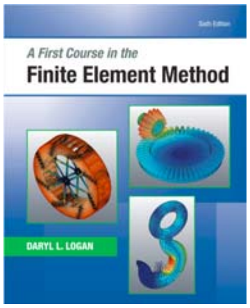
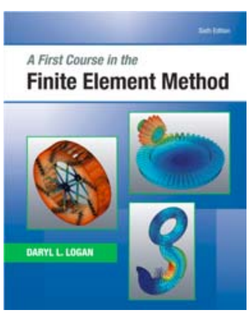
Chapter 6a – Plane Stress/Strain Equations



Learning Objectives

- To review basic concepts of plane stress and plane strain.
- To derive the constant-strain triangle (CST) element stiffness matrix and equations.
- To demonstrate how to determine the stiffness matrix and stresses for a constant strain element.
- To describe how to treat body and surface forces for two-dimensional elements.

Chapter 6a – Plane Stress/Strain Equations



Learning Objectives

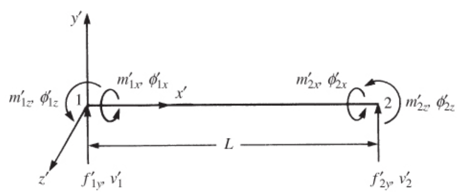
- To evaluate the explicit stiffness matrix for the constant-strain triangle element.
- To perform a detailed finite element solution of a plane stress problem.

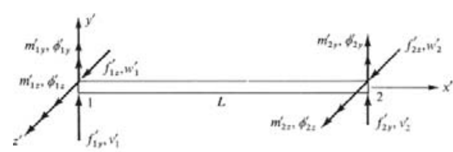
Plane Stress and Plane Strain Equations

In Chapters 2 through 5, we considered only line elements.

Line elements are connected only at common nodes, forming framed or articulated structures such as trusses, frames, and grids.

Line elements have geometric properties such as cross-sectional area and moment of inertia associated with their cross sections.



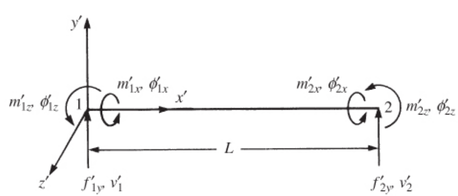


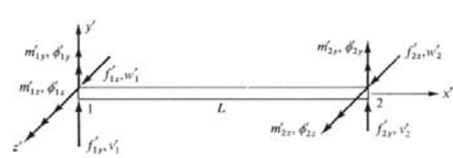
Plane Stress and Plane Strain Equations

However, only one local coordinate along the length of the element is required to describe a position along the element (hence, they are called ***line elements***).

Nodal compatibility is then enforced during the formulation of the nodal equilibrium equations for a line element.

This chapter considers the two-dimensional finite element.

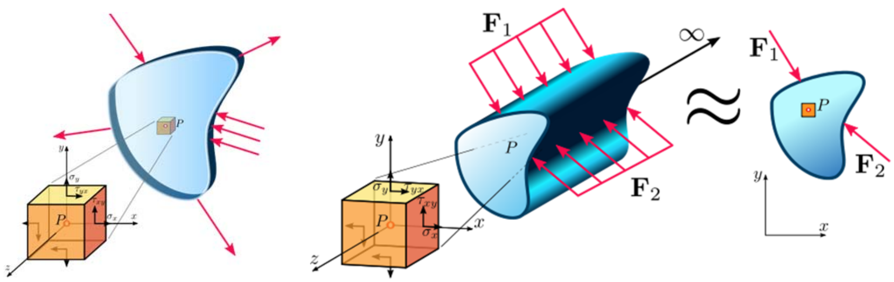




Plane Stress and Plane Strain Equations

Two-dimensional (planar) elements are thin-plate elements such that two coordinates define a position on the element surface.

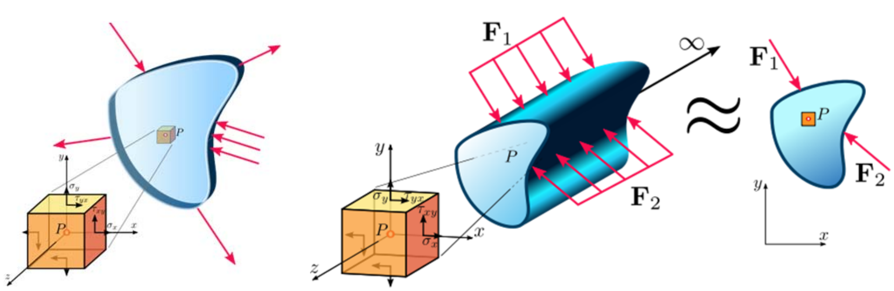
The elements are connected at common nodes and/or along common edges to form continuous structures.



Plane Stress and Plane Strain Equations

Nodal compatibility is then enforced during the formulation of the nodal equilibrium equations for two-dimensional elements.

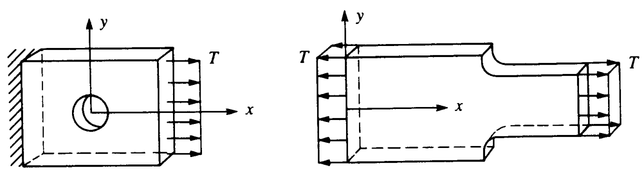
If proper displacement functions are chosen, compatibility along common edges is also obtained.



Plane Stress and Plane Strain Equations

The two-dimensional element is extremely important for:

(1) ***Plane stress analysis***, which includes problems such as plates with holes, fillets, or other changes in geometry that are loaded in their plane resulting in local stress concentrations.



Plane Stress Problems

Plane Stress and Plane Strain Equations

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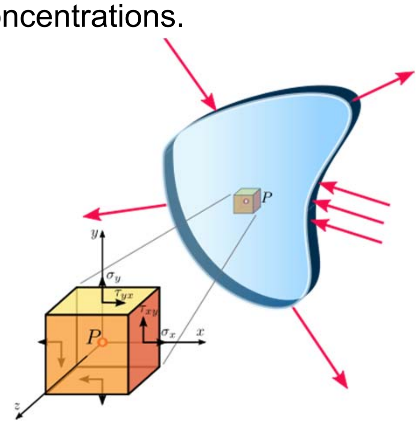

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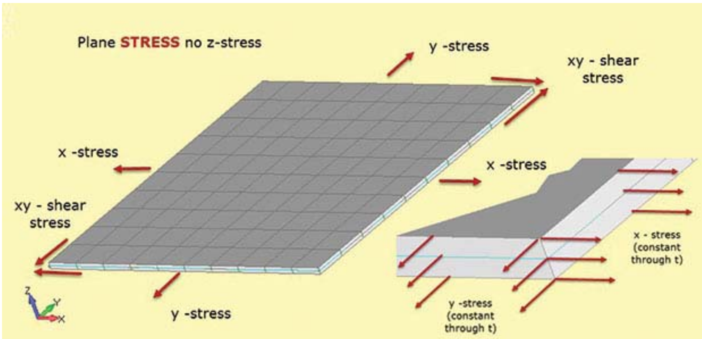

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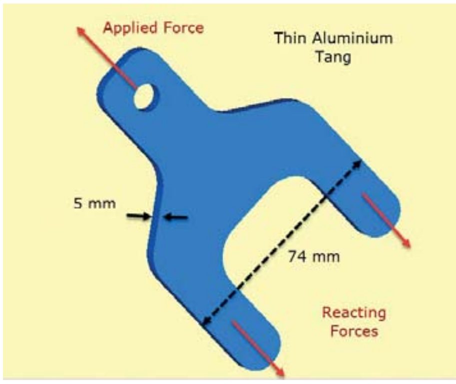

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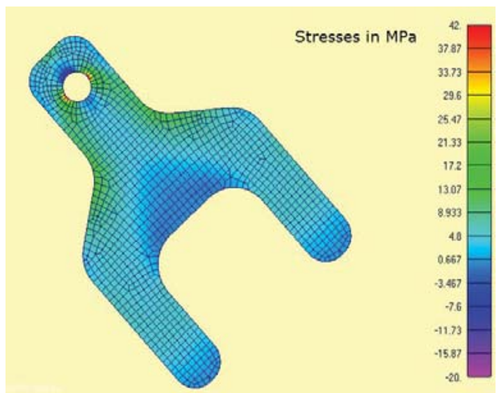
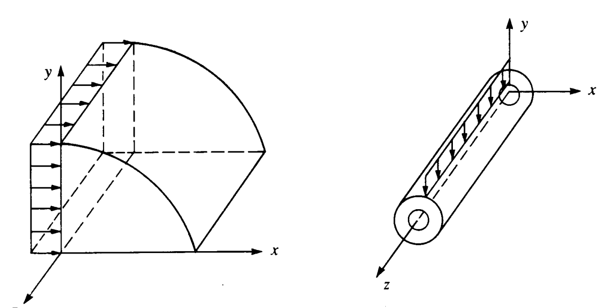


Plane Stress and Plane Strain Equations

The two-dimensional element is extremely important for:

(2) ***Plane strain analysis***, which includes problems such as a long underground box culvert subjected to a uniform load acting constantly over its length or a long cylindrical control rod subjected to a load that remains constant over the rod length (or depth).



Plane Strain Problems

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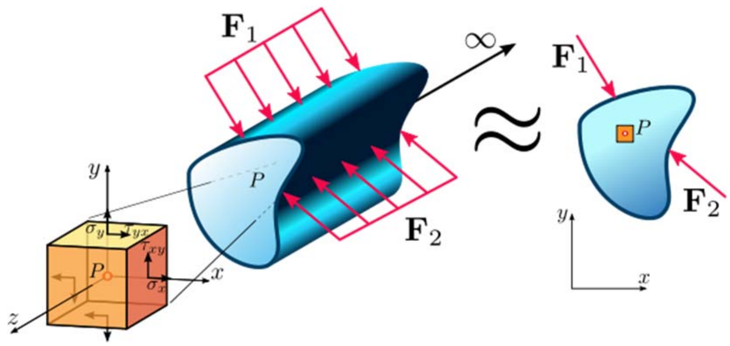

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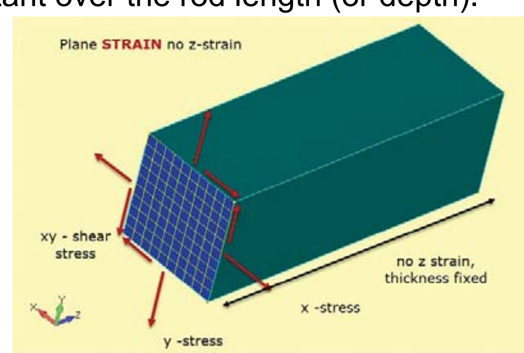

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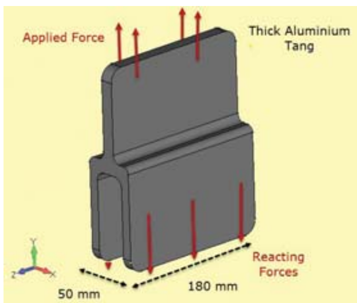

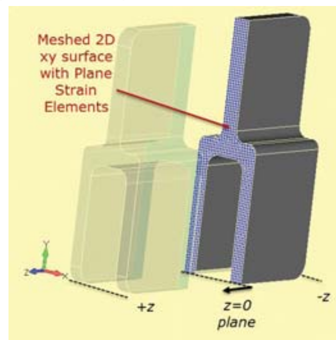

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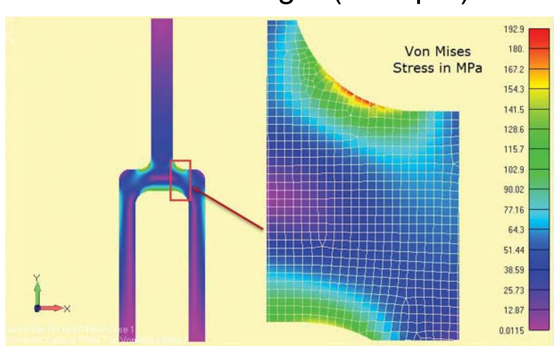

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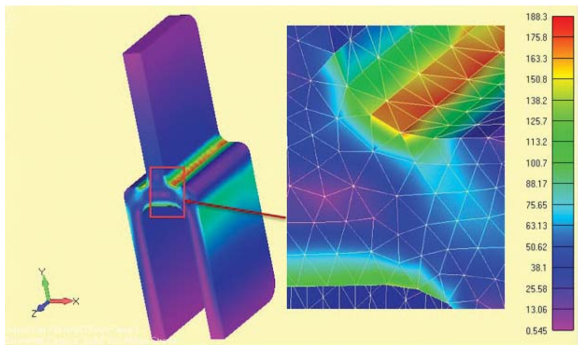


Plane Stress and Plane Strain Equations

We begin this chapter with the development of the stiffness matrix for a basic two-dimensional or plane finite element, called the ***constant-strain triangular element***.

