

#### Learning Objectives

• To discuss the dynamics of a single-degree-of freedom spring-mass system.

- To derive the finite element equations for the timedependent stress analysis of the one-dimensional bar, including derivation of the lumped and consistent mass matrices.
- To introduce procedures for numerical integration in time, including the central difference method, Newmark's method, and Wilson's method.
- To describe how to determine the natural frequencies of bars by the finite element method.
- To illustrate the finite element solution of a timedependent bar problem.

# Chapter 16 – Structural Dynamics



### Learning Objectives

- To develop the beam element lumped and consistent mass matrices.
- To illustrate the determination of natural frequencies for beams by the finite element method.
- To develop the mass matrices for truss, plane frame, plane stress, plane strain, axisymmetric, and solid elements.
- To report some results of structural dynamics problems solved using a computer program, including a fixed-fixed beam for natural frequencies, a bar, a fixed-fixed beam, a rigid frame, and a gantry crane-all subjected to time-dependent forcing functions.

### Introduction

This chapter provides an elementary introduction to timedependent problems.

We will introduce the basic concepts using the singledegree-of-freedom spring-mass system.

We will include discussion of the stress analysis of the onedimensional bar, beam, truss, and plane frame.

## Structural Dynamics

### Introduction

- We will provide the basic equations necessary for structural dynamic analysis and develop both the lumped- and the consistent-mass matrices involved in the analyses of a bar, beam, truss, and plane frame.
- We will describe the assembly of the global mass matrix for truss and plane frame analysis and then present numerical integration methods for handling the time derivative.
- We will provide longhand solutions for the determination of the natural frequencies for bars and beams, and then illustrate the time-step integration process involved with the stress analysis of a bar subjected to a time dependent forcing function.

### Dynamics of a Spring-Mass System

In this section, we will discuss the motion of a single-degree-offreedom spring-mass system as an introduction to the dynamic behavior of bars, trusses, and frames.

Consider the single-degree-of-freedom spring-mass system subjected to a time-dependent force F(t) as shown in the figure below.



The term *k* is the stiffness of the spring and *m* is the mass of the system.

## **Structural Dynamics**

### **Dynamics of a Spring-Mass System**

The free-body diagram of the mass is shown below.



The spring force T = kx and the applied force F(t) act on the mass, and the mass-times-acceleration term is shown separately.

Applying Newton's second law of motion, f = ma, to the mass, we obtain the equation of motion in the *x* direction:

$$F(t) - kx = m\ddot{x}$$

where a dot ( ` ) over a variable indicates differentiation with respect to time.

### **Dynamics of a Spring-Mass System**

The standard form of the equation is:  $m\ddot{x} + kx = F(t)$ 

The above equation is a second-order linear differential equation whose solution for the displacement consists of a homogeneous solution and a particular solution.

The homogeneous solution is the solution obtained when the right-hand-side is set equal to zero.

A number of useful concepts regarding vibrations are available when considering the free vibration of a mass; that is when F(t) = 0.

## **Structural Dynamics**

### **Dynamics of a Spring-Mass System**

Let's define the following term:  $\omega^2 = \frac{k}{m}$ 

The equation of motion becomes:  $\ddot{x} + \omega^2 x = 0$ 

where  $\omega$  is called the *natural circular frequency* of the free vibration of the mass (radians per second).

Note that the natural frequency depends on the spring stiffness k and the mass m of the body.

### Dynamics of a Spring-Mass System

The motion described by the homogeneous equation of motion is called *simple harmonic motion*. A typical displacement -



where  $x_m$  denotes the maximum displacement (or **amplitude** of the vibration).

## **Structural Dynamics**

### Dynamics of a Spring-Mass System

The time interval required for the mass to complete one full cycle of motion is called the **period** of the vibration  $\tau$  (in seconds) and is defined as:

$$\tau = \frac{2\pi}{\omega}$$

The frequency in hertz (Hz = 1/s) is  $f = 1/\tau = \omega/(2\pi)$ .

# Dynamics of a Spring-Mass System



# Structural Dynamics

## Dynamics of a Spring-Mass System



### **Dynamics of a Spring-Mass System**



# Structural Dynamics

## Dynamics of a Spring-Mass System



## **Dynamics of a Spring-Mass System**



# Structural Dynamics

## Dynamics of a Spring-Mass System



### **Direct Derivation of the Bar Element**

Let's derive the finite element equations for a time-dependent (dynamic) stress analysis of a one-dimensional bar.

### Step 1 - Select Element Type

We will consider the linear bar element shown below.



where the bar is of length *L*, cross-sectional area *A*, and mass density  $\rho$  (with typical units of *Ib-s*<sup>2</sup>/*in*<sup>4</sup>), with nodes 1 and 2 subjected to external time-dependent loads:  $f_x^{e}(t)$ 

## Structural Dynamics

### **Direct Derivation of the Bar Element**

### Step 2 - Select a Displacement Function

A linear displacement function is assumed in the *x* direction.

$$u = a_1 + a_2 x$$

The number of coefficients in the displacement function,  $a_i$ , is equal to the total number of degrees of freedom associated with the element.

We can express the displacement function in terms of the shape functions:

$$u = \begin{bmatrix} N_1 & N_2 \end{bmatrix} = \begin{cases} u_1 \\ u_2 \end{cases} \qquad \qquad N_1 = 1 - \frac{x}{L} \qquad \qquad N_2 = \frac{x}{L}$$

### **Direct Derivation of the Bar Element**

### Step 3 - Define the Strain/Displacement and Stress/Strain Relationships

The stress-displacement relationship is:

$$\{\varepsilon_x\} = \frac{du}{dx} = [B]\{d\}$$

where:  $[B] = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \quad \{d\} = \begin{cases} u_1 \\ u_2 \end{cases}$ 

The stress-strain relationship is given as:

$$\{\sigma_x\} = [D]\{\varepsilon_x\} = [D][B]\{d\}$$

## Structural Dynamics

### **Direct Derivation of the Bar Element**

### Step 4 - Derive the Element Stiffness Matrix and Equations

The bar element is typically not in equilibrium under a timedependent force; hence,  $f_{1x} \neq f_{2x}$ .

We must apply Newton's second law of motion, *f* = *ma*, to each node.

Write the law of motion as the external force  $f_x^e$  minus the internal force equal to the nodal mass times acceleration.

### **Direct Derivation of the Bar Element**

### Step 4 - Derive the Element Stiffness Matrix and Equations

Therefore:

$$f_{1x}^{e} = f_{1x} + m_1 \frac{\partial^2 u_1}{\partial t^2} \qquad f_{2x}^{e} = f_{2x} + m_2 \frac{\partial^2 u_2}{\partial t^2}$$

where:

$$m_{1} = \frac{\rho AL}{2} \qquad m_{2} = \frac{\rho AL}{2}$$
Mass lumped to the nodes
$$\begin{cases} f_{1x}^{e} \\ f_{2x}^{e} \end{cases} = \begin{cases} f_{1x} \\ f_{2x} \end{cases} + \begin{bmatrix} m_{1} & 0 \\ 0 & m_{2} \end{bmatrix} \begin{cases} \frac{\partial^{2} u_{1}}{\partial t^{2}} \\ \frac{\partial^{2} u_{2}}{\partial t^{2}} \end{cases}$$

## Structural Dynamics

#### **Direct Derivation of the Bar Element**

### Step 4 - Derive the Element Stiffness Matrix and Equations

If we replace {*f*} with [*k*]{*d*} we get:  $\{f^{e}(t)\} = [k]\{d\} + [m]\{\ddot{d}\}$ 

Where the elemental stiffness matrix is:

$$\begin{bmatrix} k \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \qquad \left\{ \ddot{d} \right\} = \frac{\partial^2 \left\{ d \right\}}{\partial t^2}$$

and the *lumped-mass matrix* is:

$$[m] = \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

### **Direct Derivation of the Bar Element**

### Step 4 - Derive the Element Stiffness Matrix and Equations

Let's derive the *consistent-mass matrix* for a bar element.

The typical method for deriving the consistent-mass matrix is the principle of virtual work; however, an even simpler approach is to use *D'Alembert's principle*.

The effective body force is:  $\{X^e\} = -\rho\{\ddot{u}\}$ 

The nodal forces associated with  $\{X^e\}$  are found by using the following:

$$\left\{f_b\right\} = \int_V [N]^T \{X\} \, dV$$

## Structural Dynamics

### **Direct Derivation of the Bar Element**

### Step 4 - Derive the Element Stiffness Matrix and Equations

Substituting {*X*<sup>e</sup>} for {*X*} gives:  $\{f_b\} = -\int_V \rho[N]^T \{\ddot{u}\} dV$ 

The second derivative of the *u* with respect to time is:

$$\{\dot{u}\} = [N]\{\dot{d}\} \qquad \qquad \{\ddot{u}\} = [N]\{\ddot{d}\}$$

where  $\dot{u}$  and  $\ddot{u}$  are the nodal velocities and accelerations, respectively.

### **Direct Derivation of the Bar Element**

### Step 4 - Derive the Element Stiffness Matrix and Equations

Therefore:  $\{f_b\} = -\int_V \rho[N]^T [N] \{\ddot{a}\} dV = -[m] \{\ddot{a}\}$ 

where:

 $[m] = \int_{V} \rho[N]^{T} [N] dV$ 

The mass matrix is called the *consistent mass matrix* because it is derived using the same shape functions use to obtain the stiffness matrix.

### Structural Dynamics

### **Direct Derivation of the Bar Element**

### Step 4 - Derive the Element Stiffness Matrix and Equations

Substituting the shape functions in the above mass matrix equations give:

$$[m] = \int_{V} \rho \begin{cases} 1 - \frac{x}{L} \\ \frac{x}{L} \end{cases} \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} dV$$
$$[m] = A\rho \int_{0}^{L} \begin{cases} 1 - \frac{x}{L} \\ \frac{x}{L} \end{cases} \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} dx$$

### **Direct Derivation of the Bar Element**

### Step 4 - Derive the Element Stiffness Matrix and Equations

Substituting the shape functions in the above mass matrix equations give:

$$[m] = \rho A \int_{0}^{L} \begin{bmatrix} \left(1 - \frac{x}{L}\right)^{2} & \left(1 - \frac{x}{L}\right)\frac{x}{L} \\ \left(1 - \frac{x}{L}\right)\frac{x}{L} & \left(\frac{x}{L}\right)^{2} \end{bmatrix} dx$$

Evaluating the above integral gives:

$$[m] = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

### Structural Dynamics

#### **Direct Derivation of the Bar Element**

### Step 5 - Assemble the Element Equations and Introduce Boundary Conditions

The global stiffness matrix and the global force vector are assembled using the nodal force equilibrium equations, and force/deformation and compatibility equations.

$$\{F(t)\} = [K]\{d\} + [M]\{\ddot{d}\}$$

where

$$[K] = \sum_{e=1}^{N} [k^{(e)}] \qquad [M] = \sum_{e=1}^{N} [m^{(e)}] \qquad \{F\} = \sum_{e=1}^{N} \{f^{(e)}\}$$

### **Numerical Integration in Time**

We now introduce procedures for the discretization of the equations of motion with respect to time.

These procedures will allow the nodal displacements to be determined at different time increments for a given dynamic system.

The general method used is called *direct integration*. There are two classifications of direct integration: *explicit* and *implicit*.

We will formulate the equations for two direct integration methods.

## Structural Dynamics

### Numerical Integration in Time

The first, and simplest, is an explicit method known as the *central difference method*.

