instead of \( \sigma \) in Eqs. (177) and (178) may shorten the analysis greatly. Naturally, Eqs. (178a) are again identical with formulae (93).

The above method presumes that the strains \( \varepsilon \) and \( \eta \) in the redundant basic system are known. Such information may be available either from previous calculations or the literature. Alternatively, we may have to analyse first the basic system for the external loads (and/or imposed strains) and the \( r \) \( X_0 \) by the method given previously. From a mathematical point of view the selection of a redundant basic system means that we solve the problem of \( n \) equations with \( n \) unknowns in two steps involving respectively the solution of \( r \) equations with \( r \) unknowns and \((n-r)\) equations with \((n-r)\) unknowns. This method is particularly useful if we have available information on the stress distribution of the redundant basic system or if the number \( n \) is very high.

Consider, for example, Fig. (33) showing a fuselage ring with transverse beam under uniform load \( p \). The loading is equilibrated by tangential shear flows \( \varepsilon \) supplied by the fuselage to the ring. The structure is now twice redundant and as redundancies we select the two groups \( X_0 \), \( X_0 \), \( X_0 \), \( X_0 \), \( X_0 \) at the intersection of the axis of symmetry with the upper part of the ring and the transverse beam. Due to symmetry of loading and structure
\[ X_0 = X_0 = 0 \]
and hence the problem reduces to finding the remaining four redundancies. We may solve the system by direct application of Eqs. (182), which in the present case take the form,
\[ \begin{align*}
\delta_{01} X_0 + \delta_{02} X_0 + \delta_{03} X_0 + \delta_{04} X_0 &= 0 \\
\delta_{12} X_0 + \delta_{13} X_0 + \delta_{14} X_0 &= 0 \\
\delta_{23} X_0 + \delta_{24} X_0 &= 0 \\
\delta_{34} X_0 &= 0
\end{align*} \]  \hspace{2cm} (207)

where the \( \delta_{ik} \) are calculated with formulae (203) and (204) in which the integrals extend over ring and transverse beam. Note that if the unknowns \( X_0 \), \( X_0 \), \( X_0 \), \( X_0 \) are referred to the elastic centre of the ring the coefficients \( \delta_{ik} \) vanishes. Having solved Eqs. (207) we find \( N \), \( S \), \( M \) in the actual structure from
\[ \begin{align*}
N &= N_0 - N_0 + N_0 + N_0 - N_0 \\
S &= S_0 + S_0 - S_0 + S_0 - S_0 \\
M &= M_0 - M_0 + M_0 - M_0 + M_0
\end{align*} \]  \hspace{2cm} (208)

The use of a redundant basic system arises continuously in wing theory. Thus, following Eberholz and Köhler and Argyris and Dunnett it is customary in wing analysis to express the actual stresses in the form,
\[ \sigma = \sigma_0 + \sigma \]  \hspace{2cm} (213)
where the stress system \( \sigma_0 \)—the choice of which is at our discretion—satisfies both the external and internal equilibrium condition and is therefore the stably equivalent stress system \( \sigma \), the self-equilibrating stress systems (in general, infinite in number), necessary to ensure external and internal compatibility. In our present terminology \( \sigma_0 \) is the basic stress system and \( \sigma \), the redundant stress systems which for practical purposes are approximated to a finite number. In fact,
\[ \sigma = \sigma_0 + \sigma_1 + \sigma_2 + \cdots + \sigma_N \]  \hspace{2cm} (214)

It is advantageous in the selection of the basic system \( \sigma_0 \) to try and satisfy the two, at times conflicting, requirements of simplicity and not too great difference from the exact system \( \sigma \). For it is obvious that small \( \sigma_0 \)-systems are highly desirable from both the theoretical and practical point of view. Now for wings with not too small an aspect ratio an excellent choice for the direct stresses of \( \sigma_0 \) is given by the Engineers' theory of bending for beams since it combines simplicity with reasonable accuracy. If the wing forms a single cell tube we deduce the shear stresses from the boom load gradients, the undetermined constant of integration being found from the overall torque equilibrium; thus, in this case the basic system \( \sigma_0 \) is statically determinate. If on the other hand the wing is an N-cell tube we see that whereas there is still only one torque equilibrium condition there are \( N \) undetermined constants of integration. To calculate them we must introduce conditions of deformation and those are the equality of rate of twist of all cells. Hence, our basic system is redundant, the degree of redundancy being \( N-1 \). The solution to this problem is reproduced in Example (a) of Section 9. General considerations on the calculation of the redundant self-equilibrating stress systems \( \sigma \) are given later in this Section in matrix form (see also Example (b) of Section 9). The example of the tube is useful also to illustrate another point. We stated on p. 27 that the basic structure is obtained by cutting redundant members but mentioned that in continuous structures the idea of a physical...
cut is not always applicable. Thus, in the case of the tube in the last paragraph, when obtaining the basic stresses we do not actually cut any redundant member but rather select the engineers' theory of bending direct stresses and the associated shear flows as statically equivalent to the applied bending moment. In general, all members are found to be load carrying. The stresses $\sigma$, only exceptionally satisfy the elastic compatibility conditions—for example, due to warping varying parallel to the axis of the tube and to rib deformability. We may give some physical reality to the basic structure in which $\sigma$, is true by releasing in the actual structure the warping restraints at every cross-section and by assuming the ribs as rigid; the former idealization does no doubt require a complicated mechanism for its realization. The idea of selecting $\sigma$, as any suitable statically equivalent stress system without reference to actual cuts may, of course, also be applied to frameworks.

The use of a redundant basic structure is important also from a further point of view. Consider the wing of Fig. (34) the main portion $\text{I}$ of which is swept and attached to some root structure $\text{II}$. It is assumed that the ribs of $\text{I}$ are taken perpendicular to a longitudinal axis approximately parallel to the spars. The necessity may arise of investigating alternative angles of sweep obtained merely by changing the root structure $\text{II}$. Thus, in Fig. (34) we show two alternative arrangements. In such instances it is obviously advantageous to have as much as possible of the stress analysis in common in the two alternative calculations. To this purpose we release at the junction of $\text{I}$ and $\text{II}$ all redundant forces or groups of forces $X_1$ to $X_s$ appearing there. The tube $\text{I}$ is then connected to tube $\text{II}$ by some statically determinate arrangement and this new structure is taken as the basic system. The scheme of the analysis is as follows. Analyse first tube $\text{I}$ for all external forces and also for $X_1=1$ to $X_s=1$ respectively. This investigation involves, of course, the solution of a highly redundant system. Irrespective, however, of the form tube $\text{II}$ takes the analysis of tube $\text{I}$ remains unaffected. Next we analyse tube $\text{II}$ for the external forces (which include the statically determinate reactions $P$ from tube $\text{I}$), and also for $X_1=1$ to $X_s=1$ respectively. Again this may involve the solution of a redundant problem. Finally we can write down equations of the type (182) for the unknowns $X_1=1$ to $X_s=1$ and note that the $\delta$-coefficients are in each case

$$\delta = \delta_I + \delta_{II} \quad \ldots \quad \ldots \quad \ldots \quad (215)$$

Hence if we change structure $\text{II}$ only $\delta_{II}$ but not $\delta_I$ is altered—an obvious advantage. The solution of Eqs. (182) yields ultimately the stress distribution in the actual wing.

Fig. 34.—Alternative root structures for swept-back wing. Redundant basic systems
HAVING discussed in the standard longhand notation the main ideas and methods for the calculation of redundant structures on the basis of forces as unknowns we now turn our attention to the matrix formulation of the analysis. Consider a system consisting of structural elements with a total number \( n \) of redundancies which may be forces (stresses), moments or any generalized forces. We select a basic system by 'cutting' a number \( r \) of redundancies where \( r < n \). Thus, the simple idea of a statically determinate basic system \((r=n)\) is but a particular case of our investigations.

The structure is assumed subjected to a system of \( m \) loads (generalized forces)

\[
\mathbf{R} = (R_1, R_2, \ldots, R_m)
\]

We denote by \( \mathbf{X} \) the column matrix of the \( r \) cut (unknown) redundancies,

\[
\mathbf{X} = (X_1, X_2, \ldots, X_r)
\]

(216)

The column matrix \( \mathbf{S} \) of the stresses and forces in the actual (un-cut) structure can always be written in the simple form

\[
\mathbf{S} = \mathbf{b}_1 \mathbf{R} + \mathbf{b}_2 \mathbf{X}
\]

(217)

where \( \mathbf{b}_1 \) and \( \mathbf{b}_2 \) are rectangular matrices with \( m \) and \( r \) columns respectively and the same number of rows as \( \mathbf{S} \). In fact, the elements of \( \mathbf{b}_1 \) or \( \mathbf{b}_2 \) correspond to the stresses \( \sigma_i \) given previously (see Eq. (176)). If the basic system is statically determinate the two matrices \( \mathbf{b}_1 \) and \( \mathbf{b}_2 \) are found merely by static reasoning.

For a redundant basic system we may obtain the necessary data either by analysing it first for the loads \( \mathbf{R} \) and the \( r \) forces \( X_i = 1 \) or in many cases by using existing standard information.

### Additional Notation

- \( \mathbf{X}, \mathbf{Y}, \mathbf{Z} \): column matrices of \( X_i, Y_i, Z_i \) respectively.
- \( \mathbf{H} \): column matrix of initial strains (displacements).
- \( \mathbf{C} \): rectangular matrix of forces (moments) \( C_i \) (see p. 31).
- \( B, C \): areas of actual longitudinal and transverse flanges.
- \( B_{\text{e}}, C_{\text{e}} \): areas of effective longitudinal and transverse flanges.
- \( l \): length of longitudinal flanges between nodal points.
- \( d \): length of transverse flanges between nodal points.
- \( h \): height of web.
- \( \Omega \): area enclosed by cell.
- \( \Phi, \Phi_{\text{e}} \): areas of cover panel and web panel respectively.
- \( L_{\text{n}}, L_{\text{h}} \): 2 × 2 matrices.
- \( \mathbf{L} = [L_{\text{n}}, L_{\text{h}}] \): effective longitudinal and transverse flange load respectively.
- \( k_n, k_s, k_t \): Partial stiffnesses due to shear strains in sheet, direct strains in sheet and direct strains in flanges respectively.
- \( \mathbf{U}, \mathbf{W} \): column matrix of kinematically indeterminate joint displacements \( U, W \) respectively.
- \( r \): column matrix of strain of elements due to unit \( r \)'s and \( U = 0 \).
- \( r_i, \bar{r}_i \): column matrix of actual and kinematically equivalent (virtual) strain of elements due to unit \( U_i \) when \( r = 0 \) respectively.
- \( C = \bar{r}_i k_i \bar{r} = \bar{a}_i k_i \bar{a}_i \)
- \( C_0 = \bar{a}_i k_i a_r \bar{a}_i = \bar{a}_i k_i a_r \)
- \( J \): column matrix of initial stresses.

---

Example for the \( \mathbf{b}_1 \) and \( \mathbf{b}_2 \) matrices.

FIG. (35a) shows a five times redundant structure, assumed symmetrical, subjected to the loads \( R_i \) and \( R_{\text{e}} \). Due to symmetry of loading and structure the system is effectively three times redundant. For the basic system we select the statically determinate structure of FIG. (35b). The \( \mathbf{b}_1 \) and \( \mathbf{b}_2 \) matrices for half the structure including the central vertical member (11) are found easily as

\[
\mathbf{b}_1' = \begin{bmatrix}
0 & -a/h & a/h & a/h & 0 & 0 & -d/h & 0 & 0 & 1 & 0 \\
0 & -a/2h & a/2h & a/h & 0 & 0 & -d/2h & -d/2h & 0 & 1/2 & 1 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11
\end{bmatrix}
\]

(218)

and

\[
\mathbf{b}_2' = \begin{bmatrix}
-a/d & 0 & -a/d & 0 & 1 & 0 & 1 & 0 & -h/d & -h/d & 0 \\
0 & -a/d & 0 & -a/d & 0 & 1 & 0 & 1 & 0 & -h/d & -2h/d \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11
\end{bmatrix}
\]

(219)

where the numbers under the columns refer to the numbered bars of FIG. 35b.

---

Figs. 35 (a and b).—Statically indeterminate pin-jointed framework. Illustration of \( \mathbf{b}_1 \) and \( \mathbf{b}_2 \) matrices.
Following our previous analysis of structures when the basic system is redundant we introduce now the matrix

\[ b_1 \]  

(220)

to denote any suitable statically equivalent stress (force) matrix corresponding to

\[ x = \begin{bmatrix} 1 & 1 & \ldots & 1 & \ldots & 1 \end{bmatrix} \]

Thus, \( b_1 \) corresponds to the stress system \( \delta_1 \) introduced before. When the basic system is statically determinate only one \( b_1 \) can be found,

\[ b_1 = b_1 \]  

(220a)

We define also by

\[ b_2 \]  

(221)

any suitable matrix statically equivalent to

\[ R = \begin{bmatrix} 1 & 1 & \ldots & 1 & \ldots & 1 \end{bmatrix} \]

\( b_2 \) may, in fact, be determined in a different statically determinate system from \( b_1 \).

Next we derive the matrix equation for compatibility of deformation in the actual structure. Denoting the relative displacements at the \( r \) cuts of the basic system due to loads \( R \) and the \( s \) redundancies \( X_i \) by \( v \), the compatibility condition is

\[ v = 0 \]  

(222)

where \( v \) is a column matrix with \( r \) elements. To express Eq. (222) in terms of \( R \) and \( X \) we note from Eq. (122) that the relative deformations \( v \) (these may be elongations of bars or flanges, shearing angles of plates), at the ends or boundaries of the \( s \) elements are,

\[ v = f_b \cdot R + f_b \cdot X \]

(223)

if the flexibility matrix of the \( s \) elements, is the partitioned diagonal matrix of Eq. (123). We find now \( v \), directly from the argument leading to Eq. (125) as

\[ v = b_1 \cdot R + f_b \cdot X \]

(224)

and hence

\[ b_1 \cdot f_b \cdot x + b_1 \cdot f_b \cdot R = 0 \]  

(225a)

These are the required equations in the \( r \) unknown \( X_i \) and are, in fact, equivalent to formulae (182). The symmetrical square matrix

\[ D = b_1 \cdot f_b \cdot \]

(to use the notation of Eq. (182)) is the flexibility matrix for the directions of the \( r \) unknown \( X_i \) in the basic system. Also in the notation of Eq. (182a)

\[ D = b_1 \cdot f_b \cdot R \]  

(225a)

Eqs. (224) are the most general formulation in matrix algebra of the equations for the \( r \) unknown \( X_i \) in a structure with a redundant basic system. Solving for \( X \) we find

\[ X = \left( b_1 \cdot f_b \right)^{-1} \cdot b_1 \cdot f_b \cdot R \]  

(226)

Substituting (226) in (217) we determine \( S \) soley as a function of the \( R \)'s. Thus,

\[ S = \left( b_1 \cdot b_1 \cdot b_1 \cdot f_b \right)^{-1} \cdot b_1 \cdot f_b \cdot R \]  

(227)

Comparing (227) with Eq. (121) we can write

\[ S = b \cdot R \]

where

\[ b = b_1 \cdot b_1 \cdot b_1 \cdot b_1 \cdot f_b \]  

(227a)

Naturally, it is always possible to substitute \( b_1 \) for \( b_1 \) in Eqs. (224) to (227a). However, the introduction of the statically determinate matrix \( b_1 \) when the basic system is redundant simplifies the calculations, often considerably.

We can apply now Eq. (227a) to derive the flexibility \( F \)

of the actual structure for the \( m \) points and directions of the applied loads. Eq. (126) gives

\[ F = b \]  

(126)

For \( F \) we may use

\[ b = b_1 \]  

or

\[ b = b_1 \cdot \]  

(126)

We obtain

\[ F = b_1 \cdot f_b \cdot b_1 \cdot b_1 \cdot f_b \]  

(229)

where

\[ F = b_1 \cdot f_b \cdot b_1 \cdot f_b \]  

(230)

is the flexibility of the basic system for the loads \( R \).

---

Fig. 36.—Doubly connected ring. Analysis with redundant basic system

Simple example of Eq. (224)

Consider the symmetrical fuselage ring with transverse beam and central load \( R \) shown in Fig. 36. As in page 32 we select as a basic system the structure with the beam cut at the centre. For the components \( s \) of the basic system we take the two statically determinate cantilever beams and the closed ring. It is assumed that we know the stress distribution and hence the flexibilities due to the pairs of loads applied to the ring (fig. 36). The basic system is thrice redundant but due to symmetry \( X_i = 0 \).

The load transformation matrices \( b_2 \) and \( b_1 \) are

\[ b_2 = \begin{bmatrix} b_{2h} \\ b_{2h} \end{bmatrix} \]  

(231)

where

\[ b_{2h} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \]

and

\[ b_1 = \begin{bmatrix} b_{1h} \\ b_{1h} \end{bmatrix} \]  

(232)

where

\[ b_{1h} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

\[ b_{1h} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

Note that in the present case \( b_1 = b_1 \) since the loadings on the two elements of the structure are statically determinate.

