Chapter 9 – Axisymmetric Elements

Learning Objectives

• To review the basic concepts and theory of elasticity equations for axisymmetric behavior.

• To derive the axisymmetric element stiffness matrix, body force, and surface traction equations.

• To demonstrate the solution of an axisymmetric pressure vessel using the stiffness method.

• To compare the finite element solution to an exact solution for a cylindrical pressure vessel.

• To illustrate some practical applications of axisymmetric elements.

Axisymmetric Elements

Introduction

In previous chapters, we have been concerned with line or one-dimensional elements (Chapters 2 through 5) and two-dimensional elements (Chapters 6 through 8).

In this chapter, we consider a special two-dimensional element called the axisymmetric element.

This element is quite useful when symmetry with respect to geometry and loading exists about an axis of the body being analyzed.

Problems that involve soil masses subjected to circular footing loads or thick-walled pressure vessels can often be analyzed using the element developed in this chapter.
Axisymmetric Elements

Introduction

We begin with the development of the stiffness matrix for the simplest axisymmetric element, the triangular torus, whose vertical cross section is a plane triangle.

We then present the longhand solution of a thick-walled pressure vessel to illustrate the use of the axisymmetric element equations.

This is followed by a description of some typical large-scale problems that have been modeled using the axisymmetric element.

Axisymmetric Elements

Derivation of the Stiffness Matrix

In this section, we will derive the stiffness matrix and the body and surface force matrices for the axisymmetric element.

However, before the development, we will first present some fundamental concepts prerequisite to the understanding of the derivation.
**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

Axisymmetric elements are triangular tori such that each element is symmetric with respect to geometry and loading about an axis such as the $z$ axis.

Hence, the $z$ axis is called the *axis of symmetry* or the *axis of revolution*.

Each vertical cross section of the element is a plane triangle.

The nodal points of an axisymmetric triangular element describe circumferential lines.

**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

In plane stress problems, stresses exist only in the $x$-$y$ plane.

In axisymmetric problems, the radial displacements develop circumferential strains that induce stresses $\sigma_r$, $\sigma_\theta$, $\sigma_z$ and $\tau_{rz}$ where $r$, $\theta$, and $z$ indicate the radial, circumferential, and longitudinal directions, respectively.

Triangular torus elements are often used to idealize the axisymmetric system because they can be used to simulate complex surfaces and are simple to work with.
Axisymmetric Elements

Derivation of the Stiffness Matrix

For instance, the axisymmetric problem of a semi-infinite half-space loaded by a circular area (circular footing) can be solved using the axisymmetric element developed in this chapter.

Axisymmetric Elements

Derivation of the Stiffness Matrix

For instance, the axisymmetric problem of a domed pressure vessel can be solved using the axisymmetric element developed in this chapter.
**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

For instance, the axisymmetric problem of stresses acting on the barrel under an internal pressure loading.

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**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

For instance, the axisymmetric problem of an engine valve stem can be solved using the axisymmetric element developed in this chapter.
**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

For instance, an axisymmetric specimen loaded under tension-compression.
**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

Because of symmetry about the $z$ axis, the stresses are independent of the $\theta$ coordinate.

Therefore, all derivatives with respect to $\theta$ vanish, and the displacement component $v$ (tangent to the $\theta$ direction), the shear strains $\gamma_{r\theta}$ and $\gamma_{\theta z}$, and the shear stresses $\tau_{r\theta}$ and $\tau_{\theta z}$ are all zero.
**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

The displacements can be expressed for element $ABCD$ in the plane of a cross-section in cylindrical coordinates.

We then let $u$ and $w$ denote the displacements in the radial and longitudinal directions, respectively.

![Diagram of element ABCD](image1)

**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

The side $AB$ of the element is displaced an amount $u$, and side $CD$ is then displaced an amount $u + \left(\frac{\partial u}{\partial r}\right)dr$ in the radial direction.

The normal strain in the radial direction is then given by: $\varepsilon_r = \frac{\partial u}{\partial r}$

![Diagram of element ABCD](image2)
**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

The strain in the tangential direction depends on the tangential displacement $v$ and on the radial displacement $u$.

However, for axisymmetric deformation behavior, recall that the tangential displacement $v$ is equal to zero.

Having only radial displacement $u$, the new length of the arc $AB$ is $(r + u)d\theta$, and the tangential strain is then given by:

$$\varepsilon_\theta = \frac{(r + u)d\theta - rd\theta}{rd\theta} = \frac{u}{r}$$
**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

Consider the longitudinal element $BDEF$ to obtain the longitudinal strain and the shear strain.

The element displaces by amounts $u$ and $w$ in the radial and longitudinal directions at point $E$.

The element displaces additional amounts:

$(\frac{\partial w}{\partial z})dz$ along line $BE$ and

$(\frac{\partial u}{\partial r})dr$ along line $EF$.

**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

Furthermore, observing lines $EF$ and $BE$, we see that point $F$ moves upward an amount $(\frac{\partial w}{\partial r})dr$ with respect to point $E$ and point $B$ moves to the right an amount $(\frac{\partial u}{\partial z})dz$ with respect to point $E$.

The longitudinal normal strain is given by:

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

The shear strain in the $r$-$z$ plane is:

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$
**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

Summarizing the strain-displacement relationships gives:

\[
\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{u}{r}, \quad \varepsilon_z = \frac{\partial w}{\partial z}, \quad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}
\]

The isotropic stress-strain relationship, obtained by simplifying the general stress-strain relationships, is:

\[
\begin{bmatrix}
\sigma_r \\
\sigma_z \\
\sigma_\theta \\
\tau_{rz}
\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & 0 & 0 \\
\nu & 1-\nu & 0 & 0 \\
0 & 0 & 1-\nu & 0 \\
0 & 0 & 0 & 0.5-\nu
\end{bmatrix} \begin{bmatrix}
\varepsilon_r \\
\varepsilon_z \\
\varepsilon_\theta \\
\gamma_{rz}
\end{bmatrix}
\]

**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

The procedure to derive the element stiffness matrix and element equations is identical to that used for the plane-stress in Chapter 6.

**Step 1 - Discretize and Select Element Types**

An axisymmetric solid is shown discretized below, along with a typical triangular element.
**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

The procedure to derive the element stiffness matrix and element equations is identical to that used for the plane-stress in Chapter 6.

**Step 1 - Discretize and Select Element Types**

The stresses in the axisymmetric problem are:

\[
\begin{align*}
\sigma_r &= \sigma \cdot r \\
\sigma_\theta &= 0 \\
\tau_{rz} &= \tau \\
\end{align*}
\]

**Step 2 - Select Displacement Functions**

The element displacement functions are taken to be:

\[
\begin{align*}
\mathbf{u}(r,z) &= \mathbf{a}_1 + \mathbf{a}_2 r + \mathbf{a}_3 z \\
\mathbf{w}(r,z) &= \mathbf{a}_4 + \mathbf{a}_5 r + \mathbf{a}_6 z
\end{align*}
\]

The nodal displacements are:

\[
\{d\} = \begin{bmatrix} d_i \\ d_j \\ d_m \end{bmatrix} = \begin{bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{bmatrix}
\]
**Axisymmetric Elements**

Derivation of the Stiffness Matrix

**Step 2 - Select Displacement Functions**

The function $u$ evaluated at node $i$ is: $u(r_i, z_i) = a_i + a_2 r_i + a_3 z_i$

The general displacement function is then expressed in matrix form as:

$$\{\psi_i\} = \begin{bmatrix} a_i + a_2 r_i + a_3 z_i \\ a_4 + a_5 r_i + a_6 z_i \end{bmatrix} = \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$$