The constant-strain triangle (CST) stiffness matrix derivation is the simplest among the available two-dimensional elements.

We will derive the CST stiffness matrix by using the principle of minimum potential energy because the energy formulation is the most feasible for the development of the equations for both two- and three-dimensional finite elements.

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

We will now follow the steps described in Chapter 1 to formulate the governing equations for a plane stress/plane strain triangular element.

First, we will describe the concepts of plane stress and plane strain.

Then we will provide a brief description of the steps and basic equations pertaining to a plane triangular element.

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Plane Stress

Plane stress is defined to be ***a state of stress in which the normal stress and the shear stresses directed perpendicular to the plane are assumed to be zero***.

That is, the normal stress σ_z and the shear stresses τ_{xz} and τ_{yz} are assumed to be zero.

Generally, members that are thin (those with a small *z* dimension compared to the in-plane *x* and *y* dimensions) and whose loads act only in the *x*-*y* plane can be considered to be under plane stress.

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Plane Strain

Plane strain is defined to be ***a state of strain in which the strain normal to the x-y plane ε_z and the shear strains γ_{xz} and γ_{yz} are assumed to be zero***.

The assumptions of plane strain are realistic for long bodies (say, in the *z* direction) with constant cross-sectional area subjected to loads that act only in the *x* and/or *y* directions and do not vary in the *z* direction.

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Two-Dimensional State of Stress and Strain

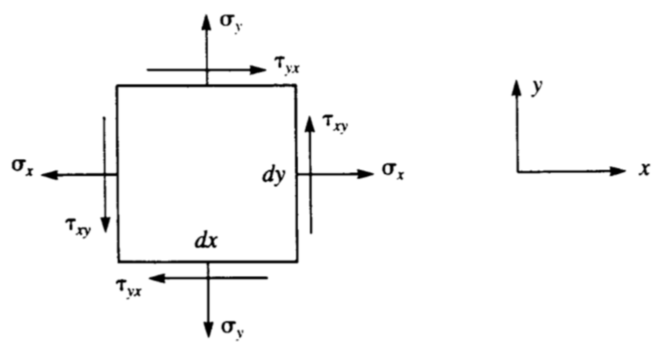
The concept of two-dimensional state of stress and strain and the stress/strain relationships for plane stress and plane strain are necessary to understand fully the development and applicability of the stiffness matrix for the plane stress/plane strain triangular element.

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Two-Dimensional State of Stress and Strain

A two-dimensional state of stress is shown in the figure below.

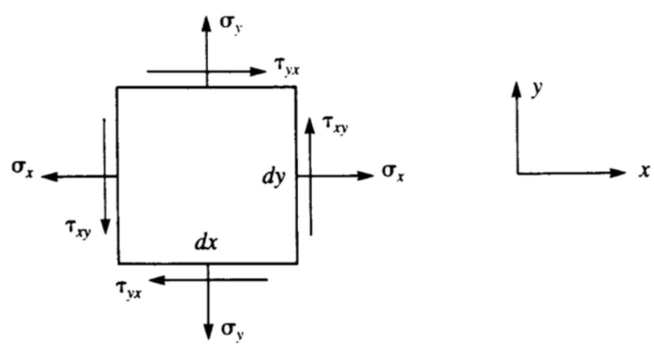


Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Two-Dimensional State of Stress and Strain

The infinitesimal element with sides *dx* and *dy* has normal stresses σ_x and σ_y acting in the *x* and *y* directions (here on the vertical and horizontal faces), respectively.

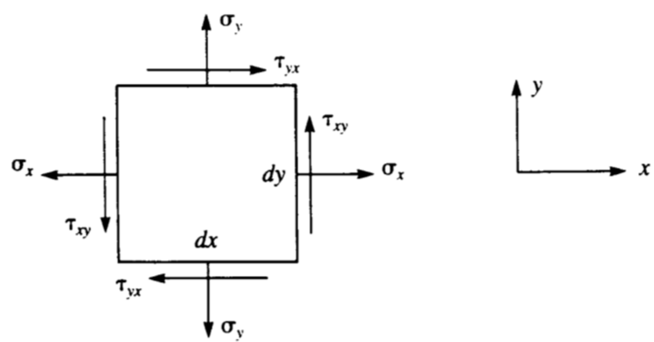


Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Two-Dimensional State of Stress and Strain

The shear stress τ_{xy} acts on the x edge (vertical face) in the y direction. The shear stress τ_{yx} acts on the y edge (horizontal face) in the x direction.



Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Two-Dimensional State of Stress and Strain

Since τ_{xy} equals τ_{yx}, three independent stress exist:

$$\{\sigma\}^T = \begin{bmatrix} \sigma_x & \sigma_y & \tau_{xy} \end{bmatrix}$$

Recall, the relationships for ***principal stresses*** in two-dimensions are:

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} = \sigma_{\max}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} = \sigma_{\min}$$

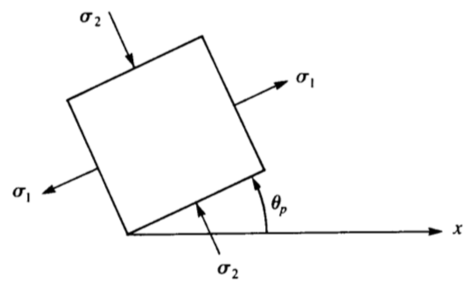
Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Two-Dimensional State of Stress and Strain

Also, θ_p is the ***principal angle*** which defines the normal whose direction is perpendicular to the plane on which the maximum or minimum principle stress acts.

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

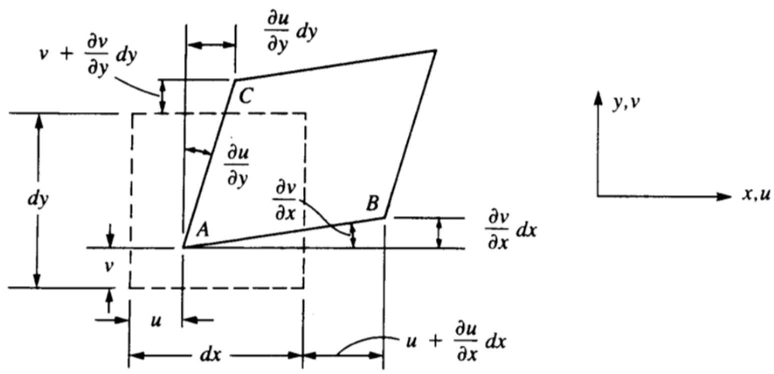


Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Two-Dimensional State of Stress and Strain

The general two-dimensional state of strain at a point is show below.

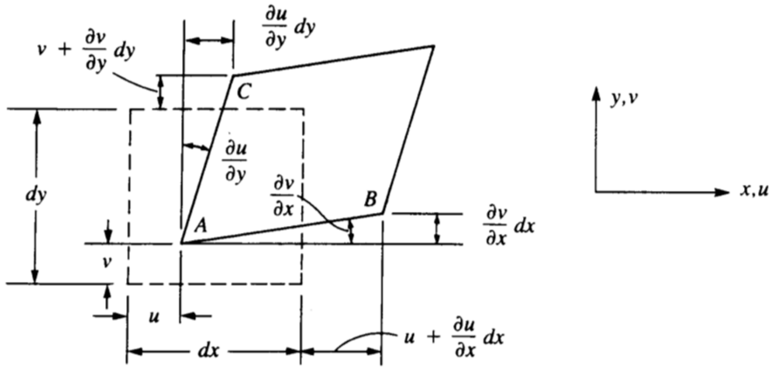


Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Two-Dimensional State of Stress and Strain

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$



Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Two-Dimensional State of Stress and Strain

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

The strain may be written in matrix form as:

$$\{\varepsilon\}^T = \begin{bmatrix} \varepsilon_x & \varepsilon_y & \gamma_{xy} \end{bmatrix}$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Two-Dimensional State of Stress and Strain

For **plane stress,** the stresses σ_z, τ_{xz}, and τ_{yz} are assumed to be zero. The stress-strain relationship is:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad [D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}$$

is called the ***stress-strain matrix*** (or the ***constitutive matrix***), E is the modulus of elasticity, and ν is Poisson's ratio.

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Two-Dimensional State of Stress and Strain

For **plane strain**, the strains ε_z, γ_{xz}, and γ_{yz} are assumed to be zero. The stress-strain relationship is:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad [D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix}$$

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Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Two-Dimensional State of Stress and Strain

The partial differential equations for plane stress are:

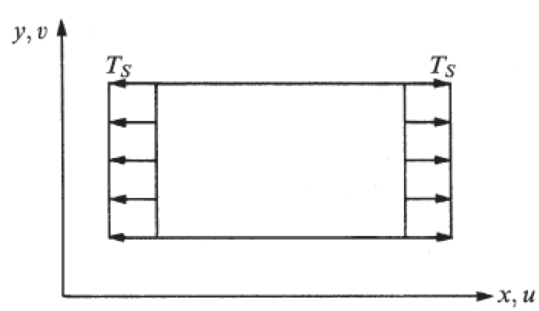
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1+\nu}{2}\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x \partial y}\right)$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1+\nu}{2}\left(\frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u}{\partial x \partial y}\right)$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Consider the problem of a thin plate subjected to a tensile load as shown in the figure below:

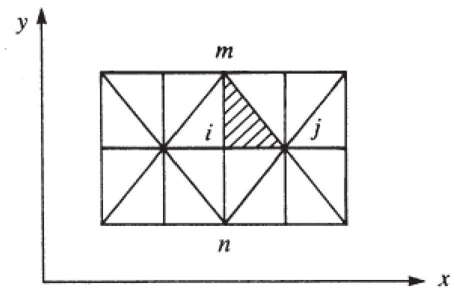


Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 1 - Discretize and Select Element Types

Discretize the thin plate into a set of triangular elements. Each element is define by nodes *i*, *j*, and *m*.

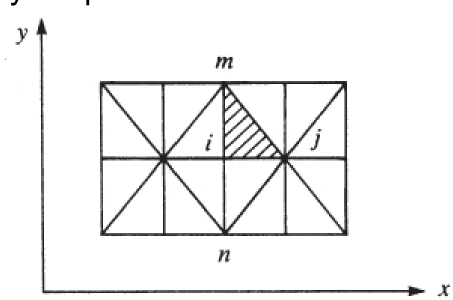


Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 1 - Discretize and Select Element Types

We use triangular elements because boundaries of irregularly shaped bodies can be closely approximated, and because the expressions related to the triangular element are comparatively simple.



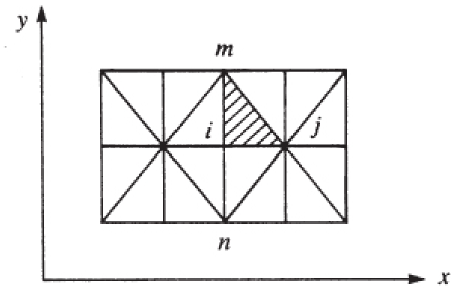
Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 1 - Discretize and Select Element Types

This discretization is called a ***coarse-mesh generation*** if few large elements are used.

Each node has two degrees of freedom: displacements in the *x* and *y* directions.

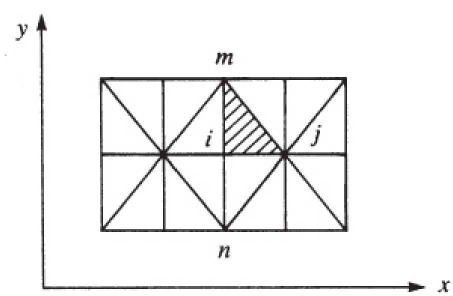


Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 1 - Discretize and Select Element Types

We will let u_i and v_i represent the node *i* displacement components in the *x* and *y* directions, respectively.



