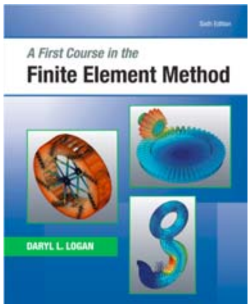
Chapter 8 – Linear-Strain Triangle Equations



Learning Objectives

- To develop the linear-strain triangular (LST) element stiffness matrix.
- To describe how the LST stiffness matrix can be determined.
- To compare the differences in results using the CST and LST elements.

Development of the Linear-Strain Triangle Equations

Introduction

In this section we will develop a higher-order triangular element, called the ***linear-strain triangle*** **(LST)**.

This element has many advantages over the constant-strain triangle (CST).

The LST element has six nodes and twelve displacement degrees of freedom.

The displacement function for the triangle is quadratic.

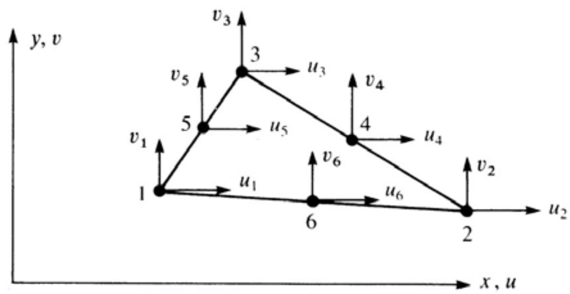
Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

The procedure to derive the LST element stiffness matrix and element equations is identical to that used for the CST element.

Step 1 - Discretize and Select Element Types

Consider the triangular element shown in the figure below:

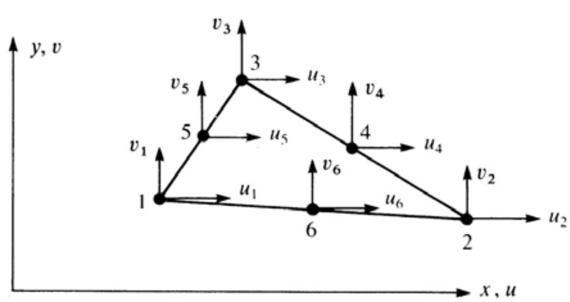


Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

Each node has two degrees of freedom: displacements in the *x* and *y* directions.

We will let u_i and v_i represent the node *i* displacement components in the *x* and *y* directions, respectively.

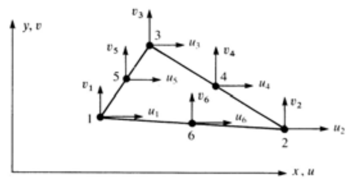


Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

The nodal displacements for an LST element are:

$$\{d\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{Bmatrix}$$



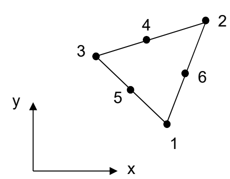
The nodes are ordered ***counterclockwise*** around the element

Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

Step 2 - Select Displacement Functions

Consider a straight-sided triangular element shown below:



The variation of the displacements over the element may be expressed as:

$$u(x,y) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2$$

$$v(x,y) = a_7 + a_8x + a_9y + a_{10}x^2 + a_{11}xy + a_{12}y^2$$

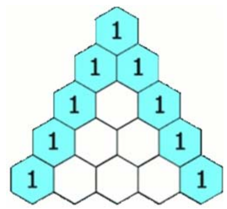
Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

The displacement compatibility among adjoining elements is satisfied because the three nodes defining adjacent sides define a unique a parabola.

The CST and LST triangles are variations of the Pascal triangles as show below.

Terms in Pascal Triangle	*Polynomial Degree*	*Number of Terms*	*Triangle*
1	0 (constant)	1	
$x \quad y$	1 (linear)	3	CST (Chap. 6)
$x^2 \quad xy \quad y^2$	2 (quadratic)	6	LST (Chap. 8)
$x^3 \quad x^2y \quad xy^2 \quad y^3$	3 (cubic)	10	QST



Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

The general element displacement functions are:

$$\{\Psi\} = \left[\begin{array}{cccccc:cccccc} 1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2 \end{array}\right] \begin{Bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_{11} \\ a_{12} \end{Bmatrix}$$

$$\{\Psi\} = [M^*]\{a\}$$

Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

To obtain the values for the ***a***'s substitute the coordinates of the nodal points into the above equations:

$$\begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \\ v_1 \\ \vdots \\ v_5 \\ v_6 \end{Bmatrix} = \left[\begin{array}{cccccc|cccccc} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_5 & y_5 & x_5^2 & x_5y_5 & y_5^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 \end{array}\right] \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \\ a_7 \\ \vdots \\ a_{11} \\ a_{12} \end{Bmatrix}$$

Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

Solving for the ***a***'s and writing the results in matrix form gives

$$\begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \\ a_7 \\ \vdots \\ a_{11} \\ a_{12} \end{Bmatrix} = \left[\begin{array}{cccccc|cccccc} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_5 & y_5 & x_5^2 & x_5y_5 & y_5^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 \end{array}\right]^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \\ v_1 \\ \vdots \\ v_5 \\ v_6 \end{Bmatrix}$$

$$\{a\} = [x]^{-1}\{u\}$$

where [*x*] is the 12 x 12 matrix on the right-hand-side of the above equation.

Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

The “best” way to invert [*x*] is to use a computer.