By substituting the coordinates of the nodal points into the equation we can solve for the $a$'s:

$$\begin{bmatrix} u_i \\ u_j \\ u_m \end{bmatrix} = \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_m & z_m \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = [x]^{-1} \{u\}$$

$$\begin{bmatrix} w_i \\ w_j \\ w_m \end{bmatrix} = \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_m & z_m \end{bmatrix} \begin{bmatrix} a_4 \\ a_5 \\ a_6 \end{bmatrix} \Rightarrow \begin{bmatrix} a_4 \\ a_5 \\ a_6 \end{bmatrix} = [x]^{-1} \{w\}$$
Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions

Performing the inversion operations we have:

\[
[x]^{-1} = \frac{1}{2A} \begin{bmatrix}
\alpha_i & \alpha_j & \alpha_m \\
\beta_i & \beta_j & \beta_m \\
\gamma_i & \gamma_j & \gamma_m
\end{bmatrix}
2A = \begin{bmatrix}
1 & r_i & z_i \\
1 & r_j & z_j \\
1 & r_m & z_m
\end{bmatrix}
\]

\[
2A = r_i (z_j - z_m) + r_j (z_m - z_i) + r_m (z_i - z_j)
\]

where \( A \) is the area of the triangle

\[
\alpha_i = r_j z_m - z_j r_m \quad \beta_i = z_j - z_m \quad \gamma_i = r_m - r_j
\]

\[
\alpha_j = r_m z_j - z_m r_j \quad \beta_j = z_m - z_i \quad \gamma_j = r_i - r_m
\]

\[
\alpha_m = r_i z_j - z_i r_j \quad \beta_m = z_i - z_j \quad \gamma_m = r_j - r_i
\]
**Axisymmetric Elements**

Derivation of the Stiffness Matrix

**Step 2 - Select Displacement Functions**

The values of $a$ may be written matrix form as:

\[
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6
\end{bmatrix} = \frac{1}{2A} \begin{bmatrix}
\alpha_i & \alpha_j & \alpha_m \\
\beta_i & \beta_j & \beta_m \\
\gamma_i & \gamma_j & \gamma_m
\end{bmatrix} \begin{bmatrix}
u_i \\
u_j \\
u_m \\
w_i \\
w_j \\
w_m
\end{bmatrix}
\]

Expanding the above equations:

\[
\begin{bmatrix} u \end{bmatrix} = \begin{bmatrix} 1 & r & z \end{bmatrix} \begin{bmatrix} a_1 \\
a_2 \\
a_3 \end{bmatrix}
\]

Substituting the values for $a$ into the above equation gives:

\[
\begin{bmatrix} u \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} 1 & r & z \end{bmatrix} \begin{bmatrix}
\alpha_i & \alpha_j & \alpha_m \\
\beta_i & \beta_j & \beta_m \\
\gamma_i & \gamma_j & \gamma_m
\end{bmatrix} \begin{bmatrix}
u_i \\
u_j \\
u_m \\
w_i \\
w_j \\
w_m
\end{bmatrix}
\]
**Axisymmetric Elements**

Derivation of the Stiffness Matrix

**Step 2 - Select Displacement Functions**

We will now derive the $u$ displacement function in terms of the coordinates $r$ and $z$.

$$
\{u\} = \frac{1}{2A} \begin{bmatrix} r \\ z \end{bmatrix} \begin{bmatrix}
\alpha_i u_i + \alpha_j u_j + \alpha_m u_m \\
\beta_i u_i + \beta_j u_j + \beta_m u_m \\
\gamma_i u_i + \gamma_j u_j + \gamma_m u_m
\end{bmatrix}
$$

Multiplying the matrices in the above equations gives:

$$
u(r,z) = \frac{1}{2A} \left\{ \left( \alpha_i + \beta_i r + \gamma_i z \right) u_i + \left( \alpha_j + \beta_j r + \gamma_j z \right) u_j \right. \\
+ \left. \left( \alpha_m + \beta_m r + \gamma_m z \right) u_m \right\}
$$

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**Axisymmetric Elements**

Derivation of the Stiffness Matrix

**Step 2 - Select Displacement Functions**

We will now derive the $w$ displacement function in terms of the coordinates $r$ and $z$.

$$
\{w\} = \frac{1}{2A} \begin{bmatrix} r \\ z \end{bmatrix} \begin{bmatrix}
\alpha_i w_i + \alpha_j w_j + \alpha_m w_m \\
\beta_i w_i + \beta_j w_j + \beta_m w_m \\
\gamma_i w_i + \gamma_j w_j + \gamma_m w_m
\end{bmatrix}
$$

Multiplying the matrices in the above equations gives:

$$
w(r,z) = \frac{1}{2A} \left\{ \left( \alpha_i + \beta_i r + \gamma_i z \right) w_i + \left( \alpha_j + \beta_j r + \gamma_j z \right) w_j \right. \\
+ \left. \left( \alpha_m + \beta_m r + \gamma_m z \right) w_m \right\}
$$
**Axisymmetric Elements**

Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions

The displacements can be written in a more convenience form as:

\[
u(r, z) = N_i u_i + N_j u_j + N_m u_m\]

\[
w(r, z) = N_i w_i + N_j w_j + N_m w_m\]

where:

\[
N_i = \frac{1}{2A} (\alpha_i + \beta_i r + \gamma_i z)\]

\[
N_j = \frac{1}{2A} (\alpha_j + \beta_j r + \gamma_j z)\]

\[
N_m = \frac{1}{2A} (\alpha_m + \beta_m r + \gamma_m z)\]

**Axisymmetric Elements**

Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions

The elemental displacements can be summarized as:

\[
\{\psi_i\} = \begin{bmatrix} u(r, z) \\ w(r, z) \end{bmatrix} = \begin{bmatrix} N_i u_i + N_j u_j + N_m u_m \\ N_i w_i + N_j w_j + N_m w_m \end{bmatrix}
\]

\[
\{\psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{bmatrix}
\]

\[
\{\psi\} = [N]\{d\}\]
**Axisymmetric Elements**

Derivation of the Stiffness Matrix

**Step 2 - Select Displacement Functions**

In another form the equations are:

\[
[N] = \begin{bmatrix}
N_i & 0 & N_j & 0 & N_m & 0 \\
0 & N_i & 0 & N_j & 0 & N_m
\end{bmatrix}
\]

The linear triangular shape functions are illustrated below:

![Linear triangular shape functions](image)

**Axisymmetric Elements**

Derivation of the Stiffness Matrix

**Step 2 - Select Displacement Functions**

The linear triangular shape functions are illustrated below:

![Linear triangular shape functions](image)
**Axisymmetric Elements**

Derivation of the Stiffness Matrix

**Step 2 - Select Displacement Functions**

So that $u$ and $w$ will yield a constant value for rigid-body displacement, $N_i + N_j + N_m = 1$ for all $r$ and $z$ locations on the element.

The linear triangular shape functions are illustrated below:

For example, assume all the triangle displaces as a rigid body in the $x$ direction: $u = u_0$

$$\{\Psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{bmatrix} u_0 \\ 0 \\ u_0 \\ 0 \\ u_0 \\ 0 \end{bmatrix} \Rightarrow N_i + N_j + N_m = 1$$
Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions

So that \( u \) and \( w \) will yield a constant value for rigid-body displacement, \( N_i + N_j + N_m = 1 \) for all \( r \) and \( z \) locations on the element.

