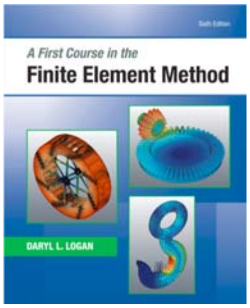
Chapter 6b – Plane Stress/Strain Equations



Learning Objectives

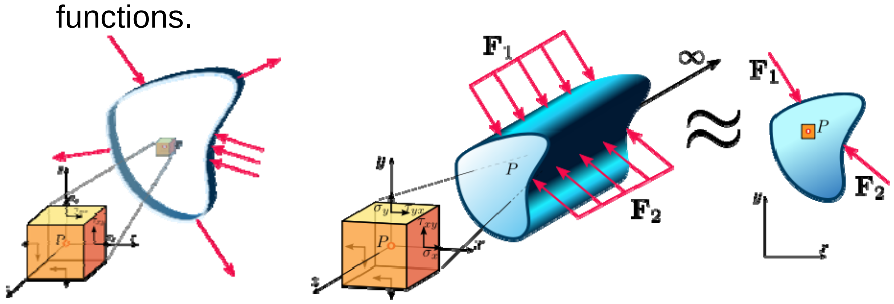
- To derive the bilinear four-noded rectangular (Q4) element stiffness matrix.
- To compare the CST and Q4 model results for a beam bending problem and describe some of the CST and Q4 elements

Plane Stress and Plane Strain Equations

We will now develop the four-noded rectangular plane element stiffness matrix.

This element is an isoparametric formulation of a general quadrilateral element (see Chapter 10).

This element is also called the bilinear rectangle because of the linear terms in *x* and y for the *x* and y displacement functions.

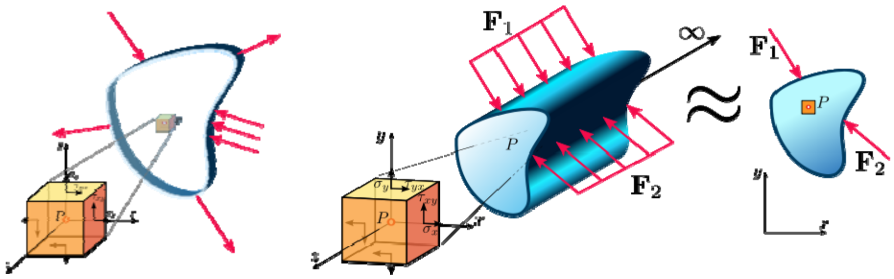


Plane Stress and Plane Strain Equations

The "Q4" symbol represents the element as a quadrilateral with four corner nodes.

Two advantages of the rectangular element over the triangular element are:

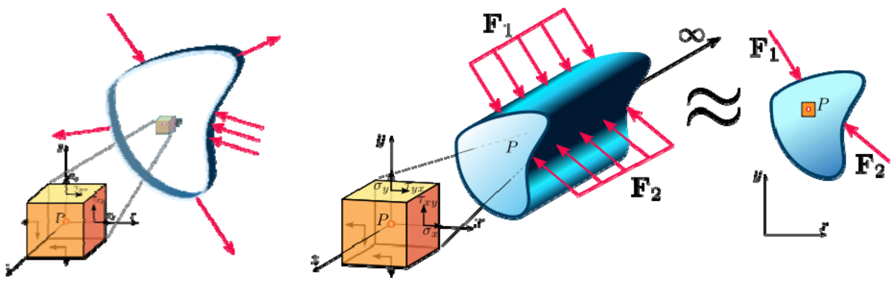
1. ease of data input and
2. simpler interpretation of output stresses.



Plane Stress and Plane Strain Equations

The "Q4" symbol represents the element as a quadrilateral with four corner nodes.