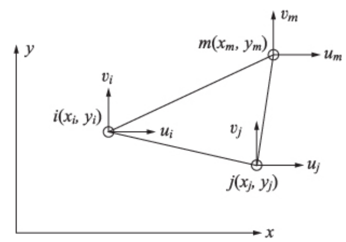
Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 1 - Discretize and Select Element Types

The nodal displacements for an element with nodes *i*, *j*, and *m* are:

$$\{d\} = \begin{Bmatrix} d_i \\ d_j \\ d_m \end{Bmatrix}$$



where the nodes are ordered ***counterclockwise*** around the element, and

$$\{d_i\} = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 1 - Discretize and Select Element Types

The nodal displacements for an element with nodes *i*, *j*, and *m* are:

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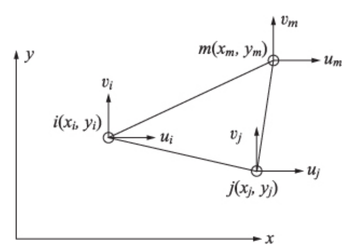


Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions

The general displacement function is: $\{\Psi_i\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}$

The functions $u(x, y)$ and $v(x, y)$ must be compatible with the element type.

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

The general definitions of normal and shear strains are:

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

For **plane stress,** the stresses σ_z, τ_{xz}, and τ_{yz} are assumed to be zero. The stress-strain relationship is:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

For **plane strain**, the strains ε_z, γ_{xz}, and γ_{yz} are assumed to be zero. The stress-strain relationship is:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 4 - Derive the Element Stiffness Matrix and Equations

Using the principle of minimum potential energy, we can derive the element stiffness matrix.

$$\{f\} = [k]\{d\}$$

This approach is better than the direct methods used for one-dimensional elements.

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 5 - Assemble the Element Equations and Introduce Boundary Conditions

The final assembled or global equation written in matrix form is:

$$\{F\} = [K]\{d\}$$

where **{*F*}** is the equivalent global nodal loads obtained by lumping distributed edge loads and element body forces at the nodes and **[*K*]** is the global structure stiffness matrix.

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 6 - Solve for the Nodal Displacements

Once the element equations are assembled and modified to account for the boundary conditions, a set of simultaneous algebraic equations that can be written in expanded matrix form as:

$$\{F\} = [K]\{d\}$$

Step 7 - Solve for the Element Forces (Stresses)

For the structural stress-analysis problem, important secondary quantities of strain and stress (or moment and shear force) can be obtained in terms of the displacements determined in Step 6.

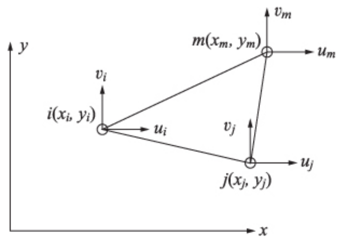
Plane Stress and Plane Strain Equations

Derivation of the Constant-Strain Triangular Element Stiffness Matrix and Equations

Step 1 - Discretize and Select Element Types

Consider the problem of a thin plate subjected to a tensile load as shown in the figure below:

$$\{d\} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

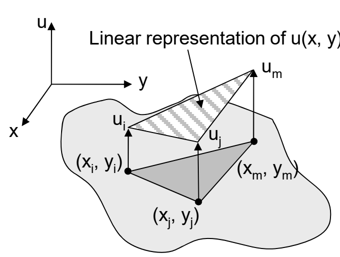


Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions

We will select a linear displacement function for each triangular element, defined as:



$$\{\Psi_i\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{Bmatrix} a_1 + a_2 x + a_3 y \\ a_4 + a_5 x + a_6 y \end{Bmatrix}$$

A linear function ensures that the displacements along each edge of the element and the nodes shared by adjacent elements are equal.

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions

We will select a linear displacement function for each triangular element, defined as:

$$\{\Psi_i\} = \begin{Bmatrix} a_1 + a_2 x + a_3 y \\ a_4 + a_5 x + a_6 y \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions

To obtain the values for the *a*'s substitute the coordinated of the nodal points into the above equations:

$$u_i = a_1 + a_2 x_i + a_3 y_i \qquad v_i = a_4 + a_5 x_i + a_6 y_i$$

$$u_j = a_1 + a_2 x_j + a_3 y_j \qquad v_j = a_4 + a_5 x_j + a_6 y_j$$

$$u_m = a_1 + a_2 x_m + a_3 y_m \qquad v_m = a_4 + a_5 x_m + a_6 y_m$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions

Solving for the ***a***'s and writing the results in matrix forms gives:

$$\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \quad \Rightarrow \quad \{a\} = [x]^{-1}\{u\}$$

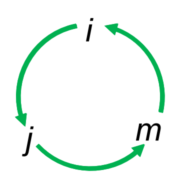
Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions

The inverse of the [x] matrix is:

$$[x]^{-1} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}$$



$$\alpha_i = x_j y_m - y_j x_m \qquad \beta_i = y_j - y_m \qquad \gamma_i = x_m - x_j$$

$$\alpha_j = x_m y_i - y_m x_i \qquad \beta_j = y_m - y_i \qquad \gamma_j = x_i - x_m$$

$$\alpha_m = x_i y_j - y_i x_j \qquad \beta_m = y_i - y_j \qquad \gamma_m = x_j - x_i$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions

The inverse of the [*x*] matrix is:

$$[x]^{-1} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \qquad 2A = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix}$$

$$2A = x_i\left(y_j - y_m\right) + x_j\left(y_m - y_i\right) + x_m\left(y_i - y_j\right)$$

where *A* is the area of the triangle

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions

The values of ***a*** may be written matrix form as:

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

$$\begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} v_i \\ v_j \\ v_m \end{Bmatrix}$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions

Expanding the above equations

$$\{u\} = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Substituting the values for ***a*** into the above equation gives:

$$\{u\} = \frac{1}{2A}\begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions

We will now derive the *u* displacement function in terms of the coordinates *x* and *y*.

$$\{u\} = \frac{1}{2A}\begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} \alpha_i u_i + \alpha_j u_j + \alpha_m u_m \\ \beta_i u_i + \beta_j u_j + \beta_m u_m \\ \gamma_i u_i + \gamma_j u_j + \gamma_m u_m \end{bmatrix}$$

Multiplying the matrices in the above equations gives:

$$u(x,y) = \frac{1}{2A}\Big\{ (\alpha_i + \beta_i x + \gamma_i y) u_i + (\alpha_j + \beta_j x + \gamma_j y) u_j + (\alpha_m + \beta_m x + \gamma_m y) u_m \Big\}$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions

We will now derive the v displacement function in terms of the coordinates x and y.

$$\{v\} = \frac{1}{2A}\begin{bmatrix} 1 & x & y \end{bmatrix}\begin{bmatrix} \alpha_i v_i + \alpha_j v_j + \alpha_m v_m \\ \beta_i v_i + \beta_j v_j + \beta_m v_m \\ \gamma_i v_i + \gamma_j v_j + \gamma_m v_m \end{bmatrix}$$

Multiplying the matrices in the above equations gives:

$$v(x,y) = \frac{1}{2A}\left\{(\alpha_i + \beta_i x + \gamma_i y)v_i + (\alpha_j + \beta_j x + \gamma_j y)v_j + (\alpha_m + \beta_m x + \gamma_m y)v_m\right\}$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions

The displacements can be written in a more convenience form as:

$$u(x,y) = N_i u_i + N_j u_j + N_m u_m$$

$$v(x,y) = N_i v_i + N_j v_j + N_m v_m$$

where:

$$N_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y)$$

$$N_j = \frac{1}{2A}(\alpha_j + \beta_j x + \gamma_j y)$$

$$N_m = \frac{1}{2A}(\alpha_m + \beta_m x + \gamma_m y)$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions

The elemental displacements can be summarized as:

$$\{\Psi_i\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{Bmatrix} N_i u_i + N_j u_j + N_m u_m \\ N_i v_i + N_j v_j + N_m v_m \end{Bmatrix}$$

In another form the above equations are:

$$\{\Psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

$$\{\Psi\} = [N]\{d\}$$

Plane Stress and Plane Strain Equations

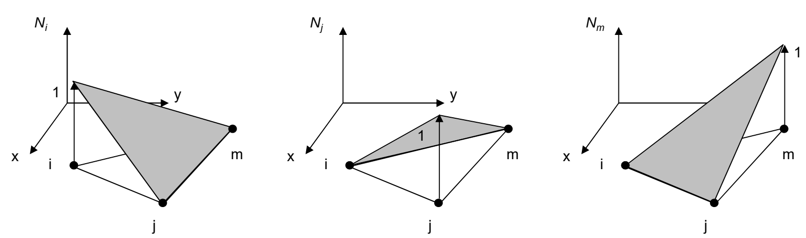
Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions

In another form the equations are: $\{\Psi\} = [N]\{d\}$

$$[N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix}$$

The linear triangular shape functions are illustrated below:



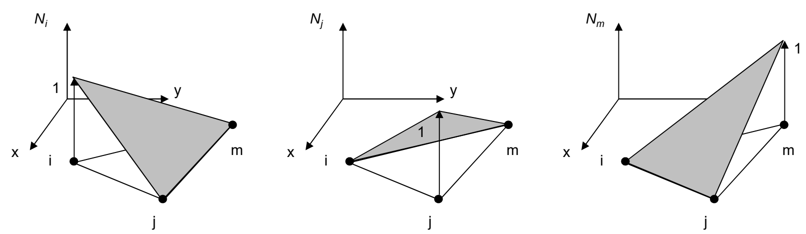
Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions

So that *u* and *v* will yield a constant value for rigid-body displacement, $N_i + N_j + N_m = 1$ for all x and y locations on the element.

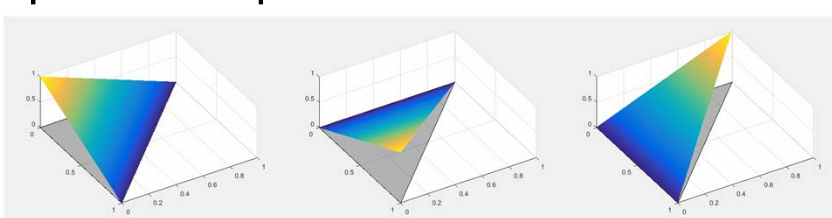
The linear triangular shape functions are illustrated below:



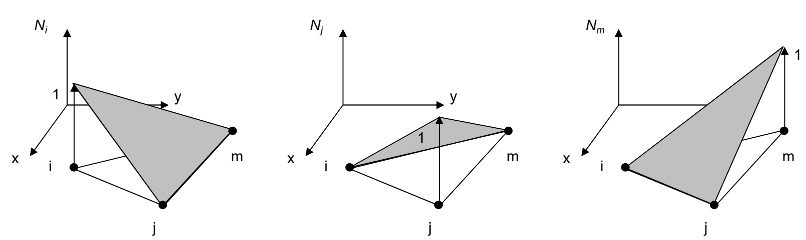
Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions



The linear triangular shape functions are illustrated below:



Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions

So that *u* and *v* will yield a constant value for rigid-body displacement, $N_i + N_j + N_m = 1$ for all x and y locations on the element.

For example, assume all the triangle displaces as a rigid body in the *x* direction: $u = u_0$

$$\{\Psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_0 \\ 0 \\ u_0 \\ 0 \\ u_0 \\ 0 \end{Bmatrix}$$

$$u_0 = u_0\left(N_i + N_j + N_m\right)$$

$$\Rightarrow N_i + N_j + N_m = 1$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions

So that *u* and *v* will yield a constant value for rigid-body displacement, $N_i + N_j + N_m = 1$ for all x and y locations on the element.