For example, assume all the triangle displaces as a rigid body in the \( z \) direction: \( w = w_0 \)

\[
\begin{bmatrix}
N_i & 0 & N_j & 0 & N_m & 0 \\
0 & N_i & 0 & N_j & 0 & N_m
\end{bmatrix}
\begin{bmatrix}
0 \\
w_0 \\
0 \\
w_0 \\
0 \\
w_0
\end{bmatrix} = w_0 \left( N_i + N_j + N_m \right)
\]

\[
\Rightarrow N_i + N_j + N_m = 1
\]

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

Elemental Strains: The strains over a two-dimensional element are:

\[
\{\varepsilon\} = \begin{bmatrix}
\varepsilon_r \\
\varepsilon_z \\
\varepsilon_\theta \\
\gamma_{rz}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u}{\partial r} \\
\frac{\partial w}{\partial z} \\
\frac{u}{r} \\
\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}
\end{bmatrix} = \begin{bmatrix}
a_2 \\
a_6 \\
a_1 + a_2 + \frac{a_3 z}{r} \\
a_3 + a_5
\end{bmatrix}
\]
Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

Elemental Strains: The strains over a two-dimensional element are:

\[
\epsilon = \begin{Bmatrix} \epsilon_r \\ \epsilon_z \\ \epsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial w}{\partial z} \\ \frac{u}{r} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & z & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}
\]

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

Substituting our approximation for the displacement gives:

\[
\frac{\partial u}{\partial r} = u_r = \frac{\partial}{\partial r} \left( N_i u_i + N_j u_j + N_m u_m \right)
\]

\[u_r = N_{i,r} u_i + N_{j,r} u_j + N_{m,r} u_m\]

where the comma indicates differentiation with respect to that variable.
Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

The derivatives of the interpolation functions are:

\[ N_{i,r} = \frac{1}{2A} \frac{\partial}{\partial r} \left( \alpha_i + \beta_i r + \gamma_i z \right) = \frac{\beta_i}{2A} \]

\[ N_{j,r} = \frac{\beta_j}{2A} \quad N_{m,r} = \frac{\beta_m}{2A} \]

Therefore:

\[ \frac{\partial u}{\partial r} = \frac{1}{2A} \left( \beta_i u_i + \beta_j u_j + \beta_m u_m \right) \]

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

In a similar manner, the remaining strain terms are approximated as:

\[ \frac{\partial w}{\partial z} = \frac{1}{2A} \left( \gamma_i w_i + \gamma_j w_j + \gamma_m w_m \right) \]

\[ \frac{u}{r} = \frac{1}{2A} \left[ \left( \frac{\alpha_i}{r} + \beta_i + \frac{\gamma_i z}{r} \right) u_i + \left( \frac{\alpha_j}{r} + \beta_j + \frac{\gamma_j z}{r} \right) u_j \right. \]

\[ \left. + \left( \frac{\alpha_m}{r} + \beta_m + \frac{\gamma_m z}{r} \right) u_m \right] \]

\[ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = \frac{1}{2A} \left( \beta_i u_i + \gamma_i w_i + \beta_j u_j + \gamma_j w_j + \beta_m u_m + \gamma_m w_m \right) \]
**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

**Step 3 - Define the Strain-Displacement and Stress-Strain Relationships**

We can write the strains in matrix form as:

\[
\{\varepsilon\} = [B]\{d\}
\]

\[
\begin{pmatrix}
\varepsilon_r \\
\varepsilon_z \\
\varepsilon_\theta \\
\gamma_{rz}
\end{pmatrix} = \frac{1}{2A} \begin{pmatrix}
\alpha_i + \frac{\gamma_i Z}{r} & 0 & \frac{\alpha_j + \gamma_j Z}{r} & 0 & \frac{\alpha_m + \gamma_m Z}{r} & 0 \\
0 & \beta_i & 0 & \beta_j & 0 & \beta_m \\
\gamma_i & \beta_j & \gamma_j & \beta_i & \gamma_m & \beta_m \\
\gamma_{rz} & \gamma_i & \gamma_j & \gamma_i & \gamma_j & \gamma_{rz}
\end{pmatrix}
\begin{pmatrix}
u_i \\
v_j \\
u_m \\
v_i \\
v_j \\
v_m
\end{pmatrix}
\]

**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

**Step 3 - Define the Strain-Displacement and Stress-Strain Relationships**

We can write the strains in matrix form as:

\[
\{\varepsilon\} = [B]\{d\}
\]

\[
\begin{pmatrix}
\varepsilon_r \\
\varepsilon_z \\
\varepsilon_\theta \\
\gamma_{rz}
\end{pmatrix} = \begin{pmatrix}
[B_i] & [B_j] & [B_m]
\end{pmatrix}
\begin{pmatrix}
u_i \\
v_j \\
u_m \\
v_i \\
v_j \\
v_m
\end{pmatrix}
\]
**Axisymmetric Elements**

Derivation of the Stiffness Matrix

**Step 3 - Define the Strain-Displacement and Stress-Strain Relationships**

**Stress-Strain Relationship:** The in-plane stress-strain relationship is:

\[
\begin{bmatrix}
\sigma_r \\
\sigma_z \\
\sigma_\theta \\
\tau_{xy} \\
\end{bmatrix} = [D] \begin{bmatrix}
\varepsilon_r \\
\varepsilon_z \\
\varepsilon_\theta \\
\gamma_{xy} \\
\end{bmatrix}
\]

\[
\{\sigma\} = [D] [B] \{d\}
\]

where

\[
[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & 0 & 0 \\
\nu & 1-\nu & 0 & 0 \\
0 & 0 & 1-\nu & 0 \\
0 & 0 & 0 & 0.5-\nu \\
\end{bmatrix}
\]

**Axisymmetric Elements**

Derivation of the Stiffness Matrix

**Step 4 - Derive the Element Stiffness Matrix and Equations**

The stiffness matrix can be defined as:

\[
[k] = \int_V [B]^T [D] [B] dV
\]

For a circumferential differential element the integral becomes:

\[
[k] = 2\pi \int_A [B]^T [D] [B] r \, dr \, dz
\]

After integrating along the circumferential boundary, the [B] matrix is a function of \( r \) and \( z \).
**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

**Step 4 - Derive the Element Stiffness Matrix and Equations**

Therefore, $[k]$ is a function of $r$ and $z$ and is of order 6 x 6.

We can evaluate $[k]$ by one of three methods:

1. Numerical integration (Gaussian quadrature) as discussed in Chapter 10.
2. Explicit multiplication and term-by-term integration.
3. Evaluate $[B]$ for a centroidal point $(\bar{r}, \bar{z})$ of the element

$$[B(\bar{r}, \bar{z})] = [\bar{B}]$$

$$r = \bar{r} = \frac{r_i + r_j + r_m}{3}$$

$$z = \bar{z} = \frac{z_i + z_j + z_m}{3}$$
**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

**Step 4 - Derive the Element Stiffness Matrix and Equations**

Therefore, \([k]\) is a function of \(r\) and \(z\) and is of order 6 x 6.

We can evaluate \([k]\) by one of three methods:

3. Evaluate \([B]\) for a centroidal point \((\bar{r}, \bar{z})\) of the element

As a first approximation: 
\[
[k] = 2\pi r A [B]^T [D] [B]
\]

If the triangular subdivisions are consistent with the final stress distribution (that is, small elements in regions of high stress gradients), then acceptable results can be obtained by Method 3.

**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

**Step 4 - Derive the Element Stiffness Matrix and Equations**

**Distributed Body Forces**

Loads such as gravity (in the direction of the \(z\) axis) or centrifugal forces in rotating machine parts (in the direction of the \(r\) axis) are considered to be body forces.