The flexibilities of the elements for the forces and moments may be written as
It is now possible to write down the equation for \( \mathbf{X} \),

\[
\mathbf{b}_f^T \mathbf{f}_b \mathbf{X} + \mathbf{b}_f^T \mathbf{f}_b \mathbf{R} = \mathbf{0}
\]

Returning to the general argument we investigate now in a system with a total number \( n \) of redundancies the effects of initial strains e.g. those due to temperature rise, excess length of bars due to manufacturing errors, 'give' of foundation at supports. Assume that the column matrix of the relative displacements at the ends or boundaries of the \( s \) elements due to \( \eta \) is

\[
\mathbf{H}
\]

We assume further that the basic system is statically determinate; then the elements of \( \mathbf{H} \) are merely the integrated effect of the imposed \( \eta \). For example, in a pin-jointed framework subjected to temperature rise the elements of \( \mathbf{H} \) are

\[
\mathbf{H} = \pmatrix{ (a(\theta_1)_1, \\
(a(\theta_1)_2) } 
\]

If the bars are of an excess length \( \Delta \) due to inaccurate manufacture, then these form directly the elements of \( \mathbf{H} \). Indepedently of the nature of \( \mathbf{H} \), however, the corresponding relative displacements at the cut redundancies are simply

\[
\mathbf{b}_f^T \mathbf{H}
\]

and the equation for the \( n \) unknowns \( \mathbf{X} \) is

\[
\mathbf{b}_f^T \mathbf{f}_b \mathbf{X} + \mathbf{b}_f^T \mathbf{H} = \mathbf{0}
\]

Note that in the present case \( \mathbf{b}_f = \mathbf{b}_b \) since the basic system is taken to be statically determinate. When the deformations arise from \( p \) 'gives' \( \Delta \) at the foundations it is advantageous to express \( \mathbf{H} \) as

\[
\mathbf{H} = \pmatrix{ (\Delta_1, \\
\Delta_2) } 
\]

Then for \( \mathbf{b}_f \) we must substitute the matrix \( \mathbf{c} \) where

\[
\mathbf{c} = \pmatrix{ C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33} } 
\]

the \( n \) rows of which are the forces due to \( \mathbf{X} = \mathbf{1} \), applied by the structure to the foundations in the directions of the gives \( \Delta \). Eq. (236) becomes now,

\[
\mathbf{b}_f^T \mathbf{f}_b \mathbf{X} + \mathbf{c}^T \mathbf{H} = \mathbf{0}
\]

If we select a redundant basic system we cannot derive the elements of \( \mathbf{H} \) immediately from the prescribed initial strain since the latter are not free to develop in a redundant structure. In this case unless we have the necessary information from previous calculations we must first analyse the basic system by the method of the previous paragraph. Having found the column matrix \( \mathbf{H} \) the \( r \) equations in the \( r \) unknowns take the form

\[
\mathbf{b}_f^T \mathbf{f}_b \mathbf{X} + \mathbf{b}_f^T \mathbf{H} = \mathbf{0}
\]

where we may write \( \mathbf{b}_f \) for \( \mathbf{b}_b \) since the basic system is now redundant.

The systematic solution of (224) and related equations was discussed on page 28 but there are a few further points arising in practical calculations which are best investigated here. Thus, we mentioned on page 20 that it is often possible and justified to neglect certain parts of the flexibilities of the elements; for example, in a ring analysis we can usually ignore the direct and shear flexibility. This applies not only to the evaluation of the external flexibility \( \mathbf{F} \) but also to the determination of the internal redundancies \( \mathbf{X} \). We write now the \( \mathbf{D} \) and \( \mathbf{D}_f \) matrices in the split form

\[
\mathbf{D} = \mathbf{D}_s + \mathbf{D}_{s \alpha} \quad \mathbf{D}_f = \mathbf{D}_{s \alpha} + \mathbf{D}_{s \alpha} + \mathbf{D}_{\alpha \alpha}
\]

where the suffixes \( s \) and \( \alpha \) refer to the two flexibilities into which we separate the total flexibility of each element. An approximate solution \( \mathbf{X} \) to the unknown column \( \mathbf{X} \) is then obtained by ignoring the flexibility \( \alpha \). Then,

\[
\mathbf{D}_s \mathbf{X} + \mathbf{D}_{s \alpha} \mathbf{x} = \mathbf{0} \quad \text{or} \quad \mathbf{x} = -\mathbf{D}_{s \alpha}^{-1} \mathbf{D}_s \mathbf{X} 
\]

Occasionally we may require subsequently the correction \( \mathbf{x} \) to \( \mathbf{X} \), to find the true column \( \mathbf{X} \),

\[
\mathbf{X} = \mathbf{X}_s + \mathbf{x}
\]

Substituting (238) and (239a) into Eq. (224) we derive easily

\[
\mathbf{X}_s = -\mathbf{D}^{-1} (\mathbf{D}_{s \alpha} + \mathbf{D}_s \mathbf{X}_s)
\]

and since \( \mathbf{x} \) is usually small in comparison with \( \mathbf{X} \), approximate methods may often be used in the evaluation of the right hand side. This technique is immediately applicable to the correction of a completed stress analysis when subsequent small changes in the flexibilities of some elements are introduced by modifications to their cross-sectional dimensions.

When the number of equations is too large for performing the matrix operation on a digital computer then we may apply the following method which is basically identical with the idea of a redundant basic system. Assume that in a structure with a number \( n \) of redundancies we select first a basic system with \( t = n-r \) redundancies and that we write down the \( r \) equations in \( \mathbf{X}_r \) in the standard form (224),

\[
\mathbf{D} \mathbf{X} + \mathbf{D}_f \mathbf{X}_f = \mathbf{0}
\]

If the number \( r \) is still too large for handling by the digital computer we solve the problem in two steps. Eqs. (224) are first put in the partitioned form

\[
\pmatrix{ \mathbf{D}_f & \mathbf{D}_{III} \\
\mathbf{D}_{III} & \mathbf{D}_{II} } \pmatrix{ \mathbf{X}_l \\
\mathbf{X}_f } + \pmatrix{ \mathbf{D}_f \\
\mathbf{D}_{II} } = \mathbf{0}
\]

where the number of rows in the matrices with suffixes \( l \) and \( II \) is \( r \) and \( r-p \) respectively and \( \mathbf{D}_f, \mathbf{D}_{II} \) are square matrices. Eq. (240) gives

\[
\begin{align*}
\mathbf{D}_l \mathbf{X}_l + \mathbf{D}_{III} \mathbf{X}_f + \mathbf{D}_f \mathbf{X}_f &= \mathbf{0} \\
\mathbf{D}_{II} \mathbf{X}_f + \mathbf{D}_{III} \mathbf{X}_l + \mathbf{D}_{III} \mathbf{X}_f &= \mathbf{0}
\end{align*}
\]

We split next the column matrix \( \mathbf{X}_l \) into matrices

\[
\mathbf{X}_l = \mathbf{x} + \mathbf{y}
\]

where \( \mathbf{x} \) satisfies the equation

\[
\mathbf{D}_l \mathbf{x} + \mathbf{D}_{III} \mathbf{y} = \mathbf{0} \quad \text{or} \quad \mathbf{y} = -\mathbf{D}_{III}^{-1} \mathbf{D}_l \mathbf{x}
\]

Hence

\[
\mathbf{D}_{III} \mathbf{x} + \mathbf{D}_{III} \mathbf{y} = \mathbf{0} \quad \text{or} \quad \mathbf{y} = -\mathbf{D}_{III}^{-1} \mathbf{D}_{III} \mathbf{x}
\]

Substituting for \( \mathbf{x} \) and \( \mathbf{y} \) into the first of Eqs. (240) we find

\[
\begin{align*}
\mathbf{D}_l \mathbf{x} + \mathbf{D}_{III} (-\mathbf{D}_{III}^{-1} \mathbf{D}_f \mathbf{X}_f ) + \mathbf{D}_f \mathbf{x}_f &= \mathbf{0} \\
\mathbf{D}_{III} \mathbf{x} + \mathbf{D}_{III} (-\mathbf{D}_{III}^{-1} \mathbf{D}_{III} \mathbf{X}_f ) + \mathbf{D}_{III} \mathbf{x}_f &= \mathbf{0}
\end{align*}
\]

from which we deduce \( \mathbf{y} \) and hence \( \mathbf{X} \). Eqs. (243) are identical with the elastic compatibility equation for a basic system with \( n-p \) redundancies.

The matrix form (224) of the equation of compatibility is particularly suitable to illustrate the transformation (see p. 29),

\[
\mathbf{X} = \mathbf{B} \mathbf{y}
\]

when the equations are ill-conditioned. Thus, by substitution of (195) into (224) we find

\[
\mathbf{b}_f^T \mathbf{f}_b \mathbf{Y} + \mathbf{b}_f^T \mathbf{f}_b \mathbf{R} = \mathbf{0}
\]

and premultiplying by \( \mathbf{B}^T \)

\[
\mathbf{B}^T \mathbf{b}_f^T \mathbf{f}_b \mathbf{Y} + \mathbf{b}_f^T \mathbf{f}_b \mathbf{R} = \mathbf{0}
\]

or

\[
\mathbf{b}_f^T \mathbf{f}_b \mathbf{Y} + \mathbf{b}_f^T \mathbf{f}_b \mathbf{R} = \mathbf{0}
\]
where (see also Eqs. (196) and (197)),
\[ b_1 = b_2 = B \text{ and } b_3 = b_r = b \] (245)
are merely the matrices for the true and statically equivalent stress systems in the basic system due to
\[ Y = [1, 1, 1] \]
The form (244) of the equations of compatibility may, of course, be written down directly when starting \( ab \) initio with the group unknown \( Y \).

**Application to a typical aircraft structure**

We present now a detailed investigation of a type of system characteristic of aircraft wings. Consider to this purpose the structure shown in Fig. 37 which consists essentially of an orthogonal or nearly orthogonal grid of spars and ribs covered with sheet material. Longitudinal and transverse flanges may be placed at the intersections of spar and rib webs with the covers. In addition the covers may be stiffened with further longitudinal and/or transverse flanges. The cross-section is assumed arbitrary and the spars may be differently in plan view and elevation but the angle of taper \( \beta \) is taken to be so small that \( \cos 2\beta \approx 1 \) and \( \sin 2\beta \approx 2\beta \). The analysis is not restricted to structures with continuous rib and spar webs, covers and flanges and includes hence any kind of minor or major cut-out. The geometry considered excludes the effects of the direct stress is taken with ribs parallel to the line of flight. On the other hand, swept back wings with ribs perpendicular to the main wing axis can be analyzed by the present method as long as we are given the necessary information for the triangular root section. Dynamic effects may also be investigated by our theory as long as the grid of ribs and spars conforms to the geometry stipulated here. Naturally, many of the restrictions imposed limit the applicability of the method. Indeed we intend our analysis only as an exploratory and tentative first attack on the more general problem. We hope to return to this and similar points in later publications.

The problem of finding the stress distribution in the shell type of structure considered is strictly infinitely redundant. Hence it is necessary to introduce for practical calculations considerable simplifications. First we adopt the standard assumption in wing stress analysis of a membrane state of stress, i.e. we exclude any bending of covers and flanges normal to the surface of the wings. For very thin wings now coming into prominence this idealization is open to grave doubts and will not doubt have to be reconsidered in future. An essential characteristic of our theory is the assumption that the longitudinal and direct stresses vary linearly between the nodal points of a three-dimensional grid of lines traced on the wing cover. This system of lines should, in general, be at least as fine as the grid of spars and ribs whose intersection with the cover forms the best minimum set of grid lines. The latter grid will often be sufficiently close if we are dealing with a multi-web structure and ribs not at too great a distance. However, many instances occur where it is necessary to select additional nodal points between which the direct stress is taken to vary linearly. For example, we may choose points intermediate between spar webs on the rib stations if the spacing of the spars and the sheet thickness of the covers are large. Similarly, if the structure has few ribs we may have to introduce no transverse stations in order to reduce the spacing of the grid in the longitudinal direction. In either case, there need not be an actual longitudinal or transverse reinforcement along the new grid lines. We call the surface enclosed between two adjoining grid lines in the \( x \) and \( y \)-direction the \( \beta \)-plane and denote by \( \beta \) a set of the wing structure which lies between two cross-sections taken through adjoining grid lines running in the \( z \)-direction (see Fig. 37). The assumption of a linear direct stress distribution along the edges of an orthogonal and flat field yields from overall equilibrium conditions a parabolic shear flow distribution the latter param along the edge parallels neither the linear direct stress nor the parabolic shear flow variation are, in general, exact and violate the internal and boundary compatibility conditions of the field. This is not serious as long as we keep the spacing of the grid lines reasonably close. Moreover, we simplify further the problem by neglecting the quadratic and linear terms in the shear flow and considering it to be constant within each field. We note that for non-orthogonal grid lines (tapered structure) the uniform shear flow offers against the equilibrium conditions even if the direct stress is considered between adjoining nodal points. The errors induced by the assumption of uniform shear flow are, however, practically insignificant for the geometry of structure considered here when the nodal point distances are small.

If the direct stresses along the grid lines were known we could calculate the fraction of shear area to be added to the reinforcements to form the equivalent or effective flanges. This applies to the cover, spar webs and rib webs and yields an idealized structure in which the fields are only shear carrying. The direct stress carrying ability is concentrated in flanges; an assumption widely used in aircraft practice. Neglecting the Poisson's ratio effect and assuming the same material for flanges and sheet material cover, the fraction of shear cross-sectional area to be added to the flanges varies linearly between \( 1/6 \) and \( 1/2 \) if the fields are flat; the former value applies when the field is in pure bending in its own plane and the latter when it is under uniform stress. Since the stress distribution is unknown we can at best only estimate the effective areas of the flanges but may use an iteration process if the first guess proves too large. The latter procedure is really only dummy and lengthy and a direct method, obviating the guessing of flange areas would evidently be useful, in particular at the root or other structural or loading discontinuities where the stress distribution is more difficult to estimate and the Poisson's ratio effect more pronounced. Such a method is given here at the end of this subsection but at first we develop the theory under the assumption that the effective flanges areas are known and that they are constant between two adjoining nodal points. For the webs, when considering torque and lift loads, it is always sufficient to add 1/6 of the web cross-sectional area to the longitudinal and transverse flanges at the intersection of the spars and ribs with the cover.

We summarize now the main assumptions underlying the idealized structure selected for analysis. Thus, our system consists of an orthogonal or nearly orthogonal grid of ribs and spars and ribs with top and bottom covers. Effective flanges of constant area between adjoining nodal points and carrying only direct stresses are assumed placed along the grid lines in longitudinal and transverse directions. For the times being we assume that the flange areas are known. The direct stresses and hence also the flange loads are taken to vary linearly between nodal points. All sheet material for covers, spars and webs is assigned a purely shear carrying role and a constant thickness within each field. The angles of taper of spars in plan view and elevation are assumed to be small. The shear deformability of covers, ribs and spars is included \( ab \) initio in the analysis. For the stresses and loads in the various elements we adopt the strain convention illustrated in Fig. 38. Naturally, the idealizations and simplifications introduced are strictly only necessary for the calculation of the redundancies. The basic or statically equivalent stress system may and should preferably be determined in the (cut) actual structure.

**Degree of redundancy of idealized structure**

We proceed next to the enumeration of the redundancies in our idealized structure. In addition to the simplifications introduced previously we ignore here the bending stiffness of the flanges for displacements tangential to the wing surface. This is, no doubt, sufficiently accurate for the present exploratory analysis. The wing structure supported at the root and free at
the tip is assumed stiffened by ribs at least at the root and the tip. These ribs need not necessarily consist of a web with flanges but may take the form of a stiff-jointed frame or ring. However, independently of the design of the ribs we may always substitute an equivalent shear web with flanges. The wing structure is subdivided into a number of bays of which we show a typical intermediate one in Fig. (39). The cross-section at the junction nearer to the tip may be stiffened by a rib carried across some or all cells. Fig. (39) indicates also those longitudinal flanges which are continuous across the same junction. It should be noted that if there is a change of transverse slope of the cover at a longitudinal flange the latter must be connected to a spar web.

We use the following notation:

\[ \beta = \text{number of longitudinal effective flanges which are continuous across the junction, i.e. are not interrupted there.} \]

\[ N = \text{number of closed cells stiffened by ribs at the tip end of the bay.} \]

Then the number of redundancies arising from the geometry of the bay is

\[ n = \frac{\beta - 3 + (N - 1)}{} \tag{247} \]

Hence, in a tubular structure of the type shown in Fig. 37, free at the tip and either fully built-in at the root or with prescribed displacements there at all longitudinal flanges, the total number of redundancies is

\[ n = \sum_{\text{bays}} \left( \frac{\beta - 3 + (N - 1)}{} \right) \tag{247} \]

If certain of the flanges are not held at the root section the number of redundancies reduces accordingly. For example, if the root-section is at the aircraft centre line and the wing is subjected to anti-symmetrical loading the number of unknowns reduces by \( \frac{\beta - 3}{2} \), \( \beta \) being the number of longitudinal flanges at the root. The number in the square brackets in (247) can, of course, vary from bay to bay since effective flanges may be interrupted at such stations. Also the number \( N \) of stiffened cells may be made different in each bay by the addition or removal of spar webs. However, when \( \beta \) and \( N \) are the same in all bays and all the flanges are held at the root formula (247) becomes simply

\[ n = a^2 \tag{247a} \]

where \( a = \text{number of bays}. \) If the sheet cover is missing between two adjoining longitudinal flanges in a bay and the cut-out is not provided with a stiff-jointed closed frame to restore partially the lost shear stiffness of the sheet then the corresponding cell is open in this bay and by definition is not included in \( N \). Similarly, if there is no rib or equivalent frame in a cell at the section considered this cell is excluded from the numbering for \( N \). Note that spar webs need not be continuous throughout the length of the wing and may be discontinued at any junction. Formula (247) still remains valid.