For example, assume all the triangle displaces as a rigid body in the *y* direction: $v = v_0$

$$\{\Psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} 0 \\ v_0 \\ 0 \\ v_0 \\ 0 \\ v_0 \end{Bmatrix}$$

$$v_0 = v_0\left(N_i + N_j + N_m\right)$$

$$\Rightarrow N_i + N_j + N_m = 1$$

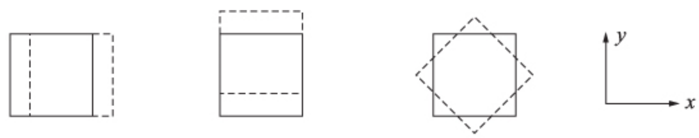
Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions

The requirement of completeness for the constant-strain triangle element used in a two-dimensional plane stress element is illustrated in figure below.

The element must be able to translate uniformly in either the *x* or *y* direction in the plane and to rotate without straining as shown



Plane Stress and Plane Strain Equations

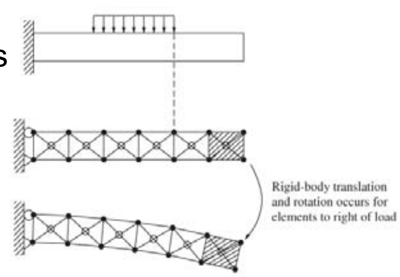
Formulation of the Plane Triangular Element Equations

Step 2 - Select Displacement Functions

The reason that the element must be able to translate as a rigid body and to rotate stress-free is illustrated in the example of a cantilever beam modeled with plane stress elements.

By simple statics, the beam elements beyond the loading are stress free.

Hence these elements must be free to translate and rotate without stretching or changing shape.



Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

Elemental Strains: The strains over a two-dimensional element are:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

Substituting our approximation for the displacement gives:

$$\frac{\partial u}{\partial x} = u_{,x} = \frac{\partial}{\partial x}\left(N_i u_i + N_j u_j + N_m u_m\right)$$

$$u_{,x} = N_{i,x} u_i + N_{j,x} u_j + N_{m,x} u_m$$

where the comma indicates differentiation with respect to that variable.

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

The derivatives of the interpolation functions are:

$$N_{i,x} = \frac{1}{2A}\frac{\partial}{\partial x}\left(\alpha_i + \beta_i x + \gamma_i y\right) = \frac{\beta_i}{2A}$$

$$N_{j,x} = \frac{\beta_j}{2A} \qquad N_{m,x} = \frac{\beta_m}{2A}$$

Therefore:

$$\frac{\partial u}{\partial x} = \frac{1}{2A}\left(\beta_i u_i + \beta_j u_j + \beta_m u_m\right)$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

In a similar manner, the remaining strain terms are approximated as:

$$\frac{\partial v}{\partial y} = \frac{1}{2A}\left(\gamma_i v_i + \gamma_j v_j + \gamma_m v_m\right)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{2A}\left(\beta_i u_i + \gamma_i v_i + \beta_j u_j + \gamma_j v_j + \beta_m u_m + \gamma_m v_m\right)$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

We can write the strains in matrix form as:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{2A} \left[\boxed{\begin{matrix} \beta_i & 0 \\ 0 & \gamma_i \\ \gamma_i & \beta_i \end{matrix}} \; \boxed{\begin{matrix} \beta_j & 0 \\ 0 & \gamma_j \\ \gamma_j & \beta_j \end{matrix}} \; \boxed{\begin{matrix} \beta_m & 0 \\ 0 & \gamma_m \\ \gamma_m & \beta_m \end{matrix}} \right] \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

$$\{\varepsilon\} = [B]\{d\} \qquad \{\varepsilon\} = \begin{bmatrix} B_i & B_j & B_m \end{bmatrix} \begin{Bmatrix} d_i \\ d_j \\ d_m \end{Bmatrix}$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

Stress-Strain Relationship: The in-plane stress-strain relationship is:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad \{\sigma\} = [D][B]\{d\}$$

For plane stress $[D]$ is:

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}$$

For plane strain $[D]$ is:

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix}$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 4 - Derive the Element Stiffness Matrix and Equations

The total potential energy is defined as the sum of the internal strain energy *U* and the potential energy of the external forces Ω:

$$\pi_p = U + \Omega_b + \Omega_p + \Omega_s$$

Where the strain energy is: $U = \frac{1}{2}\int_V \{\varepsilon\}^T\{\sigma\}dV = \frac{1}{2}\int_V \{\varepsilon\}^T[D]\{\varepsilon\}dV$

The potential energy of the ***body force*** term is:

$$\Omega_b = -\int_V \{\Psi\}^T\{X\}dV$$

where $\{\Psi\}$ is the general displacement function and $\{X\}$ is the body weight per unit volume.

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 4 - Derive the Element Stiffness Matrix and Equations

The total potential energy is defined as the sum of the internal strain energy *U* and the potential energy of the external forces Ω:

$$\pi_p = U + \Omega_b + \Omega_p + \Omega_s$$

Where the strain energy is: $U = \frac{1}{2}\int_V \{\varepsilon\}^T\{\sigma\}dV = \frac{1}{2}\int_V \{\varepsilon\}^T[D]\{\varepsilon\}dV$

The potential energy of the ***concentrated forces*** is:

$$\Omega_p = -\{d\}^T\{P\}$$

where $\{P\}$ are the concentrated forces and $\{d\}$ are the nodal displacements.

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 4 - Derive the Element Stiffness Matrix and Equations

The total potential energy is defined as the sum of the internal strain energy *U* and the potential energy of the external forces Ω:

$$\pi_p = U + \Omega_b + \Omega_p + \Omega_s$$

Where the strain energy is: $U = \frac{1}{2}\int_V \{\varepsilon\}^T \{\sigma\} dV = \frac{1}{2}\int_V \{\varepsilon\}^T [D]\{\varepsilon\} dV$

The potential energy of the ***distributed loads*** is:

$$\Omega_s = -\int_S \{\Psi\}^T \{T\} dS$$

where $\{\Psi\}$ is the general displacement function and $\{T\}$ are the surface tractions.

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 4 - Derive the Element Stiffness Matrix and Equations

Then the total potential energy expression becomes:

$$\pi_p = \frac{1}{2}\int_V \{d\}^T [B]^T [D][B]\{d\} dV - \int_V \{d\}^T [N]^T \{X\} dV - \{d\}^T \{P\} - \int_S \{d\}^T [N]^T \{T\} dS$$

The nodal displacements $\{d\}$ are independent of the general *x*-*y* coordinates, therefore

$$\pi_p = \frac{1}{2}\{d\}^T \int_V [B]^T [D][B] dV \{d\} - \{d\}^T \int_V [N]^T \{X\} dV - \{d\}^T \{P\} - \{d\}^T \int_S [N]^T \{T\} dS$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 4 - Derive the Element Stiffness Matrix and Equations

We can define the last three terms as:

$$\{f\} = \int_V [N]^T \{X\} dV + \{P\} + \int_S [N]^T \{T\} dS$$

Therefore:

$$\pi_p = \frac{1}{2}\{d\}^T \int_V [B]^T [D][B] dV \{d\} - \{d\}^T \{f\}$$

Minimization of π_p with respect to each nodal displacement requires that:

$$\frac{\partial \pi_p}{\partial \{d\}} = \int_V [B]^T [D][B] dV \{d\} - \{f\} = 0$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 4 - Derive the Element Stiffness Matrix and Equations

The above relationship requires:

$$\int_V [B]^T [D][B] dV \{d\} = \{f\}$$

The stiffness matrix can be defined as:

$$[k] = \int_V [B]^T [D][B] dV$$

For an element of constant thickness, *t*, the above integral becomes:

$$[k] = t \int_A [B]^T [D][B]\, dx\, dy$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 4 - Derive the Element Stiffness Matrix and Equations

The integrand in the above equation is not a function of *x* or *y* (global coordinates); therefore, the integration reduces to:

$$[k] = t[B]^T [D][B]\int_A dx\, dy$$

$$[k] = tA\, [B]^T [D][B]$$

where *A* is the area of the triangular element.

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 4 - Derive the Element Stiffness Matrix and Equations

Expanding the stiffness relationship gives:

$$[k] = \begin{bmatrix} [k_{ii}] & [k_{ij}] & [k_{im}] \\ [k_{ji}] & [k_{jj}] & [k_{jm}] \\ [k_{mi}] & [k_{mj}] & [k_{mm}] \end{bmatrix}$$

where each $[k_{ii}]$ is a 2 x 2 matrix define as:

$$[k_{ii}] = [B_i]^T [D][B_i]\, tA \qquad [k_{ij}] = [B_i]^T [D][B_j]\, tA$$

$$[k_{im}] = [B_i]^T [D][B_m]\, tA$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 4 - Derive the Element Stiffness Matrix and Equations

Recall:

$$[B_i] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \gamma_i & \beta_i \end{bmatrix} \qquad [B_j] = \frac{1}{2A}\begin{bmatrix} \beta_j & 0 \\ 0 & \gamma_j \\ \gamma_j & \beta_j \end{bmatrix}$$

$$[B_m] = \frac{1}{2A}\begin{bmatrix} \beta_m & 0 \\ 0 & \gamma_m \\ \gamma_m & \beta_m \end{bmatrix}$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 5 - Assemble the Element Equations to Obtain the Global Equations and Introduce the Boundary Conditions

The global stiffness matrix can be found by the direct stiffness method.

$$[K] = \sum_{e=1}^{N}[k^{(e)}]$$

The global equivalent nodal load vector is obtained by lumping body forces and distributed loads at the appropriate nodes as well as including any concentrated loads.

$$\{F\} = \sum_{e=1}^{N}\{f^{(e)}\}$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 5 - Assemble the Element Equations to Obtain the Global Equations and Introduce the Boundary Conditions

The resulting global equations are: $\{F\} = [K]\{d\}$

where $\{d\}$ is the total structural displacement vector.

In the above formulation of the element stiffness matrix, the matrix has been derived for a general orientation in global coordinates.

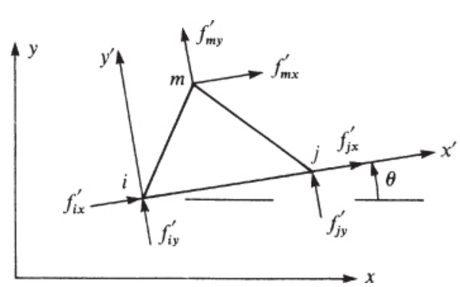
Therefore, no transformation form local to global coordinates is necessary.

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 5 - Assemble the Element Equations to Obtain the Global Equations and Introduce the Boundary Conditions

However, for completeness, we will now describe the method to use if the local axes for the constant-strain triangular element are not parallel to the global axes for the whole structure.

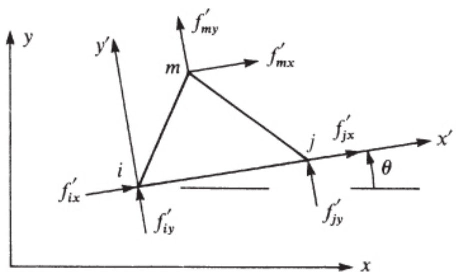


Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 5 - Assemble the Element Equations to Obtain the Global Equations and Introduce the Boundary Conditions

To relate the local to global displacements, force, and stiffness matrices we will use: $d' = Td \qquad f' = Tf \qquad k = T^T k' T$



Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 5 - Assemble the Element Equations to Obtain the Global Equations and Introduce the Boundary Conditions

The transformation matrix *T* for the triangular element is:

$$T = \left[\begin{array}{cc:cc:cc} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & C & S & 0 & 0 \\ 0 & 0 & -S & C & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & C & S \\ 0 & 0 & 0 & 0 & -S & C \end{array}\right] \qquad \begin{array}{l} C = \cos\theta \\ S = \sin\theta \end{array}$$

Plane Stress and Plane Strain Equations

Formulation of the Plane Triangular Element Equations

Step 6 - Solve for the Nodal Displacements

Step 7 - Solve for Element Forces and Stress

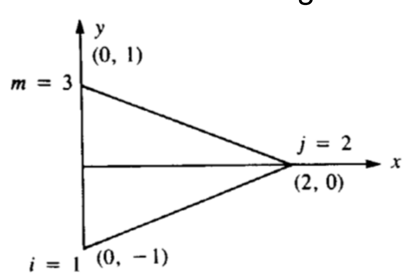
Having solved for the nodal displacements, we can obtain strains and stresses in *x* and *y* directions in the elements by using:

$$\{\varepsilon\} = [B]\{d\} \qquad \{\sigma\} = [D][B]\{d\}$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

Consider the structure shown in the figure below.