The body forces can be found by:

\[
\{f_b\} = 2\pi \int_{\Lambda} \left[ [N]^T \begin{pmatrix} R_b \\ Z_b \end{pmatrix} \right] r \, dr \, dz
\]
**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

**Step 4 - Derive the Element Stiffness Matrix and Equations**

**Distributed Body Forces**

Where \( R_b = \omega^2 \rho r \) for a machine part moving with a constant angular velocity \( \omega \) about the z axis, with material mass density \( \rho \) and radial coordinate \( r \), and \( Z_b \) is the body force per unit volume due to the force of gravity.

\[
\{f_b\} = 2\pi \int_A \{N\}^T \begin{bmatrix} R_b \\ Z_b \end{bmatrix} r \, dr \, dz
\]

The body forces can be found by:

\[
\{f_b\} = 2\pi \int_A \{N\}^T \begin{bmatrix} R_b \\ Z_b \end{bmatrix} r \, dr \, dz
\]

**Axisymmetric Elements**

**Derivation of the Stiffness Matrix**

**Step 4 - Derive the Element Stiffness Matrix and Equations**

**Distributed Body Forces**

Considering the body force at node \( i \), we have

\[
\{f_{bi}\} = 2\pi \int_A \{N_i\}^T \begin{bmatrix} R_b \\ Z_b \end{bmatrix} r \, dr \, dz
\]

\[
\{N\}^T = \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix}
\]

Multiplying and integrating yields

\[
\{f_{bi}\} = \frac{2\pi}{3} \begin{bmatrix} R_b \\ Z_b \end{bmatrix} \bar{A} \bar{r}
\]

The origin of the coordinates is the centroid of the element, and \( R_b \) is the radially directed body force per unit volume evaluated at the centroid of the element.
Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 4 - Derive the Element Stiffness Matrix and Equations

Distributed Body Forces

The body forces at nodes \( j \) and \( m \) are identical to those given for node \( i \). Hence, for an element, we have

\[
\{f_b\} = \frac{2\pi A \bar{R}}{3} \begin{bmatrix} \bar{R}_b \\ Z_b \\ \bar{R}_b \\ Z_b \end{bmatrix} \quad \bar{R}_b = \omega^2 \rho \bar{R}
\]

Surface Forces

Surface forces can be found by

\[
\{f_s\} = \int_S [N_s]^T \{T\} \, dS
\]

Where again \([N_s]\) denotes the shape function matrix evaluated along the surface where the surface traction acts.

For example, along the vertical face \( jm \) of an element, let uniform loads \( p_r \) and \( p_z \) be applied along surface \( r = r_j \).
**Axisymmetric Elements**

Derivation of the Stiffness Matrix

Step 4 - Derive the Element Stiffness Matrix and Equations

**Surface Forces**

For instance, for node \( j \), substituting \( N_j \) gives

\[
\{f_{si}\} = \int_{z_i}^{z_j} \frac{1}{2A} \begin{bmatrix}
\alpha_j + \beta_j r + \gamma_j z & 0 \\
0 & \alpha_j + \beta_j r + \gamma_j z
\end{bmatrix} \begin{bmatrix}
p_r \\
p_z
\end{bmatrix} 2\pi r_j \, dz
\]

Evaluated at \( r = r_j \) and \( z \)

---

**Axisymmetric Elements**

Derivation of the Stiffness Matrix

Step 4 - Derive the Element Stiffness Matrix and Equations

**Surface Forces**

Integrating the equations explicitly along with similar evaluations for \( f_{si} \) and \( f_{sm} \) the total distribution of surface force to nodes \( i \), \( j \), and \( m \) is

\[
\{f_s\} = \frac{2\pi r_j (z_m - z_j)}{2} \begin{bmatrix}
0 \\
0 \\
p_r \\
p_r \\
p_z \\
p_z
\end{bmatrix}
\]
Axisymmetric Elements

Derivation of the Stiffness Matrix

Steps 5 - 7

Steps 5 through 7, which involve assembling the total stiffness matrix, total force matrix, and total set of equations; solving for the nodal degrees of freedom; and calculating the element stresses, are analogous to those of Chapter 6 for the CST element.

The stresses are not constant in each element.

They are usually determined by one of two methods that we use to determine the LST element stresses.

1. Either we determine the centroidal element stresses, or
2. We determine the nodal stresses for the element and then average them.

The latter method has been shown to be more accurate in some cases.
**Axisymmetric Elements**

**Example 1**

For the element of an axisymmetric body rotating with a constant angular velocity $\omega = 100$ rev/min, evaluate the approximate body force matrix.

Include the weight of the material, where the weight density $\rho_w = 0.283$ lb./in.$^3$. Dimensions are inches.

$$Z_b = 0.283 \text{ lb./in.}^3$$

**Axisymmetric Elements**

**Example 1**

Let evaluate the approximate body force matrix.

The body forces per unit volume evaluated at the centroid of the element are:

$$\bar{R}_b = \omega^2 \rho \bar{r}$$

$$= \left[(100 \text{ rpm})(2\pi \text{ rev/rot})\left(\frac{1 \text{ min}}{60 \text{ sec}}\right)\right]^2 \frac{0.283 \text{ lb./in.}^3}{32.2 \frac{\text{lb}}{\text{ft.}^3}}(2.333 \text{ in.})$$

$$= 0.187 \text{ lb./in.}^3$$

$$\frac{2\pi A\bar{r}}{3} = 2\pi \left(0.5 \text{ in.}^2\right)(2.333 \text{ in.}) = 2.44 \text{ in.}^2$$
**Axisymmetric Elements**

Example 1

Let evaluate the approximate body force matrix.

The body forces per unit volume evaluated at the centroid of the element are:

\[
\{f_b\} = \frac{2\pi Af}{3} = 2.44 \text{ in.}^3
\]

\[
\begin{bmatrix}
R_b \\
Z_b \\
R_b \\
Z_b
\end{bmatrix}
\begin{bmatrix}
0.187 \\
-0.283 \\
0.187 \\
-0.283
\end{bmatrix}
\begin{bmatrix}
\text{lb.} \\
\text{in.}^3
\end{bmatrix} = \begin{bmatrix}
0.457 \\
-0.691 \\
0.457 \\
-0.691
\end{bmatrix}
\]

Example 2

For the long, thick-walled cylinder under internal pressure \( p \) equal to 1 psi, determine the displacements and stresses.
**Axisymmetric Elements**

**Example 2**

First discretize the cylinder into four triangular elements.

A horizontal slice of the cylinder represents the total cylinder behavior.

A coarse mesh of elements is used for simplicity's sake.

**[K]** is a matrix of order 10 x 10

\[
\begin{bmatrix}
F_{4r} \\
F_{1z} \\
F_{2r} \\
F_{2z} \\
F_{3r} \\
F_{3z} \\
F_{4r} \\
F_{4z} \\
F_{5r} \\
F_{5z}
\end{bmatrix} = [K]
\begin{bmatrix}
u_1 \\
w_1 \\
u_2 \\
w_2 \\
u_3 \\
w_3 \\
u_4 \\
w_4 \\
u_5 \\
w_5
\end{bmatrix}
\]
Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix

The $[K]$ matrix is assembled in the usual manner by superposition of the individual element stiffness matrices.