If the cross-section is singly symmetrical the \( n \) redundancies of Eq. (247) split into two groups:

\[ n = \sum_{\text{bays}} \left[ \left( \frac{\beta - 3}{2} \right) + (N - 1) \right] \tag{248} \]

of which \( n_1 \) applies for the lift and torque loads and \( n_2 \) for the drag loads.

If all cells are closed, with the same number \( N \) in all bays and effective flanges are only placed at the corners of the cells, then

\[ \beta = 2(N + 1) \]

and from (247a) the total number of redundancies is

\[ n = a(3N - 2) \tag{249} \]

which formula again assumes that all the flanges are held at the root.

Of considerable importance in modern aircraft structural practice are the multispar systems with few, often only two, end ribs. A typical wing of the latter type is shown in Fig. (40). To analyse this structure we subdivide it into a number of bays whose length should not exceed five times the spar pitch. Effective flanges will by virtue of our idealization process be acting at the junction of these bays although no ribs are provided there. For such a system the number \( n \) of redundancies when there are no cut-outs in the sheet, when all spars are continuous for the full length of the wing and all flanges are held at the root, is given by

\[ n = (N - 1) + 1 + a(\beta - 4) + N + a(\beta - 4) \tag{250} \]

The last system to be considered is a flat panel which is of special importance for diffusion investigations (see Fig. 41). It is assumed built-in at \( z = 0 \) or held with prescribed displacements and free at the other three edges. Here the number \( n \) of redundancies when there are no unstiffened cut-outs is simply

\[ n = \sum_{\text{bays}} (\beta - 2) \tag{251} \]

where \( \beta \) is defined as in the case of the wing. When \( M \) fields are removed without being replaced by stiff-jointed frames the number of redundancies reduces by \( M \).

The next step in our investigations is the discussion of suitable self-equilibrating systems which may be chosen as redundancies. Consider first the simple case of a rectangular flat panel shown in Fig. 41. For the redundancies we may select \( N \) systems of the type \( X = 1 \) illustrated in the figure. All information as to flange loads and shear flows is given there. The corresponding equations (182) or (224) for the unknown \( X \) are easily seen to be reasonably well conditioned. Naturally, we can further improve the conditioning by introducing group loads

\[ X = B \]

where \( B \) is a suitable square matrix. We do not enter at this stage into the choice of \( B \) but hope to discuss these points in Part III. When the panel is symmetrical about its middle line it is preferable to combine the \( X \)-systems into symmetrical and antisymmetrical groups.

In a wing structure of the type investigated previously we can describe three simple types of self-equilibrating internal systems. They are shown in Figs. 42, 43 and 44 and denoted by

\[ X = 1, \quad Y = 1, \quad Z = 1 \tag{252} \]

respectively. The first is the generalization of the \( X \)-system used in the flat panel and the second and third may be considered as slightly modified four boom load systems taken in the longitudinal and transverse directions respectively. The longitudinal four-boom load systems are applied extensively in standard wing analysis. The three figures are self-explanatory and give all flange loads and shear flows associated with the unit systems. Note, however, that the effect of taper is neglected except that we introduce the true local dimensions in the evaluation of the self-equilibrating systems.

When the cross-section is singly symmetrical the number of redundancies for lift and torque loads reduces to \( n \), given by the first of Eqs. (248). The \( N-1 \) \( Z \)-systems must still be included in the analysis for such loading cases. If, in addition, we use all \( N \) \( Y \)-systems the necessary number of \( X \)-systems becomes

\[
\beta = \frac{1}{2} - N
\]

and is zero when the effective flanges are arranged merely at the corners of the \( N \)-cells.

For the multispar wing of Fig. (40) with ribs only at the root and tip the \( n \) redundancies of Eq. (250) may be selected as

\[(N-1) \quad Z \text{-systems at the root}
\]
\[N \quad Y \text{-systems at the root}
\]

and

\[(\beta - 4) \quad X \text{-systems at each junction of bays and at the root.}
\]

The \( Y \)-system may involve a considerable length of the tube and if the latter is tapered a more accurate estimate of the longitudinal variation of the flange loads may become necessary.

Having selected a suitable system \( X, Y, Z \) of redundancies we can write down the \( b \)-matrix with the information given in Figs. 42, 43, 44. To obtain the \( b \)-matrix we may use any suitable statically equivalent stress system in the actual or idealized structure, but preferably the former. It was mentioned on page 32 that it is advantageous to select a basic stress system which, while being simple, approximates as closely as possible to the true stress system and reference was made to the method of example (a) of Section 9. Nevertheless, if the work in finding such a \( b \)-matrix proves excessive it may be preferable—since the choice of \( b \)-matrix does not affect the conditioning of the \( D \)-matrix—to sacrifice the closeness to the true stress system and to select a \( b \)-matrix as simple as possible. Thus, we can calculate a \( b \)-matrix for a basic system in which the spars act independently; a choice differing, in general, widely from the final \( b \)-matrix.
Consider the simple grid shown in Fig. 45 and denote by \( P_i(q_i), Q_i(q_i), Q_o(q_i) \) the longitudinal flange loads, transverse loads and shear flows corresponding to some \( R = 1 (X_i = 1) \). The corresponding columns in the \( b_0 \) and \( b_1 \) matrices are

\[
\begin{align*}
\{ P_{01} P_{10} P_{02} P_{20} P_{03} P_{30} P_{04} P_{40} Q_{01} Q_{10} Q_{02} Q_{20} Q_{03} Q_{30} Q_{04} Q_{40} \} \\
\{ P_{01} P_{10} P_{02} P_{20} P_{03} P_{30} P_{04} P_{40} Q_{01} Q_{10} Q_{02} Q_{20} Q_{03} Q_{30} Q_{04} Q_{40} \}
\end{align*}
\]

respectively.

To find the \( D_b \) and \( D_s \) matrices it only remains to give the flexibility matrix \( f \) of the elements. We write it in the partitioned form associated with the \( b_0 \) and \( b_1 \) matrices of Eqs. (253),

\[
f = \begin{bmatrix}
f_i & 0 & 0 & 0 & 0 \\
0 & f_i & 0 & 0 & 0 \\
0 & 0 & f_i & 0 & 0 \\
0 & 0 & 0 & f_i & 0 \\
0 & 0 & 0 & 0 & f_i
\end{bmatrix}
\]

where the suffices have the same meaning as in Eqs. (253). The matrices \( f_i \) and \( f_f \) are themselves partitioned diagonal matrices, the sub-matrices being the flexibility matrices of the longitudinal and transverse flange elements respectively. Since the flange loads vary linearly and the effective flange area of each element is assumed constant within each element the flexibility of the flange elements is that given on p. 22. Thus, for the grid of Fig. 45 the \( f_i \) and \( f_f \) are,

\[
f_i = \begin{bmatrix}
f_i & 0 & 0 & 0 \\
0 & f_i & 0 & 0 \\
0 & 0 & f_i & 0 \\
0 & 0 & 0 & f_i
\end{bmatrix}, \quad f_f = \begin{bmatrix}
f_i & 0 \\
0 & f_i
\end{bmatrix}
\]

The flexibility matrices \( f_i, f_o, f_r \) are diagonal matrices with elements \( \Phi/G_t, \Phi/G_o, \text{ and } \Omega/G_t \), respectively.

\[
f_i = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \Phi/G_t & 0 & 0 \\
0 & 0 & \Phi/G_o & 0 \\
0 & 0 & 0 & \Omega/G_t
\end{bmatrix}, \quad f_o = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \Phi/G_o & 0 & 0 \\
0 & 0 & \Phi/G_t & 0 \\
0 & 0 & 0 & \Omega/G_o
\end{bmatrix}
\]

where the suffices have the same meaning as in Eqs. (253).
where \( \Phi, \Phi_w, \Omega \) are the areas of the shear fields in the wing surface, webs and ribs respectively and \( t, t_w, t_r \) are the corresponding thicknesses.

We have now all the information to form the matrices

\[
D = b_f f_b R \quad \text{and} \quad D_0 = b_f f_b R
\]

can hence solve the system of equations

\[
b_f f_b (X \ Y \ Z) + b_f f_b R = O
\]  

for the unknowns \( X, Y, Z \).

Finally, we find the true flange loads and shear flows of the idealized structure from

\[
S = b_f f_b R + b_f (X \ Y \ Z)
\]  

It will then be necessary to translate the results (260) into stresses of the actual structure. Finally the flexibility \( F \) of the structure at the points and directions of the loads \( R \) may be determined from Eq. (229).

When the number of equations (259) is too large to be dealt with by our digital computer we may proceed in two (or more) steps by the method given on page 36. Essentially this introduces into our analysis a redundant basic system.

If initial strains \( H \) are imposed on the structure in addition to the loads \( R \) we have merely to add the column matrix \( b_i H \) on the left hand side of (259). Thus the analysis includes inter alia the complete calculation of wings under thermal loading.

**A new approach to the problem of cut-outs**

We emphasize that our above analysis is valid in the presence of any kind of cut-out stiffened or unstiffened by closed frames as long as the overall geometry and idealization conforms with the initial assumptions. Nevertheless, when we have a structure which is essentially continuous with only minor unstiffened cut-outs it may be worthwhile to apply an artifice which avoids the lack of uniformity in the pattern of the equations inevitably associated with cut-outs. Moreover, it is the ideal method of finding the alteration in the stresses due to the subsequent introduction of cut-outs in our system without having to repeat all the computations ab initio.

The method is as follows. To preserve the pattern of equations disturbed by missing shear panels or flanges we eliminate the cut-outs by introducing fictitious shear panels or flanges with an arbitrary thickness or area. Naturally, it is usually preferable to select for the latter dimensions those of the surrounding structure. To obtain nevertheless the same flange loads and shear flows in our altered structure as in the original system initial strains are imposed on the additional elements of such a magnitude that their stresses become zero. The effect of the fictitious elements is thus nullified whilst the uniform pattern of our equations is retained.

Let the column matrix of the unknown initial strains, in the additional elements only, be

\[
H
\]

In the new structure (i.e. without the cut-outs) we determine the flexibility matrix \( F \) and the matrices \( b_i \) and \( b_i \), which we write in the partitioned form

\[
b_i = \begin{bmatrix} b_{0r} \\ b_{0a} \end{bmatrix}, \quad b_i = \begin{bmatrix} b_{0r} \\ b_{0a} \end{bmatrix}
\]

where the sub-matrices with the suffixes \( g \) and \( h \) refer to the forces in the elements of the original structure and the fictitious new elements respectively.

Denoting the column matrix \( (X \ Y \ Z) \) simply by \( X \) and taking the initial strains in the original structure as zero the Eqs. (259) in the unknowns \( X \) become,

\[
\begin{bmatrix} b_f f_b X + b_f f_b R + b_f \end{bmatrix} \begin{bmatrix} \Omega \\ H \end{bmatrix} = O
\]

and hence using the second equation of (261)

\[
X = -D^{-1} b_f f_b R - D^{-1} b_f f_b H
\]

where

\[
D = b_f f_b R
\]

The stress matrix \( S \) follows as,

\[
S = (b_f - b_f D^{-1} b_f f_b R) R - b_f D^{-1} b_f f_b H
\]

The expression in the square bracket is the matrix \( b_f \) which we write in the partitioned form

\[
b_f = \begin{bmatrix} b_f \\ b_f \\ \end{bmatrix}
\]

To find now the column matrix \( H \) we put the stresses in the additional elements to zero. Thus, the matrix \( S \) must be

\[
\begin{bmatrix} S_f \\ O \end{bmatrix} = \begin{bmatrix} b_f \\ b_f \\ \end{bmatrix}
\]

where \( S_f \) are the true stresses (forces) in the original structure. Applying Eqs. (261), (264) and (265) in (263) we find

\[
\begin{bmatrix} S_f \\ O \end{bmatrix} = \begin{bmatrix} b_f \\ b_f \\ \end{bmatrix} R - \begin{bmatrix} b_f \\ b_f \\ \end{bmatrix} D^{-1} b_f f_b H
\]

Hence

\[
\begin{bmatrix} S_f \\ O \end{bmatrix} = \begin{bmatrix} b_f \\ b_f \\ \end{bmatrix} R - \begin{bmatrix} b_f \\ b_f \\ \end{bmatrix} D^{-1} b_f f_b H
\]

or

\[
H = (b_f - b_f D^{-1} b_f f_b R) R - b_f D^{-1} b_f f_b H
\]

which solves our problem completely. As mentioned already the method is ideally suited for finding the alteration of the stresses in a structure through a subsequent introduction of cut-outs, such as access doors which usually seem to materialize at a late stage of design. Another particularly useful application of the new approach may be found in the analysis of fuselage with window-openings. Naturally, the degree of redundancy is increased by the ‘filling-in’ of the cut-outs but this is of no importance for the automatic computations envisaged here.

**A more refined wing stress analysis**

The above general method of wing stress analysis suffers from the serious defect mentioned initially that the effective flange areas have first to be guessed since the stress distribution on which they depend is unknown. It is certainly feasible to apply an iteration technique but this is not only necessarily lengthy but also rather uninspiring.

To obviate these difficulties we develop a method which eliminates the determination of the effective flange areas and works directly with effective flange loads. The method has the further virtue that it takes full account of the Poisson’s ratio effect which may be important at the root and at other structural and loading discontinuities. The addition of 1/6 of the web area to the flanges is always sufficiently accurate for axial and torque loads and is retained here. Hence, our problem is restricted to the wing surface alone.

The previously introduced assumptions that the loads are carried in the idealized structure by a grid system of effective flange loads and fields purely shear-carrying form also the basis of the new method. We assume also that both the direct stress distribution and the effective flange loads vary linearly between consecutive nodal points. However, our analysis does not presume that the so-called effective flange areas—which do not enter into our development—are constant between nodal points. The shear flow is again taken to be constant within each field. When replacing the linearly varying direct stresses across a grid line by effective flange loads at the nodal points we introduce the additional assumption that the flange areas and thicknesses do not vary across this grid line. Since in wing structures plate thicknesses and possibly flange areas may vary just there it is suggested to take for this particular calculation the mean values of areas and thicknesses on either side of the grid line; on the other hand when there is a cut-out on one side of the grid line or the flange is interrupted the corresponding values should be taken as zero. These simplifications are not necessary for the purpose of the analysis but ease the problem of notation; moreover they do not affect seriously the accuracy of the computations.

Continuing our approach of numbering the flange elements with letters we number here only the nodal points with numerals.

We derive now the equation connecting the effective flange loads at nodal points in the \( z \) and \( s \) directions of the idealized structure and the direct stress distribution in the plate material. It is more convenient to fix a particular point and for this purpose we select the point 9 in the grid-system shown in Fig. 46.
The conditions of equilibrium for the actual and idealized systems yield in conjunction with Eq. (269)

\[ P_9 = \sigma_{9y}(B_9 + 2(A_{9,9} + A_{9,10})) + \sigma_{9x}A_{9,9} + \sigma_{9z}A_{9,10} \]

\[ = \sigma_{9y}2(A_{9,9} + A_{9,10}) + \sigma_{9y}(A_{9,9} + \sigma_{9z}A_{9,10}) + \sigma_{9z}2v(A_{9,9} + A_{9,10}) + \sigma_{9z}v(A_{9,9} + \sigma_{9y}A_{9,10}) \]

\[ Q_9 = \sigma_{9x}(C_9 + 2(A_{9,9} + A_{9,10})) + \sigma_{9x}A_{9,9} + \sigma_{9y}A_{9,10} \]

\[ = \sigma_{9x}2v(A_{9,9} + A_{9,10}) + \sigma_{9z}v(A_{9,9} + \sigma_{9y}A_{9,10}) \]

These equations are expressed more concisely in the form,

\[
\begin{bmatrix}
P_9 \\
Q_9 \\
\end{bmatrix}
= 
\begin{bmatrix}
L_{9,9} & L_{9,10} \\
L_{9,10} & L_{9,10} \\
\end{bmatrix}
\begin{bmatrix}
\sigma_{9y} \\
\sigma_{9x} \\
\end{bmatrix}
+ 
\begin{bmatrix}
L_{9,9} & L_{9,10} \\
L_{9,10} & L_{9,10} \\
\end{bmatrix}
\begin{bmatrix}
\sigma_{9z} \\
\sigma_{9y} \\
\end{bmatrix}
\]

where the matrices \( L_{9,9} \) are as follows:

\[
L_{9,9} = \begin{bmatrix}
B_9 + 2(A_{9,9} + A_{9,10}) \\
2v(A_{9,9} + A_{9,10}) \\
C_9 + 2(A_{9,9} + A_{9,10}) \\
\end{bmatrix}
\]

\[
L_{9,10} = \begin{bmatrix}
A_{9,9} & vA_{9,9} \\
vA_{9,9} & A_{9,9} \\
0 & 0 \\
\end{bmatrix}
\]

The matrix \( L_{9,10} \) is obtained from \( L_{9,9} \) by substituting \( 10(15) \) for \( 8(3) \). Equations corresponding to (272) may be written down for any other nodal point. We see immediately that

\[
L_{rr} = L_{rr}
\]

and that,

\[
L_{rs} = 0
\]

when \( r \) and \( s \) are not adjoining nodal points of the grid. We deduce also that Eqs. (273a) are the general formulae of the \( L \)-matrices for adjoining nodal points in the \( z \)- and \( s \)-directions respectively.