The second more complicated but more versatile than the central difference method, is an implicit method known as the *Newmark-Beta (or Newmark's)* method.

The versatility of Newmark's method is evidenced by its adaptation in many commercially available computer programs.

#### **Central Difference Method**

The central difference method is based on finite difference expressions for the derivatives in the equation of motion.

For example, consider the velocity and the acceleration at time t:



where the subscripts indicate the time step for a given time increment of  $\Delta t$ .

### Structural Dynamics

#### **Central Difference Method**

The acceleration can be expressed in terms of the displacements (using a Taylor series expansion).

The first two terms of Taylor series of a function d(t) is:

$$d_{i+1} = d_i + \dot{d}_i \Delta t + \frac{\ddot{d}_i}{2} (\Delta t)^2 + \cdots$$

Solving for the acceleration gives:



Substitute the central difference for the first derivation

### **Central Difference Method**

Therefore, accelerations can be expressed in terms of the displacements as:

$$\left\{\ddot{d}_{i}\right\} = \frac{\left\{d_{i+1}\right\} - 2\left\{d_{i}\right\} + \left\{d_{i-1}\right\}}{\left(\Delta t\right)^{2}}$$

We generally want to evaluate the nodal displacements; therefore, we rewrite the above equation as:

 $\left\{\boldsymbol{d}_{i+1}\right\} = 2\left\{\boldsymbol{d}_{i}\right\} - \left\{\boldsymbol{d}_{i-1}\right\} + \left\{\ddot{\boldsymbol{d}}_{i}\right\}\left(\Delta t\right)^{2}$ 

The acceleration can be expressed as:

$$\left\{ \ddot{\boldsymbol{d}}_{i} \right\} = \left[ \mathbf{M} \right]^{-1} \left( \left\{ \mathbf{F}_{i} \right\} - \left[ \mathbf{K} \right] \left\{ \boldsymbol{d}_{i} \right\} \right)$$

## Structural Dynamics

### **Central Difference Method**

To develop an expression of  $d_{i+1}$ , first multiply the nodal displacement equation by **M** and substitute the above equation for  $\{\ddot{a}_i\}$  into this equation.

$$[\mathbf{M}]\{\mathbf{d}_{i+1}\} = \mathbf{2}[\mathbf{M}]\{\mathbf{d}_i\} - [\mathbf{M}]\{\mathbf{d}_{i-1}\} + (\{\mathbf{F}_i\} - [\mathbf{K}]\{\mathbf{d}_i\})(\Delta t)^2$$

Combining terms in the above equations gives:

$$[\mathbf{M}]\{\mathbf{d}_{i+1}\} = (\Delta t)^2 \{\mathbf{F}_i\} + [2[\mathbf{M}] - (\Delta t)^2 [\mathbf{K}]]\{\mathbf{d}_i\} - [\mathbf{M}]\{\mathbf{d}_{i-1}\}$$

To start the computation to determine  $\{d_{i+1}\}, \{\dot{d}_{i+1}\}, \text{and }\{\ddot{d}_{i+1}\}\)$  we need the displacement at time step *i*-1.

### **Central Difference Method**

Using the central difference equations for the velocity and acceleration:

$$\{ \dot{d}_i \} = \frac{\{ d_{i+1} \} - \{ d_{i-1} \}}{2(\Delta t)} \implies \{ d_{i+1} \} = 2(\Delta t) \{ \dot{d}_i \} + \{ d_{i-1} \}$$

$$\{ d_{i+1} \} = 2\{ d_i \} - \{ d_{i-1} \} + \{ \ddot{d}_i \} (\Delta t)^2$$

Solving for  $\{d_{i-1}\}$  gives:

$$\left\{\boldsymbol{d}_{i-1}\right\} = \left\{\boldsymbol{d}_{i}\right\} - (\Delta t)\left\{\dot{\boldsymbol{d}}_{i}\right\} + \left\{\ddot{\boldsymbol{d}}_{i}\right\} \frac{(\Delta t)^{2}}{2}$$

## Structural Dynamics

### **Central Difference Method**

### **Procedure for solution:**

- 1. Given:  $\{d_0\}, \{\dot{d}_0\}, \text{and}\{F_i(t)\}$
- 2. If the acceleration is not given, solve for  $\{\ddot{d}_0\}$

$$\left\{\boldsymbol{d}_{0}\right\} = \left[\mathsf{M}\right]^{-1} \left(\left\{\mathsf{F}_{0}\right\} - \left[\mathsf{K}\right]\left\{\boldsymbol{d}_{0}\right\}\right)$$

3. Solve for  $\{d_{-1}\}$  at  $t = -\Delta t$ 

$$\left\{\boldsymbol{d}_{-1}\right\} = \left\{\boldsymbol{d}_{0}\right\} - (\Delta t)\left\{\dot{\boldsymbol{d}}_{0}\right\} + \left\{\ddot{\boldsymbol{d}}_{0}\right\} \frac{(\Delta t)^{2}}{2}$$

### **Central Difference Method**

### Procedure for solution:

4. Solve for  $\{d_1\}$  at  $t = \Delta t$  using the value of  $\{d_{-1}\}$  from Step 3

$$[\mathbf{M}]\{d_{i+1}\} = (\Delta t)^{2}\{\mathbf{F}_{i}\} + [2[\mathbf{M}] - (\Delta t)^{2}[\mathbf{K}]]\{d_{i}\} - [\mathbf{M}]\{d_{i-1}\}$$
$$\{d_{1}\} = [\mathbf{M}]^{-1}\{(\Delta t)^{2}\{\mathbf{F}_{0}\} + [2[\mathbf{M}] - (\Delta t)^{2}[\mathbf{K}]]\{d_{0}\} - [\mathbf{M}]\{d_{-1}\}\}$$

### Structural Dynamics

#### **Central Difference Method**

#### Procedure for solution:

5. With  $\{d_0\}$  given and  $\{d_1\}$  determined in Step 4 solve for  $\{d_2\}$ 

$$\{\boldsymbol{d}_{2}\} = \left[\boldsymbol{\mathsf{M}}\right]^{-1}\left\{\left(\Delta t\right)^{2}\left\{\boldsymbol{\mathsf{F}}_{1}\right\} + \left[\boldsymbol{2}\left[\boldsymbol{\mathsf{M}}\right] - \left(\Delta t\right)^{2}\left[\boldsymbol{\mathsf{K}}\right]\right]\left\{\boldsymbol{d}_{1}\right\} - \left[\boldsymbol{\mathsf{M}}\right]\left\{\boldsymbol{d}_{0}\right\}\right\}$$

6. Solve for  $\{\ddot{d}_1\}$ :  $\{\ddot{d}_1\} = [M]^{-1}(\{F_1\} - [K]\{d_1\})$ 

- 7. Solve for  $\{\dot{d}_1\}$ :  $\{\dot{d}_1\} = \frac{\{d_2\} \{d_0\}}{2(\Delta t)}$
- 8. Repeat Steps 5, 6, and 7 to obtain the displacement, acceleration, and velocity for other time steps.



### Central Difference Method – Example Problem

Determine the displacement, acceleration, and velocity at 0.05 second time intervals for up to 0.2 seconds for the onedimensional spring-mass system shown in the figure below.



Consider the above spring-mass system as a single degree of freedom problem represented by the displacement *d*.

### Central Difference Method – Example Problem

#### **Procedure for solution:**

1. At time t = 0:  $\{d_0\} = 0$   $\{\dot{d}_0\} = 0$ 

2. If the acceleration is not given, solve for  $\left\{ \ddot{d}_{_{0}} \right\}$ 

$$\left\{ \ddot{d}_{0} \right\} = \left[ \mathsf{M} \right]^{-1} \left\{ \left\{ \mathsf{F}_{0} \right\} - \left[ \mathsf{K} \right] \left\{ d_{0} \right\} \right\}$$
$$\left\{ \ddot{d}_{0} \right\} = \frac{2,000 - 100(0)}{31.83} = 62.83 \text{ in/s}^{2}$$

## Structural Dynamics

#### Central Difference Method – Example Problem

#### Procedure for solution:

- 3. Solve for  $\{d_{-1}\}$  at  $t = -\Delta t$   $\{d_{-1}\} = \{d_0\} - (\Delta t)\{\dot{d}_0\} + \{\ddot{d}_0\}\frac{(\Delta t)^2}{2}$  $\{d_{-1}\} = 0 - (0.05)0 + \frac{(0.05)^2}{2}(62.83) = 0.0785$  in
- 4. Solve for  $\{d_1\}$  at t = 0.05 s using the value of  $\{d_{-1}\}$  from Step 3  $\{d_1\} = [\mathbf{M}]^{-1} \{ (\Delta t)^2 \{\mathbf{F}_0\} + [2[\mathbf{M}] - (\Delta t)^2 [\mathbf{K}]] \{d_0\} - [\mathbf{M}] \{d_{-1}\} \}$   $\{d_1\} = \frac{1}{31.82} \{ (0.05)^2 2,000 + [2(31.83) - (0.05)^2 (100)] 0 - (31.83) (0.0785) \}$ = 0.0785 in

### **Central Difference Method – Example Problem**

### **Procedure for solution:**

5. With {*d*<sub>0</sub>} given and {*d*<sub>1</sub>} determined in Step 4 solve for {*d*<sub>2</sub>}  
{*d*<sub>2</sub>} = [**M**]<sup>-1</sup>{(
$$\Delta t$$
)<sup>2</sup> {**F**<sub>1</sub>} + [2[**M**] - ( $\Delta t$ )<sup>2</sup> [**K**]]{*d*<sub>1</sub>} - [**M**]{*d*<sub>0</sub>}}  
{*d*<sub>2</sub>} =  $\frac{1}{31.82}$ {(0.05)<sup>2</sup> 1,500 + [2(31.83) - (0.05)<sup>2</sup> (100)](0.0785) - (31.83)(0)}  
= 0.274 in  
6. Solve for { $\ddot{d}_1$ }: { $\ddot{d}_1$ } = [**M**]<sup>-1</sup>({**F**<sub>1</sub>} - [**K**]{*d*<sub>1</sub>})  
{ $\ddot{d}_1$ } =  $\frac{1}{31.83}$ [1,500 - 100(0.0785)] = 46.88  $\frac{in}{s^2}$ 

## Structural Dynamics

### Central Difference Method – Example Problem

### Procedure for solution:

- 7. Solve for  $\{\dot{d}_1\}$ :  $\{\dot{d}_1\} = \frac{\{d_2\} \{d_0\}}{2(\Delta t)}$  $\{\dot{d}_1\} = \frac{0.274 - 0}{2(0.05)} = 2.74 \text{ in/s}$
- 8. Repeat Steps 5, 6, and 7 to obtain the displacement, acceleration, and velocity for other time steps.