For simplicity's sake, we will evaluate $[B]$ for a centroidal point $(\bar{r}, \bar{z})$ of the element.

$$[k] = 2\pi \bar{r}A [B]^T [D] [B]$$
**Axisymmetric Elements**

**Example 2**

Assemblage of the Stiffness Matrix: Element 1

\[
\begin{align*}
\alpha_j &= r_j z_m - z_f r_m = (1.0)(0.25) - (0.0)(0.75) = 0.25\text{in.}^2 \\
\alpha_j &= r_m z_i - z_m r_i = (0.75)(0.0) - (0.25)(0.5) = -0.125\text{in.}^2 \\
\alpha_m &= r_z z_i - z_r r_j = (0.5)(0.0) - (0.0)(1.0) = 0
\end{align*}
\]

**Axisymmetric Elements**

**Example 2**

Assemblage of the Stiffness Matrix: Element 1

\[
\begin{align*}
\beta_i &= z_j - z_m = -0.25\text{in}^2 \\
\gamma_i &= r_m - r_j = -0.25\text{in}^2 \\
\beta_j &= z_m - z_i = 0.25\text{in}^2 \\
\gamma_j &= r_i - r_m = -0.25\text{in}^2 \\
\beta_m &= z_i - z_j = 0 \\
\gamma_m &= r_j - r_i = 0.5\text{in}^2
\end{align*}
\]
**Axisymmetric Elements**

**Example 2**

Assemblage of the Stiffness Matrix: Element 1

\[
\begin{align*}
\mathbf{r} &= \bar{r} \quad \mathbf{z} = \bar{z} \\
\begin{bmatrix}
\mathbf{r}_i \\
\mathbf{r}_j \\
\mathbf{r}_m
\end{bmatrix} &= 
\begin{bmatrix}
0.50 \\
1.00 \\
0.75
\end{bmatrix} \text{ in.} \\
\begin{bmatrix}
\mathbf{z}_i \\
\mathbf{z}_j \\
\mathbf{z}_m
\end{bmatrix} &= 
\begin{bmatrix}
0.00 \\
0.00 \\
0.25
\end{bmatrix} \text{ in.}
\end{align*}
\]

\[
\bar{r} = \frac{3}{3} \sum r_i = 0.75 \text{ in.} \\
\bar{z} = \frac{3}{3} \sum z_i = 0.0833 \text{ in.}
\]

\[
A = \frac{1}{2} (0.5)(0.25) = 0.0625 \text{ in.}^2
\]

**Axisymmetric Elements**

**Example 2**

Assemblage of the Stiffness Matrix: Element 1

\[
\begin{align*}
\mathbf{r} &= \bar{r} \quad \mathbf{z} = \bar{z} \\
\begin{bmatrix}
\mathbf{r}_i \\
\mathbf{r}_j \\
\mathbf{r}_m
\end{bmatrix} &= 
\begin{bmatrix}
0.50 \\
1.00 \\
0.75
\end{bmatrix} \text{ in.} \\
\begin{bmatrix}
\mathbf{z}_i \\
\mathbf{z}_j \\
\mathbf{z}_m
\end{bmatrix} &= 
\begin{bmatrix}
0.00 \\
0.00 \\
0.25
\end{bmatrix} \text{ in.}
\end{align*}
\]

\[
\mathbf{B} = \frac{1}{0.125} \begin{bmatrix}
-0.25 & 0 & 0.25 & 0 & 0 & 0 \\
0 & -0.25 & 0 & -0.25 & 0 & 0.5 \\
0.0556 & 0 & 0.0556 & 0 & 0.0556 & 0 \\
-0.025 & -0.25 & -0.25 & 0.25 & 0.5 & 0
\end{bmatrix} \text{ in.}
\]
**Axisymmetric Elements**

**Example 2**

Assemblage of the Stiffness Matrix: Element 1

Assume that $E = 30 \times 10^6$ psi and $\nu = 0.3$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 & 0 \\ \nu & 1-\nu & 0 & 0 \\ 0 & 0 & 1-\nu & 0 \\ 0 & 0 & 0 & 0.5-\nu \end{bmatrix}$$
**Axisymmetric Elements**

**Example 2**

Assemblage of the Stiffness Matrix: Element 1

\[
\begin{align*}
& r = \bar{r} \quad z = \bar{z} \\
& \{ r_i \} = \begin{bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{bmatrix} \text{ in.} \quad \{ z_i \} = \begin{bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{bmatrix} \text{ in.}
\end{align*}
\]

\[
\left[ B \right]_{3x4}^T \left[ D \right]_{4x4} = \frac{57.7 \left(10^6\right)}{0.125} = \begin{bmatrix}
-0.158 & -0.0583 & -0.0361 & -0.05 \\
-0.075 & -0.175 & -0.075 & -0.05 \\
0.192 & 0.0917 & 0.114 & 0.05 \\
-0.075 & -0.175 & -0.075 & 0.05 \\
0.0167 & 0.0166 & 0.0388 & 0.1 \\
0.15 & 0.35 & 0.15 & 0
\end{bmatrix}
\]

**Axisymmetric Elements**

**Example 2**

Assemblage of the Stiffness Matrix: Element 1

\[
\begin{align*}
& r = \bar{r} \quad z = \bar{z} \\
& \{ r_i \} = \begin{bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{bmatrix} \text{ in.} \quad \{ z_i \} = \begin{bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{bmatrix} \text{ in.}
\end{align*}
\]

\[
\left[ k^{(1)} \right] = \left(10^6\right) = \begin{bmatrix}
54.46 & 29.45 & -31.63 & 2.26 & -29.37 & -31.71 \\
29.45 & 61.17 & -11.33 & 33.98 & -31.72 & -95.15 \\
-31.63 & -11.33 & 72.59 & -38.52 & -20.31 & 49.84 \\
2.26 & 33.98 & -38.52 & 61.17 & 22.66 & -95.15 \\
-29.37 & -31.72 & -20.31 & 22.66 & 56.72 & 9.06 \\
-31.71 & -95.15 & 49.84 & -95.15 & 9.06 & 190.31
\end{bmatrix} \text{ lb. in.}
\]
**Axisymmetric Elements**

**Example 2**

Assemblage of the Stiffness Matrix: Element 2

\[
\begin{bmatrix}
\beta_i & 0 & \beta_j & 0 \\
0 & \gamma_i & 0 & \gamma_j \\
\alpha_i & \beta_i & \gamma_i & 0 \\
\gamma_i & \beta_i & \gamma_i & \beta_j
\end{bmatrix}
\]

\[
[\mathbf{B}] = \frac{1}{2A} \begin{bmatrix}
\frac{\alpha_i}{r} + \beta_i + \frac{\bar{Z}}{r} & 0 & \frac{\alpha_j}{r} + \beta_j + \frac{\bar{Z}}{r} & 0 \\
0 & \gamma_i & 0 & \gamma_j \\
\gamma_i & \beta_i & \gamma_i & 0 \\
\beta_i & \gamma_i & \beta_j & \gamma_j
\end{bmatrix}
\]

\[
r = \bar{r} \quad z = \bar{z}
\]

Input:
\[
\begin{align*}
\{r_i\} &= \begin{bmatrix} 1.00 \end{bmatrix} \text{in.} \\
\{r_j\} &= \begin{bmatrix} 1.00 \end{bmatrix} \text{in.} \\
\{r_m\} &= \begin{bmatrix} 0.75 \end{bmatrix} \\
\{z_i\} &= \begin{bmatrix} 0.00 \end{bmatrix} \text{in.} \\
\{z_j\} &= \begin{bmatrix} 0.50 \end{bmatrix} \text{in.} \\
\{z_m\} &= \begin{bmatrix} 0.25 \end{bmatrix}
\end{align*}
\]

Output:
\[
\begin{align*}
\alpha_i &= r_j z_m - z_j r_m = (1.0)(0.25) - (0.5)(0.75) = -0.125 \text{in.}^2 \\
\alpha_j &= r_m z_i - z_m r_i = (0.75)(0.0) - (0.25)(1.0) = -0.25 \text{in.}^2 \\
\alpha_m &= r_i z_j - z_i r_j = (1.0)(0.5) - (0.0)(1.0) = 0.5 \text{in.}^2
\end{align*}
\]
**Axisymmetric Elements**