Note that only the 6 x 6 part of [*x*] really need be inverted.

$$\{\Psi\} = [M^*]\{a\} \qquad \{a\} = [x]^{-1}\{u\} \quad \Rightarrow \quad \{\Psi\} = [M^*][x]^{-1}\{u\}$$

The general displacement expressions in terms of interpolation functions and the nodal degrees of freedom are:

$$\{\Psi\} = [N]\{u\} \qquad \text{where} \qquad [N] = [M^*][x]^{-1}$$

Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

Elemental Strains*:* The strains over a two-dimensional element are:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

Elemental Strains: The strains over a two-dimensional element are:

$$\{\varepsilon\} = \begin{array}{c} \qquad\qquad u \qquad\qquad\qquad\qquad\qquad\qquad\qquad v \\ \left[\begin{array}{cccccc:cccccc} 0 & 1 & 0 & 2x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & x & 2y \\ 0 & 0 & 1 & 0 & x & 2y & 0 & 1 & 0 & 2x & y & 0 \end{array}\right] \end{array} \begin{Bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_{11} \\ a_{12} \end{Bmatrix}$$

Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

Observe that the strains are linear over the triangular element; therefore, the element is called a ***linear-strain triangle*** **(LST)**.

The above equation may be written in matrix form as

$$\{\varepsilon\} = [M']\{a\}$$

where [*M* ’] is based on derivatives of [*M**].

$$[M'] = \begin{bmatrix} 0 & 1 & 0 & 2x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & x & 2y \\ 0 & 0 & 1 & 0 & x & 2y & 0 & 1 & 0 & 2x & y & 0 \end{bmatrix}$$

Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

If we substitute the values of ***a***'s into the above equation gives:

$$\{\varepsilon\} = [M']\{a\} \qquad \{a\} = [x]^{-1}\{u\} \qquad \Rightarrow \{\varepsilon\} = [B]\{d\}$$

where [B] is a function of the nodal coordinates (x_1, y_1) through (x_6, y_6).

$$[B] = [M'][x]^{-1}$$

Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

The stresses are given as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D]\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

For **plane stress**, **[*D*]** is:

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}$$

For **plane strain**, **[*D*]** is:

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix}$$

Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

Step 4 - Derive the Element Stiffness Matrix and Equations

The stiffness matrix can be defined as:

$$[k] = \int_V [B]^T [D][B] dV$$

However, $[B]$ is now a function of x and y; therefore, we must integrate the above expression to develop the element stiffness matrix.

Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

The $[B]$ matrix is:

$$[B] = \frac{1}{2A}\left[\begin{array}{cc:cc:cc:cc:cc:cc} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 & \beta_4 & 0 & \beta_5 & 0 & \beta_6 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 & 0 & \gamma_4 & 0 & \gamma_5 & 0 & \gamma_6 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 & \gamma_4 & \beta_4 & \gamma_5 & \beta_5 & \gamma_6 & \beta_6 \end{array}\right]$$

The β's and the γ's are functions of x and y as well as the nodal coordinates.

Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

The $[B]$ matrix is:

$$[B] = \frac{1}{2A}\left[\begin{array}{cc:cc:cc:cc:cc:cc} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 & \beta_4 & 0 & \beta_5 & 0 & \beta_6 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 & 0 & \gamma_4 & 0 & \gamma_5 & 0 & \gamma_6 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 & \gamma_4 & \beta_4 & \gamma_5 & \beta_5 & \gamma_6 & \beta_6 \end{array}\right]$$

The stiffness matrix is a 12 x 12 matrix and is very cumbersome to compute in explicit form.

However, if the origin of the coordinates is the centroid of the element, the integrations become more amenable.

Typically, the integrations are computed numerically.

Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

The element body forces and surface forces should ***not*** be automatically lumped at the nodes.

The following integration should be computed:

$$\{f_b\} = \int_V [N]^T \{X\} dV$$

$$\{f_s\} = \int_S [N]^T \{T\} dS$$

Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

The element equations are:

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ \vdots \\ f_{6y} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1,12} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2,12} \\ k_{31} & k_{32} & k_{33} & \cdots & k_{3,12} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{12,1} & k_{12,2} & k_{12,3} & \cdots & k_{12,12} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ \vdots \\ v_6 \end{Bmatrix}$$

$$[12 \times 1] \qquad [12 \times 12] \qquad [12 \times 1]$$

Development of the Linear-Strain Triangle Equations

Derivation of the Linear-Strain Triangular Elemental Stiffness Matrix and Equations

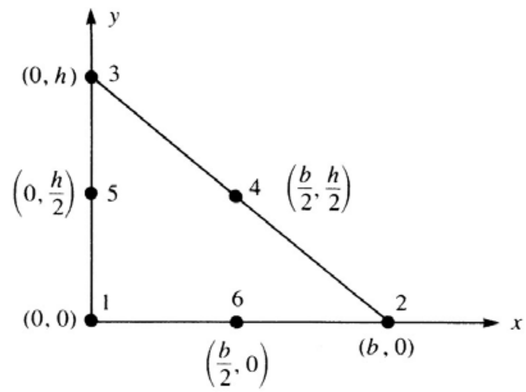
Steps 5, 6, and 7

Assembling the global stiffness matrix, determining the global displacements, and calculating the stresses, are identical to the procedures used for CST elements.

Development of the Linear-Strain Triangle Equations

Example LST Stiffness Determination

Consider a straight-sided triangular element shown below:



Development of the Linear-Strain Triangle Equations

Example LST Stiffness Determination

The triangle has a base dimension of *b* and a height *h*, with mid-side nodes.