Consider now the column matrices for the flange loads and stresses at all \( p \) nodal points

\[
S = \begin{bmatrix}
P_1Q_1, P_2Q_2, \ldots, P_pQ_p \\
\end{bmatrix}
\]

\[
\sigma = \begin{bmatrix}
\sigma_{1y}, \sigma_{1x}, \sigma_{1z}, \ldots, \sigma_{py}, \sigma_{px}, \sigma_{pz} \\
\end{bmatrix}
\]

From the set of equations of the type (272) we find,

\[
S = LS
\]

where \( L \) is the symmetrical partitioned matrix

\[
L = \begin{bmatrix}
L_{11} & L_{12} & \ldots & L_{1p} \\
L_{12} & L_{11} & \ldots & L_{1p} \\
\vdots & \vdots & \ddots & \vdots \\
L_{1p} & L_{1p} & \ldots & L_{11} \\
\end{bmatrix}
\]

From (274a) the submatrices with suffixes referring to non-adjacent nodal points are zero. Solving Eq. (276) for \( S \) we find

\[
S = L^{-1}S
\]

and hence also

\[
e = \frac{1}{E}L^{-1}S
\]

where \( e \) is the column matrix of the flange strains at the nodal points, i.e.

\[
e = \begin{bmatrix}
\epsilon_{1y}, \epsilon_{1x}, \epsilon_{1z}, \ldots, \epsilon_{py}, \epsilon_{px}, \epsilon_{pz} \\
\end{bmatrix}
\]

Thus, once we have determined the effective flange loads the flange stresses and strains follow from Eqs. (279) and the direct stresses in the sheet from Eqs. (269). No guessing of effective flange areas is involved in this procedure but we have on the other hand to invert the matrix \( L \) with \( 2p \) rows and columns. It is apparent that if we knew the effective flange areas \( B_n, C_n \) at the nodal points we could immediately write down the inverted matrix as a diagonal matrix whose elements are the unit flange flexibilities at the nodal points. In fact, then...
Consider now the $b_0$ and $b_1$ matrices of the basic system. We emphasize that the $b_0$ system can in the present method only be derived from pure static considerations since the effective flange areas are unknown. Thus, $b_0$ cannot be derived for a multicell wing from an Engineers' theory of bending cum Bredt-Batho analysis unless we assign arbitrary areas to the flanges. To find, in fact, the $b_0$ system in this example it is probably best to select a set of longitudinal flange loads in equilibrium with the applied loads and derive a statically consistent set of shear flows and transverse flange loads in equilibrium with the applied shear forces and torque. Contrary to the system adopted in the previous method we write here the statically determinate $b_0$ and $b_1$ matrices in the partitioned form

$$
\mathbf{b}_0 = \begin{bmatrix}
\mathbf{b}_{0r} \\
\mathbf{b}_{0t}
\end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix}
\mathbf{b}_{1r} \\
\mathbf{b}_{1t}
\end{bmatrix}
$$

where the submatrices $b_{0r}$, $b_{0t}$ give the flange loads at the nodal points. The rows in $b_{0r}$, $b_{0t}$ are arranged in $p$ pairs corresponding to the $p$ nodal points; the first and second row in each pair refers to longitudinal and transverse flange loads respectively.

Returning now to Eq. (279) we apply it in the basic system for the external loads $\mathbf{R}$ and the redundancies $(\mathbf{X} \mathbf{Y} \mathbf{Z})$ respectively. We find with the notation of Eq. (280) the flange strains

$$
\epsilon_0 = \frac{1}{L} \mathbf{L}^{-1} \mathbf{b}_0 \mathbf{R} \quad \text{and} \quad \epsilon_1 = \frac{1}{L} \mathbf{L}^{-1} \mathbf{b}_1 \mathbf{X} \mathbf{Y} \mathbf{Z}
$$

We seek next the contribution of the flange strains $\epsilon_0$ and $\epsilon_1$ to the matrices $\mathbf{D}$ and $\mathbf{D}_h$ of the relative displacements $\delta_a$ and $\delta_b$. For this purpose we apply a self-equilibrating unit load system $X_t = 1$ (which may be also a $Y$ or $Z$-system) and denote by $P_0$ and $P_0'\mathbf{P}_1$ its longitudinal effective flange loads at the points 3 and 9 and by $Q_8$ and $\mathbf{Q}_8'$ its transverse effective flange loads at the points 8 and 9. Let $\delta_i$ be the relative displacement at the points and directions of $X_t = 1$ due to given flange strains $\epsilon_0$ and $\epsilon_1$. The contribution to $\delta_i$ of the straining of the flanges (3, 9) and (8, 9) is then

$$
\delta_i = \ldots + \frac{9}{3} P_0 \epsilon_0 \mathbf{d} + \frac{9}{3} Q_8 \epsilon_1 \mathbf{d} + 
$$

and since we assume that both flange loads and strains vary linearly between nodal points we find

$$
\delta_i = \ldots + \left( P_0 \epsilon_0 \mathbf{d} + Q_8 \epsilon_1 \mathbf{d} \right)
$$

The complete expression of $\delta_i$ is arranged by pairing the terms involving the longitudinal and transverse strains at the same nodal point. Thus, showing only the typical terms involving $(\epsilon_0, \epsilon_1)$,

$$
\delta_i = \ldots + \left( [P_0 \mathbf{Q}_0] \mathbf{b}_{0r} + [P_0 \mathbf{Q}_0'] \mathbf{b}_{0t} + [P_0 \mathbf{Q}_0] \mathbf{b}_{1r} + [P_0 \mathbf{Q}_0'] \mathbf{b}_{1t} \right)
$$

where the $\mathbf{L}$ matrices are,

$$
\mathbf{b}_0 = \begin{bmatrix}
1/3 & 0 & 0 \\
0 & d_{89}/3
\end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix}
1/3 & 0 & 0 \\
0 & d_{89}/3
\end{bmatrix}
$$

The matrix $b_{0i}(\mathbf{b}_{0r})$ is obtained from $b_{0i}(\mathbf{b}_{0r})$ by substituting $10(15)$ for $8(3)$. It is simple now to write down in Eq. (282) the terms for any other pairs of strains $(\epsilon_0, \epsilon_1)$. We deduce immediately that,

$$
\mathbf{L} = \mathbf{L}_r
$$

and that all $\mathbf{L}_r$ matrices are zero when they do not refer to adjoining points. Moreover, matrices (283a) are typical for adjoining nodal points in the $z$ and $x$ directions respectively.

Introducing the matrix

$$
\mathbf{1} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}
$$

we can express Eq. (282) concisely as

$$
\delta_i = [P_0 \mathbf{Q}_0] \mathbf{b}_{0r} + [P_0 \mathbf{Q}_0'] \mathbf{b}_{0t} + [P_0 \mathbf{Q}_0] \mathbf{b}_{1r} + [P_0 \mathbf{Q}_0'] \mathbf{b}_{1t}
$$

Observing that the matrix

$$
[P_0 \mathbf{Q}_0] \quad [P_0 \mathbf{Q}_0']
$$

is the $ith$ row of the $b_{0r}$ matrix and noting Eqs. (281) we find that the contributions of the flange strains to the $\mathbf{D}$ and $\mathbf{D}_h$ matrices are

$$
\mathbf{D}_h = \mathbf{b}_1 \mathbf{L}^{-1} \mathbf{b}_0 \mathbf{R}, \quad \mathbf{D}_0 = \mathbf{b}_1 \mathbf{L}^{-1} \mathbf{b}_0 \mathbf{R}
$$

We conclude that the flexibility of the flanges at the nodal points is given by

$$
\mathbf{f}_1 = \mathbf{L}^{-1} \mathbf{b}_0 \mathbf{R}
$$

Note the structural similarity between the $\mathbf{L}$ and $\mathbf{L}_r$ matrices. It is particularly pronounced when $n = 0$. The total flexibility of the elements of the structure is now

$$
\mathbf{f} = \mathbf{L}^{-1} \mathbf{b}_0 \mathbf{R}
$$

where the flexibilities of the cover, webs and ribs are as before. We find for the matrices $\mathbf{D}$ and $\mathbf{D}_h$

$$
\mathbf{D} = \mathbf{b}_1 \mathbf{L}^{-1} \mathbf{b}_0 \mathbf{R}, \quad \mathbf{D}_0 = \mathbf{b}_1 \mathbf{L}^{-1} \mathbf{b}_0 \mathbf{R}
$$

D. The Analysis of Structures by the Displacement Method

The analogue between the developments for the flexibilities and stiffnesses given under A and B and summarized in Table 1 shows clearly that parallel to the analysis of structures with forces as unknowns there must be a corresponding theory with deformations as unknowns. As mentioned in the introduction to this section Ostendorf* when investigating frameworks was the first to draw attention to such an analogy. In fact, his equations are the exact counterpart of the classic $\delta$ equations given by Mueller-Breslau for forces as unknowns. In more recent times Southwell and his pupils have used his relaxation technique to solve the elasticity equations in the fine difference form with displacements as unknowns for a great number of problems. Hoff† has applied the latter method to diffusion and related problems in aircraft structures and has solved also the corresponding equations directly. Lately Williams‡ has outlined an analysis of wing-structures of the standard or solid type by introducing the deflections at a finite grid of points as unknowns; his technique, which is intended for use in combination with the automatic digital computer, neglects however the shear deflections, which may have an important influence.

* loc. cit. p. 43.
† loc. cit. p. 43.
Notably, a theory using displacements as unknowns would only be of value if it could show some concrete advantages. It is clear that such an advantage may possibly arise when the stiffer the stiffnesses are smaller than the flexibility values, which is, as we have seen previously, very often the case. In particular in the egg-box structure characteristic of aircraft wings, the stiffnesses $k_{ii}$ are much easier to find than the influence or flexibility coefficients $\delta_{i\alpha}$. Another obvious advantage arises when the number of unknowns is smaller for the displacement analysis. This may occur in framework, especially the stiff-jointed type with few degrees of freedom at the joints. The equations in the displacements for stiff-jointed frameworks are almost invariably well conditioned: a further point in their favour, not only for iteration techniques but also for the direct solution. On the other hand in continuous structures, like wings and fuselages, this is not the case. Here, in fact, the equations in the displacements are nearly always ill-conditioned and it then becomes necessary to introduce generalized or group displacements as unknowns in order to improve the conditioning. This is a pronounced drawback of the displacement method when applied to aircraft structures. Furthermore, in such continuous systems the displacement method will usually involve a considerably greater number of unknowns than the force analysis in order to achieve a comparable degree of accuracy. It is apparent then that the choice between the two parallel techniques must be made on an ad hoc basis after careful consideration of the possible advantages and disadvantages of each method for a particular problem. Itwould, however, appear that at least with the present types of construction the force method is to be preferred for aircraft structures.

Before proceeding to the general development of the displacement analysis we introduce first a simple example to familiarize ourselves with the idea. Consider the framework shown in Fig. 47, symmetrical both in structure and loading. The number of unknown forces or moments when the engineer’s theory of bending is assumed to hold is evidently six. On the other hand, if we neglect the deformations due to shear and end load, two deformations alone, the rotations $r_1$ and $r_2$ at the stiff joints, suffice to specify completely the deformation of the system. The analysis may proceed as follows:

$$r_1 = r_2 = 0$$

Then, due to the loading on the upper member, moments $M_2$ are applied at the joints and are, with the notation of Fig. 47,

$$M_{200} = \frac{-p_t l^2}{12}, \quad M_{210} = \frac{p_t l^2}{12}$$

The out-of-balance moments on the joints are then

$$M_1 = \frac{-p_t l^2}{12}, \quad M_2 = \frac{-p_t l^2}{12} + \frac{p_t l^2}{12}, \quad M_3 = 0$$

Consider next the system with free joints and no transverse loading subjected to the loading by the joint-moments

$$R_1 = -M_1 = \frac{p_t l^2}{12}, \quad R_2 = -M_2 = \frac{p_t l^2}{12} + \frac{p_t l^2}{12}$$

The superposition of this and the previous case yields the true solution of the given system under the transverse loading. To analyse the second problem we apply Eqs. (138) which take the form

$$k_{11}r_1 + k_{12}r_2 = R_1$$

$$k_{21}r_1 + k_{22}r_2 = R_2$$

The stiffnesses $k_{ii}$ are easily found as (see also Eqs. (144))

$$k_{11} = 4EI_{12}l_1 + 4EI_{12}l_2, \quad k_{22} = 2EI_{12}l_1$$

which assumes that the horizontal beams and supporting struts have the constant bending stiffness $EI_1$ and $EI_2$ respectively.

![Fig. 47: Displacement analysis of stiff-jointed frame](image)

**Fig. 47:** Displacement analysis of stiff-jointed frame

We obtain from Eqs. (292)

$$r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = K^{-1} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$$

(292a)

where

$$K^{-1} = \frac{1}{k_{11}k_{22} - k_{12}k_{21}} \begin{pmatrix} k_{22} & -k_{12} \\ -k_{12} & k_{11} \end{pmatrix}$$

(293a)

Having the rotations $r_1$ and $r_2$ and using the stiffnesses of the individual elements contained in (293) we easily derive the actual moments in the structure. Thus, introducing again the usual sign convention giving positive bending moment when upper fibres are in compression, we find for the bending moment $M_{12}$ at the junction (1) of element (1, 2)

$$M_{12} = -\frac{p_t l^2}{12} + \frac{4EI_{12}l_1}{k_{12}} + \frac{2EI_{12}l_1}{k_{21}}$$

(294a)

This method forms also the basis of the Hardy-Cross or the more general Southwell relaxation technique in stiff-jointed frameworks.

We develop next the general theory of the displacement method. We introduce immediately the matrix notation and assume that the structure consists of a finite number $n$ of elements whose stiffnesses $k_{ii}$ due to relative displacements at the ends or boundaries of each element, are known. In order to show most clearly and concisely the striking analogy between the force and displacement methods we present them side by side in the following table:

<table>
<thead>
<tr>
<th>Method of Forces</th>
<th>Method of Displacements</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>External forces</strong></td>
<td><strong>Joint displacements</strong></td>
</tr>
<tr>
<td><strong>Flexibility</strong></td>
<td><strong>Stiffness</strong></td>
</tr>
<tr>
<td><strong>Displacements</strong></td>
<td><strong>Forces</strong></td>
</tr>
<tr>
<td>$R = FR$</td>
<td>$K = KR$</td>
</tr>
<tr>
<td>$FK = I$</td>
<td></td>
</tr>
</tbody>
</table>

See also Table 1

**TABLE II**

A COMPARATIVE PRESENTATION OF STRUCTURAL ANALYSIS BY THE FORCE AND DISPLACEMENT METHODS
TABLE II (continued)

Unit Load Method

Given the true strains $\varepsilon$ in a structure the kinematically related displacement $\rho$ at a given point and direction can be calculated from

\[ (295a) \quad 1 - \rho = \int \sigma \varepsilon dV \]

where $\sigma$ is a virtual or otherwise statically equivalent stress system due to unit load in given direction. Statically equivalent stresses ignore compatibility conditions. (See also Section 6 and FIG. 15.)

Cf. Eq. (84b)

Statically Determinate System

Internal forces (stresses) on elements determined from

\[ (296a) \quad S = bR \]

where matrix $b$ is obtained by static reasoning alone.

Flexibility of individual (unassembled) elements

\[ (297a) \quad v = fS = fbR \]

Flexibility

\[ (298a) \quad r = b'v = b'bR \]

Cf. Eqs. (121), (122), (125), (126)

Unit Displacement Method

Given the true stresses $\sigma$ in a structure the equilibrating force $R$ at a given point and direction can be calculated from

\[ (295b) \quad 1 - R = \int \sigma \varepsilon dV \]

where $\sigma$ is a virtual or otherwise statically equivalent stress system due to unit displacement in given direction. In what follows we denote virtual stresses as kinematically equivalent stresses. Kinematically equivalent stresses ignore equilibrium conditions. (See also Section 4 and FIG. 8.)

Cf. Eq. (146)

Statically Determinate System

Internal relative displacements (strains) of elements determined from

\[ (296b) \quad v = ar \]

where matrix $a$ is obtained by kinematic reasoning alone by displacing one joint at a time whilst keeping all others fixed.

Stiffness of individual (unassembled) elements

\[ (297b) \quad S = kv = kar \]

Stiffness

\[ (298b) \quad R = a'S = a'kar \]

Cf. Eqs. (153), (154), (159), (160)

Statically Indeterminate System

In the relation

\[ (300a) \quad r = b'v \]

where $b$ is merely a statically equivalent (virtual) matrix due to unit $R$'s.

Hence flexibility

\[ (300a) \quad F = b'bfb \]

Eq. (300a) is a special form of the Unit Load method (Principle of Virtual Forces).

FIG. 48b illustrates the matrices $b$ and $S$ on a particularly simple example of a singly redundant system.