**Example 2**

Assemblage of the Stiffness Matrix: Element 2

\[
\begin{align*}
    r &= \bar{r} \\
    z &= \bar{z}
\end{align*}
\]

\[
\begin{bmatrix}
    r_i \\
    r_j \\
    r_m
\end{bmatrix} = \begin{bmatrix}
    1.00 \\
    1.00 \\
    0.75
\end{bmatrix} \text{ in.} \quad \begin{bmatrix}
    z_i \\
    z_j \\
    z_m
\end{bmatrix} = \begin{bmatrix}
    0.00 \\
    0.50 \\
    0.25
\end{bmatrix} \text{ in.}
\]

\[\beta_i = z_j - z_m = 0.25\text{in}^2\]
\[\gamma_i = r_m - r_j = -0.25\text{in}^2\]

\[\beta_j = z_m - z_i = 0.25\text{in}^2\]
\[\gamma_j = r_j - r_m = 0.25\text{in}^2\]

\[\beta_m = z_i - z_j = -0.5\text{in}^2\]
\[\gamma_m = r_j - r_i = 0\]

\[
\bar{r} = \sum_{i=1}^{3} \frac{r_i}{3} = 0.9167\text{in.} \quad \bar{z} = \sum_{j=1}^{3} \frac{z_j}{3} = 0.25\text{in.}
\]

\[A = \frac{1}{2} (0.5)(0.25) = 0.0625\text{in}^2\]
**Axisymmetric Elements**

**Example 2**

Assemblage of the Stiffness Matrix: Element 2

\[
\begin{align*}
    r &= \bar{r} \quad z = \bar{z} \\
    \left\{ \begin{array}{c}
        r_i \\
        r_j \\
        r_m
    \end{array} \right. &= \left\{ \begin{array}{c}
        1.00 \\
        1.00 \\
        0.75
    \end{array} \right. \text{ in.} \\
    \left\{ \begin{array}{c}
        z_i \\
        z_j \\
        z_m
    \end{array} \right. &= \left\{ \begin{array}{c}
        0.00 \\
        0.50 \\
        0.25
    \end{array} \right. \text{ in.}
\end{align*}
\]

\[
\begin{bmatrix}
    85.75 & -46.07 & 52.52 & 12.84 & -118.92 & 33.23 \\
    -46.07 & 74.77 & -12.84 & -41.54 & 45.32 & -33.23 \\
    52.52 & -12.84 & 85.74 & 46.07 & -118.92 & -33.23 \\
    12.84 & -41.54 & 46.07 & 74.77 & -45.21 & -33.23 \\
    -118.92 & 45.32 & -118.92 & -45.21 & 216.41 & 0 \\
    33.23 & -33.23 & -33.23 & -33.23 & 0 & 66.46
\end{bmatrix} = (10^6)
\]

\[
\left[ k^{(2)} \right] = \begin{bmatrix}
    85.75 & -46.07 & 52.52 & 12.84 & -118.92 & 33.23 \\
    -46.07 & 74.77 & -12.84 & -41.54 & 45.32 & -33.23 \\
    52.52 & -12.84 & 85.74 & 46.07 & -118.92 & -33.23 \\
    12.84 & -41.54 & 46.07 & 74.77 & -45.21 & -33.23 \\
    -118.92 & 45.32 & -118.92 & -45.21 & 216.41 & 0 \\
    33.23 & -33.23 & -33.23 & -33.23 & 0 & 66.46
\end{bmatrix}
\]

**Axisymmetric Elements**

**Example 2**

Assemblage of the Stiffness Matrix: Element 3

\[
\begin{align*}
    r &= \bar{r} \quad z = \bar{z} \\
    \left\{ \begin{array}{c}
        r_i \\
        r_j \\
        r_m
    \end{array} \right. &= \left\{ \begin{array}{c}
        1.00 \\
        0.50 \\
        0.75
    \end{array} \right. \text{ in.} \\
    \left\{ \begin{array}{c}
        z_i \\
        z_j \\
        z_m
    \end{array} \right. &= \left\{ \begin{array}{c}
        0.50 \\
        0.50 \\
        0.25
    \end{array} \right. \text{ in.}
\end{align*}
\]

\[
\begin{bmatrix}
    \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\
    0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\
    \alpha_i + \beta_i + \frac{\bar{Z}}{r} & 0 & \alpha_j + \beta_j + \frac{\bar{Z}}{r} & 0 & \alpha_m + \beta_m + \frac{\bar{Z}}{r} & 0 \\
    \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m
\end{bmatrix}
\]

\[
\left[ \bar{B} \right] = \frac{1}{2A} \begin{bmatrix}
    \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\
    0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\
    \alpha_i + \beta_i + \frac{\bar{Z}}{r} & 0 & \alpha_j + \beta_j + \frac{\bar{Z}}{r} & 0 & \alpha_m + \beta_m + \frac{\bar{Z}}{r} & 0 \\
    \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m
\end{bmatrix}
\]
Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 3

\[ r = \bar{r} \quad z = \bar{z} \]

\[
\begin{aligned}
\{ r_i \} &= \begin{bmatrix} 1.00 \\ 0.50 \\ 0.75 \end{bmatrix} \text{ in.} \\
\{ z_i \} &= \begin{bmatrix} 0.50 \\ 0.50 \\ 0.25 \end{bmatrix} \text{ in.}
\end{aligned}
\]

\[ \alpha_j = r_j z_m - z_j r_m = (0.5)(0.25) - (0.5)(0.75) = -0.25 \text{ in.}^2 \]

\[ \alpha_j = r_m z_i - z_m r_i = (0.75)(0.5) - (0.25)(1.0) = 0.125 \text{ in.}^2 \]

\[ \alpha_m = r_i z_j - z_i r_j = (1.0)(0.5) - (0.5)(0.5) = 0.25 \text{ in.}^2 \]

Asymmetry Elements

Example 2

Assemblage of the Stiffness Matrix: Element 3

\[ r = \bar{r} \quad z = \bar{z} \]

\[
\begin{aligned}
\{ r_i \} &= \begin{bmatrix} 1.00 \\ 0.50 \\ 0.75 \end{bmatrix} \text{ in.} \\
\{ z_i \} &= \begin{bmatrix} 0.50 \\ 0.50 \\ 0.25 \end{bmatrix} \text{ in.}
\end{aligned}
\]

\[ \beta_i = z_j - z_m = 0.25 \text{ in}^2 \quad \gamma_i = r_m - r_j = 0.25 \text{ in}^2 \]

\[ \beta_j = z_m - z_i = -0.25 \text{ in}^2 \quad \gamma_j = r_i - r_m = 0.25 \text{ in}^2 \]

\[ \beta_m = z_j - z_j = 0 \quad \gamma_m = r_j - r_i = -0.5 \text{ in}^2 \]
**Axisymmetric Elements**

**Example 2**

Assemblage of the Stiffness Matrix: Element 3

\[
\begin{align*}
\{ r_i \} &= \begin{bmatrix} 1.00 \\ 0.50 \\ 0.75 \end{bmatrix} \text{ in.} \\
\{ z_i \} &= \begin{bmatrix} 0.50 \\ 0.50 \end{bmatrix} \text{ in.} \\
\{ r_m \} &= \begin{bmatrix} 0.75 \end{bmatrix} \\
\{ z_m \} &= \begin{bmatrix} 0.25 \end{bmatrix}
\end{align*}
\]

\[
\bar{r} = \sum_{i=1}^{3} \frac{r_i}{3} = 0.75 \text{ in.} \quad \bar{z} = \sum_{i=1}^{3} \frac{z_i}{3} = 0.417 \text{ in.}
\]

\[
A = \frac{1}{2} (0.5)(0.25) = 0.0625 \text{ in.}^2
\]
**Axisymmetric Elements**