We can calculate the coefficients a_1 through a_6 by evaluating the displacement *u* at each node.

$$u(x,y) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2$$

$$u_1 = u(0,0) = a_1$$

$$u_2 = u(b,0) = a_1 + a_2b + a_4b^2$$

$$u_3 = u(0,h) = a_1 + a_3h + a_6h^2$$

Development of the Linear-Strain Triangle Equations

Example LST Stiffness Determination

The triangle has a base dimension of b and a height h, with mid-side nodes.

We can calculate the coefficients a_1 through a_6 by evaluating the displacement u at each node.

$$u_4 = u\left(\frac{b}{2},\frac{h}{2}\right) = a_1 + a_2\frac{b}{2} + a_3\frac{h}{2} + a_4\left(\frac{b}{2}\right)^2 + a_5\frac{bh}{4} + a_6\left(\frac{h}{2}\right)^2$$

$$u_5 = u\left(0,\frac{h}{2}\right) = a_1 + a_3\frac{h}{2} + a_6\left(\frac{h}{2}\right)^2$$

$$u_6 = u\left(\frac{b}{2},0\right) = a_1 + a_2\frac{b}{2} + a_4\left(\frac{b}{2}\right)^2$$

Development of the Linear-Strain Triangle Equations

Example LST Stiffness Determination

Solving the above equations simultaneously for the ***a***'s gives:

$$a_1 = u_1 \qquad a_2 = \frac{4u_6 - 3u_1 - u_2}{b}$$

$$a_3 = \frac{4u_5 - 3u_1 - u_3}{h} \qquad a_4 = \frac{2(u_2 - 2u_6 + u_1)}{b^2}$$

$$a_5 = \frac{4(u_1 + u_4 - u_5 - u_6)}{bh} \qquad a_6 = \frac{2(u_3 - 2u_5 + u_1)}{h^2}$$

Development of the Linear-Strain Triangle Equations

Example LST Stiffness Determination

The *u* displacement equation is:

$$u(x,y) = u_1 + \left[\frac{4u_6 - 3u_1 - u_2}{b}\right]x + \left[\frac{4u_5 - 3u_1 - u_3}{h}\right]y$$

$$+ \left[\frac{2(u_2 - 2u_6 + u_1)}{b^2}\right]x^2 + \left[\frac{4(u_1 + u_4 - u_5 - u_6)}{bh}\right]xy$$

$$+ \left[\frac{2(u_3 - 2u_5 + u_1)}{h^2}\right]y^2$$

Development of the Linear-Strain Triangle Equations

Example LST Stiffness Determination

The *v* displacement equation can be determined in a manner identical to that used for the *u* displacement:

$$v(x,y) = v_1 + \left[\frac{4v_6 - 3v_1 - v_2}{b}\right]x + \left[\frac{4v_5 - 3v_1 - v_3}{h}\right]y$$

$$+ \left[\frac{2(v_2 - 2v_6 + v_1)}{b^2}\right]x^2 + \left[\frac{4(v_1 + v_4 - v_5 - v_6)}{bh}\right]xy$$

$$+ \left[\frac{2(v_3 - 2v_5 + v_1)}{h^2}\right]y^2$$

Development of the Linear-Strain Triangle Equations

Example LST Stiffness Determination

The general form of the displacement expressions in terms of the interpolation functions is given as

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \left[\begin{array}{cc|cc|cc|cc|cc|cc} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 \end{array}\right] \begin{Bmatrix} u_1 \\ v_1 \\ \vdots \\ u_6 \\ v_6 \end{Bmatrix}$$

where the interpolation functions are:

$$N_1 = 1 - \frac{3x}{b} - \frac{3y}{h} + \frac{2x^2}{b^2} + \frac{4xy}{bh} + \frac{2y^2}{h^2} \qquad N_2 = -\frac{x}{b} + \frac{2x^2}{b^2}$$

$$N_3 = -\frac{y}{h} + \frac{2y^2}{h^2}$$

Development of the Linear-Strain Triangle Equations

Example LST Stiffness Determination

The general form of the displacement expressions in terms of the interpolation functions is given as

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \left[\begin{array}{cc|cc|cc|cc|cc|cc} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 \end{array}\right] \begin{Bmatrix} u_1 \\ v_1 \\ \vdots \\ u_6 \\ u_6 \end{Bmatrix}$$

where the interpolation functions are:

$$N_4 = \frac{4xy}{bh} \qquad N_5 = \frac{4y}{h} - \frac{4xy}{bh} - \frac{4y^2}{h^2}$$

$$N_6 = \frac{4x}{b} - \frac{4xy}{bh} - \frac{4x^2}{b^2}$$

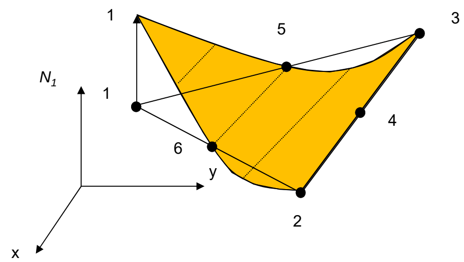
Development of the Linear-Strain Triangle Equations

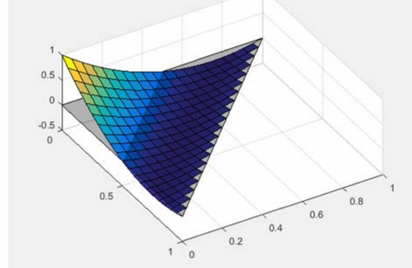
Example LST Stiffness Determination

The element interpolation functions N have two basic shapes.