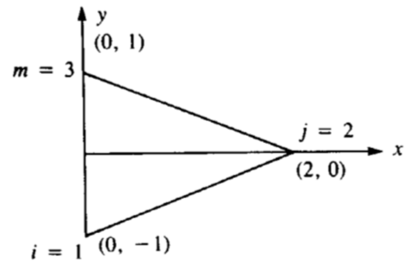
Let E = 30 x 10^6 *psi*, ν = 0.25, and t = 1 *in*.

Assume the element nodal displacements have been determined to be u_1 = 0.0, v_1 = 0.0025 *in*, u_2 = 0.0012 *in*, v_2 = 0.0, u_3 = 0.0, and v_3 = 0.0025 *in*.

Plane Stress and Plane Strain Equations

Plane Stress Example 1

First, we calculate the element β's and γ's as:



$$\beta_i = y_j - y_m = 0 - 1 = -1 \qquad \gamma_i = x_m - x_j = 0 - 2 = -2$$

$$\beta_j = y_m - y_i = 0 - (-1) = 2 \qquad \gamma_j = x_i - x_m = 0 - 0 = 0$$

$$\beta_m = y_i - y_j = -1 - 0 = -1 \qquad \gamma_m = x_j - x_i = 2 - 0 = 2$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

Therefore, the [B] matrix is:

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix} = \frac{1}{2(2)}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix}$$

$$\beta_i = y_j - y_m = 0 - 1 = -1 \qquad \gamma_i = x_m - x_j = 0 - 2 = -2$$

$$\beta_j = y_m - y_i = 0 - (-1) = 2 \qquad \gamma_j = x_i - x_m = 0 - 0 = 0$$

$$\beta_m = y_i - y_j = -1 - 0 = -1 \qquad \gamma_m = x_j - x_i = 2 - 0 = 2$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

For plane stress conditions, the [*D*] matrix is:

$$[D] = \frac{30 \times 10^{6}}{1-(0.25)^{2}} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

Substitute the above expressions for [*D*] and [*B*] into the general equations for the stiffness matrix:

$$[k] = tA\,[B]^{T}[D][B]$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

$$[k] = tA\,[B]^{T}[D][B]$$

$$\mathbf{k} = \frac{(2)30 \times 10^{6}}{4(0.9375)} \begin{bmatrix} -1 & 0 & -2 \\ 0 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \frac{1}{2(2)} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix}$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

Performing the matrix triple product gives:

$$\mathbf{k} = 4\times10^{6}\left[\begin{array}{cc:cc:cc} 2.5 & 1.25 & -2 & -1.5 & -0.5 & 0.25 \\ 1.25 & 4.375 & -1 & -0.75 & -0.25 & -3.625 \\ \hdashline -2 & -1 & 4 & 0 & -2 & 1 \\ -1.5 & -0.75 & 0 & 1.5 & 1.5 & -0.75 \\ \hdashline -0.5 & -0.25 & -2 & 1.5 & 2.5 & -1.25 \\ 0.25 & -3.625 & 1 & -0.75 & -1.25 & 4.375 \end{array}\right] {}^{lb}\!/\!_{in}$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

The in-plane stress can be related to displacements by:

$$\{\sigma\} = [D][B]\{d\}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{30\times10^{6}}{0.9375}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}\frac{1}{2(2)}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix}\begin{Bmatrix} 0.0 \\ 0.0025\ in \\ 0.0012\ in \\ 0.0 \\ 0.0 \\ 0.0025\ in \end{Bmatrix}$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

The stresses are:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} 19,200 \ psi \\ 4,800 \ psi \\ -15,000 \ psi \end{bmatrix}$$

Recall, the relationships for ***principal stresses*** and ***principal angle*** in two-dimensions are:

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}} = \sigma_{\max} \qquad \theta_p = \frac{1}{2}\tan^{-1}\left[\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right]$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}} = \sigma_{\min}$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

Therefore:

$$\sigma_1 = \frac{19,200 + 4,800}{2} + \sqrt{\left(\frac{19,200 - 4,800}{2}\right)^2 + \left(-15,000\right)^2} = 28,639 \ psi$$

$$\sigma_2 = \frac{19,200 + 4,800}{2} - \sqrt{\left(\frac{19,200 - 4,800}{2}\right)^2 + \left(-15,000\right)^2} = -4,639 \ psi$$

$$\theta_p = \frac{1}{2}\tan^{-1}\left[\frac{2(-15,000)}{19,200 - 4,800}\right] = -32.3^o$$

Plane Stress and Plane Strain Equations

Treatment of Body and Surface Forces

The general force vector is defined as:

$$\{f\} = \int_V [N]^T \{X\} dV + \{P\} + \int_S [N]^T \{T\} dS$$

Let's consider the first term of the above equation.

$$\{f_b\} = \int_V [N]^T \{X\} dV \qquad \{X\} = \begin{Bmatrix} X_b \\ Y_b \end{Bmatrix}$$

where X_b and Y_b are the weight densities in the x and y directions, respectively.

The force may reflect the effects of gravity, angular velocities, or dynamic inertial forces.

Plane Stress and Plane Strain Equations

Treatment of Body and Surface Forces

For a given thickness, t, the body force term becomes:

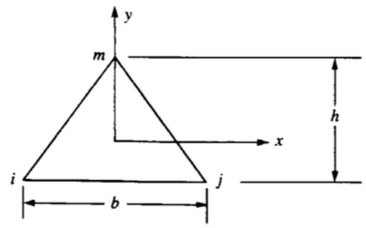
$$\{f_b\} = \int_V [N]^T \{X\} dV = t \int_A [N]^T \{X\} dA$$

$$[N]^T = \begin{bmatrix} N_i & 0 \\ 0 & N_i \\ N_j & 0 \\ 0 & N_j \\ N_m & 0 \\ 0 & N_m \end{bmatrix} \qquad \{X\} = \begin{Bmatrix} X_b \\ Y_b \end{Bmatrix}$$

Plane Stress and Plane Strain Equations

Treatment of Body and Surface Forces

The integration of the $\{f_b\}$ is simplified if the origin of the coordinate system is chosen at the centroid of the element, as shown in the figure below.



With the origin placed at the centroid, we can use the definition of a centroid.

$$\int_A y \, dA = 0 \qquad \int_A x \, dA = 0$$

Plane Stress and Plane Strain Equations

Treatment of Body and Surface Forces

Recall the interpolation functions for a plane stress/strain triangle:

$$N_i = \frac{1}{2A}\left(\alpha_i + \beta_i x + \gamma_i y\right) \qquad N_j = \frac{1}{2A}\left(\alpha_j + \beta_j x + \gamma_j y\right)$$

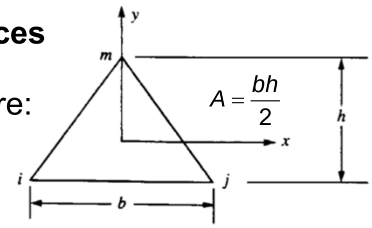
$$N_m = \frac{1}{2A}\left(\alpha_m + \beta_m x + \gamma_m y\right)$$

With the origin placed at the centroid, we can use the definition of a centroid.

$$\int_A y \, dA = 0 \qquad \int_A x \, dA = 0$$

Plane Stress and Plane Strain Equations

Treatment of Body and Surface Forces



Therefore the terms in the integrand are:

$$\int_A \beta_i x \, dA = 0 \qquad \int_A \gamma_i y \, dA = 0$$

$$\alpha_i = x_j y_m - y_j x_m = \left(\frac{b}{2}\right)\left(\frac{2h}{3}\right) - \left(-\frac{h}{3}\right)(0) = \frac{bh}{3}$$

$$\alpha_j = x_m y_i - y_m x_i = (0)\left(-\frac{h}{3}\right) - \left(-\frac{b}{2}\right)\left(\frac{2h}{3}\right) = \frac{bh}{3}$$

$$\alpha_m = x_i y_j - y_i x_j = \left(-\frac{b}{2}\right)\left(-\frac{h}{3}\right) - \left(\frac{b}{2}\right)\left(-\frac{h}{3}\right) = \frac{bh}{3}$$

$$\alpha_i = \alpha_j = \alpha_m = \frac{2A}{3} \qquad t\int_A \frac{1}{2A}\alpha_i dA = t\int_A \frac{1}{3} dA = \frac{tA}{3}$$

Plane Stress and Plane Strain Equations

Treatment of Body and Surface Forces

Therefore the terms in the integrand are:

$$\{f_b\} = \int_V [N]^T \{X\} dV = t\int_A [N]^T \{X\} dA$$

The body force at node *i* is given as:

$$\{f_{bi}\} = \frac{tA}{3}\begin{Bmatrix} X_b \\ Y_b \end{Bmatrix}$$

The general body force vector is:

$$\{f_b\} = \begin{Bmatrix} f_{bix} \\ f_{biy} \\ f_{bjx} \\ f_{bjy} \\ f_{bmx} \\ f_{bmy} \end{Bmatrix} = \frac{tA}{3}\begin{Bmatrix} X_b \\ Y_b \\ X_b \\ Y_b \\ X_b \\ Y_b \end{Bmatrix}$$

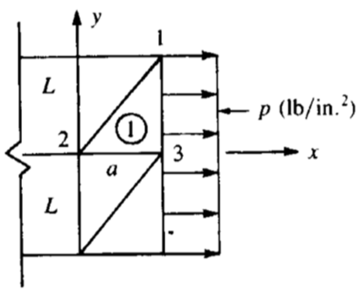
Plane Stress and Plane Strain Equations

Treatment of Body and Surface Forces

The third term in the general force vector is defined as:

$$\{f_s\} = \int_S [N]^T \{T\} dS$$

Let's consider the example of a uniform stress p acting between nodes 1 and 3 on the edge of element 1 as shown in figure below.



$$\{T\} = \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} = \begin{Bmatrix} p \\ 0 \end{Bmatrix}$$

Plane Stress and Plane Strain Equations

Treatment of Body and Surface Forces

The third term in the general force vector is defined as:

$$\{f_s\} = \int_S [N]^T \{T\} dS$$

$$[N]^T = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \qquad \{T\} = \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} = \begin{Bmatrix} p \\ 0 \end{Bmatrix}$$

evaluated at $x = a$

Plane Stress and Plane Strain Equations

Treatment of Body and Surface Forces

Therefore, the traction force vector is:

$$\{f_s\} = \int_S [N]^T \{T\} dS$$

$$\{f_s\} = \int_0^t \int_0^L \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix}_{x=a} \begin{Bmatrix} p \\ 0 \end{Bmatrix} dy\, dz \quad = t \int_0^L \begin{bmatrix} N_1 p \\ 0 \\ N_2 p \\ 0 \\ N_3 p \\ 0 \end{bmatrix}_{x=a} dy$$

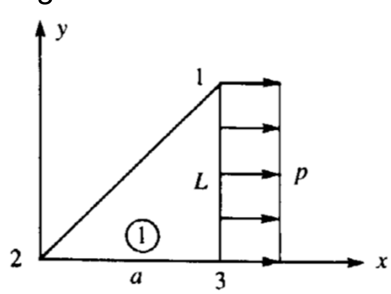
Plane Stress and Plane Strain Equations

Treatment of Body and Surface Forces

The interpolation function for i = 1 is:

$$N_i = \frac{1}{2A}\left(\alpha_i + \beta_i x + \gamma_i y\right)$$

For convenience, let's choose the coordinate system shown in the figure below.



$$\alpha_i = x_j y_m - y_j x_m$$

with i = 1, j = 2, and m = 3

$$\alpha_1 = x_2 y_3 - y_2 x_3$$

$$= (0)(0) - (0)(a) = 0$$

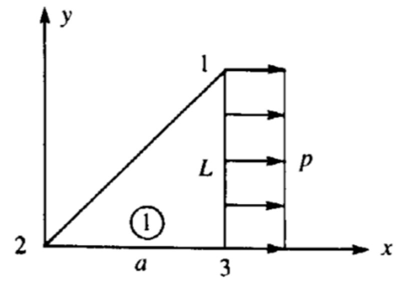
Plane Stress and Plane Strain Equations

Treatment of Body and Surface Forces

The interpolation function for i = 1 is:

$$N_i = \frac{1}{2A}\left(\alpha_i + \beta_i x + \gamma_i y\right)$$

For convenience, let's choose the coordinate system shown in the figure below.