## Central Difference Method – Example Problem

### **Procedure for solution:**

5. With {*d*<sub>1</sub>} given and {*d*<sub>2</sub>} determined in Step 4 solve for {*d*<sub>3</sub>}  
{*d*<sub>3</sub>} = [**M**]<sup>-1</sup>{(
$$\Delta t$$
)<sup>2</sup> {**F**<sub>2</sub>} + [2[**M**] - ( $\Delta t$ )<sup>2</sup> [**K**]]{*d*<sub>2</sub>} - [**M**]{*d*<sub>1</sub>}}  
{*d*<sub>3</sub>} =  $\frac{1}{31.82}$ {(0.05)<sup>2</sup>1,000 + [2(31.83) - (0.05)<sup>2</sup>(100)](0.274) - (31.83)(0.0785)}  
= 0.546 *in*  
6. Solve for { $\ddot{d}_2$ }: { $\ddot{d}_2$ } = [**M**]<sup>-1</sup>({**F**<sub>2</sub>} - [**K**]{*d*<sub>2</sub>})  
{ $\ddot{d}_2$ } =  $\frac{1}{31.83}$ [1,000 - 100(0.274)] = 30.56 *in*/s<sup>2</sup>

## Structural Dynamics

## Central Difference Method – Example Problem

### Procedure for solution:

7. Solve for 
$$\{\dot{d}_2\}$$
:  $\{\dot{d}_2\} = \frac{\{d_3\} - \{d_1\}}{2(\Delta t)}$   
 $\{\dot{d}_2\} = \frac{0.546 - 0.0785}{2(0.05)} = 4.68 \text{ in/s}$ 

8. Repeat Steps 5, 6, and 7 to obtain the displacement, acceleration, and velocity for other time steps.

### **Central Difference Method – Example Problem**

The following table summarizes the results for the remaining time steps as compared with the exact solution.

<i>t</i> (s)	F(t) (lb)	$\ddot{d}_i$ (in/s <sup>2</sup> )	ḋ <sub>i</sub> (in∕s)	$d_i(in)$	$d_i$ (exact)
0.00	2,000	62.8338	0.0000	0.0000	0.0000
0.05	1,500	46.8786	2.7428	0.0785	0.0718
0.10	1,000	30.5552	4.6787	0.2743	0.2603
0.15	500	13.9918	5.7923	0.5464	0.5252
0.20	0	-2.6815	6.0751	0.8535	0.8250
0.25	0	-3.6252	5.9174	1.1539	1.132

## Structural Dynamics

### **Central Difference Method – Example Problem**

Plotting the motion for about 4 s gives:



#### **Newmark's Method**

Newmark's equations are given as:

$$\left\{ \dot{\boldsymbol{d}}_{i+1} \right\} = \left\{ \dot{\boldsymbol{d}}_{i} \right\} + (\Delta t) \left[ (1-\gamma) \left\{ \ddot{\boldsymbol{d}}_{i} \right\} + \gamma \left\{ \ddot{\boldsymbol{d}}_{i+1} \right\} \right]$$

$$\left\{ \boldsymbol{d}_{i+1} \right\} = \left\{ \boldsymbol{d}_{i} \right\} + (\Delta t) \left\{ \dot{\boldsymbol{d}}_{i} \right\} + (\Delta t)^{2} \left[ \left( \frac{1}{2} - \beta \right) \left\{ \ddot{\boldsymbol{d}}_{i} \right\} + \beta \left\{ \ddot{\boldsymbol{d}}_{i+1} \right\} \right]$$

where  $\beta$  and  $\gamma$  are parameters.

- The parameter  $\beta$  is typically between 0 and  $\frac{1}{4}$ , and  $\gamma$  is often taken to be  $\frac{1}{2}$ .
- For example, if  $\beta = 0$  and  $\gamma = \frac{1}{2}$  the above equation reduce to the central difference method.

### Structural Dynamics

#### Newmark's Method

To find  $\{d_{i+1}\}$  first multiply the above equation by the mass matrix [M] and substitute the result into this the expression for acceleration. Recall the acceleration is:

$$\left\{ \ddot{\boldsymbol{\mathcal{O}}}_{i}\right\} = \left[\mathbf{M}\right]^{-1}\left(\left\{\mathbf{F}_{i}\right\} - \left[\mathbf{K}\right]\left\{\boldsymbol{\mathcal{O}}_{i}\right\}\right)$$

The expression  $[M]{d_{i+1}}$  is:

$$\begin{bmatrix} \mathbf{M} \end{bmatrix} \{ \mathbf{d}_{i+1} \} = \begin{bmatrix} \mathbf{M} \end{bmatrix} \{ \mathbf{d}_i \} + (\Delta t) \begin{bmatrix} \mathbf{M} \end{bmatrix} \{ \dot{\mathbf{d}}_i \} + (\Delta t)^2 \begin{bmatrix} \mathbf{M} \end{bmatrix} (\frac{1}{2} - \beta) \{ \ddot{\mathbf{d}}_i \} \\ + \beta (\Delta t)^2 \begin{bmatrix} \{ \mathbf{F}_{i+1} \} - \begin{bmatrix} \mathbf{K} \end{bmatrix} \{ \mathbf{d}_{i+1} \} \end{bmatrix}$$

Combining terms gives:

$$\left( \left[ \mathbf{M} \right] + \beta (\Delta t)^2 \left[ \mathbf{K} \right] \right) \left\{ \mathbf{d}_{i+1} \right\} = \beta (\Delta t)^2 \left\{ \mathbf{F}_{i+1} \right\} + \left[ \mathbf{M} \right] \left\{ \mathbf{d}_i \right\}$$
$$+ (\Delta t) \left[ \mathbf{M} \right] \left\{ \dot{\mathbf{d}}_i \right\} + (\Delta t)^2 \left[ \mathbf{M} \right] \left( \frac{1}{2} - \beta \right) \left\{ \ddot{\mathbf{d}}_i \right\}$$

### Newmark's Method

Dividing the above equation by  $\beta(\Delta t)^2$  gives:  $[K']\{d_{i+1}\} = \{F'_{i+1}\}$  where:

$$\begin{bmatrix} \mathbf{K}' \end{bmatrix} = \begin{bmatrix} \mathbf{K} \end{bmatrix} + \frac{1}{\beta (\Delta t)^2} \begin{bmatrix} \mathbf{M} \end{bmatrix}$$
$$\{ \mathbf{F}'_{i+1} \} = \{ \mathbf{F}_{i+1} \} + \frac{\begin{bmatrix} \mathbf{M} \end{bmatrix}}{\beta (\Delta t)^2} \begin{bmatrix} \{ \mathbf{d}_i \} + (\Delta t) \{ \dot{\mathbf{d}}_i \} + (\frac{1}{2} - \beta) (\Delta t)^2 \{ \ddot{\mathbf{d}}_i \} \end{bmatrix}$$

The advantages of using Newmark's method over the central difference method are that Newmark's method can be made unconditionally stable (if  $\beta = \frac{1}{4}$  and  $\gamma = \frac{1}{2}$ ) and that larger time steps can be used with better results.



### Newmark's Method

### Procedure for solution:

- 1. Given:  $\{d_0\}, \{\dot{d}_0\}, \text{and} \{F_i(t)\}$
- 2. If the acceleration is not given, solve for  $\{\ddot{d}_0\}$

$$\left\{ \ddot{\boldsymbol{d}}_{0} \right\} = \left[ \mathbf{M} \right]^{-1} \left( \left\{ \mathbf{F}_{0} \right\} - \left[ \mathbf{K} \right] \left\{ \boldsymbol{d}_{0} \right\} \right)$$

3. Solve for  $\{d_1\}$  at t = 0

$$[K']{d_1} = {F'_1}$$

## Structural Dynamics

### **Newmark's Method**

### Procedure for solution:

4. Solve for  $\{\ddot{d}_{i+1}\}$  -- original Newmark equation for  $\{d_{i+1}\}$  rewritten for  $\{\ddot{d}_{i+1}\}$ :

$$\left\{\ddot{\boldsymbol{d}}_{1}\right\} = \frac{1}{\beta(\Delta t)^{2}} \left[\left\{\boldsymbol{d}_{1}\right\} - \left\{\boldsymbol{d}_{0}\right\} - (\Delta t)\left\{\dot{\boldsymbol{d}}_{0}\right\} - (\Delta t)^{2}\left(\frac{1}{2} - \beta\right)\left\{\ddot{\boldsymbol{d}}_{0}\right\}\right]$$

5. Solve for  $\{\dot{d}_1\}$ 

$$\left\{\dot{d}_{1}\right\} = \left\{\dot{d}_{0}\right\} + (\Delta t)\left[(1-\gamma)\left\{\ddot{d}_{0}\right\} + \gamma\left\{\ddot{d}_{1}\right\}\right]$$

6. Repeat Steps 3, 4, and 5 to obtain the displacement, acceleration, and velocity for the next time step.

### Newmark's Method – Example Problem

Determine the displacement, acceleration, and velocity at 0.1 second time intervals for up to 0.5 seconds for the onedimensional spring-mass system shown in the figure below.



Consider the above spring-mass system as a single degree of freedom problem represented by the displacement *d*.

Use Newmark's method with  $\beta = 1/6$  and  $\gamma = \frac{1}{2}$ .

## **Structural Dynamics**

### Newmark's Method – Example Problem

### **Procedure for solution:**

1. Given:  $\{d_0\}, \{\dot{d}_0\}, \text{and}\{F_i(t)\}$ 

2. If the acceleration is not given, solve for  $\{\ddot{d}_0\}$ 

$$\begin{aligned} \left\{ \ddot{d}_0 \right\} &= \left[ \mathsf{M} \right]^{-1} \left( \left\{ \mathsf{F}_0 \right\} - \left[ \mathsf{K} \right] \left\{ d_0 \right\} \right) \\ &= \frac{1}{1.77} \left[ 100 - 70(0) \right] = 56.5 \text{ in / } s^2 \end{aligned}$$

Newmark's Method – Example Problem

**Procedure for solution:** 

3. Solve for {d<sub>1</sub>} at t = 0.1 s [K']{d<sub>1</sub>} = {F'\_1}  

$$\begin{bmatrix} \mathbf{K}' \end{bmatrix} = \begin{bmatrix} \mathbf{K} \end{bmatrix} + \frac{1}{\beta(\Delta t)^2} \begin{bmatrix} \mathbf{M} \end{bmatrix} = 70 + \frac{1}{\frac{1}{6}(0.1)^2} (1.77) = 1,132 \ lb \ ln \\ \{\mathbf{F}'_1\} = \{\mathbf{F}_1\} + \frac{\begin{bmatrix} \mathbf{M} \end{bmatrix}}{\beta(\Delta t)^2} \begin{bmatrix} \{d_0\} + (\Delta t) \{\dot{d}_0\} + (\frac{1}{2} - \beta)(\Delta t)^2 \{\ddot{d}_0\} \end{bmatrix} \\ = 80 + \frac{1.77}{\frac{1}{6}(0.1)^2} \begin{bmatrix} 0 + (0.1)0 + (\frac{1}{2} - \frac{1}{6})(0.1)^2 (56.5) \end{bmatrix} = 280 \ lb \\ \{d_1\} = \frac{\{\mathbf{F}'_1\}}{\begin{bmatrix} \mathbf{K}' \end{bmatrix}} = \frac{280 \ lb}{1,132 \ lb/in} = 0.248 \ in$$

## Structural Dynamics

### Newmark's Method – Example Problem

### **Procedure for solution:**

4. Solve for  $\{\ddot{d}_1\}$  at t = 0.1 s

$$\left\{ \ddot{d}_{1} \right\} = \frac{1}{\beta (\Delta t)^{2}} \left[ \left\{ d_{1} \right\} - \left\{ d_{0} \right\} - (\Delta t) \left\{ \dot{d}_{0} \right\} - (\Delta t)^{2} \left( \frac{1}{2} - \beta \right) \left\{ \ddot{d}_{0} \right\} \right]$$

$$= \frac{1}{\frac{1}{6} (0.1)^{2}} \left[ 0.248 - 0 - (0.1)0 - (0.1)^{2} \left( \frac{1}{2} - \frac{1}{6} \right) 56.5 \right] = 35.4 \frac{i\eta_{s}^{2}}{s^{2}}$$