The behavior of the functions N_1, N_2, and N_3 is similar except referenced at different nodes.

The shape function N_1 is shown below:

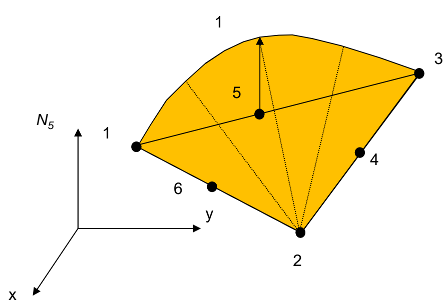


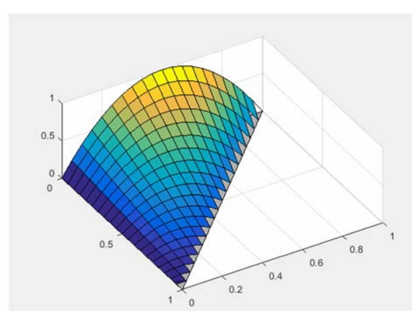


Development of the Linear-Strain Triangle Equations

Example LST Stiffness Determination

The second type of interpolation function is valid for functions N_4, N_5, and N_6. The function N_5 is shown below:





Development of the Linear-Strain Triangle Equations

Example LST Stiffness Determination

The element strain is given as:

$$\{\varepsilon\} = [B]\{d\}$$

where the [B] matrix is:

$$[B] = \frac{1}{2A}\left[\begin{array}{cc:cc:cc:cc:cc:cc} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 & \beta_4 & 0 & \beta_5 & 0 & \beta_6 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 & 0 & \gamma_4 & 0 & \gamma_5 & 0 & \gamma_6 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 & \gamma_4 & \beta_4 & \gamma_5 & \beta_5 & \gamma_6 & \beta_6 \end{array}\right]$$

$$\beta_i = 2A\left(\frac{\partial N_i}{\partial x}\right) \qquad \gamma_i = 2A\left(\frac{\partial N_i}{\partial y}\right)$$

Development of the Linear-Strain Triangle Equations

Example LST Stiffness Determination

Therefore, since $\beta_i = 2A\left(\frac{\partial N_i}{\partial x}\right)$ $\qquad A = \frac{bh}{2}$

$$N_1 = 1 - \frac{3x}{b} - \frac{3y}{h} + \frac{2x^2}{b^2} + \frac{4xy}{bh} + \frac{2y^2}{h^2} \qquad \beta_1 = -3h + \frac{4hx}{b} + 4y$$

$$N_2 = -\frac{x}{b} + \frac{2x^2}{b^2} \qquad \beta_2 = -h + \frac{4hx}{b}$$

$$N_3 = -\frac{y}{h} + \frac{2y^2}{h^2} \qquad \beta_3 = 0$$

Development of the Linear-Strain Triangle Equations

Example LST Stiffness Determination

Therefore, since $\beta_i = 2A\left(\frac{\partial N_i}{\partial x}\right)$ $\qquad A = \frac{bh}{2}$

$$N_4 = \frac{4xy}{bh} \qquad \beta_4 = 4y$$

$$N_5 = \frac{4y}{h} - \frac{4xy}{bh} - \frac{4y^2}{h^2} \qquad \beta_5 = -4y$$

$$N_6 = \frac{4x}{b} - \frac{4xy}{bh} - \frac{4x^2}{b^2} \qquad \beta_6 = 4h - \frac{8hx}{b} - 4y$$

Development of the Linear-Strain Triangle Equations

Example LST Stiffness Determination

Therefore, since $\gamma_i = 2A\left(\frac{\partial N_i}{\partial y}\right)$ $\qquad A = \frac{bh}{2}$

$$N_1 = 1 - \frac{3x}{b} - \frac{3y}{h} + \frac{2x^2}{b^2} + \frac{4xy}{bh} + \frac{2y^2}{h^2} \qquad \gamma_1 = -3b + \frac{4by}{h} + 4x$$

$$N_2 = -\frac{x}{b} + \frac{2x^2}{b^2} \qquad \gamma_2 = 0$$

$$N_3 = -\frac{y}{h} + \frac{2y^2}{h^2} \qquad \gamma_3 = -b + \frac{4by}{h}$$

Development of the Linear-Strain Triangle Equations

Example LST Stiffness Determination

Therefore, since $\gamma_i = 2A\left(\frac{\partial N_i}{\partial y}\right)$ $\qquad A = \frac{bh}{2}$

$$N_4 = \frac{4xy}{bh} \qquad \gamma_4 = 4x$$

$$N_5 = \frac{4y}{h} - \frac{4xy}{bh} - \frac{4y^2}{h^2} \qquad \gamma_5 = 4b - 4x - \frac{8by}{h}$$

$$N_6 = \frac{4x}{b} - \frac{4xy}{bh} - \frac{4x^2}{b^2} \qquad \gamma_6 = -4x$$

Development of the Linear-Strain Triangle Equations

Example LST Stiffness Determination

The stiffness matrix for a constant thickness element can be obtained by substituting the β's and the γ's into the [B] and then substituting [B] into the following expression and evaluating the integral numerically.