Fig. 48(a).—True and statically equivalent stress systems in singly redundant, pin-jointed framework

Cf. Eqs. (125a), (126)

Kinematically Determinate System

In the relation

\[ (296b) \quad v = ar \]

where $a$ cannot be determined by kinematics alone.

Stiffness of structure needs to be considered, entering as equilibrium conditions.

On the other hand if internal stresses $S$ are known the equilibrating external forces may be derived from

\[ (300b) \quad R = a'S = a'kar \]

where $a'$ is merely a kinematically equivalent (virtual) matrix due to unit $r$'s.

Hence stiffness

\[ (301b) \quad K = a'ka \]

Eq. (300b) is a special form of the Unit Displacement method (Principle of Virtual Displacements).

Fig. 48b illustrates the matrices $a$ and $a'$ on the same example as in FIG. 48a.

Fig. 48(b).—True and kinematically equivalent displacement systems in pin-jointed framework

Cf. Eqs. (159a), (160b)
TABLE II (continued)

Problem a

Given a set of forces $R$, determine a set of statically indeterminate forces $X$ necessary to satisfy the compatibility conditions. Find also the displacements $r$ in the directions of $R$.

Complete force matrix

$$S_0 = b_0 R$$

Stresses due to $X$ (with $R = O$)

$$S_1 = b_1 X$$

where $b_0$ and $b_1$ are obtained from statics alone.

True stresses in actual structure

$$S = S_0 + S_1 = b_0 R + b_1 X$$

Strains of elements

$$\nu = fS = f_b R + f_b X$$

Compatibility condition in actual system at points of application of forces $X$

$$b_1 \nu = b_1 f_b R + b_1 f_b X = O$$

or

$$DX + D_0 = O$$

(308a)

where

$$D = b_1 f_b , \quad D_0 = b_1 f_b R$$

(309a)

Hence

$$X = -D^{-1} D_0 = -(b_1 f_b)^{-1} b_1 f_b R$$

(310a)

True stresses

$$S = bR$$

where

$$b = b_0 - b_1 (b_1 f_b)^{-1} b_1 f_b$$

(311a)

True strains

$$\nu = fS = f_b R$$

(297a)

where

$$F_0 = b_0 f_b (b_1 f_b)^{-1} b_1 f_b$$

(312a)

and

$$F_b = b_0 f_b$$

(313a)

Cf. Eqs. (217), (223), (222a), (224), (225), (226), (227a), (228), (229), (230)

Problem b

Given a set of displacements $r$ find forces $R$, stresses $S$ and strains $\nu$

From Eqs. (300c)

$$r = FR$$

(314a)

Hence

$$R = F^{-1} r$$

(315a)

and

$$S = bR = bF^{-1} r$$

(316a)

$$\nu = fS = f_b R$$

(317a)

Once $F$ is known the question of statical determinacy or indeterminacy is irrelevant in this problem.

Problem c

Given a set of initial strains $H$ imposed on free unassembled elements due to temperature, lack of fit, 'give' at foundations, find stresses $S$ and total strains $\nu$ when forces $R = O$.

Total strains of elements

$$\nu = f_b X + H$$

(318a)

Compatibility condition in actual system at points of application of forces $X$

$$b_1 \nu = b_1 f_b X + b_1 H = O$$

(319a)

Hence,

$$X = -(b_1 f_b)^{-1} b_1 H$$

(320a)

and

$$S = -b_0 (b_1 f_b)^{-1} b_1 H$$

(321a)

$$\nu = -(b_1 f_b)^{-1} b_1 H + H$$

(322a)

Note,

$$H = f$$

Cf. Eq. (236)

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Problem a

Given a set of joint displacement $r$, determine the set of kinematically indeterminate joint displacements $U$ necessary to satisfy the equilibrium conditions. Find also the forces $R$ in the directions of $r$.

Complete displacement matrix

$$\nu = a_0 r$$

(302b)

By putting $U = O$ we obtain the so-called basic system which is kinematically determinate within limits of idealization.

Strains in basic system

$$\nu = a_0 r$$

(303b)

where $a_0$ and $a_1$ are obtained by kinematics alone.

True strains in actual structure

$$\nu = -C^{-1} C_0 = -(a_1^t f) a_1' k a_1$$

(304b)

Hence

$$U = -C^{-1} C_0 a_0 u$$

(305b)

True strains

$$\nu = a r$$

(296b)

where

$$a = a_0 - a_1^t f$$

(311a)

True stresses

$$S = k v = k a r$$

(297b)

Forces $R$ due to $r$

$$R = k S = k a r$$

(300d)

where

$$k = K_0 - a_1^t f a_1 k a_0$$

(312b)

and

$$K_0 = a_0 k a_0$$

is the stiffness of the basic system since we may choose

$$a = a_0$$

(313b)

Problem b

Given a set of forces $R$ find joint displacements $r$, strains $\nu$ and stresses $S$

From Eqs. (300d)

$$R = k S = k a r = k$$

(314b)

Hence

$$r = k^{-1} R$$

(315b)

and

$$\nu = a r = a K^{-1} R$$

(316b)

$$S = k v = k a K^{-1} R$$

(317b)

Once $K$ is known the question of kinematical determinacy or indeterminacy is irrelevant in this problem.

Problem c

Given a set of initial stresses $J$ imposed on elements with frozen joints (i.e. all joint displacements zero) due to temperature, lack of fit, 'give' at foundations, find strains $\nu$ and stresses $S$ when displacements $r = O$.

Total stresses on elements

$$S = k a U + J$$

(318b)

Note that the column matrix $U$ must here include all unknown joint displacements.

Equilibrium condition in actual system in the directions of $U$

$$a_0 = a k a_0 + a_2 = 0$$

(319b)

Hence,

$$U = -a_1^t f a_1 J$$

(320b)

and

$$\nu = a_1 a_1' k a_1 J$$

(321b)

$$S = -k a_1 a_1' k a_1 J$$

(322b)

Note,

$$J = -k H$$
TABLE II (continued)

**Problem d**
Assume that we write the total set of forces (including the statically indeterminate forces) in the partitioned form

\[
\begin{bmatrix}
R \\
X \\
Z
\end{bmatrix}
\]

in which \( Z \) is known in terms of \( R \) and \( X \). We set now the modified problem (a): Given the set of forces \( R \) determine the set of forces \( X \) necessary to satisfy the compatibility conditions.

Here in the basic system obtained by putting \( X = 0 \) the stresses

\[
S_b = b_bR
\]

are completely known although the system is statically indeterminate.

Similarly we know the stresses

\[
S_t = b_tX
\]

when \( R = 0 \).

Strains in actual structure,

\[
S = S_0 + S_t = b_tR + b_tX
\]

Compatibility condition in actual system at points of application of \( X \)

\[
F = F_0 - b_tY(\bar{Y}_1Y_b - 1)^{-1}b_tY_bR
\]

where \( b_t \) is a set of stresses statically equivalent to unit \( X \)’s (and \( R = 0 \)) preferably found for \( Z = 0 \). In the latter case the rows of \( b_t \) are the same as the corresponding rows of \( b_b \) of Problem (a).

Thus,

\[
S = b_tR, \quad v = f_bR
\]

where

\[
F = F_0 - b_tY_1Y_b(\bar{Y}_1Y_b - 1)^{-1}b_tY_bR
\]

and

\[
F_0 = \bar{b}_tY_b
\]

is the flexibility of the basic system.

The matrix \( b_t \) is a set of stresses statically equivalent to unit \( R \)’s preferably found for \( Z = 0 \) and \( X = 0 \). In the latter case \( b_t \) is identical with \( b_b \) of problem (a).

Cf. Eqs. (226), (227a), (229), (230).

**Problem d**
Assume that we write the total set of joint displacements in the partitioned form

\[
\begin{bmatrix}
r \\
u \\
w
\end{bmatrix}
\]

in which \( U \) is known in terms of \( r \) and \( U \). We set now the modified problem (a): Given the set of displacements \( r \) determine the set of displacements \( U \) necessary to satisfy the equilibrium conditions.

Here in the basic system obtained by putting \( U = 0 \) the strains

\[
v_0 = a_0r
\]

are completely known although the system is kinematically indeterminate.

Similarly we know the strains

\[
v_1 = a_1U
\]

when \( r = 0 \).

True strains in actual structure,

\[
v = v_0 + v_1 = a_0r + a_1U
\]

forces on elements

\[
S = kv = ka_1U
\]

Equilibrium condition in actual system at displacements \( U \)

\[
\bar{a}_1S = \bar{a}_1ka_0U + \bar{a}_1ka_0U = 0
\]

where \( \bar{a}_1 \) is a set of strains kinematically equivalent to unit \( U \)’s (and \( r = 0 \)) preferably found for \( W = 0 \). In the latter case the rows of \( \bar{a}_1 \) are the same as the corresponding rows of \( a_1 \) of Problem (a).

Thus,

\[
\begin{bmatrix}
U \\
v \\
w
\end{bmatrix} = (\bar{a}_1'ka_0)^{-1}a_0'ka_0r
\]

True strains and stresses

\[
v = a_0r, \quad S = ka_0
\]

where

\[
a_0 = a_0 - a_1(\bar{a}_1'ka_0)^{-1}\bar{a}_1'ka_0
\]

Forces \( R \) due to \( r \) (see Eq. (300b))

\[
R = \bar{a}_0S
\]

where

\[
K = K_0 - a_0'ka_0(\bar{a}_1'ka_0)^{-1}\bar{a}_1'ka_0
\]

and

\[
K_0 = \bar{a}_0'ka_0
\]

is the stiffness of the basic system.

The matrix \( \bar{a}_0 \) is a set of strains kinematically equivalent to unit \( r \)’s preferably found for \( W = 0 \) and \( U = 0 \). In the latter case \( \bar{a}_0 \) is identical with \( a_0 \) of problem (a).

**Condensation of flexibility matrix**

The calculation of the flexibility matrix \( F \) given under problem (a) can be developed concisely as a condensation of the complete flexibility matrix for the forces \( R \) and \( X \).

This matrix may be written as

\[
\begin{bmatrix}
F_{II} & F_{III} \\
F_{III} & F_{II}
\end{bmatrix}
\]

where \( I(II) \) is for forces \( R \) (\( X \)) only and was denoted by \( F_X \) (\( D_X \)) in problem (a). Evidently \( F_{II} R = D_{II} \).

The flexibility matrix \( F \) of the actual structure under the forces \( R \) is then

\[
(F - \bar{F}) = \frac{F_{III} - F_{II} - F_{II} F_{II}^{-1} F_{III}}{(1 - \bar{F}^{-1}) F_{II}^{-1} F_{III} - F_{II}^{-1} F_{III} - F_{II}^{-1} F_{III} (1 - \bar{F}^{-1}) F_{III}^{-1} F_{II}}
\]

Naturally this condensation may be performed in two or more stages and is then equivalent to the method of problem (d).

**Condensation of stiffness matrix**

The calculation of the stiffness matrix \( K \) given under problem (a) can be developed concisely as a condensation of the complete stiffness matrix for the displacements \( r \) and \( U \).

This stiffness may be written as

\[
\begin{bmatrix}
K_{II} & K_{III} \\
K_{III} & K_{II}
\end{bmatrix}
\]

where \( I(II) \) is for displacements \( r \) (\( U \)) only and was denoted by \( K_U \) (\( C_U \)) in problem (a). Evidently \( K_{III} = C_{III} \).

The stiffness matrix \( K \) of the actual structure for the displacements \( r \) is then

\[
K = K_R - K_{III} - K_{II} F_{II}^{-1} F_{III}
\]

Naturally this condensation may be performed in two or more stages and is then equivalent to the method of problem (d).

Cf. Eq. (175)
TABLE II (continued)

Elimination and rigidification of structural elements

Assume a set of initial strains, written as column matrix \( \mathbf{H} \), in the structural elements to be removed, of such magnitude as to give zero stress in resultant system.

Write the \( \mathbf{b} \) and \( \mathbf{b}_1 \) matrices of the complete structure in the partitioned form

\[
\begin{bmatrix}
\mathbf{b} \\
\mathbf{b}_1
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{b}_2 \\
\mathbf{b}_3
\end{bmatrix},
\begin{bmatrix}
\mathbf{a} \\
\mathbf{a}_1
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{a}_2 \\
\mathbf{a}_3
\end{bmatrix}
\]

where the suffix \( h \) refers to those elements that are to be removed.

We find

\[
\mathbf{H} = (\mathbf{b}_2 \mathbf{D}^{-1} \mathbf{b}_3 )^{-1} \mathbf{b}_1 \mathbf{R} \tag{332a}
\]

and hence forces in the new structure

\[
\mathbf{S}_i = (\mathbf{b}_2 \mathbf{D}^{-1} \mathbf{b}_3 )^{-1} (\mathbf{b}_2 \mathbf{D}^{-1} \mathbf{b}_3 )^{-1} \mathbf{b}_1 \mathbf{R} \tag{333a}
\]

In this process the number of statically indeterminate forces \( \mathbf{X} \) has been reduced to a degree depending on the number of elements removed.

In the inverse process of making infinitely rigid certain of the structural elements we have merely to put \( \mathbf{b}_1 = \mathbf{0} \) for the affected elements. The number of statically indeterminate forces remains the same.

Cf. Eqs. (264), (267), (268)

Rigidification and elimination of structural elements

Assume a set of initial stresses, written as column matrix \( \mathbf{J} \), in the structural elements to be made infinitely rigid, of such magnitude as to give zero strain in resultant system.

Write the \( \mathbf{a} \) and \( \mathbf{a}_1 \) matrices of the complete structure in the partitioned form

\[
\begin{bmatrix}
\mathbf{a} \\
\mathbf{a}_1
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{a}_2 \\
\mathbf{a}_3
\end{bmatrix} \tag{332b}
\]

where the suffix \( h \) refers to those elements that are to be made infinitely rigid.

We find

\[
\mathbf{J} = (\mathbf{a}_2 \mathbf{C}^{-1} \mathbf{a}_3 )^{-1} \mathbf{a}_1 \mathbf{r} \tag{333b}
\]

and hence strains in the new structure

\[
\mathbf{v} = (\mathbf{a}_2 - \mathbf{a}_2 \mathbf{C}^{-1} \mathbf{a}_3 ) (\mathbf{a}_2 \mathbf{C}^{-1} \mathbf{a}_3 )^{-1} \mathbf{a}_1 \mathbf{r} \tag{334a}
\]

In the process of making elements infinitely rigid (stiff) we introduce kinematic relations between displacements and hence reduce the number of unknown displacements \( \mathbf{U} \) accordingly.

In the inverse process of eliminating certain of the structural elements we have merely to put \( \mathbf{b}_1 = \mathbf{0} \) for the affected elements. The number of kinematically indeterminate displacements remains the same.

Generalized Forces

Generalized forces given by

\[
\begin{bmatrix}
\mathbf{R} \\
\mathbf{X}
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{B}_0 & 0 \\
0 & \mathbf{B}_1
\end{bmatrix} 
\begin{bmatrix}
\mathbf{R} \\
\mathbf{X}
\end{bmatrix}
\]

The equation for the unknown \( \mathbf{X} \) is

\[
\mathbf{D} \mathbf{X} + \mathbf{D}_0 = \mathbf{0} \tag{335a}
\]

where

\[
\mathbf{D} = \mathbf{B}_1 \mathbf{D} \mathbf{B}_1 \text{ and } \mathbf{D}_0 = \mathbf{B}_1 \mathbf{D} \mathbf{B}_0 \tag{336a}
\]

Then

\[
\mathbf{S} = \mathbf{D} \mathbf{R} \tag{337a}
\]

where

\[
\mathbf{S} = \mathbf{b}_1 \mathbf{B}_1 - \mathbf{b}_1 \mathbf{B}_1 (\mathbf{B}_1 \mathbf{D} \mathbf{B}_1 )^{-1} \mathbf{b}_1^T \mathbf{f} \mathbf{b}_1 \mathbf{B}_0
\]

and the flexibility of the actual structure for the forces \( \mathbf{R} \) is

\[
\mathbf{F} = \mathbf{F}_0 - \mathbf{b}_1 \mathbf{B}_1 (\mathbf{B}_1 \mathbf{D} \mathbf{B}_1 )^{-1} \mathbf{b}_1^T \mathbf{f} \mathbf{b}_1 \mathbf{B}_0 \tag{339a}
\]

where

\[
\mathbf{F}_0 = \mathbf{b}_0 \mathbf{B}_0 \mathbf{f} \mathbf{b}_0 \mathbf{B}_0
\]

is the flexibility of the basic system under the forces \( \mathbf{R} \)

Generalized Displacements

Generalized displacements given by

\[
\begin{bmatrix}
\mathbf{r} \\
\mathbf{U}
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{A}_0 & 0 \\
0 & \mathbf{A}_1
\end{bmatrix} 
\begin{bmatrix}
\mathbf{r} \\
\mathbf{U}
\end{bmatrix} \tag{335b}
\]

The equation for the unknown \( \mathbf{U} \) is

\[
\mathbf{C} \mathbf{U} + \mathbf{C}_0 = \mathbf{0} \tag{336b}
\]

where

\[
\mathbf{C} = \mathbf{A}_1 \mathbf{C} \mathbf{A}_1 \text{ and } \mathbf{C}_0 = \mathbf{A}_1 \mathbf{C} \mathbf{A}_0 \tag{337b}
\]

Then

\[
\mathbf{v} = \mathbf{a} \mathbf{r} \tag{338b}
\]

where

\[
\mathbf{a} = \mathbf{a}_1 \mathbf{A}_1 - \mathbf{a}_1 \mathbf{A}_1 (\mathbf{A}_1 \mathbf{C} \mathbf{A}_1 )^{-1} \mathbf{A}_1 \mathbf{a}_0 \mathbf{A}_0
\]

and the stiffness of the actual structure for the displacements \( \mathbf{r} \) is

\[
\mathbf{K} = \mathbf{K}_0 - \mathbf{A}_0 \mathbf{a}_0^T \mathbf{a}_0 \mathbf{A}_0 \tag{340b}
\]

where,

\[
\mathbf{K}_0 = \mathbf{A}_0 \mathbf{a}_0^T \mathbf{a}_0 \mathbf{A}_0
\]

is the stiffness of the basic structure for the displacements \( \mathbf{r} \)
The application of the general theory with displacements as unknowns to frameworks—both of the pin-jointed and stiff-jointed type—is straightforward. For the stiff-jointed system the method is particularly simple when direct and shear deformations are ignored. In fact, for all frameworks the determination of the matrices $C$ and $C_0$ is trivial once we consider all possible degrees of freedom of the joints. See for example, the systems of refs. (23), (24) and (48) investigated on pp. 23, and 45, which show clearly how elementary the matrices $a$ and stiffness $k$ are when we break up the structure into its simplest constituent components. We need not therefore concern ourselves any more here with frameworks, and we turn our attention to the membrane type of system characteristic of aircraft applications. Essentially, a major aircraft structure like a wing consists of an assembly of plates (fields) stiffened by flanges along their edges. The field may be a curved and/or tapered surface but we ignore here both these effects and consider only rectangular flat elements of constant thickness. For convenience the element formed by the plate (sheet) and its four-edge members is denoted by the term unit panel. It is assumed that the flange areas are constant along each edge.