5. Solve for  $\left\{ \dot{d}_{1} \right\}$ 

$$\left\{ \dot{d}_1 \right\} = \left\{ \dot{d}_0 \right\} + (\Delta t) \left[ (1 - \gamma) \left\{ \ddot{d}_0 \right\} + \gamma \left\{ \ddot{d}_1 \right\} \right]$$
  
= 0 + (0.1)  $\left[ (1 - \frac{1}{2}) 56.5 + \frac{1}{2} (35.4) \right] = 4.59 \text{ in/s}$ 

### Newmark's Method – Example Problem

#### **Procedure for solution:**

6. Repeat Steps 3, 4, and 5 to obtain the displacement, acceleration, and velocity for the next time step.

Repeating Steps 3, Solve for  $\{d_1\}$  at t = 0.2 s

$$\{\mathbf{F'}_2\} = \{\mathbf{F}_2\} + \frac{\left[\mathbf{M}\right]}{\beta(\Delta t)^2} \left[ \{d_1\} + (\Delta t)\{\dot{d}_1\} + \left(\frac{1}{2} - \beta\right)(\Delta t)^2\{\ddot{d}_1\} \right]$$
$$= 60 + \frac{1.77}{\frac{1}{6}(0.1)^2} \left[ 0.248 + (0.1)4.59 + \left(\frac{1}{2} - \frac{1}{6}\right)(0.1)^2(35.4) \right] = 934 \ lb$$

$$\{d_2\} = \frac{\{\mathbf{F'}_2\}}{[\mathbf{K'}]} = \frac{934}{1,132} = 0.825 \text{ in}$$

## Structural Dynamics

#### Newmark's Method – Example Problem

### **Procedure for solution:**

Repeating Step 4: solve for  $\{\ddot{d}_1\}$  at t = 0.2 s

$$\begin{aligned} \left\{ \ddot{d}_{2} \right\} &= \frac{1}{\beta (\Delta t)^{2}} \left[ \left\{ d_{2} \right\} - \left\{ d_{1} \right\} - (\Delta t) \left\{ \dot{d}_{1} \right\} - (\Delta t)^{2} \left( \frac{1}{2} - \beta \right) \left\{ \ddot{d}_{1} \right\} \right] \\ &= \frac{1}{\frac{1}{6} (0.1)^{2}} \left[ 0.825 - 0.248 - (0.1)4.59 - (0.1)^{2} \left( \frac{1}{2} - \frac{1}{6} \right) 35.4 \right] \\ &= 1.27 \ in/s^{2} \end{aligned}$$

Newmark's Method – Example Problem

**Procedure for solution:** 

5. Solve for 
$$\{\dot{d}_2\}$$
  
 $\{\dot{d}_2\} = \{\dot{d}_1\} + (\Delta t) \Big[ (1 - \gamma) \{\ddot{d}_1\} + \gamma \{\ddot{d}_2\} \Big]$   
 $= 4.59 + (0.1) \Big[ (1 - \frac{1}{2}) 35.4 + \frac{1}{2} (1.27) \Big]$   
 $= 6.42 in/s$ 

## **Structural Dynamics**

## Newmark's Method – Example Problem

### **Procedure for solution:**

The following table summarizes the results for the time steps through t = 0.5 seconds.

t (s)	<b>F</b> (t) lb	ä₁(in/s²)	ḋ,(in/s)	d <sub>i</sub> (in)
0.0	100	56.4972	0.0000	0.0000
0.1	80	35.4155	4.5956	0.2473
0.2	60	1.1939	6.4261	0.8270
0.3	48.5714	-28.9276	5.0394	1.4253
0.4	45.7143	-43.7731	1.4044	1.7599
0.5	42.8571	-42.3776	-2.9031	1.6838

### Newmark's Method – Example Problem

### **Procedure for solution:**

Plotting the motion for about 4 *s* gives:



## Structural Dynamics

### Natural Frequencies of a One-Dimensional Bar

- Before solving the structural stress dynamic analysis problem, let's consider how to determine the natural frequencies of continuous elements.
- Natural frequencies are necessary in vibration analysis and important when choosing a proper time step for a structural dynamics analysis.
- Natural frequencies are obtained by solving the following equation:

$$\left[\mathbf{M}\right]\left\{\ddot{\mathbf{d}}\right\} + \left[\mathbf{K}\right]\left\{\mathbf{d}\right\} = \mathbf{0}$$

### Natural Frequencies of a One-Dimensional Bar

The standard solution for  $\{d\}$  is given as:  $\{d(t)\} = \{d'\}e^{i\omega t}$ 

where  $\{d'\}$  is the part of the nodal displacement matrix called *natural modes* that is assumed to independent of time, *i* is the standard imaginary number, and  $\omega$  is a natural frequency.

Differentiating the above equation twice with respect to time

gives:

 $\left\{ \ddot{\boldsymbol{d}} \right\} = \left\{ \boldsymbol{d'} \right\} \left( -\omega^2 \right) \boldsymbol{e}^{i\omega t}$ 

Substituting the above expressions for  $\{d\}$  and  $\{\ddot{d}\}$  into the equation of motion gives:

$$-[\mathbf{M}]\omega^{2}\{d'\}e^{i\omega t}+[\mathbf{K}]\{d'\}e^{i\omega t}=0$$

## Structural Dynamics

### Natural Frequencies of a One-Dimensional Bar

Combining terms gives:  $e^{i\omega t} \left( [\mathbf{K}] - \omega^2 [\mathbf{M}] \right) \{ d' \} = 0$ 

Since  $e^{i\omega t}$  is not zero, then:  $([\mathbf{K}] - \omega^2 [\mathbf{M}]) = 0$ 

The above equations are a set of linear homogeneous equations in terms of displacement mode  $\{d'\}$ .

There exists a non-trivial solution if and only if the determinant of the coefficient matrix of is zero.

$$\left\| \left[ \mathbf{K} \right] - \omega^2 \left[ \mathbf{M} \right] \right\| = 0$$

#### **One-Dimensional Bar - Example Problem**

Determine the first two natural frequencies for the bar shown in the figure below.



Assume the bar has a length 2*L*, modulus of elasticity *E*, mass density  $\rho$ , and cross-sectional area *A*.

## Structural Dynamics

### **One-Dimensional Bar - Example Problem**

Let's discretize the bar into two elements each of length *L* as shown below.



We need to develop the stiffness matrix and the mass matrix (either the lumped- mass of the consistent-mass matrix).

In general, the consistent-mass matrix has resulted in solutions that compare more closely to available analytical and experimental results than those found using the lumped-mass matrix.

#### **One-Dimensional Bar - Example Problem**

Let's discretize the bar into two elements each of length *L* as shown below.



However, when performing a long hand solution, the consistent-mass matrix is more difficult and tedious to compute; therefore, we will use the lumped-mass matrix in this example.

## Structural Dynamics

#### **One-Dimensional Bar - Example Problem**

Let's discretize the bar into two elements each of length *L* as shown below.



The elemental stiffness matrices are:  $1 - 2^2$ 

$$\begin{bmatrix} k_{(1)} \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} k_{(2)} \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The global stiffness matrix is:

$$[K] = \frac{AE}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

1

2 3

2

3

#### **One-Dimensional Bar - Example Problem**

Let's discretize the bar into two elements each of length *L* as shown below.



The lumped-mass matrices are:



## Structural Dynamics

#### **One-Dimensional Bar - Example Problem**

Substituting the above stiffness and lumped-mass matrices into the natural frequency equation:

$$\left( \left[ \mathbf{K} \right] - \omega^2 \left[ \mathbf{M} \right] \right) \left\{ d' \right\} = 0$$

and applying the boundary condition  $\{u_1\} = 0$  (or  $\{d'_1\} = 0$ ) gives:

$$\left(\frac{AE}{L}\begin{bmatrix}2&-1\\-1&1\end{bmatrix}-\omega^2\frac{\rho AL}{2}\begin{bmatrix}2&0\\0&1\end{bmatrix}\right)\left\{\frac{d'_2}{d'_3}\right\}=\left\{\begin{matrix}0\\0\end{matrix}\right\}$$

Set the determinant of the coefficient matrix equal to zero as:

$$\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \frac{\rho AL}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = 0 \quad \text{where } \lambda = \omega^2$$
### **One-Dimensional Bar - Example Problem**

Dividing the above equation by  $\rho AL$  and letting  $\mu = \frac{E}{\rho L^2}$  gives:

$$\begin{vmatrix} 2\mu - \lambda & -\mu \\ -\mu & \mu - \frac{\lambda}{2} \end{vmatrix} = 0 \qquad \qquad \mu^2 - 2\mu\lambda + \frac{\lambda^2}{2} = 0$$

Evaluating the determinant of the above equations gives:

 $\lambda = 2\mu \pm \mu \sqrt{2}$   $\lambda_1 = 0.5858\mu$   $\lambda_2 = 3.4142\mu$ 

For comparison, the exact solution gives  $\lambda = 0.616 \mu$ , whereas the consistent-mass approach yields  $\lambda = 0.648 \mu$ .

### Structural Dynamics

#### **One-Dimensional Bar - Example Problem**

- Therefore, for bar elements, the lumped-mass approach can yield results as good as, or even better than, the results from the consistent-mass approach.
- However, the consistent-mass approach can be mathematically proven to yield an upper bound on the frequencies, whereas the lumped-mass approach has no mathematical proof of boundedness.

The first and second natural frequencies are given as:

$$\omega_1 = \sqrt{\lambda_1} = 0.7654\sqrt{\mu}$$
  $\omega_2 = \sqrt{\lambda_2} = 1.8478\sqrt{\mu}$ 

### **One-Dimensional Bar - Example Problem**

The term  $\mu$  may be computed as:

$$\mu = \frac{E}{\rho L^2} = \frac{30 \times 10^6 \text{psi}}{(0.00073 \frac{\text{lb.s}^2}{\text{in.}^4})(100 \text{ in.})^2} = 4.12 \times 10^6 \text{ s}^{-2}$$

Therefore, first and second natural frequencies are:

$$\omega_1 = 1.56 \times 10^3 \text{ rad / s}$$
  $\omega_2 = 3.76 \times 10^3 \text{ rad / s}$ 

### Structural Dynamics

#### **One-Dimensional Bar - Example Problem**

In general, an *n*-degree-of-freedom discrete system has *n* natural modes and frequencies.

A continuous system actually has an infinite number of natural modes and frequencies.

The lowest modes and frequencies are approximated most often; the higher frequencies are damped out more rapidly and are usually less important.

### **One-Dimensional Bar - Example Problem**

Substituting  $\lambda_1$  into the following equation

( <u>AE</u> [2	-1	$\rho AL [2]$	$0 ] \int d'_2 ]$	_∫0
$\left( L \right) - 1$	$1 \int_{-\omega}^{-\omega}$	2 0	$1 \int d'_3 \int d'$	_∫0∫

Gives:  $1.4142\mu d'_{2}^{(1)} - \mu d'_{3}^{(1)} = 0$  $-\mu d'_{2}^{(1)} + 0.7071\mu d'_{3}^{(1)} = 0$ 

where the superscripts indicate the natural frequency.

It is customary to specify the value of one of the natural modes  $\{d'\}$  for a given  $\mu_i$  or  $\omega_i$  and solve for the remaining values.