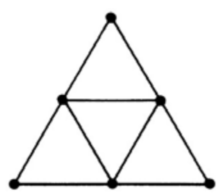
$$[k] = \int_V [B]^T [D][B] dV$$

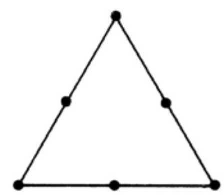
Development of the Linear-Strain Triangle Equations

Comparison of Elements

For a given number of nodes, a better representation of true stress and displacement is generally obtained using LST elements than is obtained using the same number of nodes a finer subdivision of CST elements.

For example, a single LST element gives better results than four CST elements.

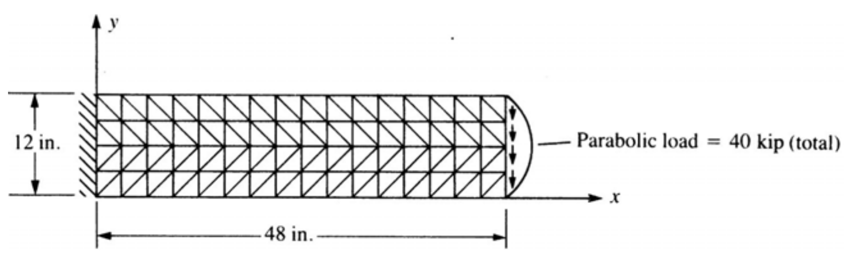




Development of the Linear-Strain Triangle Equations

Comparison of Elements

Consider the following cantilever beam with $E = 30 \times 10^6$ *psi*, $\nu = 0.25$, and $t = 1$ *in*.



Development of the Linear-Strain Triangle Equations

Comparison of Elements

Table 1 lists the series of tests run to compare results using the CST and LST elements.

Table 1. Comparison of CST and LST results

Series of Test Runs	Number of Nodes	Degrees of Freedom, n_d	Number of Elements
A-1 4 x 16	85	160	128 CST
A-2 8 x 32	297	576	512 CST
B-1 2 x 8	85	160	32 LST
B-2 4 x 16	297	576	128 LST

Development of the Linear-Strain Triangle Equations

Comparison of Elements

Table 2 shows comparisons of free-end (tip) deflection and stress for each element type used to model the cantilever beam.

Table 2. Comparison of CST and LST results

Runs	n_d	Bandwidth, n_d	Tip Deflection (*in*)	σ_x (*ksi*)	Location (*x, y*)
A-1	160	14	-0.29555	67.236	(2.250,11.250)
A-2	576	22	-0.33850	81.302	(1.125,11.630)
B-1	160	18	-0.33470	58.885	(4.500,10.500)
B-2	576	22	-0.35159	69.956	(2.250,11.250)
		Exact Solution	-0.36133	80.000	(0,12)

Development of the Linear-Strain Triangle Equations

Comparison of Elements

The larger the number of degrees of freedom for a given type of triangular element, the closer the solution converges to the exact one (compare run A-1 to run A-2, and B-1 to B-2).

Table 2. Comparison of CST and LST results

Runs	n_d	Bandwidth, n_d	Tip Deflection (*in*)	σ_x (*ksi*)	Location (*x, y*)
A-1	160	14	-0.29555	67.236	(2.250,11.250)
A-2	576	22	-0.33850	81.302	(1.125,11.630)
B-1	160	18	-0.33470	58.885	(4.500,10.500)
B-2	576	22	-0.35159	69.956	(2.250,11.250)
		Exact Solution	-0.36133	80.000	(0,12)

Development of the Linear-Strain Triangle Equations

Comparison of Elements

For a given number of nodes, the LST analysis yields somewhat better results than the CST analysis (compare run A-1 to run B-1).

Table 2. Comparison of CST and LST results

Runs	n_d	Bandwidth, n_d	Tip Deflection (*in*)	σ_x (*ksi*)	Location (*x, y*)
A-1	160	14	-0.29555	67.236	(2.250,11.250)
A-2	576	22	-0.33850	81.302	(1.125,11.630)
B-1	160	18	-0.33470	58.885	(4.500,10.500)
B-2	576	22	-0.35159	69.956	(2.250,11.250)
		Exact Solution	-0.36133	80.000	(0,12)

Development of the Linear-Strain Triangle Equations

Comparison of Elements

Although the CST element is rather poor in modeling bending, we observe that the element can be used to model a beam in bending if sufficient number of elements is used.

Table 2. Comparison of CST and LST results

Runs	n_d	Bandwidth, n_d	Tip Deflection (*in*)	σ_x (*ksi*)	Location (*x, y*)
A-1	160	14	-0.29555	67.236	(2.250,11.250)
A-2	576	22	-0.33850	81.302	(1.125,11.630)
B-1	160	18	-0.33470	58.885	(4.500,10.500)
B-2	576	22	-0.35159	69.956	(2.250,11.250)
		Exact Solution	-0.36133	80.000	(0,12)

Development of the Linear-Strain Triangle Equations

Comparison of Elements

In general, both the LST and CST analyses yield sufficient results for most plane stress/strain problems provided a sufficient number of elements are used.