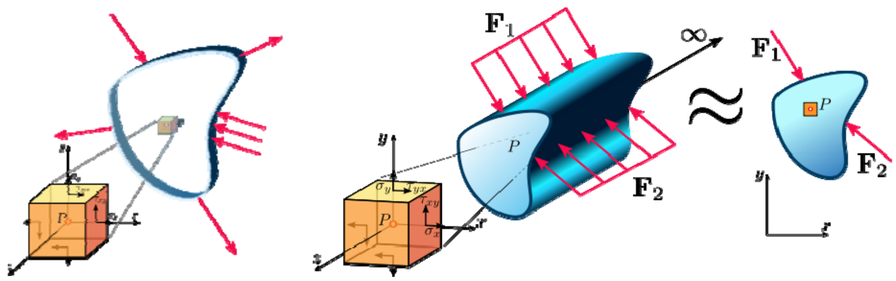
A disadvantage of the rectangular element is that the simple linear-displacement rectangle with its associated straight sides poorly approximates the real boundary condition edges.



Plane Stress and Plane Strain Equations

The usual steps outlined in Chapter 1 will be followed to obtain the element stiffness matrix and related equations.

1. ***Discretize and Select Element Type***
2. ***Select a Displacement Function*** - Assume a variation of the displacements over each element.
3. ***Define the Strain/Displacement and Stress/Strain Relationships*** - use elementary concepts of equilibrium and compatibility.

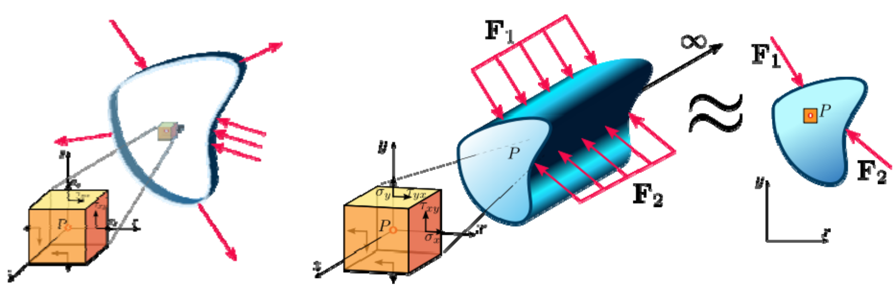


Plane Stress and Plane Strain Equations

The usual steps outlined in Chapter 1 will be followed to obtain the element stiffness matrix and related equations.

4. ***Derive the Element Stiffness Matrix and Equations*** - Define the stiffness matrix for an element and then consider the derivation of the stiffness matrix for a linear-elastic spring element.

5. ***Assemble the Element Equations to Obtain the Global or Total Equations and Introduce Boundary Conditions***



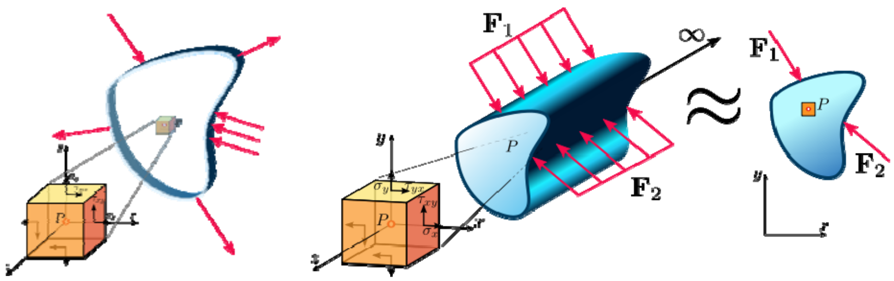
Plane Stress and Plane Strain Equations

The usual steps outlined in Chapter 1 will be followed to obtain the element stiffness matrix and related equations.

6. ***Solve for the Unknown Degrees of Freedom (or Generalized Displacements)*** - Solve for the nodal displacements.

7. ***Solve for the Element Strains and Stresses*** - The reactions and internal forces association with the bar element.