For example, if  $\{d'_{3}^{(1)}\} = 1$  than the solution for  $\{d'_{2}^{(1)}\} = 0.7071$ 

### Structural Dynamics

### **One-Dimensional Bar - Example Problem**

Similarly, if we substitute  $\lambda_2$  and let  $\{d'_3^{(2)}\} = 1$  the solution of the above equations gives  $\{d'_2^{(2)}\} = -0.7071$ 

The modal responses for the first and second natural frequencies are shown in the figure below.



The first mode means that the bar is completely in tension or compression, depending on the excitation direction.

### **One-Dimensional Bar - Example Problem**

Similarly, if we substitute  $\lambda_2$  and let  $\left\{ d'_3^{(2)} \right\} = 1$  the solution of the above equations gives  $\left\{ d'_2^{(2)} \right\} = -0.7071$ 

The modal responses for the first and second natural frequencies are shown in the figure below.



The second mode means that bar is in compression and tension or in tension and compression.

## **Structural Dynamics**

### **One-Dimensional Bar - Example Problem**

Similarly, if we substitute  $\lambda_2$  and let  $\{d'_3^{(2)}\} = 1$  the solution of the above equations gives  $\{d'_2^{(2)}\} = -0.7071$ 

The modal responses for the first and second natural frequencies are shown in the figure below.



### **Time-Dependent One-Dimensional Bar - Example**

Consider the one-dimensional bar system shown in the figure below. F(t)



Assume the boundary condition  $\{d_{1x}\} = 0$  and the initial conditions  $\{d_0\} = 0$  and  $\{\dot{d}_0\} = 0$ 

Let  $\rho = 0.00073$  lb-s<sup>2</sup>/in.<sup>4</sup>, A = 1 in.<sup>2</sup>,  $E = 30 \times 10^6$  *psi*, and L = 100 in.

### Structural Dynamics

#### Time-Dependent One-Dimensional Bar - Example

The bar will be discretized into two elements as shown below.

The elemental stiffness matrices are:

$$\begin{bmatrix} k_{(1)} \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} k_{(2)} \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The global stiffness matrix is:

$$[K] = \frac{AE}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

### **Time-Dependent One-Dimensional Bar - Example**

The bar will be discretized into two elements as shown below.



The lumped-mass matrices are:



### Structural Dynamics

### Time-Dependent One-Dimensional Bar - Example

Substitute the global stiffness and mass matrices into the global dynamic equations gives:

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1\\ u_2\\ u_3 \end{bmatrix} + \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{u}_1\\ \ddot{u}_2\\ \ddot{u}_3 \end{bmatrix} = \begin{bmatrix} R_1\\ 0\\ F_3(t) \end{bmatrix}$$

where  $R_1$  denotes the unknown reaction at node 1.

#### **Time-Dependent One-Dimensional Bar - Example**

For this example, we will use the central difference method, because it is easier to apply, for the numerical time integration.

It has been mathematically shown that the time step  $\Delta t$  must be less than or equal to two divided by the highest natural frequency.

$$\Delta t \leq \frac{2}{\omega_{max}}$$

For practical results, we should use a time step defined by:

$$\Delta t \leq \frac{3}{4} \left( \frac{2}{\omega_{max}} \right)$$

### Structural Dynamics

### Time-Dependent One-Dimensional Bar - Example

An alternative guide (used only for a bar) for choosing the approximate time step is:  $\Delta t = \frac{L}{c_x}$ 

where *L* is the element length, and  $c_x = J$  is the *longitudinal wave velocity*.

$$c_x = \sqrt{\frac{E_x}{\rho}}$$

Evaluating the time step estimates gives:

$$\Delta t = \frac{3}{4} \left( \frac{2}{\omega_{max}} \right) = \frac{1.5}{3.76 \times 10^3} = 0.40 \times 10^{-3} s$$
$$\Delta t = \frac{L}{c_x} = \frac{100 \text{ in.}}{\sqrt{\frac{30 \times 10^6 \text{ psi}}{0.00073 \frac{\text{lb. s}^2}{\text{in.}^4}}}} = 0.48 \times 10^{-3} s$$

### **Time-Dependent One-Dimensional Bar - Example**

Guided by these estimates for time step, we will select  $\Delta t = 0.25 \text{ x } 10^{-3} \text{ s.}$ 

### **Procedure for solution:**

1. At time t = 0:  $\{d_0\} = 0$   $\{\dot{d}_0\} = 0$ 

2. If the acceleration is not given, solve for  $\{\ddot{d}_0\}$ 

$$\left\{ \ddot{d}_0 \right\} = \left[ \mathsf{M} \right]^{-1} \left\{ \left\{ \mathsf{F}_0 \right\} - \left[ \mathsf{K} \right] \left\{ d_0 \right\} \right)$$

$$\left\{ \ddot{d}_0 \right\} = \left\{ \ddot{u}_2 \\ \ddot{u}_3 \right\}_{t=0} = \frac{2}{\rho AL} \begin{bmatrix} \frac{\gamma_2}{0} & 0 \\ 0 & 1 \end{bmatrix} \left( \begin{cases} 0 \\ 1,000 \end{cases} - \frac{AE}{L} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{cases} 0 \\ 0 \end{bmatrix} \right)$$

### Structural Dynamics

#### **Time-Dependent One-Dimensional Bar - Example**

Applying the boundary conditions  $u_1 = 0$  and  $\ddot{u}_1 = 0$  and simplifying gives:

$$\left\{\ddot{d}_{0}\right\} = \left\{\begin{matrix} \ddot{u}_{2} \\ \ddot{u}_{3} \end{matrix}\right\}_{t=0} = \frac{2000}{\rho AL} \left\{\begin{matrix} 0 \\ 1 \end{matrix}\right\} = \left\{\begin{matrix} 0 \\ 27,400 \end{matrix}\right\} in/s^{2}$$

#### 3. Solve for $d_{-1}$ at $t = -\Delta t$

$$\{d_{-1}\} = \{d_0\} - (\Delta t)\{\dot{d}_0\} + \{\ddot{d}_0\}\frac{(\Delta t)^2}{2} \\ \begin{cases}u_2\\u_3\}_{-1} = \{0\\0\} - (0.25 \times 10^{-3})\{0\\0\} + \frac{(0.25 \times 10^{-3})^2}{2}\{0\\27,400\} \\ = \{0\\0.856 \times 10^{-3}\}in \end{cases}$$

### Time-Dependent One-Dimensional Bar - Example

4. Solve for  $d_1$  at  $t = \Delta t$  using the value of  $d_{-1}$  from Step 3

$$\begin{cases} d_1 \\ = \begin{bmatrix} \mathbf{M} \end{bmatrix}^{-1} \left\{ \left( \Delta t \right)^2 \left\{ \mathbf{F}_0 \right\} + \begin{bmatrix} 2 \begin{bmatrix} \mathbf{M} \end{bmatrix} - \left( \Delta t \right)^2 \begin{bmatrix} \mathbf{K} \end{bmatrix} \end{bmatrix} \left\{ d_0 \\ \right\} - \begin{bmatrix} \mathbf{M} \end{bmatrix} \left\{ d_{-1} \\ \right\} \right\} \\ \begin{cases} u_2 \\ u_3 \\ 1 \end{bmatrix} = \frac{2}{0.073} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \left\{ \left( 0.25 \times 10^{-3} \right)^2 \left\{ \begin{array}{c} 0 \\ 1,000 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{2(0.073)}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right\} \\ - \left( 0.25 \times 10^{-3} \right)^2 \left( 30 \times 10^4 \right) \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \right] \left\{ \begin{array}{c} 0 \\ 0 \\ -1 & 1 \end{bmatrix} \right\} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \frac{0.073}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} 0 \\ 0.856 \times 10^{-3} \\ 0 \end{bmatrix} \right\} \\ \begin{cases} u_2 \\ u_3 \\ 1 \end{bmatrix} = \frac{2}{0.073} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} 0 \\ 0.0625 \times 10^{-3} \\ 0 \end{bmatrix} - \left\{ \begin{array}{c} 0 \\ 0.0312 \times 10^{-3} \\ 0 \end{bmatrix} \right\} \\ \begin{cases} u_2 \\ u_3 \\ 1 \end{bmatrix} = \left\{ \begin{array}{c} 0 \\ 0.858 \times 10^{-3} \\ 0 \end{bmatrix} in \end{cases}$$

## Structural Dynamics

### Time-Dependent One-Dimensional Bar - Example

5. With  $d_0$  given and  $d_1$  determined in Step 4 solve for  $d_2$ 

$$\begin{split} \left\{ d_{2} \right\} &= \left[ \mathbf{M} \right]^{-1} \left\{ \left( \Delta t \right)^{2} \left\{ \mathbf{F}_{1} \right\} + \left[ 2 \left[ \mathbf{M} \right] - \left( \Delta t \right)^{2} \left[ \mathbf{K} \right] \right] \left\{ d_{1} \right\} - \left[ \mathbf{M} \right] \left\{ d_{0} \right\} \right\} \\ &\left\{ u_{2} \\ u_{3} \right\}_{2} = \frac{2}{0.073} \begin{bmatrix} y_{2} & 0 \\ 0 & 1 \end{bmatrix} \left\{ \left( 0.25 \times 10^{-3} \right)^{2} \left\{ \begin{array}{c} 0 \\ 1000 \end{array} \right\} + \left[ \frac{2(0.073)}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right] \\ &- \left( 0.25 \times 10^{-3} \right)^{2} \left( 30 \times 10^{4} \right) \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \right] \left\{ \begin{array}{c} 0 \\ 0.858 \times 10^{-3} \end{array} \right\} - \frac{0.073}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \left\{ 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} u_{2} \\ u_{3} \\ u_{3} \\ 2 \end{bmatrix} = \frac{2}{0.073} \begin{bmatrix} y_{2} & 0 \\ 0 & 1 \end{bmatrix} \left[ \left\{ \begin{array}{c} 0 \\ 0.0625 \times 10^{-3} \\ 0.0466 \times 10^{-3} \\ 0.0466 \times 10^{-3} \\ 2.99 \times 10^{-3} \\ 2.99 \times 10^{-3} \\ \end{array} \right\} in \end{split}$$

### Time-Dependent One-Dimensional Bar - Example

6. Solve for 
$$\{\ddot{d}_1\}$$
  
 $\{\ddot{d}_1\} = [\mathbf{M}]^{-1}(\{\mathbf{F}_1\} - [\mathbf{K}]\{d_1\})$   
 $\{\ddot{u}_2\\\ddot{u}_3\}_1 = \frac{2}{0.073} \begin{bmatrix} \frac{1}{2} & 0\\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0\\ 1000 \end{bmatrix} - (30 \times 10^4) \begin{bmatrix} 2 & -1\\ -1 & 1 \end{bmatrix} \{ 0.858 \times 10^{-3} \} \end{bmatrix}$   
 $\{\ddot{u}_2\\\ddot{u}_3\}_1 = \{ 3,526\\20,345 \} in/s^2$ 

$$\ddot{u}_{2} \\ \ddot{u}_{3} \\ 1 = \begin{cases} 3,526 \\ 20,345 \end{cases} in/s^{2}$$

## **Structural Dynamics**

### **Time-Dependent One-Dimensional Bar - Example**

7. Solve for  $\left\{ \dot{d}_{1} \right\}$ 

$$\left\{\dot{d}_{1}\right\} = \frac{\left\{d_{2}\right\} - \left\{d_{0}\right\}}{2(\Delta t)}$$