Table 2. Comparison of CST and LST results

Runs	n_d	Bandwidth, n_d	Tip Deflection (*in*)	σ_x (*ksi*)	Location (*x, y*)
A-1	160	14	-0.29555	67.236	(2.250,11.250)
A-2	576	22	-0.33850	81.302	(1.125,11.630)
B-1	160	18	-0.33470	58.885	(4.500,10.500)
B-2	576	22	-0.35159	69.956	(2.250,11.250)
		Exact Solution	-0.36133	80.000	(0,12)

Development of the Linear-Strain Triangle Equations

Comparison of Elements

The results of Table 2 indicate:

That the LST model might be preferred over the CST model for plane stress applications when a relatively small number of nodes is used.

The use of triangular elements of higher order, such as the LST, is not visibly more advantageous when large numbers of nodes are used, particularly when the cost of the formation of element stiffnesses, equation bandwidth, and overall complexities involved in the computer modeling are considered.

Development of the Linear-Strain Triangle Equations

Comparison of Elements

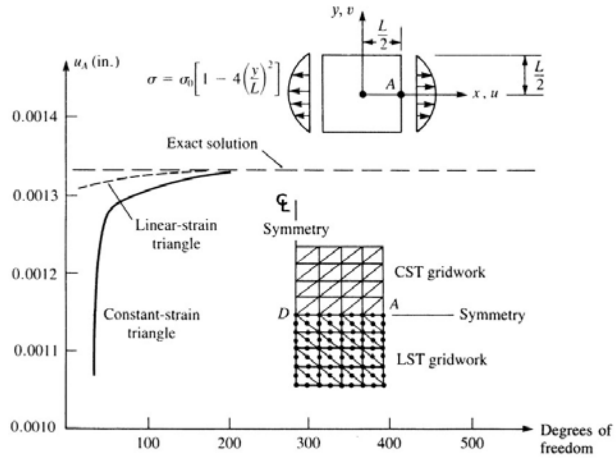
Most commercial programs incorporate the use of CST and/or LST elements for plane stress/strain problems although these elements are used primarily as transition elements (usually during mesh generation).

Also, recall that finite element displacements will always be less than the exact ones, because finite element models are always predicted to be stiffer than the actual structures when using the displacement formulation of the finite element method.

Development of the Linear-Strain Triangle Equations

Comparison of Elements

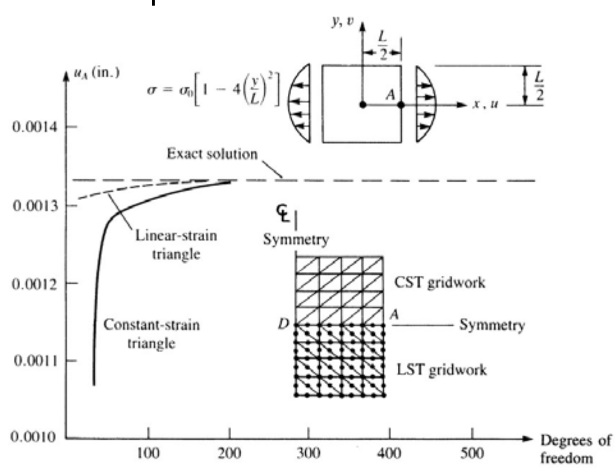
A comparison of CST and LST models of a plate subjected to parabolically distributed edge loads is shown below.



Development of the Linear-Strain Triangle Equations

Comparison of Elements

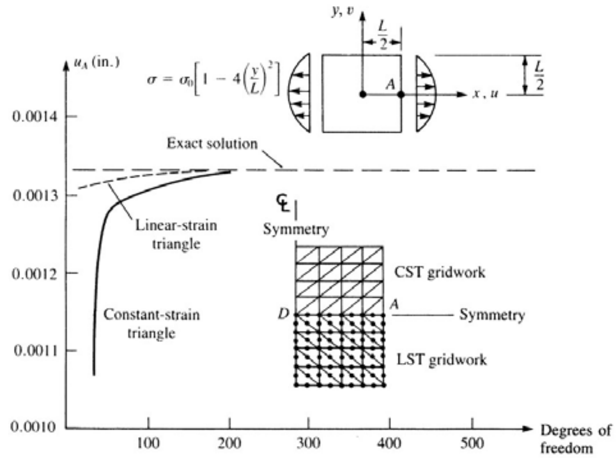
The LST model converges to the exact solution for horizontal displacement at point *A* faster than does the CST model.



Development of the Linear-Strain Triangle Equations

Comparison of Elements

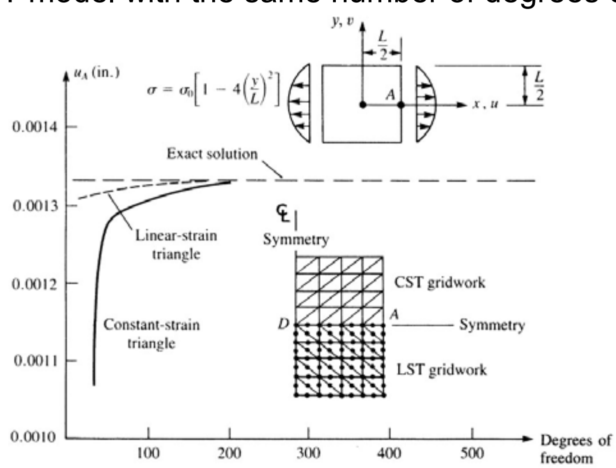
However, the CST model is quite acceptable even for modest numbers of degrees of freedom.