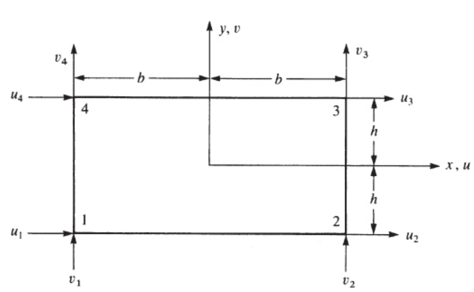
8. ***Interpret the Results***



Plane Stress and Plane Strain Equations

1. ***Discretize and Select Element Type***

Consider the rectangular element shown below (all interior angles are 90°) with corner nodes 1-4 (again labeled counterclockwise) and base and height dimensions of *2b* and *2h,* respectively



$$\{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

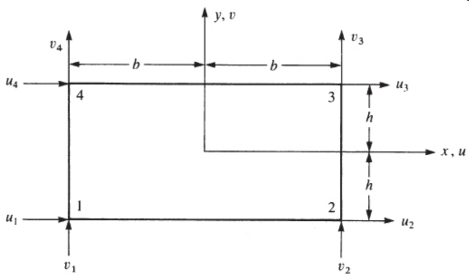
Plane Stress and Plane Strain Equations

2. Select a Displacement Function

For a compatible displacement field, the element displacement functions *u* and *v* must be linear along each edge because only two points (the corner nodes) exist along each edge.

$$\{\Psi_i\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}$$

$$= \begin{Bmatrix} a_1 + a_2 x + a_3 y + a_4 xy \\ a_5 + a_6 x + a_7 y + a_8 xy \end{Bmatrix}$$



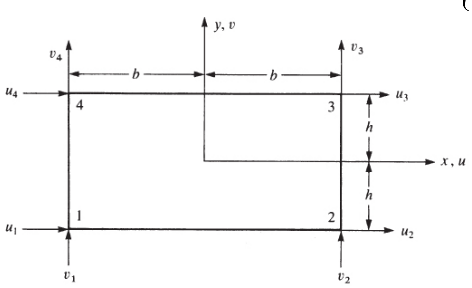
Plane Stress and Plane Strain Equations

2. Select a Displacement Function

There are a total of eight generalized degrees of freedom (***a***'s) and a total of eight specific degrees of freedom (u_1, v_1 at node 1 through u_4, v_4 at node 4) for the element.

$$\{\Psi_i\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}$$

$$= \begin{Bmatrix} a_1 + a_2 x + a_3 y + a_4 xy \\ a_5 + a_6 x + a_7 y + a_8 xy \end{Bmatrix}$$



Plane Stress and Plane Strain Equations

2. Select a Displacement Function

We can proceed in the usual manner to solve for the ***a***'s and obtain:

$$u(x,y) = \frac{1}{4bh}\Big[(b-x)(h-y)u_1 + (b+x)(h-y)u_2 + (b+x)(h+y)u_3 + (b-x)(h+y)u_4\Big]$$

$$v(x,y) = \frac{1}{4bh}\Big[(b-x)(h-y)v_1 + (b+x)(h-y)v_2 + (b+x)(h+y)v_3 + (b-x)(h+y)v_4\Big]$$

Plane Stress and Plane Strain Equations

2. Select a Displacement Function

These displacement expressions, can be expressed equivalently in terms of the interpolation functions and unknown nodal displacements as:

$$u(x,y) = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$
$$v(x,y) = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

where:

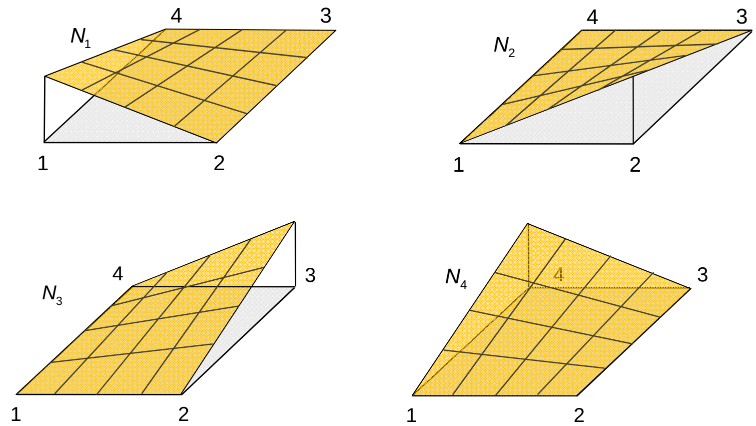
$$N_1 = \frac{(b-x)(h-y)}{4bh} \qquad N_2 = \frac{(b+x)(h-y)}{4bh}$$

$$N_3 = \frac{(b+x)(h+y)}{4bh} \qquad N_4 = \frac{(b-x)(h+y)}{4bh}$$

Plane Stress and Plane Strain Equations

2. Select a Displacement Function

The shape functions are visually deceiving. There is no curvature in directions parallel to any side; however, there is a twist due to the *xy* term in the element representation.



Plane Stress and Plane Strain Equations

2. Select a Displacement Function

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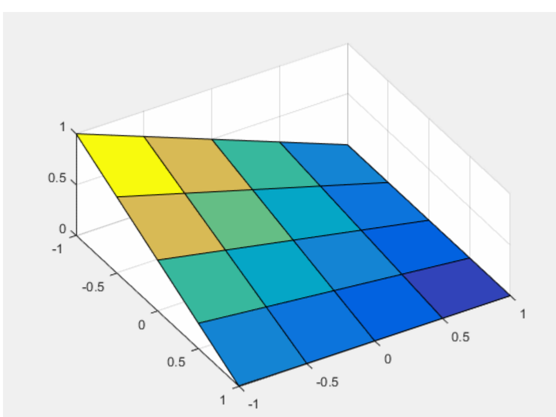


Plane Stress and Plane Strain Equations

2. Select a Displacement Function

In expanded form, the equations become:

$$\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\{\Psi\} = [N]\{d\}$$

Plane Stress and Plane Strain Equations

2. Select a Displacement Function

So that u and v will yield a constant value for rigid-body displacement, $N_1 + N_2 + N_3 + N_4 = 1$ for all x and y locations on the element.

For example, assume all the triangle displaces as a rigid body in the y direction: $v = v_0$

$$\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} 0 \\ v_0 \\ 0 \\ v_0 \\ 0 \\ v_0 \\ 0 \\ v_0 \end{Bmatrix}$$

$$v(x,y) = v_0 = v_0\left(N_1 + N_2 + N_3 + N_4\right)$$

$$\Rightarrow N_1 + N_2 + N_3 + N_4 = 1$$

Plane Stress and Plane Strain Equations

3. Define the Strain-Displacement and Stress-Strain Relationships

The general definitions of normal and shear strains are:

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

The strains over a two-dimensional element are:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

Plane Stress and Plane Strain Equations

3. Define the Strain-Displacement and Stress-Strain Relationships

Substituting our approximation for the displacement gives:

$$\frac{\partial u}{\partial x} = u_{,x} = \frac{\partial}{\partial x}\left(N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4\right)$$

$$u_{,x} = N_{1,x} u_1 + N_{2,x} u_2 + N_{3,x} u_3 + N_{4,x} u_4$$

where the comma indicates differentiation with respect to that variable.