Similarly, we can find:

$$\beta_1 = 0 \qquad \gamma_1 = a$$

$$N_1 = \frac{ay}{2A}$$

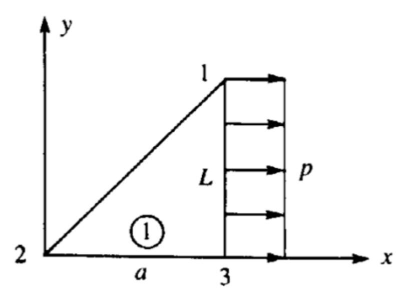
Plane Stress and Plane Strain Equations

Treatment of Body and Surface Forces

The remaining interpolation function, N_2 and N_3 are:

$$N_1 = \frac{ay}{2A} \qquad N_2 = \frac{L(a-x)}{2A} \qquad N_3 = \frac{Lx-ay}{2A}$$

Evaluating N_i along the 1-3 edge of the element (x = a) gives:



$$N_1 = \frac{ay}{2A}$$

$$N_2 = 0$$

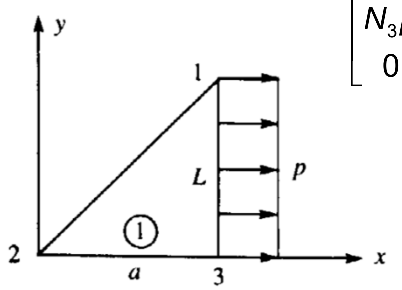
$$N_3 = \frac{a\left(L-y\right)}{2A}$$

Plane Stress and Plane Strain Equations

Treatment of Body and Surface Forces

Substituting the interpolation function in the traction force vector expression gives:

$$\{f_s\} = \int_S [N]^T \{T\} dS = t\int_0^L \begin{bmatrix} N_1 p \\ 0 \\ N_2 p \\ 0 \\ N_3 p \\ 0 \end{bmatrix} dy = \frac{atp}{2A}\int_0^L \begin{bmatrix} y \\ 0 \\ 0 \\ 0 \\ L-y \\ 0 \end{bmatrix} dy$$



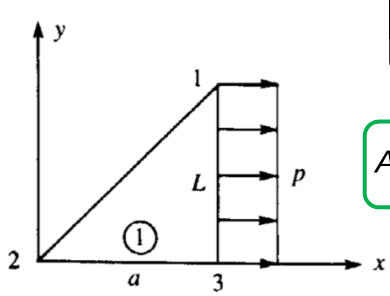
Plane Stress and Plane Strain Equations

Treatment of Body and Surface Forces

Therefore, the traction force vector is:

$$\{f_s\} = \int_S [N]^T \{T\} dS = \begin{Bmatrix} f_{s1x} \\ f_{s1y} \\ f_{s2x} \\ f_{s2y} \\ f_{s3x} \\ f_{s3y} \end{Bmatrix} = \frac{atp}{4A} \begin{bmatrix} L^2 \\ 0 \\ 0 \\ 0 \\ L^2 \\ 0 \end{bmatrix} = \frac{pLt}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

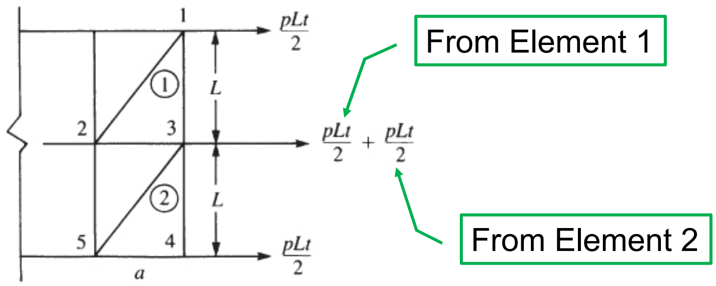
$$\boxed{A = \frac{aL}{2}}$$



Plane Stress and Plane Strain Equations

Treatment of Body and Surface Forces

The figure below shows the results of the surface load equivalent nodal for both elements 1 and 2:



Plane Stress and Plane Strain Equations

Treatment of Body and Surface Forces

For the CS triangle, a distributed load on the element edge can be treated as concentrated loads acting at the nodes associated with the loaded edge.

However, for higher-order elements, like the linear strain triangle (discussed in Chapter 8), load replacement should be made by using the principle of minimum potential energy.

For higher-order elements, load replacement by potential energy is not equivalent to the apparent statically equivalent one.

Plane Stress and Plane Strain Equations

Explicit Expression for the Constant-Strain Triangle Stiffness Matrix

Usually the stiffness matrix is computed internally by computer programs, but since we are not computers, we need to explicitly evaluate the stiffness matrix.

For a constant-strain triangular element, considering the plane strain case, recall that: $[k] = tA[B]^T[D][B]$

where [D] for plane strain is:

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix}$$

Plane Stress and Plane Strain Equations

Explicit Expression for the Constant-Strain Triangle Stiffness Matrix

Substituting the appropriate definition into the above triple product gives:

$$[k] = tA[B]^T[D][B]$$

$$[k] = \frac{tE}{4A(1+\nu)(1-2\nu)} \begin{bmatrix} \beta_i & 0 & \gamma_i \\ 0 & \gamma_i & \beta_i \\ \beta_j & 0 & \gamma_j \\ 0 & \gamma_j & \beta_j \\ \beta_m & 0 & \gamma_m \\ 0 & \gamma_m & \beta_m \end{bmatrix} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix} \begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

Plane Stress and Plane Strain Equations

Explicit Expression for the Constant-Strain Triangle Stiffness Matrix

Substituting the appropriate definition into the above triple product gives:

$$\underline{k} = \frac{tE}{4A(1+\nu)(1-2\nu)}$$

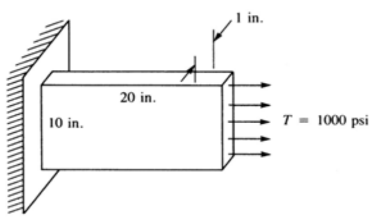
$$\times \begin{bmatrix} \beta_i^2(1-\nu)+\gamma_i^2\left(\frac{1-2\nu}{2}\right) & \beta_i\gamma_i\nu+\beta_i\gamma_i\left(\frac{1-2\nu}{2}\right) & \beta_i\beta_j(1-\nu)+\gamma_i\gamma_j\left(\frac{1-2\nu}{2}\right) & \beta_i\gamma_j\nu+\beta_j\gamma_i\left(\frac{1-2\nu}{2}\right) & \beta_i\beta_m(1-\nu)+\gamma_i\gamma_m\left(\frac{1-2\nu}{2}\right) & \beta_i\gamma_m\nu+\beta_m\gamma_i\left(\frac{1-2\nu}{2}\right) \\ & \gamma_i^2(1-\nu)+\beta_i^2\left(\frac{1-2\nu}{2}\right) & \beta_j\gamma_i\nu+\beta_i\gamma_j\left(\frac{1-2\nu}{2}\right) & \gamma_i\gamma_j(1-\nu)+\beta_i\beta_j\left(\frac{1-2\nu}{2}\right) & \beta_m\gamma_i\nu+\beta_i\gamma_m\left(\frac{1-2\nu}{2}\right) & \gamma_i\gamma_m(1-\nu)+\beta_i\beta_m\left(\frac{1-2\nu}{2}\right) \\ & & \beta_j^2(1-\nu)+\gamma_j^2\left(\frac{1-2\nu}{2}\right) & \beta_j\gamma_j\nu+\beta_j\gamma_j\left(\frac{1-2\nu}{2}\right) & \beta_j\beta_m(1-\nu)+\gamma_j\gamma_m\left(\frac{1-2\nu}{2}\right) & \beta_j\gamma_m\nu+\gamma_j\beta_m\left(\frac{1-2\nu}{2}\right) \\ & & & \gamma_j^2(1-\nu)+\beta_j^2\left(\frac{1-2\nu}{2}\right) & \beta_m\gamma_j\nu+\beta_j\gamma_m\left(\frac{1-2\nu}{2}\right) & \gamma_j\gamma_m(1-\nu)+\beta_j\beta_m\left(\frac{1-2\nu}{2}\right) \\ & & & & \beta_m^2(1-\nu)+\gamma_m^2\left(\frac{1-2\nu}{2}\right) & \gamma_m\beta_m\nu+\beta_m\gamma_m\left(\frac{1-2\nu}{2}\right) \\ \text{Symmetry} & & & & & \gamma_m^2(1-\nu)+\beta_m^2\left(\frac{1-2\nu}{2}\right) \end{bmatrix}$$

The stiffness matrix is a function of the global coordinates *x* and *y*, the material properties, and the thickness and area of the element.

Plane Stress and Plane Strain Equations

Plane Stress Problem 2

Consider the thin plate subjected to the surface traction shown in the figure below.



Assume plane stress conditions. Let E = 30 x 10^6 *psi*, ν = 0.30, and t = 1 *in*.

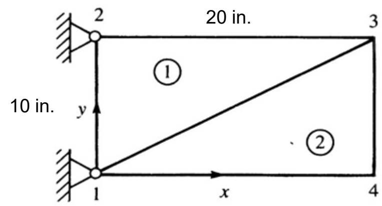
Determine the nodal displacements and the element stresses.

Plane Stress and Plane Strain Equations

Plane Stress Problem 2

Discretization

Let's discretize the plate into two elements as shown below:



This level of discretization will probably not yield practical results for displacement and stresses; however, it is useful example for a longhand solution.

Plane Stress and Plane Strain Equations

Plane Stress Problem 2

Discretization

For element 2, The tensile traction forces can be converted into nodal forces as follows:

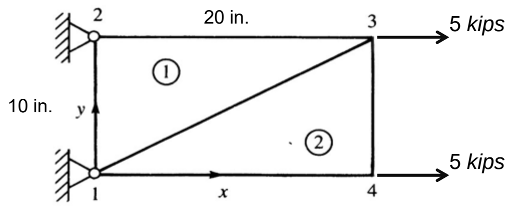
$$\{f_s\}_2 = \begin{Bmatrix} f_{s3x} \\ f_{s3y} \\ f_{s1x} \\ f_{s1y} \\ f_{s4x} \\ f_{s4y} \end{Bmatrix} = \frac{pLt}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1{,}000\ psi(10\ in)1\ in}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5{,}000\ lb \\ 0 \\ 0 \\ 0 \\ 5{,}000\ lb \\ 0 \end{bmatrix}$$

Plane Stress and Plane Strain Equations

Plane Stress Problem 2

Discretization

For element 2, The tensile traction forces can be converted into nodal forces as follows:



$$F = \sigma A = 1{,}000 psi\,(10\,in)(1\,in)$$
$$= 10{,}000\,lb$$

Plane Stress and Plane Strain Equations

Plane Stress Problem 2

The governing global matrix equations are: $\{F\} = [K]\{d\}$

Expanding the above matrices gives:

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{Bmatrix} = \begin{Bmatrix} R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \\ 5{,}000\ lb \\ 0 \\ 5{,}000\ lb \\ 0 \end{Bmatrix} = [K]\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix} = [K]\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix}$$

Plane Stress and Plane Strain Equations

Plane Stress Problem 2

Assemblage of the Stiffness Matrix

The global stiffness matrix is assembled by superposition of the individual element stiffness matrices.