### **Time-Dependent One-Dimensional Bar - Example**

8. Repeat Steps 5, 6, and 7 to obtain the displacement, acceleration, and velocity for other time steps.

Repeating Step 5:

$$\left\{ \boldsymbol{d}_{3} \right\} = \left[ \mathbf{M} \right]^{-1} \left\{ \left( \Delta t \right)^{2} \left\{ \mathbf{F}_{2} \right\} + \left[ 2 \left[ \mathbf{M} \right] - \left( \Delta t \right)^{2} \left[ \mathbf{K} \right] \right] \left\{ \boldsymbol{d}_{2} \right\} - \left[ \mathbf{M} \right] \left\{ \boldsymbol{d}_{1} \right\} \right\}$$

$$\left\{ \begin{matrix} u_{2} \\ u_{3} \end{matrix} \right\}_{3} = \frac{2}{0.073} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \left\{ \left( 0.25 \times 10^{-3} \right)^{2} \begin{cases} 0 \\ 1000 \end{cases} + \left[ \frac{2(0.073)}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

$$- \left( 0.25 \times 10^{-3} \right)^{2} \left( 30 \times 10^{4} \right) \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \right] \left\{ \begin{matrix} 0.221 \times 10^{-3} \\ 2.99 \times 10^{-3} \end{cases} - \frac{0.073}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{matrix} 0.858 \times 10^{-3} \\ 0.858 \times 10^{-3} \end{matrix} \right\} \right\}$$

$$\left\{ \begin{matrix} u_{2} \\ u_{3} \end{matrix} \right\}_{2} = \left\{ \begin{matrix} 1.096 \times 10^{-3} \\ 5.397 \times 10^{-3} \end{matrix} \right\} in$$

## Structural Dynamics

### Time-Dependent One-Dimensional Bar - Example

Repeating Step 6. Solve for  $\{\ddot{d}_2\}$  $\{\ddot{d}_2\} = [\mathbf{M}]^{-1}(\{\mathbf{F}_2\} - [\mathbf{K}]\{d_2\})$  $\{\ddot{u}_2\}_2 = \frac{2}{0.073} \begin{bmatrix} 1/_2 & 0\\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0\\ 1000 \end{bmatrix} - (30 \times 10^4) \begin{bmatrix} 2 & -1\\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0.221 \times 10^{-3}\\ 2.99 \times 10^{-3} \end{bmatrix} \end{bmatrix}$  $\{\ddot{u}_2\}_1 = \{10,500\\ \ddot{u}_3\}_1 = \{10,500\\ 4,600 \} in/_{S^2}$ 

### Time-Dependent One-Dimensional Bar - Example

Repeat Step 7: Solve for  $\{\dot{d}_2\}$ 

$$\left\{ \dot{d}_{2} \right\} = \frac{\left\{ d_{3} \right\} - \left\{ d_{1} \right\}}{2(\Delta t)}$$

$$\left\{ \dot{u}_{2} \right\}_{2} = \frac{\left[ \left\{ 1.096 \times 10^{-3} \right\} - \left\{ 0 \\ 5.397 \times 10^{-3} \right\} - \left\{ 0.858 \times 10^{-3} \right\} \right]}{2(0.25 \times 10^{-3})} = \left\{ \begin{array}{c} 2.192 \\ 9.078 \end{array} \right\} in/s$$

## Structural Dynamics

#### **Time-Dependent One-Dimensional Bar - Example**

The following table summarizes the results for the remaining time steps as compared with the exact solution.

<i>t</i> (10 <sup>-3</sup> s)	d <sub>2</sub>	<i>d</i> <sub>3</sub>	$\dot{d}_2$ (in/s)	$\dot{d}_{3}$ (in/s)	$\ddot{d}_{2}(10^{4}in/s^{2})$	$\ddot{d}_{3}(10^{4}in/s^{2})$
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	2.7397
0.25	0.0000	0.0009	0.4398	5.9697	0.3518	2.0360
0.50	0.0002	0.0030	2.1870	9.0987	1.0459	0.4672
0.75	0.0011	0.0054	5.1477	8.6772	1.3227	-0.8044
1.00	0.0028	0.0073	7.6928	6.4426	0.7134	-0.9833
1.25	0.0049	0.0086	7.9409	4.8502	-0.5149	-0.2906
1.50	0.0068	0.0097	5.3555	4.8455	-1.5535	0.2868

### Time-Dependent One-Dimensional Bar - Example

Plotting the displacements for 0.015 *s* gives:



## **Structural Dynamics**

### Time-Dependent One-Dimensional Bar - Example

Plotting the velocities for 0.015 *s* gives:



### Time-Dependent One-Dimensional Bar - Example

Plotting the accelerations for 0.015 *s* gives:



## **Structural Dynamics**

Time-Dependent One-Dimensional Bar - Example



### **Beam Element Mass Matrices and Natural Frequencies**

We will develop the lumped- and consistent-mass matrices for time-dependent beam analysis.

Consider the beam element shown in the figure below.



The basic equations of motion are:  $\{F(t)\} = [K]\{d\} + [M]\{\ddot{d}\}$ 

### Structural Dynamics

#### **Beam Element Mass Matrices and Natural Frequencies**

The stiffness matrix is:

$$[k] = \frac{EI}{L^3} \begin{bmatrix} v_1 & v_1 & v_2 & v_2 \\ 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

The lumped-mass matrix is:

IS:  

$$\begin{bmatrix} m \end{bmatrix} = \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

### **Beam Element Mass Matrices and Natural Frequencies**

- The mass is lumped equally into each transitional degree of freedom; however, the inertial effects associated with any possible rotational degrees of freedom is assumed to be zero.
- A value for these rotational degrees of freedom could be assigned by calculating the mass moment of inertia about each end node using basic dynamics as:

$$I = \frac{1}{3}mL^{2} = \frac{1}{3}\left(\frac{\rho AL}{2}\right)\left(\frac{L}{2}\right)^{2} = \frac{\rho AL^{3}}{24}$$

### Structural Dynamics

#### **Beam Element Mass Matrices and Natural Frequencies**

The consistent-mass matrix can be obtained by applying

$$[m] = \int_{V} \rho[N]^{T} [N] dV$$
$$[m] = \int_{0}^{L} \int_{A} \rho \begin{cases} N_{1} \\ N_{2} \\ N_{3} \\ N_{4} \end{cases} [N_{1} \quad N_{2} \quad N_{3} \quad N_{4}] dA dx$$

where

$$N_{1} = \frac{1}{L^{3}} \left( 2x^{3} - 3x^{2}L + L^{3} \right) \qquad N_{2} = \frac{1}{L^{3}} \left( x^{3}L - 2x^{2}L^{2} + xL^{3} \right)$$
$$N_{3} = \frac{1}{L^{3}} \left( -2x^{3} + 3x^{2}L \right) \qquad N_{4} = \frac{1}{L^{3}} \left( x^{3}L - x^{2}L^{2} \right)$$

### **Beam Element Mass Matrices and Natural Frequencies**

The shape functions are shown below:



## **Structural Dynamics**

#### **Beam Element Mass Matrices and Natural Frequencies**

Substituting the shape functions into the above mass expression and integrating gives:

	∫ 156	22L	54	-13 <i>L</i> ]
$[m] = \rho AL$	22L	4 <i>L</i> <sup>2</sup>	13 <i>L</i>	$-3L^2$
$[m] = \frac{1}{420}$	54	13 <i>L</i>	156	-22L
	–13L	-3L <sup>2</sup>	-22L	4 <i>L</i> <sup>2</sup>

### **Beam Element - Example 1**

Determine the first natural frequency for the beam shown in the figure below. Assume the bar has a length 2*L*, modulus of elasticity *E*, mass density  $\rho$ , and cross-sectional area *A*.



Let's discretize the beam into two elements each of length L.

We will use the lumped-mass matrix.

### Structural Dynamics

### Beam Element - Example 1

We can obtain the natural frequencies by using the following equation.

$$\left\| \left[ \mathbf{K} \right] - \omega^2 \left[ \mathbf{M} \right] \right\| = \mathbf{0}$$

The boundary conditions are  $v_1 = \phi_1 = 0$  and  $v_3 = \phi_3 = 0$ .

$$\begin{bmatrix} k^{(1)} \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} k^{(2)} \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

### **Beam Element - Example 1**

We can obtain the natural frequencies by using the following equation.

 $\left\| \left[ \mathbf{K} \right] - \omega^2 \left[ \mathbf{M} \right] \right\| = 0$ 

The boundary conditions are  $v_1 = \phi_1 = 0$  and  $v_3 = \phi_3 = 0$ .

	<sup>v</sup> 1	<sup>¢</sup> 1	<sup>v</sup> 2	<sup>\$\$</sup> 2		<sup>v</sup> 2	<sup>¢</sup> 2	<sup>v</sup> 3	<sup>\$\$</sup> 3
	1	0	0	0		<b>⊺</b> 1	0	0	0
$[m^{(1)}] - \rho AL$	0	0	0	0	$\lceil m^{(2)} \rceil - \rho AL$	0	0	0	0
$\begin{bmatrix} m^{*} \end{bmatrix} = \frac{1}{2}$	0	0	1	0	$\begin{bmatrix} m \\ - \end{bmatrix} = \frac{1}{2}$	0	0	1	0
	0	0	0	0		0	0	0	0

### Structural Dynamics

#### **Beam Element - Example 1**

We can obtain the natural frequencies by using the following equation.

$$\left[ \left[ \mathbf{K} \right] - \omega^2 \left[ \mathbf{M} \right] \right] = \mathbf{0}$$

The boundary conditions are  $v_1 = \phi_1 = 0$  and  $v_3 = \phi_3 = 0$ .

Therefore, the global stiffness and lumped-mass matrices are:

$$\begin{bmatrix} \mathbf{K} \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{M} \end{bmatrix} = \frac{\rho AL}{2} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

### **Beam Element - Example 1**

Substituting the global stiffness and mass matrices into the global dynamic equations gives:

$$\left\| \begin{bmatrix} \mathbf{K} \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M} \end{bmatrix} \right\| = 0 \qquad \left| \begin{array}{cc} \frac{EI}{L^3} \begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix} - \omega^2 \rho AL \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right| = 0$$

Dividing by  $\rho AL$  and simplify

$$\omega^2 = \frac{24EI}{\rho AL^4} \qquad \omega = \frac{4.8990}{L^2} \sqrt{\frac{EI}{\rho A}}$$

The exact solution for the first natural frequency is:

$$\omega = \frac{5.59}{L^2} \sqrt{\frac{EI}{\rho A}}$$

### Structural Dynamics

### Beam Element - Example 2

Determine the first natural frequency for the beam shown in the figure below. Assume the bar has a length 3L, modulus of elasticity *E*, mass density  $\rho$ , and cross-sectional area *A*.



Let's discretize the beam into three elements each of length L.

We will use the lumped-mass matrix.

### **Beam Element - Example 2**

We can obtain the natural frequencies by using the following equation.

 $\left\| \left[ \mathbf{K} \right] - \omega^2 \left[ \mathbf{M} \right] \right\| = 0$ 

The boundary conditions are  $v_1 = \phi_1 = 0$  and  $v_4 = \phi_4 = 0$ .