Plane Stress and Plane Strain Equations

3. Define the Strain-Displacement and Stress-Strain Relationships

The derivatives of the interpolation functions with respect to *x* are:

$$N_{1,x} = \frac{1}{4bh}\frac{\partial}{\partial x}(b-x)(h-y) = -\frac{(h-y)}{4bh}$$

$$N_{2,x} = \frac{1}{4bh}\frac{\partial}{\partial x}(b+x)(h-y) = \frac{(h-y)}{4bh}$$

$$N_{3,x} = \frac{1}{4bh}\frac{\partial}{\partial x}(b+x)(h+y) = \frac{(h+y)}{4bh}$$

$$N_{4,x} = \frac{1}{4bh}\frac{\partial}{\partial x}(b-x)(h+y) = -\frac{(h+y)}{4bh}$$

Plane Stress and Plane Strain Equations

3. Define the Strain-Displacement and Stress-Strain Relationships

The derivatives of the interpolation functions with respect to *y* are:

$$N_{1,y} = \frac{1}{4bh}\frac{\partial}{\partial y}(b-x)(h-y) = -\frac{(b-x)}{4bh}$$

$$N_{2,y} = \frac{1}{4bh}\frac{\partial}{\partial y}(b+x)(h-y) = -\frac{(b+x)}{4bh}$$

$$N_{3,y} = \frac{1}{4bh}\frac{\partial}{\partial y}(b+x)(h+y) = \frac{(b+x)}{4bh}$$

$$N_{4,y} = \frac{1}{4bh}\frac{\partial}{\partial y}(b-x)(h+y) = \frac{(b-x)}{4bh}$$

Plane Stress and Plane Strain Equations

3. Define the Strain-Displacement and Stress-Strain Relationships

We can write the strains in matrix form as: $\{\varepsilon\} = [B]\{d\}$

$$[B] = \frac{1}{4bh}\begin{bmatrix} -(h-y) & 0 & (h-y) & 0 & (h+y) & 0 & -(h+y) & 0 \\ 0 & -(b-x) & 0 & -(b+x) & 0 & (b+x) & 0 & (b-x) \\ -(b-x) & -(h-y) & -(b+x) & (h-y) & (b+x) & (h+y) & (b-x) & -(h+y) \end{bmatrix}$$

Plane Stress and Plane Strain Equations

3. Define the Strain-Displacement and Stress-Strain Relationships

From equations, we observe that ε_x is a function of *y*, ε_y is a function of *x*, and γ_{xy} is a function of both *x* and *y*.

The stresses are again given as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D]\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad \{\sigma\} = [D][B]\{d\}$$

For plane stress [*D*] is:

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}$$

For plane strain [*D*] is:

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix}$$

Plane Stress and Plane Strain Equations

4. Derive the Element Stiffness Matrix and Equations

The above relationship requires:

$$\int_{-h}^{h}\int_{-b}^{b}[B]^T[D][B]\,t\,dx\,dy\,\{d\}=\{f\}$$

The stiffness matrix can be defined as:

$$[k]=\int_{-h}^{h}\int_{-b}^{b}[B]^T[D][B]\,t\,dx\,dy$$

For an element of constant thickness, *t*, the above integral becomes:

$$[k]=t\int_{-h}^{h}\int_{-b}^{b}[B]^T[D][B]\,dx\,dy$$

Plane Stress and Plane Strain Equations

4. Derive the Element Stiffness Matrix and Equations

Because the [*B*] matrix is a function of *x* and *y*, integration must be performed.

The [*k*] matrix for the rectangular element is now of order 8 x 8.

A numerical evaluation for [*k*] using *b* = 4 in., *h* = 2 in., *t* = 1 in., *E* = 30 x 10^6 psi, and ν = 0.3.

This double integral was solved using Mathcad.