**Example 2**

Assemblage of the Stiffness Matrix: Element 4

$r = \bar{r}$  \hspace{1cm}  $z = \bar{z}$

\[
\begin{bmatrix}
\beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\
0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\
\alpha_i + \beta_i + \frac{z}{r} & 0 & \alpha_j + \beta_j + \frac{z}{r} & 0 & \alpha_m + \gamma_m + \frac{z}{r} & 0 \\
\gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m
\end{bmatrix}
\]

$\alpha_i = r_i z_m - z_i r_m = (0.75)(0.5) - (0.25)(0.5) = 0.25 \text{ in}^2$

$\alpha_j = r_m z_i - z_m r_j = (0.5)(0.0) - (0.5)(0.5) = -0.25 \text{ in}^2$

$\alpha_m = r_i z_j - z_i r_j = (0.5)(0.25) - (0.0)(0.75) = 0.125 \text{ in}^2$
**Axisymmetric Elements**

Example 2

Assemblage of the Stiffness Matrix: Element 4

\[
\begin{align*}
\beta_i &= z_j - z_m = -0.25\text{in}^2 \\
\gamma_i &= r_m - r_j = -0.25\text{in}^2 \\
\beta_j &= z_m - z_i = 0.5\text{in}^2 \\
\gamma_j &= r_i - r_m = 0 \\
\beta_m &= z_i - z_j = -0.25\text{in}^2 \\
\gamma_m &= r_j - r_i = 0.25\text{in}^2
\end{align*}
\]

\[
\begin{align*}
\bar{r} &= \sum_{i=1}^{3} \frac{r_i}{3} = 0.5833\text{in.} \\
\bar{z} &= \sum_{j=1}^{3} \frac{z_j}{3} = 0.25\text{in.} \\
A &= \frac{1}{2} (0.5)(0.25) = 0.0625\text{in}^2
\end{align*}
\]
**Axisymmetric Elements**

**Example 2**

Assemblage of the Stiffness Matrix: Element 4

\[
\begin{align*}
\mathbf{r} &= \bar{r} \\
\mathbf{z} &= \bar{z}
\end{align*}
\]

\[
\begin{bmatrix}
\mathbf{r}_i \\
\mathbf{r}_j \\
\mathbf{r}_m
\end{bmatrix} = \begin{bmatrix}
0.50 \\
0.75 \\
0.50
\end{bmatrix} \text{ in.}
\begin{bmatrix}
\mathbf{z}_i \\
\mathbf{z}_j \\
\mathbf{z}_m
\end{bmatrix} = \begin{bmatrix}
0.00 \\
0.25 \\
0.50
\end{bmatrix} \text{ in.}
\]

\[
\begin{bmatrix}
41.53 & 21.90 & -66.45 & -21.14 & 20.39 & -0.75 \\
21.92 & 47.57 & -36.24 & -21.14 & 0.75 & -26.43 \\
20.39 & 0.75 & -66.45 & 21.14 & 41.53 & -21.90 \\
-0.75 & -26.43 & 36.24 & -21.14 & -21.90 & 47.57
\end{bmatrix} \times 10^6 \text{ lb. in.}
\]

**Axisymmetric Elements**

**Example 2**

Using superposition of the element stiffness matrices, where we rearrange the elements of each stiffness matrix in order of increasing nodal degrees of freedom, we can obtain the global stiffness matrix.
**Axisymmetric Elements**

**Example 2**

The applied nodal forces are given as:

\[
\begin{bmatrix}
\rho_r & \rho_z \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
0.785 \\
0 \\
0 \\
0.785 \\
0
\end{bmatrix}
= 0.785 \text{ lb.}
\]

\[
F_{tr} = F_{4r} = \frac{2\pi (0.5 \text{ in.})(0.5 \text{ in.})}{2}(1 \text{ psi}) = 0.785 \text{ lb.}
\]
### Axisymmetric Elements

**Example 2**

The resulting equations are:

\[
\begin{bmatrix}
95.99 & 51.35 & -36.63 & 2.26 & 0 & 0 & 20.39 & -0.75 & -95.82 & -52.86 \\
51.35 & 108.74 & -11.33 & 33.98 & 0 & 0 & 33.98 & -26.43 & -67.96 & -116.3 \\
-36.63 & -11.33 & 159.34 & -84.59 & 52.52 & 12.84 & 0 & 0 & -139.2 & 83.07 \\
2.26 & 33.98 & -84.59 & 135.94 & 12.84 & -41.54 & 0 & 0 & -67.96 & -128.4 \\
0 & 0 & 52.52 & -12.84 & -31.63 & 11.33 & 0 & 0 & -139.2 & -83.07 \\
0 & 0 & 12.84 & -41.54 & 84.59 & 135.94 & -2.26 & 33.98 & -67.96 & -128.4 \\
20.39 & 33.98 & 0 & 0 & -31.63 & -2.26 & 95.99 & -51.35 & -95.82 & 52.86 \\
0.75 & -26.43 & 0 & 0 & 11.33 & 33.98 & 51.35 & 108.74 & 67.96 & -116.3 \\
-95.82 & -67.96 & -139.2 & 67.98 & -139.2 & -67.98 & 95.99 & 67.96 & 498.99 & 0 \\
-52.86 & -116.3 & 83.07 & -128.4 & -83.07 & -128.4 & 52.86 & -116.3 & 0 & 489.36 \\
\end{bmatrix}
\begin{bmatrix}
\frac{u_1}{(10^6) \text{ in.}} \\
\frac{w_1}{(10^6) \text{ in.}} \\
\frac{u_2}{(10^6) \text{ in.}} \\
\frac{w_2}{(10^6) \text{ in.}} \\
\frac{u_3}{(10^6) \text{ in.}} \\
\frac{w_3}{(10^6) \text{ in.}} \\
\frac{u_4}{(10^6) \text{ in.}} \\
\frac{w_4}{(10^6) \text{ in.}} \\
\frac{u_5}{(10^6) \text{ in.}} \\
\frac{w_5}{(10^6) \text{ in.}} \\
\end{bmatrix}
= \begin{bmatrix}
0.785 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} \text{ lb.}
\]

**Axisymmetric Elements**

**Example 2**

The nodal displacements are:

\[
\begin{align*}
\begin{bmatrix}
0.0322 \\
0.00115 \\
0.0219 \\
0.00206 \\
0.0219 \\
-0.00206 \\
0.0322 \\
-0.00115 \\
0.0244 \\
0.0 \\
\end{bmatrix}
\times (10^{-6}) \text{ in.}.
\end{align*}
\]
Axisymmetric Elements

Example 2

The results for nodal displacements are as expected because radial displacements at the inner edge are equal \( u_1 = u_4 \) and those at the outer edge are equal \( u_2 = u_3 \).

Axisymmetric Elements

Example 2

In addition, the axial displacements at the outer nodes and inner nodes are equal but opposite in sign \( w_1 = -w_4 \) and \( w_2 = -w_3 \) as a result of the Poisson effect and symmetry.
**Axisymmetric Elements**

**Example 2**

Finally, the axial displacement at the center node is zero ($w_5 = 0$), as it should be because of symmetry.