![Fig. 49.—Stiffnesses of unit panel](image)

We determine first the stiffnesses $k_{ij}$ of the unit panel shown in Fig. 49 for unit displacements in the $z$- and $x$-directions at the four corners or nodal points of the idealized system. The stiffness of our element is hence an $8 \times 8$ matrix. As in the case of the force method it is necessary for the practical evaluation of the $k_{ij}$ to introduce simplifying assumptions which are, naturally, concerned here with the state of deformations. Thus, we assume that the displacements vary linearly between the nodal points. Although this idealization offends against the equilibrium conditions its effect upon the stiffness is not pronounced as long as we keep the unit panels reasonably small. Nevertheless, it is inevitable that the stress distribution derived from an approximate deformation analysis should, in general, be less accurate than the one obtained from the approximate force method in the same grid system.

We denote for the purpose of the analysis of the unit panel the displacements parallel to the $x$ and $z$ axes by $u$ and $w$ respectively and introduce also the local coordinate system $\xi$, $\eta$. Consider now the state of strain and stress arising from a unit displacement

$$v_3 = 1$$

(341)

Following our assumption the internal displacements are given by

$$u_3 = 0, \quad w_3 = \xi \left(1 - \frac{\xi}{d}\right)$$

(341a)

where the suffix 3 indicates that these displacements are due to $v_3 = 1$.

The strains and stresses in the sheet are:

$$\begin{align*}
\epsilon_{11} &= -\frac{\xi}{d}, & \sigma_{11} &= -E \frac{\xi}{d} \\
\epsilon_{22} &= \frac{1}{d} \left(1 - \frac{\xi}{d}\right), & \epsilon_{12} &= 0 \\
\sigma_{12} &= \frac{E'}{d} \left(1 - \frac{\xi}{d}\right), & \sigma_{22} &= \frac{E'}{d} \left(1 - \frac{\xi}{d}\right)
\end{align*}$$

(322)

where $E' = E/(1 - \nu^2)$.

Strain $\epsilon_3$ and load $P_1$ in flange $B_1$:

$$\epsilon_{13} = \frac{1}{d}, \quad P_{13} = B_1$$

(322a)

all other flange strains and loads are zero.

Similar formulae are obtained for the strains and stresses due to any other $v_i = 1$. To derive the stiffnesses we apply the unit displacement method Eq. (293b), which takes here the form,

$$k_{ij} = \int_0^1 \int_0^1 \sigma_{ij} \, \text{d}x \, \text{d}y$$

where the integral extends over sheet and flange. For example, for the stiffnesses associated with $v_3 = 1$ we obtain,

$$\begin{align*}
k_{11} &= \frac{E'd}{2} - \frac{E'B_1}{d} + \frac{Gt}{d} \\
k_{22} &= -\frac{E'd}{6} + \frac{Gt}{d} \\
k_{33} &= -\frac{E'd}{6} + \frac{E'B_1}{d} + \frac{Gt}{3d} \\
k_{44} &= +\frac{E'd}{6} + \frac{Gt}{d}
\end{align*}$$

(333)

and

$$\begin{align*}
k_{12} &= -\frac{vE'd}{4} + \frac{Gt}{4}, & k_{21} &= +\frac{vE'd}{4} + \frac{Gt}{4} \\
k_{13} &= -\frac{vE'd}{4} + \frac{Gt}{4}, & k_{31} &= +\frac{vE'd}{4} + \frac{Gt}{4}
\end{align*}$$

(333a)

It is simple now to write down the stiffnesses corresponding to any other unit displacement. For convenience we express the total stiffness matrix in the form

$$k = k_x + k_y + k_r$$

(344)

* Contrast to our usual notation subscripts are used here to denote stresses and strains due to unit displacements.
where the suffixes \( s, d, f \) indicate the partial stiffnesses for shear strains and direct strains in sheet, and direct strains in flanges. We find

\[
\begin{bmatrix}
G_t & G_t & G_t & G_t & G_t & G_t \\
3d & 3d & 6d & 6d & 4 & 4 & 4 \\
G_t & G_t & G_t & G_t & G_t & G_t \\
3d & 3d & 6d & 6d & 4 & 4 & 4 \\
G_t & G_t & G_t & G_t & G_t & G_t \\
6d & 6d & 3d & 3d & 4 & 4 & 4 \\
G_t & G_t & G_t & G_t & G_t & G_t \\
6d & 6d & 3d & 3d & 4 & 4 & 4 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
G_t & G_t & G_t & G_t & G_t & G_t \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
G_t & G_t & G_t & G_t & G_t & G_t \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
G_t & G_t & G_t & G_t & G_t & G_t \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\]

The stiffnesses \( k_s, k_a \) contract to 6 \( \times \) 6 matrices and are

\[
\begin{bmatrix}
G_t & G_t & G_t & G_t & G_t & G_t \\
3d & 3d & 6d & 6d & 2 & 2 \\
G_t & G_t & G_t & G_t & G_t & G_t \\
3d & 3d & 6d & 6d & 2 & 2 \\
G_t & G_t & G_t & G_t & G_t & G_t \\
6d & 6d & 3d & 3d & 2 & 2 \\
G_t & G_t & G_t & G_t & G_t & G_t \\
6d & 6d & 3d & 3d & 2 & 2 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
G_t & G_t & G_t & G_t & G_t & G_t \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
G_t & G_t & G_t & G_t & G_t & G_t \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
G_t & G_t & G_t & G_t & G_t & G_t \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
E'dt & E'dt & E'dt & E'dt & E'dt & E'dt \\
3l & 6l & 3l & 6l & 4 & 4 \\
E'dt & E'dt & E'dt & E'dt & E'dt & E'dt \\
3l & 6l & 3l & 6l & 4 & 4 \\
E'dt & E'dt & E'dt & E'dt & E'dt & E'dt \\
3l & 6l & 3l & 6l & 4 & 4 \\
E'dt & E'dt & E'dt & E'dt & E'dt & E'dt \\
3l & 6l & 3l & 6l & 4 & 4 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
E'dt & E'dt & E'dt & E'dt & E'dt & E'dt \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
E'dt & E'dt & E'dt & E'dt & E'dt & E'dt \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
E'dt & E'dt & E'dt & E'dt & E'dt & E'dt \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\]

No contribution of the flanges is called for when evaluating the stiffnesses of the webs since \( k_s \) is best included in the top and bottom panels.

Further simplification of the stiffness matrices for the webs is possible when the top and bottom panels of our wing structure are identical. Then for vertical loads alone the horizontal displacements in the two covers are antisymmetrical and the stiffness matrices (346) and (346a) may be contracted to 4 \( \times \) 4 matrices.

We illustrate now the application of the unit panel stiffnesses to the diffusion problem shown in Fig. (50). The plate is reinforced longitudinally and laterally by stiffeners of area \( B \) and \( C \) respectively, and edge members of area \( B_0 \). Displacements in the \( x \) and \( z \) directions are defined at all nodes of the grid formed by lateral and longitudinal stiffening. Naturally the grid does not have to be restricted to this definition and we can always choose a finer one if the stiffeners are widely spaced so that the assumption of linear variation between adjacent nodal points can represent adequately the displacement pattern. Using the stiffness matrix of the unit panel already derived, the setting up of the complete stiffness matrix follows quite simply. It is only necessary to identify quickly and easily the displacements defined for the unit panels separately with those defined for the assembled panel. The complete stiffness matrix is obtained as (Eq. 299b).

\[
K = a'k \cdot a
\]

where \( k \) is the stiffness matrix of the unassembled unit panels and may be written in the diagonal partitioned form

\[
\begin{bmatrix}
K_s & 0 & 0 & \cdots & 0 \\
0 & k_a & 0 & \cdots & 0 \\
0 & 0 & k_a & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & k_a \\
\end{bmatrix}
\]

In assembling the panels of Fig. 50 to form a wing structure the stiffnesses (345) and (345a) may be simplified considerably when applied to rib and spar webs. Thus, for these cases we can always neglect the expansion in the direction of the height of the web. The corresponding formulae may be obtained by putting \( v_s = u_s \) and \( v_a = u_a \) which apply when \( C_1 \) and \( C_2 \) are infinite.
\[ k_{13,15} = \frac{4E'dt}{3I} + 2EB \cdot 4Glt \]
\[ k_{17,15} = k_{13,15} = -\frac{2E'dt}{3I} - \frac{E'dt}{T} + 3d \]
\[ k_{21,15} = k_{13,15} = -\frac{E'dt}{3I} - \frac{2Glt}{3d} \]
\[ k_{25,15} = k_{13,15} - k_{21,15} = \frac{k_{27,15} - k_{25,15}}{6d} \]
\[ k_{33,15} = -k_{4,15} - k_{29,15} = \frac{k_{38,15}}{4} \]

All the remaining \( k \)'s associated with 15 are here zero due to symmetry. If \( R \) is the column matrix (50 rows) of forces applied at the nodes then the displacements \( r \) are given by

\[ r = K^{-1} R \]

Naturally, loads may not be applied at all nodes (joints) in which case it may be desirable partially to solve the problem by eliminating the displacements where forces are not applied and to use the condensed matrix. (See Table II.)

Finally we apply the unit panel to the assembly and analysis of the egg box type of structure illustrated in Fig. 51 where upper and lower plates are connected together by longitudinal and transverse webs. Any stiffeners on the plates are assumed for the present example to be along the lines of web-plate intersections. The structure is taken to be symmetrical about the horizontal middle surface and we consider the application of vertical loads only. With these assumptions it is only necessary to specify three displacements at each web intersection; the vertical displacement and the two rotations of the web intersection line (Fig. 51). In many cases the webs may be too widely spaced for the assumed linear variation of displacements between them to give satisfactory accuracy. It then becomes necessary to introduce further grid lines intermediate between the actual webs, the displacements being defined at all nodal points formed by grid line intersections. Where such nodal points do not lie on a web then obviously we define there only the two rotations, since vertical displacement does not affect the cover plates. Naturally, further lateral and longitudinal reinforcement of the plates can lie along the extra grid lines.

The analysis of such a structure under vertical loads follows that given under Problems (a) and (b) in Table II. Thus, we designate the vertical displacements as \( r \) and take the rotations as the redundant displacements \( U \).

The strains of the elements are here identified as the displacements of the unit panel defined in Fig. 49 and can therefore be written (Eq. 305b).

\[ \psi = a_r f + a_u U \]

The equation for the unknowns \( U \), which is here the condition of equilibrium of the moments corresponding to the rotations \( U \) at the joints is (Table II, Eqs. (310b)),

\[ a_r'k_u U + a_u'k_a = 0 \]

or

\[ U = \frac{1}{a_u'k_u} a_u'k_a' U \]

The stiffness for displacements \( r \) only is then

\[ K = a_r' k_u - a_u'k_u(a_r'k_u)^{-1}a_u'k_a \]

---

* Suffix \( r \) refers here to the number of unit panels and should not be confused with suffix \( r \) for stiffness due to shear stresses (344).
from which the displacements \( r \) under loads \( R \) are found as
\[
r = \mathbf{K}^{-1} \mathbf{R}
\]
The total strains of the elements due to \( R \) are then
\[
\mathbf{v} = \mathbf{a} r = (\mathbf{a}_0 - \mathbf{a}_0 \mathbf{a}_0^T \mathbf{K}^{-1} \mathbf{r}) \mathbf{K}^{-1} \mathbf{R}
\]
\[\text{(316b)}\]
from which the stresses in the unit panels are calculated.

Due to the simple geometry of the structure the matrices \( \mathbf{a}_0, \mathbf{a}_0^T \) are again quite straightforward. Thus writing \( \mathbf{a}_0 \) in the partitioned form
\[
\mathbf{a}_0 = (\mathbf{a}_{01} \mathbf{a}_{02} \ldots \mathbf{a}_{09})
\]
\[\text{(351)}\]
it is apparent that for the cover plates the \( \mathbf{a}_0 \)'s are all zero since vertical displacements \( r \) can cause no strain in the plates \((\text{with } \mathbf{U} = \mathbf{0})\).

For the web \( f \), the \( \mathbf{a}_{0f} \) matrix is easily seen to be
\[
\mathbf{a}_{0f} = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
\[\text{(352)}\]
Likewise the \( \mathbf{a}_c \) matrix for unit panel \( c \) of the top cover is
\[
\begin{bmatrix}
10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
\hline
h/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & h/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & h/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & h/2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & h/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & h/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & h/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h/2
\end{bmatrix}
\]
\[\text{(353)}\]
and for the web plate \( f \)
\[
\begin{bmatrix}
10 & 11 & 12 \\
\hline
h/2 & 0 & 0 \\
0 & h/2 & 0 \\
0 & 0 & -h/2 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
\[\text{(354)}\]
All other columns are zero.

The matrices for the web plates may of course be reduced to six or even four rows by using the assumption of zero vertical direct strain and the antisymmetric character of the \( U \) displacements (see also Eqs. (346) and (346a)). However, we retain here the full eight displacements of the unit panels to show the simple formation of the \( \mathbf{a} \) matrices with a completely standard unit panel.

The stiffness matrix \( \mathbf{k} \) of the unassembled unit panels is written again in the diagonal partitioned form:

\[
\mathbf{k} = \begin{bmatrix}
k_{00} & k_{01} & \ldots & k_{08} \\
0 & k_{11} & \ldots & k_{18} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & k_{88}
\end{bmatrix}
\]

The factor 2 is introduced for the cover plates to take advantage of the symmetry of the structure by including the unit panels of the lower cover with their opposite numbers in the top cover. The \( k_0, k_0 \), etc., are of course the stiffnesses of the unit panels discussed earlier.

The method of formulation of the matrices \( a_k, k_{0k}, \) etc., given above is probably the most practical for use with the automatic digital computer since the various terms in the constituent matrices are reduced to their simplest and most standard forms. However, it is instructive to consider directly the components of \( a_k, k_{0k}, \) etc., and gain some physical insight into their formation.