Development of the Linear-Strain Triangle Equations

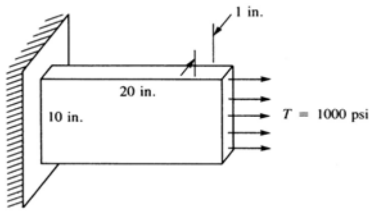
Comparison of Elements

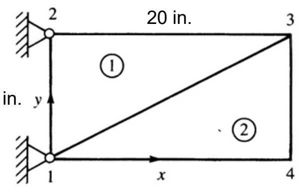
For example, a CST model with 100 nodes (200 degrees of freedom) often yields nearly as accurate a solution as does an LST model with the same number of degrees of freedom.



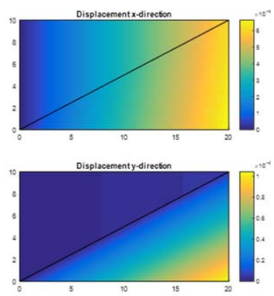
Development of the Linear-Strain Triangle Equations

Comparing CST and LST triangles – 2 elements

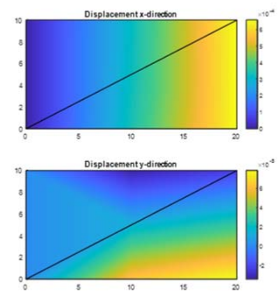




CST



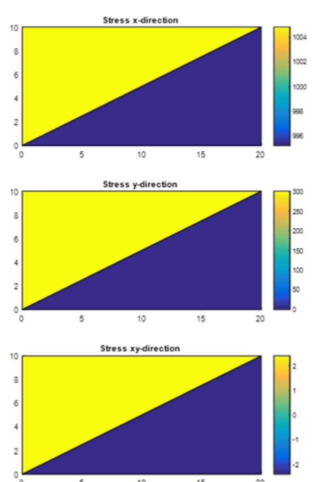
LST



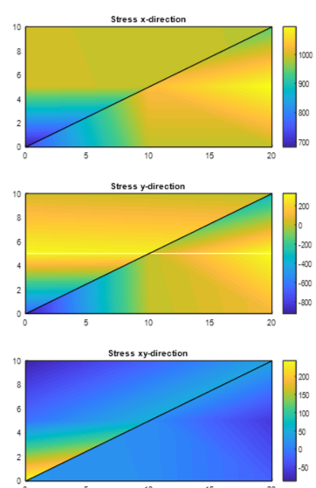
Development of the Linear-Strain Triangle Equations

Comparing CST and LST triangles – 2 elements

CST

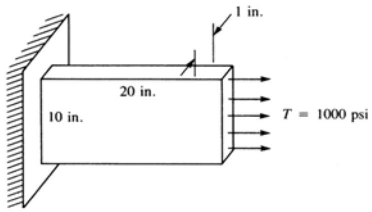


LST

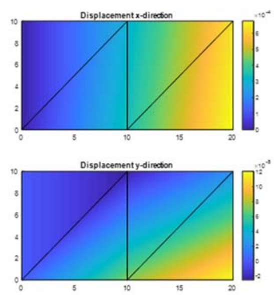


Development of the Linear-Strain Triangle Equations

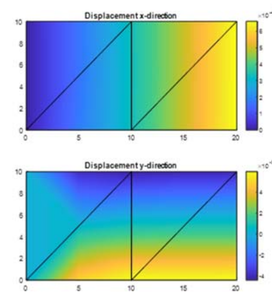
Comparing CST and LST triangles – 4 elements



CST



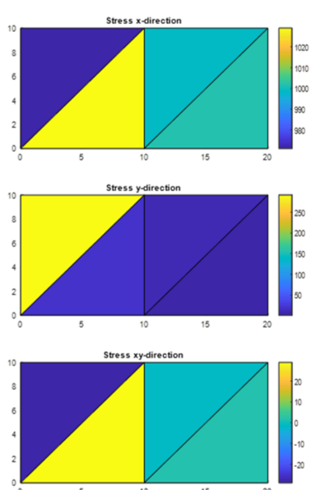
LST



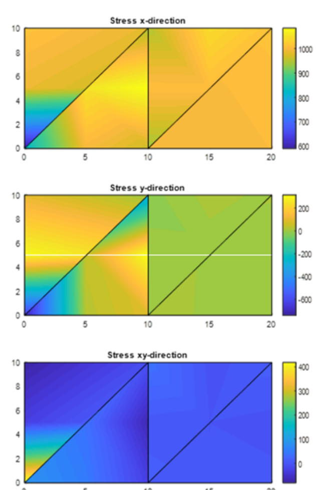
Development of the Linear-Strain Triangle Equations