Therefore the elements of the stiffness matrix for element 1 are:

	<i>V</i> <sub>1</sub>	$\phi_{_1}$	V 2	$\phi_{_2}$
	12	6L	-12	6L
$\begin{bmatrix} \mathbf{k}^{(1)} \end{bmatrix} = \mathbf{E}\mathbf{I}$	6L	4 <i>L</i> <sup>2</sup>	-6L	<b>2</b> <i>L</i> <sup>2</sup>
$\begin{bmatrix} \mathbf{n} \end{bmatrix} = \overline{L^3}$	-12	-6L	12	-6L
	6L	<b>2</b> L <sup>2</sup>	-6L	4 <i>L</i> <sup>2</sup>

## **Structural Dynamics**

#### **Beam Element - Example 2**

Element 2:		V_2	$\phi_{_2}$	V <sub>3</sub>	$\phi_{_3}$
		<b>∏</b> 12	6L	-12	6 <i>L</i> ]
	$\begin{bmatrix} k^{(2)} \end{bmatrix} = EI$	6L	4 <i>L</i> <sup>2</sup>	-6L	2 <i>L</i> <sup>2</sup>
	$\begin{bmatrix} \mathbf{k} \\ \mathbf{k} \end{bmatrix} = \overline{L^3}$		-6 <i>L</i>	12	-6L
		6 <i>L</i>	2 <i>L</i> <sup>2</sup>	-6 <i>L</i>	4 <i>L</i> <sup>2</sup>
Element 2:					
Element 5.		V <sub>3</sub>	$\phi_{_3}$	V_4	$\phi_{_4}$
Element 5.		v₃ [ 12	φ <sub>3</sub> 6L	v <sub>4</sub> -12	φ <sub>4</sub> 6L
Element 5.	[ <sub>k<sup>(3)</sup>]_ El</sub>	v₃ 「12 」 6 <i>L</i>	φ <sub>3</sub> 6L 4L <sup>2</sup>	v₄ -12 -6L	φ <sub>4</sub> 6L 2L <sup>2</sup>
Element 5.	$\left[\boldsymbol{k}^{(3)}\right] = \frac{\boldsymbol{E}\boldsymbol{h}}{\boldsymbol{L}^3}$	v <sub>3</sub> 12 6 <i>L</i> -12	<ul> <li>φ<sub>3</sub></li> <li>6L</li> <li>4L<sup>2</sup></li> <li>-6L</li> </ul>	v₄ -12 -6 <i>L</i> 12	<ul> <li>φ₄</li> <li>6L</li> <li>2L<sup>2</sup></li> <li>-6L</li> </ul>

### Beam Element - Example 2

Assembling the global stiffness matrix as:

$$\begin{bmatrix} \kappa \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} v_2 & \phi_2 & v_3 & \phi_3 \\ 12+12 & -6L+6L & -12 & 6L \\ -6L+6L & 4L^2+2L^2 & -6L & 2L^2 \\ -12 & -6L & 12+12 & -6L+6L \\ -6L & 2L^2 & -6L+6L & 4L^2+4L^2 \end{bmatrix}$$
$$\begin{bmatrix} \kappa \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 24 & 0 & -12 & 6L \\ 0 & 8L^2 & -6L & 2L^2 \\ -12 & -6L & 2L^2 \\ -12 & -6L & 2L^2 \\ -12 & -6L & 2L^2 \end{bmatrix}$$

### Structural Dynamics

### Beam Element – Example 2

We can obtain the natural frequencies by using the following equation.

$$\left\| \left[ \mathbf{K} \right] - \omega^2 \left[ \mathbf{M} \right] \right\| = \mathbf{0}$$

Therefore the elements of the mass matrix for element 1 are:

$$\begin{bmatrix} m^{(1)} \end{bmatrix} = \frac{\rho AL}{2} \begin{bmatrix} v_1 & \phi_1 & v_2 & \phi_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

### Beam Element – Example 2

Element 2:	V_2	$\phi_{_2}$	V <sub>3</sub>	$\phi_{_3}$	Element 3:	V <sub>3</sub>	$\phi_{_3}$	V_4	$\phi_{_4}$
	[1	0	0	0		[1	0	0	0
$\lceil m^{(2)} \rceil - \rho AL$	0	0	0	0	$\lceil m^{(2)} \rceil - \rho AL$	0	0	0	0
$\begin{bmatrix} m^{-1} \end{bmatrix} = \frac{1}{2}$	0	0	1	0		0	0	1	0
	0	0	0	0		0	0	0	0
The assembled n	nas	s m	atri	x is	:				

	V <sub>2</sub>	$\phi_{_2}$	V <sub>3</sub>	$\phi_{_3}$		<i>V</i> <sub>2</sub>	$\phi_{_2}$	<i>V</i> <sub>3</sub>	$\phi_{_3}$
	<b>[</b> 1+1	0	0	0		<b>[</b> 1	0	0	0
$[m] \rho AL$	0	0	0	0	$[\Lambda A] = - \Lambda A$	0	0	0	0
[ <i>m</i> ]=2	0	0	1+1	0	$[NI] = \rho AL$	0	0	1	0
	0	0	0	0		0	0	0	0

## **Structural Dynamics**

### Beam Element – Example 2

We can obtain the frequency equation as:  $\left[ [\mathbf{K}] - \omega^2 [\mathbf{M}] \right] = 0$ 

### Beam Element – Example 2

Simplifying and assuming:  $\beta = \rho AL$ 

$$\begin{vmatrix} 24\frac{EI}{L^3} - \omega^2 \beta & 0 & -12\frac{EI}{L^3} & 6\frac{EI}{L^2} \\ 0 & 8\frac{EI}{L} & -6\frac{EI}{L^2} & 2\frac{EI}{L} \\ -12\frac{EI}{L^3} & -6\frac{EI}{L^2} & 24\frac{EI}{L^3} - \omega^2 \beta & 0 \\ 6\frac{EI}{L^2} & 2\frac{EI}{L} & 0 & 8\frac{EI}{L} \end{vmatrix} = 0$$

Evaluating the 4x4 determinate gives:

### Structural Dynamics

## Beam Element – Example 2

Ignoring the negative root as it is not physically possible gives:

$$\omega_1 = \frac{2.4495}{L^2} \sqrt{\frac{EI}{A\rho}} \qquad \qquad \omega_2 = \frac{5.6921}{L^2} \sqrt{\frac{EI}{A\rho}}$$

To compare with the two-element solution, assume  $L = \frac{2}{3}L$ :

$$\omega_{1} = \frac{2.4495}{\left(\frac{2L}{3}\right)^{2}} \sqrt{\frac{EI}{A\rho}} = \frac{5.5113}{L^{2}} \sqrt{\frac{EI}{A\rho}}$$

### Beam Element – Example 1 Revisited

Determine the first natural frequency for the beam shown in the figure below. Assume the bar has a length 2L, modulus of elasticity *E*, mass density  $\rho$ , and cross-sectional area *A*.



In summary:



## Structural Dynamics

### Beam Element – Example 3

Determine the first natural frequency for the beam shown in the figure below.



Assume the bar has a length L = 30 in, modulus of elasticity  $E = 3 \times 10^7 \text{ psi}$ , mass density  $\rho = 0.00073 \text{ lb-s}^2/\text{in}^4$ , and crosssectional area A = 1 in<sup>2</sup>, moment of inertia I = 0.0833 in<sup>4</sup>, and Poisson's ratio v = 0.3.

### **Beam Element – Example 3**

Determine the first natural frequency for the beam shown in the figure below.



Let's discretize the beam into two elements each of length *L* = 15 *in*. We will use the lumped-mass matrix.

We can obtain the natural frequencies by using the following equation.

$$\left\| \left[ \mathbf{K} \right] - \omega^2 \left[ \mathbf{M} \right] \right\| = 0$$

## Structural Dynamics

## Beam Element – Example 3

In this example, the elemental stiffness matrices are:

		<i>V</i> <sub>1</sub>	$\phi_{_1}$	V_2	$\phi_{_2}$
Element 1:		12	6L	-12	6L ]
	$\begin{bmatrix} \boldsymbol{k}^{(1)} \end{bmatrix} = \boldsymbol{EI}$	6L	4 <i>L</i> <sup>2</sup>	-6L	2 <i>L</i> <sup>2</sup>
	$\begin{bmatrix} n \end{bmatrix} = \overline{L^3}$	-12	-6L	12	-6L
		6L	2 <i>L</i> <sup>2</sup>	-6L	4 <i>L</i> <sup>2</sup>
		V_2	$\phi_{_2}$	V <sub>3</sub>	$\phi_{_3}$
Element 2:		v₂ □ 12	φ <sub>2</sub> 6L	v₃ −12	<sup>ø</sup> ₃ 6 <i>L</i> ]
Element 2:	Γ <sub>κ<sup>(2)</sup>] _ ΕΙ</sub>	v₂ 「 12 6 <i>L</i>	φ <sub>2</sub> 6L 4L <sup>2</sup>	∨₃ −12 −6L	<ul> <li>φ<sub>3</sub></li> <li>6L</li> <li>2L<sup>2</sup></li> </ul>
Element 2:	$\left[k^{(2)}\right] = \frac{EI}{L^3}$	v <sub>2</sub> 12 6 <i>L</i> –12	φ <sub>2</sub> 6L 4L <sup>2</sup> –6L	v <sub>3</sub> -12 -6 <i>L</i> 12	<sup>∅</sup> ₃ 6L 2L <sup>2</sup> –6L

### Beam Element – Example 3

In this example, the elemental mass matrices are:

		1	$\boldsymbol{\varphi}_{1}$	<b>v</b> <sub>2</sub>	$\varphi_2$
Element 1:		1	0	0	0
	$\left[ m^{(1)} \right]_{-} \rho AL$	0	0	0	0
		0	0	1	0
		0	0	0	0
		<i>V</i> <sub>2</sub>	$\phi_{_2}$	$V_{_3}$	$\phi_{_3}$
Element 2:		v₂ [1	φ <sub>2</sub> 0	v <sub>3</sub>	<i>φ</i> ₃ 0 ]
Element 2:	$\left[m^{(2)}\right] - \rho AL$	v <sub>2</sub>	φ <sub>2</sub> 0 0	v <sub>3</sub> 0 0	$\left  \begin{array}{c} \phi_{3} \\ 0 \\ 0 \end{array} \right $
Element 2:	$\left[m^{(2)}\right] = \frac{\rho AL}{2}$	v <sub>2</sub> 1 0 0	φ <sub>2</sub> 0 0	<ul><li>v₃</li><li>0</li><li>0</li><li>1</li></ul>	φ <sub>3</sub> 0 0 0

## **Structural Dynamics**

### Beam Element – Example 3

The boundary conditions are  $u_1 = \phi_1 = 0$ . Therefore the global stiffness and lumped mass matrices is:

	V <sub>2</sub>	$\phi_{_2}$	V <sub>3</sub>	$\phi_{_3}$		<i>V</i> <sub>2</sub>	$\phi_{_2}$	<i>V</i> <sub>3</sub>	$\phi_{_3}$
	24	0	-12	6L ]		2	0	0	0
[K]_EI	0	8 <i>L</i> <sup>2</sup>	-6L	2 <i>L</i> <sup>2</sup>	$\rho AL$	0	0	0	0
$[\Lambda] = \overline{L^3}$	-12	-6L	12	-6L	$M = \frac{1}{2}$	0	0	1	0
	6 <i>L</i>	2 <i>L</i> <sup>2</sup>	-6L	4 <i>L</i> <sup>2</sup>		0	0	0	0

### Beam Element – Example 3

Substituting the global stiffness and mass matrices into the global dynamic equations gives:

## Structural Dynamics

#### Beam Element – Example 3

Evaluating the determinant of the above equations gives:

The solution for the first two natural frequencies are:

$$\omega_1 = \frac{0.7891}{L^2} \sqrt{\frac{EI}{\rho A}} \qquad \omega_2 = \frac{4.0647}{L^2} \sqrt{\frac{EI}{\rho A}}$$

### Beam Element – Example 3

In this example, the length of each element is actual L/2, therefore:

$$\omega_{1} = \frac{0.7891}{(\frac{1}{2})^{2}} \sqrt{\frac{EI}{\rho A}} = \frac{3.1564}{L^{2}} \sqrt{\frac{EI}{\rho A}}$$
$$\omega_{2} = \frac{4.0645}{(\frac{1}{2})^{2}} \sqrt{\frac{EI}{\rho A}} = \frac{16.2580}{L^{2}} \sqrt{\frac{EI}{\rho A}}$$

The exact solution for the first natural frequency is:

$$\omega = \frac{3.516}{L^2} \sqrt{\frac{EI}{\rho A}}$$

### Structural Dynamics

#### Beam Element – Example 3

In the exact solution of the vibration of a clamped-free beam, the higher natural frequencies to the first natural frequency can be given as:

$$\frac{\omega_2}{\omega_1} = 6.2669$$
  $\frac{\omega_3}{\omega_1} = 17.5475$ 

### Beam Element – Example 3

The figures below shows the first and second mode shapes corresponding to the first two natural frequencies for a 2-element FEA cantilever beam.