The element stiffness matrix is: $[k] = tA[B]^T[D][B]$

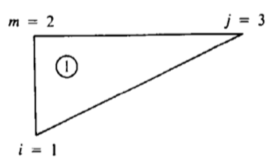
$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix}$$

Plane Stress and Plane Strain Equations

Plane Stress Problem 2

For **element 1**: the coordinates are x_i = 0, y_i = 0, x_j = 20, y_j = 10, x_m = 0, and y_m = 10. The area of the triangle is:



$$A = \frac{bh}{2} = \frac{(20)(10)}{2} = 100\ in.^2$$

$$\beta_i = y_j - y_m = 10 - 10 = 0 \qquad \gamma_i = x_m - x_j = 0 - 20 = -20$$

$$\beta_j = y_m - y_1 = 10 - 0 = 10 \qquad \gamma_j = x_i - x_m = 0 - 0 = 0$$

$$\beta_m = y_i - y_j = 0 - 10 = -10 \qquad \gamma_m = x_i - x_j = 20 - 0 = 20$$

Plane Stress and Plane Strain Equations

Plane Stress Problem 2

Therefore, the [*B*] matrix is:

$$[B]=\frac{1}{200}\begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -20 & 0 & 0 & 0 & 20 \\ -20 & 0 & 0 & 10 & 20 & -10 \end{bmatrix} \tfrac{1}{in}$$

$$\beta_i = y_j - y_m = 10 - 10 = 0 \qquad \gamma_i = x_m - x_j = 0 - 20 = -20$$

$$\beta_j = y_m - y_1 = 10 - 0 = 10 \qquad \gamma_j = x_i - x_m = 0 - 0 = 0$$

$$\beta_m = y_i - y_j = 0 - 10 = -10 \qquad \gamma_m = x_i - x_j = 20 - 0 = 20$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

For plane stress conditions, the [*D*] matrix is:

$$[D]=\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix} = \frac{30\times 10^6}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} psi$$

Substitute the above expressions for [*D*] and [*B*] into the general equations for the stiffness matrix:

$$[k] = tA\,[B]^T [D][B]$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

Therefore:

$$[B]^T[D] = \frac{30(10^6)}{200(0.91)}\begin{bmatrix} 0 & 0 & -20 \\ 0 & -20 & 0 \\ 10 & 0 & 0 \\ 0 & 0 & 10 \\ -10 & 0 & 20 \\ 0 & 20 & -10 \end{bmatrix}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} {}^{lb}\!/_{in^3}$$

$$[B]^T[D] = \frac{30(10^6)}{200(0.91)}\begin{bmatrix} 0 & 0 & -7 \\ -6 & -20 & 0 \\ 10 & 3 & 0 \\ 0 & 0 & 3.5 \\ -10 & -3 & 7 \\ 6 & 20 & -3.5 \end{bmatrix} {}^{lb}\!/_{in^3}$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

$$[k] = tA\,[B]^T[D][B]$$

$$1(100)\frac{(0.15)(10^6)}{0.91}\begin{bmatrix} 0 & 0 & -7 \\ -6 & -20 & 0 \\ 10 & 3 & 0 \\ 0 & 0 & 3.5 \\ -10 & -3 & 7 \\ 6 & 20 & -3.5 \end{bmatrix} \times \frac{1}{200}\begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -20 & 0 & 0 & 0 & 20 \\ -20 & 0 & 0 & 10 & 20 & -10 \end{bmatrix}$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

Simplifying the above expression gives:

$$
[k^{(1)}] = \frac{75{,}000}{0.91}
\begin{array}{c}
\begin{array}{cccccc} u_1 & v_1 & u_3 & v_3 & u_2 & v_2 \end{array} \\
\left[\begin{array}{rr|rr|rr}
140 & 0 & 0 & -70 & -140 & 70 \\
0 & 400 & -60 & 0 & 60 & -400 \\
\hline
0 & -60 & 100 & 0 & -100 & 60 \\
-70 & 0 & 0 & 35 & 70 & -35 \\
\hline
-140 & 60 & -100 & 70 & 240 & -130 \\
70 & -400 & 60 & -35 & -130 & 435
\end{array}\right]
\end{array}
{}^{lb}\!/_{in}
$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

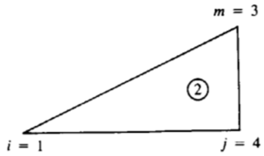
Rearranging the rows and columns gives:

$$
[k^{(1)}] = \frac{75{,}000}{0.91}
\begin{array}{c}
\begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{array} \\
\left[\begin{array}{rr|rr|rr}
140 & 0 & -140 & 70 & 0 & -70 \\
0 & 400 & 60 & -400 & -60 & -0 \\
\hline
-140 & 60 & 240 & -130 & -100 & 70 \\
70 & -400 & -130 & 435 & 60 & -35 \\
\hline
0 & -60 & -100 & 60 & 100 & 0 \\
-70 & 0 & 70 & -35 & 0 & 35
\end{array}\right]
\end{array}
{}^{lb}\!/_{in}
$$

Plane Stress and Plane Strain Equations

Plane Stress Problem 2

For **element 2**: the coordinates are $x_i = 0$, $y_i = 0$, $x_j = 20$, $y_j = 0$, $x_m = 20$, and $y_m = 10$. The area of the triangle is:



$$A = \frac{bh}{2} = \frac{(20)(10)}{2} = 100\ in.^2$$

$$\beta_i = y_j - y_m = 0 - 10 = -10 \qquad \gamma_i = x_m - x_j = 20 - 20 = 0$$

$$\beta_j = y_m - y_1 = 10 - 0 = 10 \qquad \gamma_j = x_i - x_m = 0 - 20 = -20$$

$$\beta_m = y_i - y_j = 0 - 0 = 0 \qquad \gamma_m = x_i - x_j = 20 - 0 = 20$$

Plane Stress and Plane Strain Equations

Plane Stress Problem 2

Therefore, the [*B*] matrix is:

$$[B] = \frac{1}{200}\left[\begin{array}{cc|cc|cc} -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -10 & -20 & 10 & 20 & 0 \end{array}\right] \frac{1}{in}$$

$$\beta_i = y_j - y_m = 0 - 10 = -10 \qquad \gamma_i = x_m - x_j = 20 - 20 = 0$$

$$\beta_j = y_m - y_1 = 10 - 0 = 10 \qquad \gamma_j = x_i - x_m = 0 - 20 = -20$$

$$\beta_m = y_i - y_j = 0 - 0 = 0 \qquad \gamma_m = x_i - x_j = 20 - 0 = 20$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

For plane stress conditions, the [*D*] matrix is:

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix} = \frac{30\times 10^6}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} psi$$

Substitute the above expressions for [*D*] and [*B*] into the general equations for the stiffness matrix:

$$[k] = tA\,[B]^T[D][B]$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

Therefore:

$$[B]^T[D] = \frac{30(10^6)}{200(0.91)}\begin{bmatrix} -10 & 0 & 0 \\ 0 & 0 & -10 \\ \hdashline 10 & 0 & -20 \\ 0 & -20 & 10 \\ \hdashline 0 & 0 & 20 \\ 0 & 20 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$[B]^T[D] = \frac{30(10^6)}{200(0.91)}\begin{bmatrix} -10 & -3 & 0 \\ 0 & 0 & -3.5 \\ 10 & 3 & -7 \\ 0 & -20 & 3.5 \\ -6 & 0 & 7 \\ 6 & 20 & 0 \end{bmatrix}$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

$$[k] = tA\,[B]^T[D][B]$$

$$1(100)\frac{(0.15)(10^6)}{0.91}\begin{bmatrix} -10 & -3 & 0 \\ 0 & 0 & -3.5 \\ 10 & 3 & -7 \\ 0 & -20 & 3.5 \\ -6 & 0 & 7 \\ 6 & 20 & 0 \end{bmatrix} \times \frac{1}{200}\left[\begin{array}{cc|cc|cc} -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -10 & -20 & 10 & 20 & 0 \end{array}\right]$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

Simplifying the above expression gives:

$$[k^{(2)}] = \frac{75{,}000}{0.91}\begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_4 & v_4 & u_3 & v_3 \end{array} \\ \left[\begin{array}{cc|cc|cc} 100 & 0 & -100 & 60 & 0 & -60 \\ 0 & 35 & 70 & -35 & -70 & 0 \\ \hline -100 & 70 & 240 & -130 & -140 & 60 \\ 60 & -35 & -130 & 435 & 70 & -400 \\ \hline 0 & -70 & -140 & 70 & 140 & 0 \\ -60 & 0 & 60 & -400 & 0 & 400 \end{array}\right] \end{array}$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

Rearranging the rows and columns gives:

$$[k^{(2)}] = \frac{75,000}{0.91} \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_3 & v_3 & u_4 & v_4 \end{array} \\ \left[\begin{array}{cc:cc:cc} 100 & 0 & 0 & -60 & -100 & 60 \\ 0 & 35 & -70 & 0 & 70 & -35 \\ \hdashline 0 & -70 & 140 & 0 & -140 & 70 \\ -60 & 0 & 0 & 400 & 60 & -400 \\ \hdashline -100 & 70 & -140 & 60 & 240 & -130 \\ 60 & -35 & 70 & -400 & -130 & 435 \end{array}\right] \end{array}$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

In expanded form, element 1 is:

$$[k^{(1)}] = \frac{375,000}{0.91} \begin{array}{c} \begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\ \left[\begin{array}{cccccccc} 28 & 0 & -28 & 14 & 0 & -14 & 0 & 0 \\ 0 & 80 & 12 & -80 & -12 & 0 & 0 & 0 \\ -28 & 12 & 48 & -26 & -20 & 14 & 0 & 0 \\ 14 & -80 & -26 & 87 & 12 & -7 & 0 & 0 \\ 0 & -12 & -20 & 12 & 20 & 0 & 0 & 0 \\ -14 & 0 & 14 & -7 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

In expanded form, element 2 is:

$$
\begin{array}{c}
\qquad\qquad\quad u_1 \quad\;\; v_1 \quad\;\; u_2 \quad\;\; v_2 \quad\;\; u_3 \quad\;\; v_3 \quad\;\; u_4 \quad\;\; v_4 \\
[k^{(2)}] = \dfrac{375{,}000}{0.91}\begin{bmatrix}
20 & 0 & 0 & 0 & 0 & -12 & -20 & 12 \\
0 & 7 & 0 & 0 & -14 & 0 & 14 & -7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -14 & 0 & 0 & 28 & 0 & -28 & 14 \\
-12 & 0 & 0 & 0 & 0 & 80 & 12 & -80 \\
-20 & 14 & 0 & 0 & -28 & 12 & 48 & -26 \\
12 & -7 & 0 & 0 & 14 & -80 & -26 & 87
\end{bmatrix}
\end{array}
$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

Using the superposition, the total global stiffness matrix is:

$$
\begin{array}{c}
\qquad\qquad\quad u_1 \quad\;\; v_1 \quad\;\; u_2 \quad\;\; v_2 \quad\;\; u_3 \quad\;\; v_3 \quad\;\; u_4 \quad\;\; v_4 \\
[k] = \dfrac{375{,}000}{0.91}\begin{bmatrix}
48 & 0 & -28 & 14 & 0 & -26 & -20 & 12 \\
0 & 87 & 12 & -80 & -26 & 0 & 14 & -7 \\
-28 & 12 & 48 & -26 & -20 & 14 & 0 & 0 \\
14 & 80 & -26 & 87 & 12 & -7 & 0 & 0 \\
0 & -26 & -20 & 12 & 48 & 0 & -28 & 14 \\
-26 & 0 & 14 & -7 & 0 & 87 & 12 & -80 \\
-20 & 14 & 0 & 0 & -28 & 12 & 48 & -26 \\
12 & -7 & 0 & 0 & 14 & -80 & -26 & 87
\end{bmatrix}
\end{array}
$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

Applying the boundary conditions: $d_{1x} = d_{1y} = d_{2x} = d_{2y} = 0$

$$\begin{Bmatrix} R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \\ 5{,}000\ lb \\ 0 \\ 500\ lb \\ 0 \end{Bmatrix} = \frac{375{,}000}{0.91} \begin{bmatrix} 48 & 0 & -28 & 14 & 0 & -26 & -20 & 12 \\ 0 & 87 & 12 & -80 & -26 & 0 & 14 & -7 \\ -28 & 12 & 48 & -26 & -20 & 14 & 0 & 0 \\ 14 & 80 & -26 & 87 & 12 & -7 & 0 & 0 \\ 0 & -26 & -20 & 12 & 48 & 0 & -28 & 14 \\ -26 & 0 & 14 & -7 & 0 & 87 & 12 & -80 \\ -20 & 14 & 0 & 0 & -28 & 12 & 48 & -26 \\ 12 & -7 & 0 & 0 & 14 & -80 & -26 & 87 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix}$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