Plane Stress and Plane Strain Equations

4. Derive the Element Stiffness Matrix and Equations

$$[k] = \begin{bmatrix} 1.35E+10 & 5.49E+09 & -1.69E+09 & -4.22E+08 & -6.75E+09 & -5.49E+09 & -5.06E+09 & 4.22E+08 \\ 5.49E+09 & 2.45E+10 & 4.22E+08 & 9.28E+09 & -5.49E+09 & -1.22E+10 & -4.22E+08 & -2.15E+10 \\ -1.69E+09 & 4.22E+08 & 1.35E+10 & -5.49E+09 & -5.06E+09 & -4.22E+08 & -6.75E+09 & 5.49E+09 \\ -4.22E+08 & 9.28E+09 & -5.49E+09 & 2.45E+10 & 4.22E+08 & -2.15E+10 & 5.49E+09 & -1.22E+10 \\ -6.75E+09 & -5.49E+09 & -5.06E+09 & 4.22E+08 & 1.35E+10 & 5.49E+09 & -1.69E+09 & -4.22E+08 \\ -5.49E+09 & -1.22E+10 & -4.22E+08 & -2.15E+10 & 5.49E+09 & 2.45E+10 & 4.22E+08 & 9.28E+09 \\ -5.06E+09 & -4.22E+08 & -6.75E+09 & 5.49E+09 & -1.69E+09 & 4.22E+08 & 1.35E+10 & -5.49E+05 \\ 4.22E+08 & -2.15E+10 & 5.49E+09 & -1.22E+10 & -4.22E+08 & 9.28E+09 & -5.49E+09 & 2.45E+10 \end{bmatrix}$$

Plane Stress and Plane Strain Equations

Steps 5 - 7

Steps 5 through 7, which involve assembling the global stiffness matrix and equations, determining the unknown nodal displacements, and calculating the stress, are identical to those in Section 6.2 for the CST.

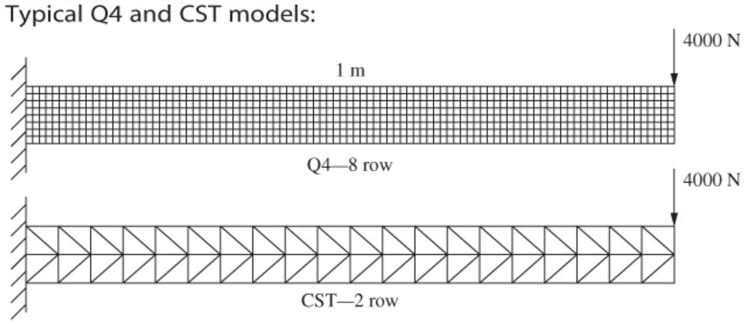
However, the stresses within each element now vary in both the x and y directions.

Plane Stress and Plane Strain Equations

Numerical Comparison of CST to Q4 Element Models and Element Defects.

Table 6-1 compares the free end deflection and maximum principal stress for a cantilevered beam modeled with various all triangular CST elements or all rectangular Q4 elements.

Typical Q4 and CST models:



Plane Stress and Plane Strain Equations

Numerical Comparison of CST to Q4 Element Models and Element Defects.

Table 6-1 compares the free end deflection and maximum principal stress for a cantilevered beam modeled with various all triangular CST elements or all rectangular Q4 elements.

Table 6–1 Table comparing free-end deflections and largest principal stresses for CST and Q4 element models (end force = 4000 N, length = 1 m, $I = 1 \times 10^{-5}$ m^4, thickness = 0.12 m, E = 200 GPa)

Plane Element Used/Rows	Number of Nodes	Number of Degrees of Freedom	Free End Displ., m	Principal Stress, MPa
Q4/2	60	120	6.708×10^{-4}	19.35
Q4/4	200	400	6.729×10^{-4}	20.30
Q4/8	720	1440	6.729×10^{-4}	21.72
CST/2	60	120	3.630×10^{-4}	7.80
CST/4	200	400	5.537×10^{-4}	13.76
CST/8	720	1440	6.385×10^{-4}	17.61
Classical beam theory			6.667×10^{-4}	20.00

Plane Stress and Plane Strain Equations

Numerical Comparison of CST to Q4 Element Models and Element Defects.

We observe from the displacement results that the CST element models produce stiffer models than the actual beam behavior, as the deflections are predicted to be smaller than classical beam theory predicts.

We also observe that the CST model converges very slowly to the classical beam theory solution.

This is partly due to the element predicting only constant stress within each element when for a bending problem; the stress actually varies linearly through the depth of the beam.