Comparing CST and LST triangles – 4 elements

CST



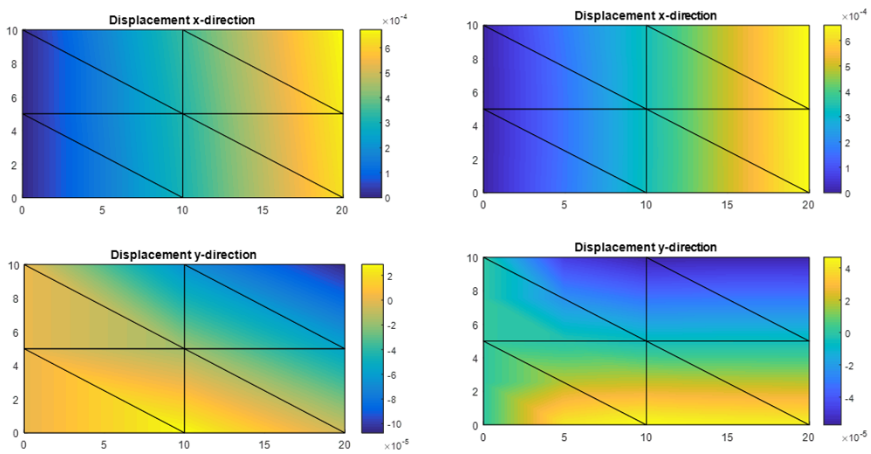
LST



Plane Stress and Plane Strain Equations

Comparing CST and LST triangles – 8 elements

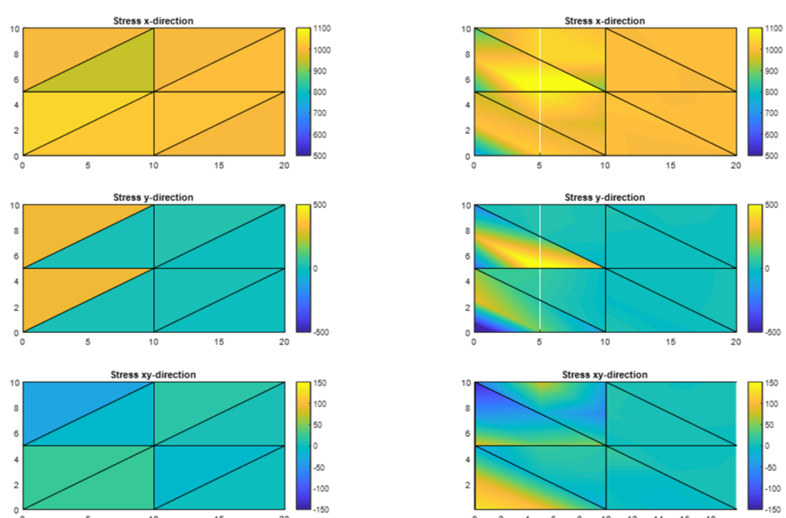
CST LST



Plane Stress and Plane Strain Equations

Comparing CST and LST triangles – 8 elements

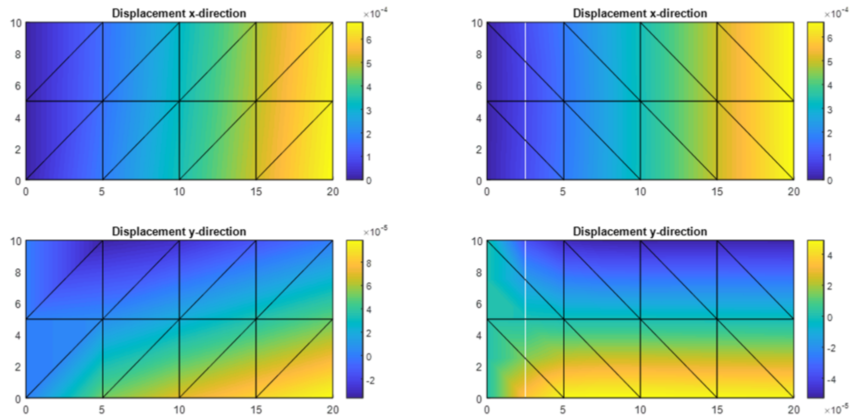
CST LST



Plane Stress and Plane Strain Equations

Comparing CST and LST triangles – 16 elements

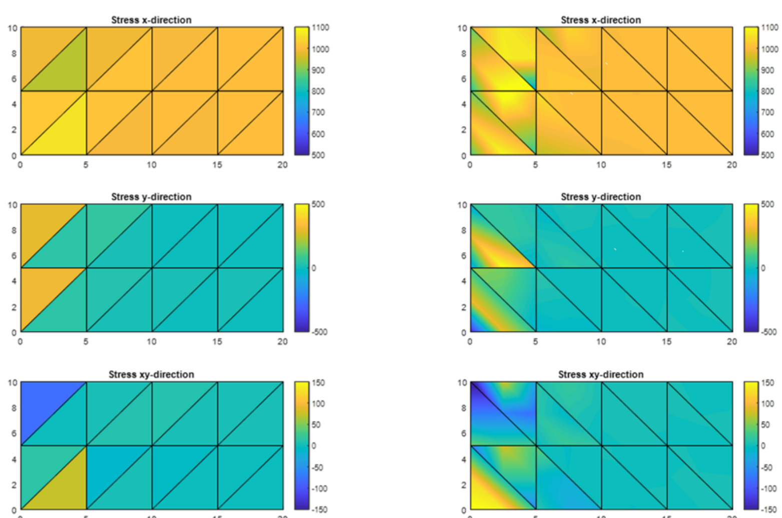
CST LST



Plane Stress and Plane Strain Equations

Comparing CST and LST triangles – 16 elements

CST LST



Development of the Linear-Strain Triangle Equations

Problems

16. Work problems ***8.3***, ***8.6***, and ***8.7*** in your textbook.

17. Rework the plane stress problem given on page 364 and **6.13** in your textbook using Camp's LST code.

    Start with the simple two element model. Continuously refine your discretization by a factor of two each time until your FEM solution is in agreement with the exact solution for both displacements and stress. How many elements did you need?

End of Chapter 8