We call
\[
\begin{align*}
\mathbf{a}_0^T \mathbf{a}_0 &= \mathbf{K}_0 \\
\mathbf{a}_0^T \mathbf{a}_k &= \mathbf{C} \\
\mathbf{a}_k^T \mathbf{a}_0 &= \mathbf{C}^T
\end{align*}
\]
\[\text{(356)}\]
Thus the complete stiffness matrix for the displacement column \( \begin{bmatrix} \mathbf{r} & \mathbf{U} \end{bmatrix} \) is
\[
\begin{bmatrix}
\mathbf{K}_0 & \mathbf{C} \\
\mathbf{C}^T & \mathbf{C}
\end{bmatrix}
\]
\[\text{(357)}\]
\( \mathbf{K}_0 \) is clearly the set of vertical forces \( \mathbf{R} \) which arise due to unit \( \mathbf{r} \) displacements when \( \mathbf{U} = \mathbf{0} \). Evidently only the webs are involved and we find easily as a typical example the vertical forces at the joints due to \( r_9 = 1 \)
\[
k_{0,9,9} = \frac{2Ght}{l} + \frac{2Ght}{d}
\]
\[\text{(358)}\]
Similarly \( \mathbf{C} \) is the set of moments arising at the joints due to unit \( \mathbf{r} \) displacements (rotations) \( \mathbf{U} \). By using the stiffnesses of the unit panel (or by carrying out the matrix multiplication \( a_k^T a_k \)) we find for the moments due to \( U_9 = 1 \)
\[
\begin{align*}
c_{0,9} &= \frac{E' \theta^2 dt}{3l} + \frac{E' \theta^2 dt}{6l} + \frac{2Ght^2}{3d} + \frac{Ghtl^2}{3} \\
c_{0,10} &= \frac{E' \theta^2 dt}{3l} + \frac{E' \theta^2 dt}{6l} + \frac{Ghtl^2}{12l} + \frac{Ghtl^2}{6d} + \frac{Ghtl^2}{6} \\
c_{10,9} &= \frac{E' \theta^2 dt}{6l} + \frac{Ghtl^2}{3d} \\
c_{11,9} &= \frac{E' \theta^2 dt}{6l} + \frac{Ghtl^2}{3d} \\
c_{11,10} &= \frac{E' \theta^2 dt}{12l} + \frac{Ghtl^2}{12l} \\
c_{12,9} &= \frac{E' \theta^2 dt}{8} + \frac{Ghtl^2}{8} \\
c_{12,10} &= \frac{E' \theta^2 dt}{8} + \frac{Ghtl^2}{8}
\end{align*}
\]
\[\text{(359)}\]
Finally \( \mathbf{C} \) is the set of moments arising at the joints due to the vertical displacements. Again only the webs are involved, and we obtain easily
\[
\begin{align*}
c_{0,9} &= -\frac{Ght}{l} \\
c_{0,10} &= -\frac{Ght}{d}
\end{align*}
\]
\[\text{(360)}\]
9. ILLUSTRATIONS TO THE ANALYSIS OF REDUNDANT STRUCTURES BY THE FORCE METHOD

In this section we present two very simple applications of the force method developed in Section 8C. The first example shows how to determine the statically equivalent stress system in an N cell tube typical of a wing structure.† With the direct stresses distributed according to E.T.B., we find using the $\delta_{mn}$ method the corresponding shear flow distribution for the multi-cell cross-section with the assumption that the ribs are rigid in their plane. Naturally, the axial constraint stresses and the effect of rib deformation remain to be investigated but the statically equivalent stress system derived here is particularly useful, being, in general, a reasonable first approximation when the structural design has to be based merely on a statically equivalent stress system. This, of necessity, has been the approach in most cases up to the present owing to the difficulty of computing highly redundant systems.

Although the problem may, with some justice, be described as trivial in relation to the powerful analytical techniques of Section 8 it is astonishing to see what an unfortunate treatment it often receives—even today.

The second example analyses—first by the $\delta_{mn}$ method—the axial constraint stresses in a four flange tube with shear carrying walls and deformable ribs under arbitrary loading at the rib stations. The solution of the same problem is also obtained by the matrix method and the effect of a cut-out is investigated by the $H$ matrix device of p. 41. The 'exact' flexibility of the structure is derived and compared with that given by E.T.B. and Bredt-Batho. A thermal loading is also investigated by the matrix method.

These two problems are only meant as preliminary illustrations of the force method. Theoretical applications of the $\delta_{mn}$ method are investigated in Part II. More complicated structures, particularly suitable for showing the power of the matrix formulation of the theory, will be analysed in a later publication.

(a) Shear Flow Distribution in a Multi-cell Tube Due to E.T.B. Direct Stresses

Consider the uniform cylindrical and multi-cell tube of the type shown in Fig. 52 subjected to shear forces $S_y, S_z$ through a force $T_x$ about the point $O$. Find the corresponding distribution of the shear flow if the direct stresses are given by the engineer's theory of bending; thus axial constraint stresses due to restrained warping are ignored. Instead of referring the loads to the arbitrary point $O$ we may alternatively give the point $O$ through which the resultant of all transverse forces is acting, i.e. $T_x = 0$.

The direct stress $\sigma$ due to bending moments $M_y, M_z$ about the axes $Gx, Gy$ through the section centroid $G$ and parallel to $Ox, Oy$, is

$$\sigma = M_y \gamma + M_z \zeta$$

where

$$M_y = M_y - M_z (\frac{I_y}{I_x} I_z - \frac{I_z}{I_x} I_y)$$

are the effective bending moments for the chosen axes which are, in general, not the principal axes of the cross-section. Physically $M_y, M_z$ are the combinations of $M_y, M_z$ which give rise to pure bending strains about $Gx, Gy$ respectively. We could alternatively restrict ourselves to principal axes of the cross-section but in practice, unless these are obvious, it is preferable to use Eqs. (a1), (a2). They are not only more convenient from the computational point of view but permit also the retention of parallel axes $Ox, Oy$ at all cross-sections of a wing regardless of the change of directions of the principal axes.

The condition of equilibrium in the $z$-direction of an element $dx dz$ of a wall gives,

$$dy + dz = 0$$

where

$$q = \frac{\sigma_y dx dz}{t_z} = \text{shear flow in the wall}$$

$t_z$=effective direct stress carrying thickness of the wall (i.e. including an allowance for the stringers).

Similarly, we find from the equilibrium of an element $dc$ of a typical flange $g$ placed at a web-cover intersection (see Fig. 52),

$$q + \frac{\partial q}{\partial s} ds dz$$

is the shear flow in the wall.

Eqs. (a3) and (a4) yield, except for a constant of integration in each of the $N$ cells, a shear flow distribution $q$ whose resultants in the $y$ and $x$ directions are $S_y$ and $S_z$ respectively. Since there remains one further equilibrium condition,

$$T_x = 0$$

for torque about $O$, the degree of redundancy is $(N-1)$; (the integration in (a5) extends over all walls and the normal $p_s$ is taken positive (negative) if movement along $s$ leads to an ant clockwise (clockwise) rotation about $O$). It follows that the shear flow distribution in a single-cell tube under prescribed transverse loading is statically determined once we stipulate that the direct stresses are distributed as per E.T.B.

For the analysis of the general case of an $N$ cell tube we find it more convenient to use a slightly different approach. Thus, for the moment, we prescribe instead of the torque equilibrium condition, the rate of twist

$$\phi = \frac{d\theta}{dz}$$

in all cells. The prescribed $\phi$ may be considered as an initial 'give' experienced by the ribs maintaining the shape of the cross-section and is subsequently determined from the torque equilibrium equation. Our modified problem has now $N$ redundancies. The basic system is obtained by cutting the wall in each of the $N$ cells and the unknowns

$$X_1, X_2, \ldots, X_N$$

are then the shear flows at the cuts. They are determined from the compatibility equations,

$$\sum_{r=1}^{N} \delta_{rr} X_r + \delta_{m} = 0$$

which express the conditions of zero relative warping $\delta$ at the cuts.*

* We denote here the warping by the unconventional symbol $\phi$ to apply directly Eqs.(282) in their original notation.
The shear flow distribution $q_s$ in the open tube forming the basic system is obtained from Eqs. (a3) and (a4). Integrating (a3) with respect to $s$ and using Eq. (a2) we find

$$q_s = \frac{S_y}{I_{y'}} + \frac{S_z}{I_{z'}}$$

where

$$S_{y} = S_{y} - S_{y}(l_{y}/l_{1}) \quad S_{z} = S_{z} - S_{z}(l_{z}/l_{1})$$

and

$$D_{x} = \frac{-f_{x}t_{d}q_{s}}{l_{d}} \quad D_{y} = \frac{-f_{y}t_{d}q_{s}}{l_{d}}$$

To determine completely the $D_s$ distributions from Eqs. (a10) we require also the equilibrium conditions of the type (a4) at each joint of spar web and cover. Using (a1) and (a8) in (a4) we have

$$\begin{align*}
(D_{x} - D_{x}') + D_{s} + B_{y} & = 0 \\
(D_{y} - D_{y}') - D_{s} & = 0
\end{align*}$$

The positive directions of $q$ and $s$ are indicated on Fig. 52.

**Choice of Basic System**

The reduction of the multi-cell tube to an open section can, of course, be achieved in a variety of ways. For example, we may cut the upper or lower cover in each cell or we may cut $N$ of the vertical walls (see Fig. 52). However, consideration of the form of the $\delta$'s shows that the compatibility equations (a7) are very much simpler for the former choice. We confirm this immediately by applying the unit load method for the calculation of the $\delta$'s which measure the relative warping at the cuts. Thus, if we apply unit shear flows at each of the cuts of the open section of Fig. 52 we produce merely constant shear flow around each of the individual cells.

**The Redundant Shear Flows**

$$\begin{align*}
&\text{Transverse loading } S_x \text{ through } D \\
&\quad S_x = 0
\end{align*}$$

$$\begin{align*}
&\text{Transverse loading } S_y \text{ through } D \\
&\quad S_y = 0
\end{align*}$$

**Fig. 53.—Effective shear forces for bending about non-principal axes**

If we consider now the total transverse loading $S_x$ and $S_y$ as acting through $D$ and split it into the two component loads $S_x'$ and $S_y'$ (see Fig. 53) we can express the statically indeterminate shear flows in the form

$$X_{M} = q_{M} = \frac{S_{x}'}{I_{x'}} + \frac{S_{y}'}{I_{y'}}$$

where the $D_{x}$ and $D_{y}$ are unknown. For the basic system of Fig. 52 the shear flows in the $M$ cell of the actual system can then be written as:

- external walls $q = q_s + Y_M$
- web between $M - 1$ and $M$ cells $q = q_s + Y_{M-1} - Y_M$
- web between $M$ and $M + 1$ cells $q = q_s + Y_M - Y_{M+1}$

We may always put the total shear flow in the form

$$q = \frac{S_{x}'}{I_{x'}} + \frac{S_{y}'}{I_{y'}}$$

and Eqs. (a12), (a13), (a14) yield for the cross-sectional function $D_e$ in the $M$ cell external walls

$$D_e = D_{x} + D_{y}$$

web between $M - 1$ and $M$ cells

$$D_e = D_{x} + D_{x-1} - D_{y}$$

web between $M$ and $M + 1$ cells

$$D_e = D_{x} + D_{x} + D_{x+1} - D_{x+1}$$

Similar equations may be written down for $D_y$.

The $\delta$ Coefficients for the Basic System of Fig. 52

The $\delta_{M}$ coefficients consist of two parts $\delta_{M}$ and $\delta_{M}$ corresponding to the shear flow $q_s$ in the basic system and the initial 'give' $\phi$ (see also Eq. (177)). We have:

$$\delta_{M} = \delta_{M} + \delta_{M}$$

Application of the unit load method yields immediately for $\delta_{M}$ -

$$\delta_{M} = \left[ \frac{q_s}{G_{M}} \right]$$

where $z_M$ is the circumferential anticlockwise co-ordinate in the $M$ cell and the integral

$$\int \left[ \frac{q_s}{G_{M}} ds_M \right]$$

denotes integration over the $M$ cell. It must be emphasized that the sign convention used for $q_s$ and $s$ in the basic (open) system is in opposition to $s_M$ in the left-hand wall of the $M$ cell and hence in evaluating the above term the sign of $q_s$ must be reversed over this wall.

Similarly, we obtain for the relative warping $\delta_{M}$ due to $\phi$ (see Fig. 54).

$$\delta_{M} = -2\Omega_{M} \phi$$

The standard derivation of Eqs. (a17) and (a18) is by kinematics. Thus, Eq. (a17) is obtained by integration of the shear strain expression corresponding to zero rate of twist. Also, we find (a18) from the condition of zero shear strain along the middle line of the wall.*

The unit load method also yields directly the coefficients of the unknown $X's$. Thus, the relative warping $\delta_{M}$ due to unit shear flow in the $M$ cell is

$$\delta_{M} = \int_{M}^{M} ds_M \beta_G$$

where

$$\beta_G = \int_{M}^{M} ds_M$$

All cross-terms but $\delta_{M}$, $M-1$, and $\delta_{M}$, $M+1$ vanish since only unit shear flows in adjoining cells act over a common wall. We find

$$\delta_{M}, M-1 = \delta_{M}, M+1 = \frac{2\Omega}{G} \phi$$

where

$$\beta_{M-1} = \int_{M-1}^{M} ds_M \quad \beta_{M+1} = \int_{M}^{M+1} ds_M$$

are the integrals extending over the common walls (spar webs) of cells $M-1$, $M$, and $M$, $M+1$ respectively. The minus sign arises since the shear flows due to unit redundancies have opposite signs in the common walls.

**Determination of Redundant Shear Flows**

We denote the unknown rates of twist associated with $S_x$ and $S_y$ by $\phi_x$ and $\phi_y$ respectively. For the loading due to $S_x$ the $M$'th compatibility Eq. (a7), which expresses the condition of zero relative warping at the $M$'th cut is obviously

$$S_x \left[ \frac{D_{x} ds_M - 2\Omega_{x} \phi_y}{S_y} \right]$$

We obtain hence the set of $N$ equations in the $N$ unknowns $D_{x}$

$$\beta_{x} D_{x} = \beta_{x} D_{x}$$

or

$$\beta_{x} D_{x} = \beta_{x} D_{x}$$

where

$$\beta_{x} D_{x} = \beta_{x} D_{x}$$

for $M = 1, 2, \ldots, N$.

**Fig. 54.—Warping in $M$th cell of basic system due to rate of twist $\phi$**

These formulae are usually derived more simply from the condition of equal rate of twist
\[
\phi = \frac{d\theta}{dz} = -\frac{1}{2\Omega_M G_i} \frac{G_i}{ds} M
\]
(a24)
in all \( N \) cells in the actual system. In this approach we specify initially zero relative warping at all cuts and express the compatibility condition by the equality of \( \phi \) in all cells. On the other hand in our present method we specify initially the same rate of twist \( \phi \) in all cells and express the compatibility by the condition of zero relative warping at the cuts.

The solution of Eqs. (a23) and the corresponding ones for \( D_i, M \) is straightforward and may be put in the form
\[
D_M = d_M + a_M G_i \phi_i
\]
\[
D_M = (a25)
\]
\[
d_M, d_M \] are the values of the redundancies corresponding to zero rate of twist. They yield hence the shear flow distribution \( q \) —commonly known as the engineers' theory of bending shear flows—due to transverse shear forces acting through the shear centre \( E_i \). \( q \) may be written
\[
q_E = \frac{d_M}{I_E} + \frac{d}{I_E}
\]
(a26)
where the cross-sectional functions \( d_M, d \) are obtained from Eqs. (a15) with \( d_M \) in place of \( D_M \). The co-ordinates \( x_E, y_E \) of \( E_i \) can be determined from a consideration of the two loading cases \( S_x \) and \( S_y \) through \( E_i \) shown in FIG. 55. We find
\[
I_{xx} = I_{yy} = I_{xy} = I_{xy} = 0
\]
\[
I_{xx} = I_{yy} = I_{xy} = I_{xy} = 0
\]
(a27)
where the integrals extend over all walls and the normal \( p_i \) is taken positive (negative) if movement along the positive \( x \) direction produces anticlockwise (clockwise) rotation about \( O \).

Transverse loading \( S \) through shear centre \( E_i \), \( S = 0 \)
Transverse loading \( S \) through shear centre \( E_i \), \( S = 0 \)

Fig. 55.—Effective shear forces applied through the shear centre \( E_i \)

The shear centre allows us to define the loading alternatively by the shear forces through and the torque
\[
T_i = \frac{S_i}{S_i} x_E + \frac{S_i}{S_i} y_E + \frac{S_i}{S_i} y_E + \frac{S_i}{S_i} y_E
\]
(a28)
about \( E_i \).

To obtain the total redundancies \( D_M, D_M, D_M, D_M, D_M \) we have still to determine the rates of twist \( \phi \) and \( \phi \). Observing that the distribution coefficients \( a_m \) are the same in the two equations (a25)—a direct result of the rigid diaphragm assumption which allows us to displace the transverse forces anywhere along their lines of action—we can combine the indeterminate shear flows due to \( S_x \), \( S_y \) into a single set
\[
q_E = \frac{S_i}{S_i} a_m G_i \phi_i
\]
(a29)
where
\[
\phi = \frac{d_M}{d_M} = \phi \phi
\]
(a30)
the total rate of twist due to the loading.

To derive the rate(s) of twist we may apply the equilibrium condition about \( D \) (see FIG. 53)
\[
[S_i p d s] = 0
\]
(a31)
where the sign of \( p_i \) is defined as for \( p_i \). Applying Eq. (a31) to the \( S_x \) loading we obtain
\[
D_M p d s + 2 S_x d_x \Omega_M + 2 S_y d_y \Omega_M \cdot G_i \phi_i
\]
or
\[
N = 2 \Sigma a_m \Omega_M \cdot G_i \phi_i
\]
(a32)
Similarly
\[
N = 2 \Sigma a_m \Omega_M \cdot G_i \phi_i
\]
(a33)
and remembering that the engineers' theory of bending shear flows have no torque contribution about \( E_i \), we obtain
\[
T_i = \frac{S_i}{S_i} - \frac{S_i}{S_i} + \frac{S_i}{S_i} - \frac{S_i}{S_i}
\]
(a34)

Hence the redundants \( q \) due to the shear force \( T_i \) are given by:
\[
q_E = \frac{S_i}{S_i} - \frac{S_i}{S_i} + \frac{S_i}{S_i}
\]
(a35)

web between \( M-1 \) and \( M \) cells
\[
q_E = \frac{S_i}{S_i} - \frac{S_i}{S_i} + \frac{S_i}{S_i}
\]
(a36)

web between \( M \) and \( M+1 \) cells
\[
q_E = \frac{S_i}{S_i} - \frac{S_i}{S_i} + \frac{S_i}{S_i}
\]
(a37)
The shear flow is, of course, constant in each wall between two consecutive joints. Thus, we have in the \( M^\prime \)th cell external walls
\[
q_E = \frac{S_i}{S_i} + \frac{S_i}{S_i}
\]
(a40)
where the engineers' theory shear flows are given by Eq. (a26).