![Diagram of axisymmetric element](image)

**Axisymmetric Elements**

**Example 2**

Determine the stresses in each element as: $\{\sigma\} = [D][\bar{B}]{d}$

For Element 1:

$$[D] = 57.7 \times 10^8 \begin{bmatrix} \sigma & 0.7 & 0.3 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix} \text{ psi}$$

$$[\bar{B}] = \begin{bmatrix} -0.25 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & -0.25 & 0 & -0.25 & 0 & 0.5 \\ 0.0556 & 0 & 0.0556 & 0 & 0.0556 & 0 \\ -0.025 & -0.25 & -0.25 & 0.25 & 0.5 & 0 \end{bmatrix} \frac{1}{\text{in.}}$$
**Axisymmetric Elements**

**Example 2**

Determine the stresses in each element as: \( \{\sigma\} = [D][\bar{B}]{d} \)

For Element 1:

\[
\begin{pmatrix}
\sigma_r \\
\sigma_z \\
\sigma_\theta \\
\tau_{xy}
\end{pmatrix} = [D][\bar{B}]
\begin{pmatrix}
u_1 \\
w_1 \\
u_2 \\
w_2 \\
u_3 \\
w_3 \\
u_5 \\
w_5
\end{pmatrix}
= \begin{pmatrix}
-0.338 \\
-0.0126 \\
0.942 \\
0.1037
\end{pmatrix}
\text{psi}
\]

For Element 2:

\[
\begin{pmatrix}
\sigma_r \\
\sigma_z \\
\sigma_\theta \\
\tau_{xy}
\end{pmatrix} = [D][\bar{B}]
\begin{pmatrix}
u_1 \\
w_1 \\
u_2 \\
w_2 \\
u_3 \\
w_3 \\
u_5 \\
w_5
\end{pmatrix}
= \begin{pmatrix}
-0.105 \\
-0.0747 \\
0.690 \\
0.0
\end{pmatrix}
\text{psi}
\]
**Axisymmetric Elements**

**Example 2**

Determine the stresses in each element as: \( \{ \sigma \} = [D][B]\{d\} \)

For Element 3:

\[
\begin{bmatrix}
\sigma_r \\
\sigma_z \\
\sigma_\theta \\
\tau_{xy}
\end{bmatrix} = [D][B]
\begin{bmatrix}
u_3 \\
w_3 \\
u_4 \\
w_4 \\
u_5 \\
w_5
\end{bmatrix}
= \begin{bmatrix}
-0.337 \\
-0.0125 \\
0.942 \\
0.1037
\end{bmatrix}
\text{ psi}
\]

---

**Axisymmetric Elements**

**Example 2**

Determine the stresses in each element as: \( \{ \sigma \} = [D][B]\{d\} \)

For Element 4:

\[
\begin{bmatrix}
\sigma_r \\
\sigma_z \\
\sigma_\theta \\
\tau_{xy}
\end{bmatrix} = [D][B]
\begin{bmatrix}
\sigma \\
\sigma \\
\sigma \\
\sigma
\end{bmatrix}
= \begin{bmatrix}
0.470 \\
0.1493 \\
1.426 \\
0.0
\end{bmatrix}
\text{ psi}
\]
**Axisymmetric Elements**

**Example 2**

Determine the stresses in each element as: \( \{\sigma\} = [D][\bar{B}]\{d\} \)

For Element 1:
\[
\begin{bmatrix}
\sigma_r \\
\sigma_z \\
\sigma_\theta \\
\tau_{xy}
\end{bmatrix} = \begin{bmatrix}
-0.338 \\
-0.0126 \\
0.942 \\
-0.1037
\end{bmatrix} \text{ psi}
\]

For Element 3:
\[
\begin{bmatrix}
\sigma_r \\
\sigma_z \\
\sigma_\theta \\
\tau_{xy}
\end{bmatrix} = \begin{bmatrix}
-0.337 \\
-0.0125 \\
0.942 \\
0.1037
\end{bmatrix} \text{ psi}
\]

For Element 4:
\[
\begin{bmatrix}
\sigma_r \\
\sigma_z \\
\sigma_\theta \\
\tau_{xy}
\end{bmatrix} = \begin{bmatrix}
-0.470 \\
0.1493 \\
1.426 \\
0.0
\end{bmatrix} \text{ psi}
\]

For Element 2:
\[
\begin{bmatrix}
\sigma_r \\
\sigma_z \\
\sigma_\theta \\
\tau_{xy}
\end{bmatrix} = \begin{bmatrix}
-0.105 \\
-0.0747 \\
0.690 \\
0.0
\end{bmatrix} \text{ psi}
\]
**Axisymmetric Elements**

Example 2 – Using Matlab code

A coarse mesh of 4 CST elements

---

**Axisymmetric Elements**

Example 2 – Using Matlab code

A coarse mesh of 4 CST elements
Axisymmetric Elements

Example 2 – Using Matlab code

A coarse mesh of 8 CST elements
**Axisymmetric Elements**

Example 2 – Using Matlab code

A coarse mesh of 16 CST elements

---

**Axisymmetric Elements**

Example 2 – Using Matlab code

A coarse mesh of 16 CST elements
**Axisymmetric Elements**

**Example 2**

The figure below shows the exact solution along with the results determined here and the other results.

![Diagram showing stress distributions](image)

**Axisymmetric Elements**

**Example 2**

Observe that agreement with the exact solution is quite good except for the limited results due to the very coarse mesh used in the longhand example.

![Diagram showing stress distributions](image)
**Axisymmetric Elements**

**Example 2**

Stresses have been plotted at the center of the quadrilaterals and were obtained by averaging the stresses in the four connecting triangles.

Applications

Consider the finite element model of a steel-reinforced concrete pressure vessel.
**Axisymmetric Elements**

**Applications**

The vessel is a thick-walled cylinder with flat heads.

An axis of symmetry (the z axis) exists such that only one-half of the r-z plane passing through the middle of the structure need be modeled.
Axisymmetric Elements

Applications

The concrete was modeled by using the axisymmetric triangular element developed in this chapter.

Axisymmetric Elements

Applications

The steel elements were laid out along the boundaries of the concrete elements so as to maintain continuity (or perfect bond assumption) between the concrete and the steel.
Axisymmetric Elements

Applications

The vessel was then subjected to an internal pressure as shown in the figure.

Note that the nodes along the axis of symmetry should be supported by rollers preventing motion perpendicular to the axis of symmetry.
**Axisymmetric Elements**

**Applications**

The figure below shows a finite element model of a high-strength steel die used in a thin-plastic-film-making process.

![Finite element model of a high-strength steel die](image)

**Axisymmetric Elements**

**Applications**

The die is an irregularly shaped disk. An axis of symmetry with respect to geometry and loading exists as shown.

![Finite element model of a high-strength steel die](image)
**Axisymmetric Elements**

**Applications**

The die was modeled by using simple quadrilateral axisymmetric elements. The locations of high stress were of primary concern.

![Stress Contours](image)

**Axisymmetric Elements**

**Applications**

The figure shows a plot of the von Mises stress contours for the die. The von Mises (or equivalent, or effective) stress is often used as a failure criterion in design.

![Stress Contours](image)
**Axisymmetric Elements**

**Applications**

The figure shows a stepped 4130 steel shaft with a fillet radius subjected to an axial pressure of 1,000 psi in tension.

![Diagram of a stepped 4130 steel shaft with a fillet radius subjected to an axial pressure of 1,000 psi in tension.]

**Axisymmetric Elements**

**Applications**

Fatigue analysis for reversed axial loading required an accurate stress concentration factor to be applied to the average axial stress of 1,000 psi.

![Diagram of stress concentrations on a stepped 4130 steel shaft subjected to axial pressure.]
**Axisymmetric Elements**

**Applications**

Fatigue analysis for reversed axial loading required an accurate stress concentration factor to be applied to the average axial stress of 1,000 psi.

![Diagram showing stress concentration](image)

**Axisymmetric Elements**

**Applications**

The figure below shows the resulting maximum principal stress plot using a computer program.

![Diagram showing stress concentration](image)
Axisymmetric Elements

Problems


End of Chapter 9