This problem is rectified by using the linear-strain triangle (LST) element as described in Chapter 8.

Plane Stress and Plane Strain Equations

Numerical Comparison of CST to Q4 Element Models and Element Defects.

The results indicate that the Q4 element model predicts more accurate deflection behavior than the CST element model.

The two-row model of Q4 elements yields deflections very close to that predicted by the classical beam deflection equation, whereas the two-row model of CST elements is quite inaccurate in predicting the deflection.

As the number of rows is increased to four and then eight, the deflections are predicted increasingly more accurately for the CST and Q4 element models.

The two-noded beam element model gives the identical deflection as the classical equation ($\delta = PL^3/3EI$) as expected (see discussion in Section 4.5) and is the most appropriate model for this problem when you are not concerned, for instance, with stress concentrations.

Plane Stress and Plane Strain Equations

Numerical Comparison of CST to Q4 Element Models and Element Defects.

It has been shown for a beam subjected to pure bending, the CST has a spurious or false shear stress and hence a spurious shear strain in parts of the model that should not have any shear stress or shear strain.

This spurious shear strain absorbs energy; therefore, some of the energy that should go into bending is lost.

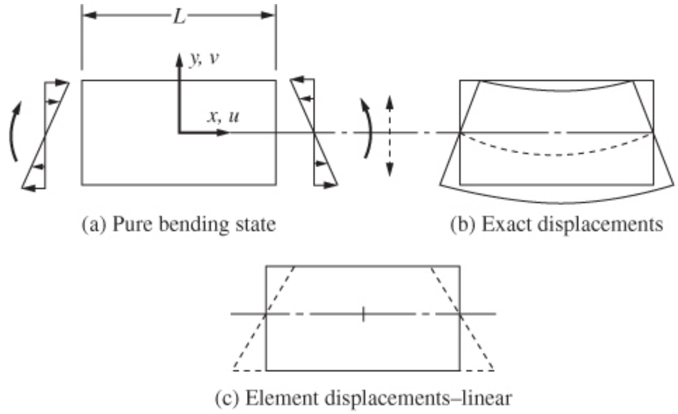
The CST is then too stiff in bending, and the resulting deformation is smaller than actually should be.

This phenomenon of excessive stiffness developing in one more modes of deformation is sometimes described as *shear locking* or *parasitic shear.*

Plane Stress and Plane Strain Equations

Numerical Comparison of CST to Q4 Element Models and Element Defects.

It should be noted that using a single row of Q4 elements with their linear edge displacement is not recommended to accurately predict the stress gradient through the depth of the beam.



(a) Pure bending state (b) Exact displacements

(c) Element displacements–linear

Plane Stress and Plane Strain Equations

Numerical Comparison of CST to Q4 Element Models and Element Defects.

As mentioned previously, the CST element has constant strain and stress within it, while the Q4 element normal strain ε_x and hence the normal stress σ_x is linear in the *y* direction.

Therefore, the CST is not able simulate the bending behavior nearly as well as the Q4 element.

The classical beam theory/bending stress equation predicts a linear stress variation through the depth the beam given by $\sigma_x = -My/I$

As shown when comparing the principal stresses for each model, as more rows are used, the stresses approach the classical bending stress of 20 MPa with the Q4 approaching the classical solution much faster as indicated by comparing the two-row solutions for Q4 and CST models.

Plane Stress and Plane Strain Equations

Numerical Comparison of CST to Q4 Element Models and Element Defects.

This brief description of some of the limitations in using the CST and Q4 elements does not prevent us from using them to model plane stress and plane strain problems.

It just requires us to use a fine mesh as opposed to a coarse one, particularly where bending occurs and where in general large stress gradients will results.

Also, we must make sure our computer program can handle Poisson's ratios that approach 0.5 (if that is desired, such as in rubber-like materials).

For common materials, such as metals, Poisson's ratio is around 0.3, so locking should not be of concern.

End of Chapter 6b