With the chosen basic system Eqs. (a23) and (a37) are particularly well conditioned since the diagonal coefficients are predominant and at the most only three unknowns are involved in a given equation. Direct solution by elimination is quite easy and may be performed by slide rule even for a high degree of redundancy; application of the relaxation technique is superfluous. An additional virtue of the basic system of FIG. 52 is that the \( D_M \) distribution is quite close to the final \( d_M \) distribution and the values of the redundancies are small.

In finding on the other hand the \( d_M \) redundancies there is a conflict in the choice of the basic system. For when we cut the external wall in each cell the \( D_M \) distribution is vastly different from the final one although the equations are as well conditioned as for the \( d_M \) since the matrix \( D \) is the same in both cases. Naturally, the criterion of good conditioning of the equations is always the most important one. The alternative basic system in which the webs are cut gives a \( D_M \) distribution close to the actual \( d_M \) but the equations are not so easy to solve since each equation involves all the unknowns. To satisfy both the above requirements it is necessary to make composite cuts, i.e. to cut both upper and lower external walls and instead of the condition \( D_M = 0 \) at a cut to take say equal and opposite values of \( D_M \) at the two cuts of each cell. In any case the \( d_M \) distribution is, in general, of small importance and a more approximate solution is acceptable.

Generalizations of the Above Analysis

The method given above for the determination of the statically equivalent stress system in uniform cylindrical tubes may be generalized and applied to tubes with conical or non-conical taper and with sheet thicknesses and boom areas varying lengthwise. The angle of taper is taken, however, to be so small that \( \cos 2\theta \approx 1 \) and \( \sin 2\theta \approx 2\theta \).
from Eqs. (27). The engineers' theory of bending shears for transverse forces through the flexural axis are now

\[ q_e = \frac{Q}{r} \frac{d_1}{r} - \frac{Q_e}{r} \frac{d_1}{r} \]  \hspace{1cm} (a45)

where \( d_1 \) are obtained for the root dimensions from Eqs. (a23) for \( \phi_2 = 0 \).

If the shear centre \( E_s \) has been found we may calculate at any cross-section the torque \( T_k \) of all applied transverse forces about the flexural axis. It is then preferable to calculate the Bredt-Batho shears \( q_B \) of the total statically equivalent shear flow

\[ q = q_e + q_B \]  \hspace{1cm} (a4a)

by a slightly modified version of the method on p. 55. Thus, Eqs. (a37) for the redundant \( q_{BM} \) become

\[ \beta q_{BM} = \beta_{B11} q_{BY} = 2\Omega_L \rho s \phi \phi \]  \hspace{1cm} (a4b)

The solution of which may be written

\[ q_{BM} = a_q G\rho s \phi \phi \]  \hspace{1cm} (a47)

Next we deduce from the equilibrium condition about the flexural axis

\[ G\rho s \phi \phi = \frac{T_k}{2\rho s N_{A_s} \Omega_L} \]  \hspace{1cm} (a48)

Hence the shears \( q_{BM} \) are

\[ q_{BM} = \frac{T_k}{2\rho s N_{A_s} \Omega_L} \]  \hspace{1cm} (a49)

Finally, we derive the shear flow \( q_B \) in all walls from Eqs. (a36).

If the torque is given initially about an arbitrary axis \( V \) we can calculate \( T_k \) with the formula

\[ T_k = T_k - \rho q_x (x_B - x_A) - \rho q_y (y_B - y_A) \]  \hspace{1cm} (a50)

where \( x_A, y_A \) are the co-ordinates of the point \( A \) at the root.

2. Conical and Cylindrical Tubes with Arbitrary Variation of Boom Areas and Wall Thicknesses

Here we investigate conical or cylindrical tubes with an arbitrary lengthwise variation of skin thicknesses and boom areas. The engineers' theory shear flows are not any longer proportional to \( Q_e \) and \( Q_s \) and Eq. (a42) does not apply. The concept of a flexural axis, either straight or curved, is also not any longer strictly true.

Under this heading all cross-sectional dimensions, areas and functions are based on the current cross-section.

For the subsequent analysis we require the modified forms of the internal equilibrium conditions (a5) and (a4). Thus, we deduce immediately from the geometry of fig. 57 the equilibrium condition on a conical element on the surface

\[ \frac{\partial f}{\partial r} + \frac{3g}{r} \]  \hspace{1cm} (a51)

where

\[ f = at \]  \hspace{1cm} (a52)

Hence

\[ \frac{\partial g}{\partial r} + \frac{\partial f}{\partial s} \]  \hspace{1cm} (a53)

In practical calculations we find \( 3g/2 \) numerically by finite differences.
The equilibrium condition Eq. (a4) for a flange $B$, at the intersection of cover and web becomes here

$$\frac{dP_x}{dz} - q_w - q_{wT} + q_{wB} = 0 \quad \cdots \quad (a54)$$

where

$$P_x = B_x \sigma_x \quad \cdots \quad (a55)$$

To find the statically equivalent stresses for an arbitrary system of transverse loads defined by the shear forces (and associated bending moments)

$$S_x(M_x) \text{ and } S_y(M_y)$$

and the torque

$$T_A$$

about an axis VA we proceed as follows. First we equilibrate the applied bending moments by the E.T.B. direct stresses

$$\sigma = M_{xT} + M_{yT} \quad \cdots \quad (a1)$$

The co-ordinates $x$, $y$ from the centre of gravity and the corresponding moments of inertia may vary arbitrarily from section to section.

The statically equivalent shear flows are calculated as before in two parts

$$q_x = q_x + q_y \quad \cdots \quad (a56)$$

where $q_x$ and $q_y$ are in equilibrium with $Q_x$, $Q_y$, with the imposed condition of zero twist throughout the tube and additional shear flow necessary to balance the applied torque. The solution for an $N$ cell tube follows closely the method given initially and the selection of a basic system derives from similar considerations. The calculations must, of course, be repeated for every section to be analysed.

**Calculation of $q_x$:**

The shear flow $q_x$ in the basic system is obtained at all sections we want to analyse by integration from (a51a)

$$q_x = \int f_x ds \quad \cdots \quad (a57)$$

where $f_x$ is calculated from Eqs. (a1) and (a52). The necessary constants of integration to compute the shear flow in the webs, etc., are derived from equations of the type (a54). The $N$ unknown shear flows $q_{xM}$ at the cuts of each section are determined by the same method leading to Eq. (a23). We find

$$\beta_{M-1} q_{xM-1} + \beta_{M+1} q_{xM+1} = q_x \quad \cdots \quad (a58)$$

Note that all dimensions are based on the current cross-section. Having solved these equations we calculate the shear flow $q_x$ with Eqs. (a13).

**Calculation of $q_y$:**

The torque $T_y$ of the shear flows $q_yB$ about the $VA$ axis is

$$T_y = \int q_yB ds = \int q_y ds + 222 \Omega_{M} \Omega_{M} \quad \cdots \quad (a59)$$

where $P_x$ is the normal from the point $A$ to the tangent at the wall; note the usual sign convention. Hence the torque $T_y$ to be resisted by the shear flows $q_y$ is

$$T_y = T_A - T_x \quad \cdots \quad (a60)$$

The shear flows $q_y$ at any section may now be determined from Eqs. (a37) and (a29) with

$$q_y = \frac{T_y}{2 \Omega_M \Omega_M} \quad \cdots \quad (a61)$$

3. **Tubes with Non-Conical Taper**

For tubes with non-conical taper we can find the statically equivalent stress system

$$f_x \text{ and } q_x + q_y$$

by the method under (2). However, when finding the torque $T_y$ to be carried by the $q_y$ shear flows we must make allowance for the torque $T_x$ carried by the direct flow $f_x$ and the load boom. Thus

$$T_y = T_A - T_x - T_f \quad \cdots \quad (a62)$$

* $q_x$ may be regarded as quasi-engineers' shear flow; see also J. H. Aitken and P. C. Donn, loc. cit., p. 53 (Handbook).
With $D_{a}$ having (constant) values in the inter-cell webs only the six compatibility equations are formed easily and systematically by means of the arrangement in Table I. All values in the table are obtained directly from the dimensional data of Fig. 58 and the $D_{a}$ distribution of Fig. 59.

The equations for the unknown $D_{aM}$ are therefore (see Eqs. (22))

$$763D_{a1} - 106D_{a4} = -7200 + 700G_{a1}$$

$$-106D_{a1} + 471D_{a2} - 125D_{a3} = -7800 + 1110G_{a2}$$

$$-125D_{a1} + 400D_{a2} - 125D_{a4} = -7200 + 1080G_{a3}$$

$$-125D_{a2} + 466D_{a4} - 100D_{a5} = -7200 + 800G_{a4}$$

$$-100D_{a1} + 404D_{a3} - 62.5D_{a5} = -7200 + 640G_{a5}$$

$$-62.5D_{a2} + 908D_{a4} = -7200 + 3120 + 420G_{a6}$$

(6a3)

Also the equilibrium Eq. (31) for torque about $D$ is

$$700D_{a2} + 1110D_{a3} + 1200D_{a5} + 1080D_{a6} + 780D_{a2} + 420D_{a4} - 94600 = 0$$

(6a4)

Writing the solution of Eqs. (6a3) in the form

$$D_{aM} = d_{aM} + \frac{G_{a1}I_{a1}}{S_{a}}$$

(6a5)

$G_{a1}I_{a1}/S_{a}$ is determined from (6a4) and the results are as in Table II.

**Table II of Example 9a**

<table>
<thead>
<tr>
<th>$M$</th>
<th>$d_{aM}$</th>
<th>$\alpha_{M}$</th>
<th>$D_{aM}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>-12.17</td>
<td>1.46</td>
<td>-20.13</td>
</tr>
<tr>
<td>II</td>
<td>-19.62</td>
<td>3.90</td>
<td>-40.92</td>
</tr>
<tr>
<td>III</td>
<td>-1.15</td>
<td>4.52</td>
<td>-25.85</td>
</tr>
<tr>
<td>IV</td>
<td>+15.79</td>
<td>4.20</td>
<td>-7.16</td>
</tr>
<tr>
<td>V</td>
<td>+20.71</td>
<td>3.08</td>
<td>+3.89</td>
</tr>
<tr>
<td>VI</td>
<td>+4.88</td>
<td>0.68</td>
<td>+1.16</td>
</tr>
</tbody>
</table>

From Eq. (6a4) $G_{a1}I_{a1}/S_{a} = 5.46$

Adding these $D_{aM}$ to the $D_{a}$ we obtain the total $D_{a}$ distribution due to $S_{a}$ through $D$. Since $D_{a}$ is zero in the outer walls we have simply (see also Eqs. (13))

for outer walls $D_{a} = D_{aM}$

for web between $M$ and $M + 1$ cells

$D_{a} = D_{aM} + D_{aM - D_{aM + 1}}$

The total actual shear flows under the 10000 kg. load at $D$ are

$q = \frac{S_{a}D_{a}}{I_{a}} = 7992$ $D_{a} = 1.25D_{a}$

(6a4)

in kg./cm., the actual values being shown diagrammatically in Fig. 60.

**Shear Centre**

Since the present cross-section is singly symmetrical, the shear centre lies on the axis of symmetry and the first of Eqs. (27) (with $I_{a2} = 0$) gives immediately the $x_{G}$ co-ordinate. Alternatively we note that the shear flows $d_{a}$ have a resultant $S_{a} = I_{a}$ through the shear centre and we can modify slightly Eq. (6a4) for torque about $D$ to give

$$700D_{a2} + 1110D_{a3} + 1200D_{a5} + 1080D_{a6} + 780D_{a2} + 420D_{a4} + 94600 = f_{a}(x_{G} - x_{D})$$

Therefore $x_{G} - x_{D} = 12.3$ cm.
(b) **The Four-Boom Tube with Deformable Ribs and some more General Structures**

In this example we determine, using the force method of analysis, the 'exact' stress distribution in the idealized four-boom tube shown in fig. 61. The investigation is carried out first by the δm-method and illustrated in a numerical example. Subsequently, we analyse the same example by the general matrix method of section 8C. We show how the matrix formulation may be used with advantage in the more interesting case of a six-boom tube with or without intermediate spar web. It is hoped that these simple applications of the matrix method will condition the reader to the new ideas and show him their power and basic simplicity.

Consider the cylindrical tube of fig. 61 with a singly symmetrical trapezoidal cross-section the flange cross-sectional areas £ and the wall thicknesses $t$ of which may vary arbitrarily length-wise. Loads are applied only at the rib positions

$$R_1, R_2, \ldots, R_i, \ldots, R_n$$

in the form of shear forces

$$R_{a1}, R_{a2}, \ldots, R_{an}, R_{bn}$$

and moments

$$M_{a1}, M_{a2}, \ldots, M_{an}, M_{bn}$$

at the front (a) and rear (b) spars. Following the general discussion on the idealization of aircraft structures given in Section 8 C (pp. 37 and 40) we assume that the walls carry only shear stresses (the direct stress carrying ability is allowed for by suitably increasing the flange areas). The shear flow is hence constant in any field of each bay since changes in the z-direction can only be brought about at the ribs. It follows then that the end loads in the flanges vary linearly between ribs and that a knowledge of the flange loads at the rib positions suffices to determine them everywhere. Having found the flange loads the corresponding shear flows are determined easily from the flange load gradients and the condition of equilibrium with the applied shear force and torque.

In the tube shown in fig. 61 there are $n$ ribs including those at the free and built-in ends. At each rib position there are four flange loads and only three equilibrium equations are available for their determination. Hence, noting that the flange loads at the tip are zero, the degree of redundancy is $(n-1)$. This trivial result is confirmed by the general Eq. (247a) on p. 38 by substituting $N = 1, B = 4, \alpha = n$.

In selecting the basic system many choices are open to us. We may, for example, make a single cut in one of the flanges at each rib station to reduce the structure to a statically determinate three-flange basic system. Here, however, we calculate the statically equivalent stress system by the E.T.B. and the Bredt-Batho theory of torsion, a more general example of which was investigated in Section 9a. In this choice, instead of making a single cut we have, in fact, cut all the flanges to allow the relative warping consistent with the statically equivalent stresses while at the same time the direct stresses are transmitted across the cuts; see also the discussion in Section 8C, pp. 32 and 33. The redundancies then consist of self-equilibrating flange load systems at the $(n-1)$ rib stations. A suitable and symmetrical measure of such a system is the boom load function $P$ introduced by Argyris and Dunne.† We prefer to use instead a slightly variant of $P$, the $Y$-system introduced on p. 39. The $(n-1)$ redundant $Y$ are determined from the compatibility conditions of warping at the $(n-1)$ rib stations.

As a further alternative procedure we could, of course, choose as basic system the very simple structure consisting only of the two spars acting independently, i.e. we cut top and bottom covers of the tube. However, except for very flexible covers and ribs our previous choice is much closer to the final correct solutions. The latter method is hence to be preferred when the design work has to be checked solely by a good statically equivalent stress system as is, in general, the case when no automatic digital computer is available. Subsequently the exact distribution, if found at all, is obtained only after completion of the design work—usually by a more clumsy version of the last method.

On the other hand when we use the matrix formulation in conjunction with a digital computer and derive the complete stress distribution as a single process it is preferable to select the simplest possible $b$ matrix (see also the discussion on p. 39). Then the basic system formed by the independent spars is obviously indicated. An important advantage of this system is the absence of rib stresses if external loads are only applied in the plane of the spars. In multispot construction the simplicity of the basic system consisting of independent spars in comparison with the multi-cell system of Example 9a is even more striking. If it is necessary to take into account external loads applied at intermediate points in the ribs and/or if spars are interrupted we may select as basic system for the matrix type of analysis the grid formed by spars and ribs (without covers)—still a very simple structure in which to find the $b$ matrix.

(1) **Analysis by the δm Method**

**The statically equivalent stress system**

Following our previous discussion we select here the E.T.B. (or quasi E.T.B.) and Bredt-Batho stresses as statically equivalent stress system. The flange loads are calculated at the rib stations with the effective area $B_i$, the $m'$th flange load at the $i$th rib is

$$P_{mi} = M_{mi} \frac{Y_{ai}}{I_i}$$

where $M_{mi}$ is the bending moment at the $i$th rib due to the applied loads $R_{ai}$.

† loc. cit. p. 38.

* Actually, we may consider this basic system as also derived by a single cut from the given system.

† The particular case of the four-flange single-cell tube under a given loading has also been treated by W. J. Goody in 'Two-jet Wing Stress Analysis', *Aircraft Engineering*, Vol. XXI, No. 427, p. 287, September 1948; No. 288, p. 352, October 1949; No. 289, p. 358, November 1949. There the author uses the Castigliano technique to formulate equations analogous to the three- and three-point equations given here with essentially the same basic idealizations but with the effect of taper included. A comparison of the theoretical results with experimental strain measurements is also given.