First mode shape

Second mode shape

## **Structural Dynamics**

### Beam Element – Example 3

The figures below shows the first and second mode shapes corresponding to the first two natural frequencies for a 8-element FEA cantilever beam.



First mode shape



### Beam Element – Example 3

The figures below shows the third and fourth mode shapes corresponding to the first two natural frequencies for a 8element FEA cantilever beam.



## **Structural Dynamics**

### Beam Element – Example 3

The table below shows various finite element solutions compared to the exact solution.

	$\omega_1$ (rad/s)	$\omega_2$ (rad/s)
Exact Solution	228	1,434
FE Solution		
Using 2 elements	205	1,286
Using 6 elements	226	1,372
Using 10 elements	227.5	1,410
Using 30 elements	228.5	1,430
Using 60 elements	228.5	1,432

### **Truss Analysis**

The dynamics of trusses and plane frames are preformed by extending the concepts of bar and beam element.

The truss element requires the same transformation of the mass matrix from local to global coordinates as that used for the stiffness matrix given as:



$$[m] = [\mathbf{T}]^T [m'] [\mathbf{T}]$$

## **Structural Dynamics**

### **Truss Analysis**

Considering two-dimensional motion, the axial and the transverse displacement are given as:

$$\begin{cases} u' \\ v' \end{cases} = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \begin{cases} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{cases} = \frac{1}{L} \begin{bmatrix} L - x' & 0 & x' & 0 \\ 0 & L - x' & 0 & x' \end{bmatrix} \begin{bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{bmatrix}$$

The shape functions for the matrix are:

$$[N] = \frac{1}{L} \begin{bmatrix} L - x' & 0 & x' & 0 \\ 0 & L - x' & 0 & x' \end{bmatrix}$$

### **Truss Analysis**

The consistent-mass matrix can be obtained by applying:

$$[m'] = \int_{V} \rho[N]^{T}[N]dV \qquad [m'] = \frac{\rho AL}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

The lumped-mass matrix for two-dimensional motion is obtained by simply lumping mass at each node (mass is the same in both the *x* and *y* directions):  $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ 

	1	0	0	0	
$[m'] - \rho AL$	0	1	0	0	
["] - 2	0	0	1	0	
	0	0	0	1	

L O

### Structural Dynamics

#### **Plane Frame Analysis**

The plane frame element requires combining the bar and beam elements to obtain the local mass matrix.

There are six degrees of freedom associated with a plane frame element.



### **Plane Frame Analysis**

The plane frame analysis requires first expanding and then combining the bar and beam mass matrices to obtain the local mass matrix.

The bar and beam mass matrices are expanded to a 6 x 6 and superimposed  $\frac{v'_2}{v'_2}$ 



## Structural Dynamics

### **Plane Frame Analysis**

Combining the local axis consistent-mass matrices for the bar and beam elements gives:

$$[m'] = \rho AL \begin{bmatrix} \frac{2}{6} & 0 & 0 & \frac{1}{6} & 0 & 0 \\ 0 & \frac{156}{420} & \frac{22L}{420} & 0 & \frac{54}{420} & -\frac{13L}{420} \\ 0 & \frac{22L}{420} & \frac{4L^2}{420} & 0 & \frac{13L}{420} & -\frac{3L^2}{420} \\ \frac{1}{6} & 0 & 0 & \frac{2}{6} & 0 & 0 \\ 0 & \frac{54}{420} & \frac{13L}{420} & 0 & \frac{156}{420} & -\frac{22L}{420} \\ 0 & -\frac{13L}{420} & -\frac{3L^2}{420} & 0 & -\frac{22L}{420} & \frac{4L^2}{420} \end{bmatrix}$$

Beam element components

### **Plane Frame Analysis**

The resulting lumped-mass matrix for a plane frame element is give as:

$[m'] - \rho A$		1	0	0	0	0	0
		0	1	0	0	0	0
	ho AL	0	0	0	0	0	0
[///]=	2	0	0	0	1	0	0
		0	0	0	0	1	0
		0	0	0	0	1 0	0

## Structural Dynamics

### **Plane Frame Analysis**

The global mass matrix for the plane frame element arbitrarily oriented in x-y coordinates is transformed by:

 $[m] = [\mathbf{T}]^T [m'] [\mathbf{T}]$ 

where the transformation matrix is given as:

$$[\mathbf{T}] = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

### Plane Stress/Strain Elements

The plane stress/strain constant-strain triangle consistent-mass matrix is obtained using the shape functions given below as:



The linear triangular shape functions are illustrated below:



## **Structural Dynamics**

### **Plane Stress/Strain Elements**



The linear triangular shape functions are illustrated below:


#### **Plane Stress/Strain Elements**

The consistent-mass matrix can be obtained by applying:

$$[m] = \int_{V} \rho[N]^{T} [N] dV$$

where dV = tdA

The CST global consistent-mass matrix is:

$[m] = \frac{\rho tA}{12}$	2	0	1	0	1	0
	0	2	0	1	0	1
	1	0	2	0	1	0
	0	1	0	2	0	1
	1	0	1	0	2	0
	0	1	0	1	0	2

## Structural Dynamics

#### Frame Example Problem 1

Use SAP2000 to determine the motion of the frame structure below.



#### Frame Example Problem 1

Assume the modulus of elasticity  $E = 3 \times 10^7 \text{ psi}$ . The nodal lumped mass values are obtained by dividing the total weight (dead loads included) of each floor or wall section by gravity.



## **Structural Dynamics**

#### Frame Example Problem 1

Below is the SAP2000 model for this example:



#### Frame Example Problem 1

For example, compute the total mass of the uniform vertical load on elements 4, 5, and 6:

$$M_{4} = \frac{W_{4}}{g} = \frac{(50 \text{ psf})(30 \text{ ft})(15 \text{ ft})}{386.04 \text{ in/s}^{2}} = 58.28 \text{ lb} \cdot \text{s}^{2}/\text{in}$$
$$M_{5} = \frac{W_{5}}{g} = \frac{(104 \text{ psf})(30 \text{ ft})(15 \text{ ft})}{386.04 \text{ in/s}^{2}} = 121.23 \text{ lb} \cdot \text{s}^{2}/\text{in}$$
$$M_{6} = \frac{W_{6}}{g} = \frac{(104 \text{ psf})(30 \text{ ft})(15 \text{ ft})}{386.04 \text{ in/s}^{2}} = 121.23 \text{ lb} \cdot \text{s}^{2}/\text{in}$$

## Structural Dynamics

#### Frame Example Problem 1

Next, lump the mass equally to each node of the beam element.

For this example calculation, a lumped mass of 29.14  $lb \cdot s^2/in$  should be added to nodes 7 and 8 and a mass of 60.62  $lb \cdot s^2/in$  should be added to nodes 3, 4, 5, and 6, all in the *x* direction.



#### Frame Example Problem 1

In an identical manner, masses for the dead loads for additional wall sections should be added to their respective nodes.

In this example, additional wall loads should be converted to mass added to the appropriate loads.



## Structural Dynamics

#### Frame Example Problem 1

For example, the additional mass due to the wall load on element 7 is:

$$M_{7} = \frac{W_{7}}{g} = \frac{(20 \text{ psf})(10 \text{ ft})(15 \text{ ft})}{386.04 \text{ in}_{g^{2}}} = 7.77 \text{ lb} \cdot \text{s}^{2}/\text{in}$$

Therefore, an additional 3.88  $lb \cdot s^2/in$  should be added to nodes 6 and 8.



#### Frame Example Problem 1

For example, the additional mass due to the wall load on element 8 is:

$$M_8 = \frac{W_8}{g} = \frac{(20 \text{ psf})(10 \text{ ft})(15 \text{ ft})}{386.04 \frac{in}{s^2}} = 7.77 \text{ lb} \cdot \text{s}^2/\text{in}$$

Therefore, an additional 3.88  $lb \cdot s^2/in$  should be added to nodes 4 and 6.



#### Structural Dynamics

#### Frame Example Problem 1

For example, the additional mass due to the wall load on element 9 is:

$$M_{9} = \frac{W_{9}}{g} = \frac{(20 \text{ psf})(15 \text{ ft})(15 \text{ ft})}{386.04 \text{ in/s}^{2}} = 11.66 \text{ lb} \cdot \text{s}^{2}/\text{in}$$

Therefore, an additional 5.83  $lb \cdot s^2/in$  should be added to node 4.



#### Frame Example Problem 1

The final values for the nodal masses on this frame are shown below.



## **Structural Dynamics**

#### Frame Example Problem 1

The deflected shape from SAP2000 is shown below:



#### Frame Example Problem 1

A trace of the displacements of nodes 8, 6, and 4 as a function of time can be generated using the SAP2000 Display Menu.



## **Structural Dynamics**

#### Frame Example Problem 1

A plot of the displacements of nodes 8, 6, and 4 over the first 5 seconds of the analysis generate by SAP2000 is shown below:



#### Frame Example Problem 1

The maximum displacement of node 8 is 3.16 in. and the period of the vibration is approximately 1.2 seconds.



## **Structural Dynamics**

## Frame Example Problem 1

The maximum velocity of node 8 is 8.08 in/s



#### Frame Example Problem 1

The maximum acceleration of node 8 is -41.9 in/s<sup>2</sup>



## **Structural Dynamics**

#### Frame Example Problem 1

A plot of the displacement, velocity, and acceleration of node 8 is show below:



#### Frame Example Problem 2

Use SAP2000 to determine the motion of the frame structure below.



## **Structural Dynamics**

#### Frame Example Problem 2

A trace of the displacements as a function of time of nodes 8, 6, and 4 over the first 5 seconds of the analysis generate by SAP2000 is shown below:



#### Frame Example Problem 2

The maximum displacement of node 8 is 0.877 in. and the period of the vibration is approximately 1.3 seconds.



## **Structural Dynamics**

## Frame Example Problem 2

The maximum velocity of node 8 is 4.68 in/s



### Frame Example Problem 2

The maximum acceleration of node 8 is -75.2 in/s<sup>2</sup>



## **Structural Dynamics**

## Frame Example Problem 2

A plot of the displacement, velocity, and acceleration of node 8 is show below:



## **Development of the Plate Bending Element**

#### Problems

- 23. Do problems *16.6* and *16.11c* in your textbook.
- 24. Do problems *16.14* and *16.16* in your textbook using SAP2000.

# End of Chapter 16