The governing equations are:

$$\begin{Bmatrix} 5{,}000\ lb \\ 0 \\ 5{,}000\ lb \\ 0 \end{Bmatrix} = \frac{375{,}000}{0.91} \begin{bmatrix} 48 & 0 & -28 & 14 \\ 0 & 87 & 12 & -80 \\ -28 & 12 & 48 & -26 \\ 14 & -80 & -26 & 87 \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix}$$

Solving the equations gives:

$$\begin{Bmatrix} d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix} = \left(10^{-6}\right) \begin{Bmatrix} 609.6 \\ 4.2 \\ 663.7 \\ 104.1 \end{Bmatrix} in$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

The exact solution for the displacement at the free end of the one-dimensional bar subjected to a tensile force is:

$$\delta = \frac{PL}{AE} = \frac{(10{,}000)20}{10(30\times10^{6})} = 670\times10^{-6}\ in$$

The two-element FEM solution is:

$$\begin{Bmatrix} d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix} = \left(10^{-6}\right)\begin{Bmatrix} 609.6 \\ 4.2 \\ 663.7 \\ 104.1 \end{Bmatrix} in$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

The in-plane stress can be related to displacements by:

$$\{\sigma\} = [D][B]\{d\}$$

$$\{\sigma\} = \frac{E}{2A(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}\begin{Bmatrix} d_{ix} \\ d_{iy} \\ d_{jx} \\ d_{jy} \\ d_{mx} \\ d_{my} \end{Bmatrix}$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

Element 1: $\{\sigma\} = [D][B]\{d\}$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{30(10^6)(10^{-6})}{0.96(200)} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -20 & 0 & 0 & 0 & 20 \\ -20 & 0 & 0 & 10 & 20 & -10 \end{bmatrix} \begin{Bmatrix} 0.0 \\ 0.0 \\ 609.6 \\ 4.2 \\ 0.0 \\ 0.0 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} 1{,}005 \; psi \\ 301 \; psi \\ 2.4 \; psi \end{bmatrix}$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

Element 2: $\{\sigma\} = [D][B]\{d\}$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{30(10^6)(10^{-6})}{0.96(200)} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \begin{bmatrix} 10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & 10 & -20 & 10 & 20 & 0 \end{bmatrix} \begin{Bmatrix} 0.0 \\ 0.0 \\ 663.7 \\ 104.1 \\ 609.6 \\ 4.2 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} 995 \; psi \\ -1.2 \; psi \\ -2.4 \; psi \end{bmatrix}$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

The ***principal*** stresses and principal angle in element 1 are:

$$\sigma_1 = \frac{1005-301}{2} + \sqrt{\left(\frac{1005+301}{2}\right)^2 + (2.4)^2} = 1{,}005\ psi$$

$$\sigma_2 = \frac{1005-301}{2} - \sqrt{\left(\frac{1005+301}{2}\right)^2 + (2.4)^2} = 301\ psi$$

$$\theta_p = \frac{1}{2} tan^{-1}\left[\frac{2(2.4)}{1005+301}\right] \approx 0^o$$

Plane Stress and Plane Strain Equations

Plane Stress Example 1

The ***principal*** stresses and principal angle in element 2 are:

$$\sigma_1 = \frac{995-1.2}{2} + \sqrt{\left(\frac{995+1.2}{2}\right)^2 + (-2.4)^2} = 995\ psi$$

$$\sigma_2 = \frac{995-1.2}{2} - \sqrt{\left(\frac{995+1.2}{2}\right)^2 + (-2.4)^2} = -1.1\ psi$$

$$\theta_p = \frac{1}{2} tan^{-1}\left[\frac{2(-2.4)}{995+1.2}\right] \approx 0^o$$

Plane Stress and Plane Strain Equations

Problems

12. Do problems ***6.6a, 6.6c, 6.7, 6.10a-c, 6.11,*** and ***6.13***.

13. Rework the plane stress problem given on page 356 in your textbook using Matlab code **FEM_2Dor3D_linelast_standard** to do analysis.

    Start with the simple two element model. Continuously refine your discretization by a factor of two each time until your FEM solution is in agreement with the exact solution.

    How many elements did you need?

Plane Stress and Plane Strain Equations

FEM_2Dor3D_linelast_standard

```
function FEM_2Dor3D_linelast_standard
%
%          Example 2D and 3D Linear elastic FEM code
%            Currently coded to run either plane strain or plane stress (2DOF) or general 3D but
%            could easily be modified for axisymmetry too.
%
%        Variables read from input file;
%        nprops              No. material parameters
%        materialprops(i)    List of material parameters
%        ncoord              No. spatial coords (2 for 2D, 3 for 3D)
%        ndof                No. degrees of freedom per node (2 for 2D, 3 for 3D)
%                            (here ndof=ncoord, but the program allows them to be different
%                            to allow extension to plate & beam elements with C^1 continuity)
%        nnode               No. nodes
%        coords(i,j)         ith coord of jth node, for i=1..ncoord; j=1..nnode
%        nelem               No. elements
%        maxnodes            Max no. nodes on any one element (used for array dimensioning)
%        nelnodes(i)         No. nodes on the ith element
%        elident(i)          An integer identifier for the ith element.  Not used
%                            in this code but could be used to switch on reduced integration,
%                            etc.
%        connect(i,j)        List of nodes on the jth element
%        nfix                Total no. prescribed displacements
%        fixnodes(i,j)       List of prescribed displacements at nodes
%                            fixnodes(1,j) Node number
%                            fixnodes(2,j) Displacement component number (1, 2 or 3)
%                            fixnodes(3,j) Value of the displacement
%        ndload              Total no. element faces subjected to tractions
%        dloads(i,j)         List of element tractions
%                            dloads(1,j) Element number
%                            dloads(2,j) face number
%                            dloads(3,j), dloads(4,j), dloads(5,j) Components of traction
%                            (assumed uniform)
%
```

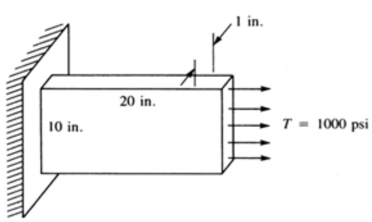
Plane Stress and Plane Strain Equations

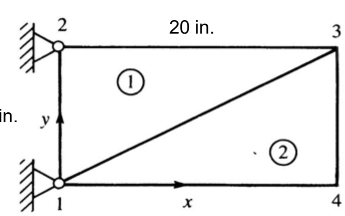
FEM_2Dor3D_linelast_standard

```
%  To run the program you first need to set up an input file, as described in
%  the lecture notes.  Then change the fopen command below to point to the file.
%  Also change the fopen command in the post-processing step (near the bottom of the
%  program) to point to a suitable output file.  Then execute the file in
%  the usual fashion (e.g. hit the green arrow at the top of the MATLAB
%  editor window)
%
%
% =================== Read data from the input file ==========================
%
%
% YOU NEED TO CHANGE THE PATH & FILE NAME TO POINT TO YOUR INPUT FILE
%
infile=fopen ('Logan_p364_2_element.txt','r');
outfile=fopen('Logan_p364_2_element_results.txt','w');
```

Plane Stress and Plane Strain Equations

FEM_2Dor3D_linelast_standard





Logan_p364_2_element.txt

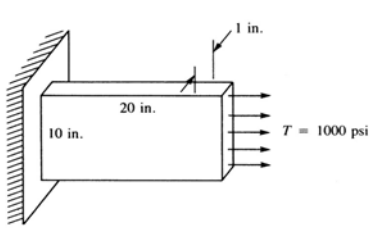
$$G = \frac{E}{2(1+\nu)}$$

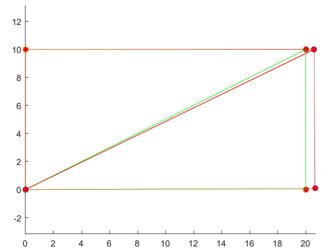
```
No._material_props:    3
    Shear_modulus:   11.5385
    Poissons_ratio:  0.3
    Plane_strain/stress: 0
No._coords_per_node:   2
No._DOF_per_node:      2
No._nodes:             4
Nodal_coords:
    0.0   0.0
    0.0  10.0
   20.0  10.0
   20.0   0.0
No._elements:                        2
Max_no._nodes_on_any_one_element:    3
element_identifier; no._nodes_on_element; connectivity:
    1  3  1 3 2
    2  3  1 4 3
No._nodes_with_prescribed_DOFs:  4
Node_#, DOF#, Value:
   1 1 0.0
   1 2 0.0
   2 1 0.0
   2 2 0.0
No._elements_with_prescribed_loads: 1
Element_#, Face_#, Traction_components
  2 2 1.0 0.0
```

In order to see the displacements, G is divided by 10^6 and the forces are divided by 10^3.

Plane Stress and Plane Strain Equations

FEM_2Dor3D_linelast_standard

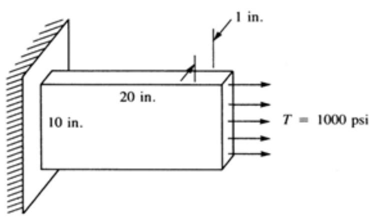


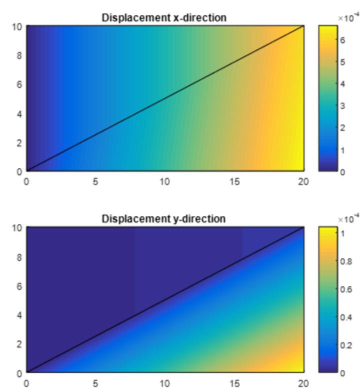


```
Nodal Displacements:
 Node       Coords          u1       u2
  1    0.0000    0.0000  -0.0000  -0.0000
  2    0.0000   10.0000  -0.0000   0.0000
  3   20.0000   10.0000   0.6096   0.0042
  4   20.0000    0.0000   0.6637   0.1041
 Strains and Stresses
 Element; 1
 int pt     Coords          e_11       e_22       e_12       s_11       s_22       s_12
     1  6.6667   6.6667    0.0305     0.0000     0.0001     1.0048     0.3014     0.0024
Element; 2
 int pt     Coords          e_11       e_22       e_12       s_11       s_22       s_12
     1 13.3333   3.3333    0.0332    -0.0100    -0.0001     0.9952    -0.0012    -0.0024
```

Plane Stress and Plane Strain Equations

FEM_2Dor3D_linelast_standard



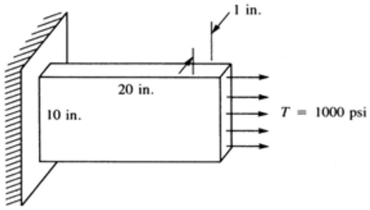


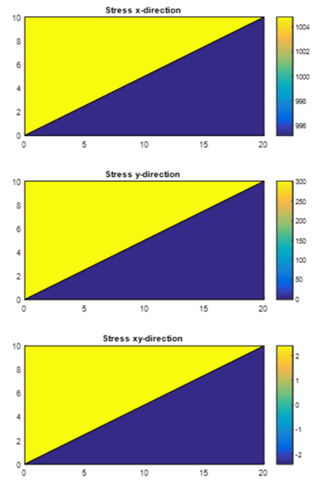

```
Nodal Displacements:
Node       Coords          u1       u2
1    0.0000    0.0000  -0.0000  -0.0000
2    0.0000   10.0000  -0.0000   0.0000
3   20.0000   10.0000   0.6096   0.0042
4   20.0000    0.0000   0.6637   0.1041

Strains and Stresses
Element; 1
int pt     Coords          e_11       e_22       e_12       s_11       s_22       s_12
   1  6.6667   6.6667    0.0305     0.0000     0.0001     1.0048     0.3014     0.0024
Element; 2
int pt     Coords          e_11       e_22       e_12       s_11       s_22       s_12
   1 13.3333   3.3333    0.0332    -0.0100    -0.0001     0.9952    -0.0012    -0.0024
```

Plane Stress and Plane Strain Equations

FEM_2Dor3D_linelast_standard



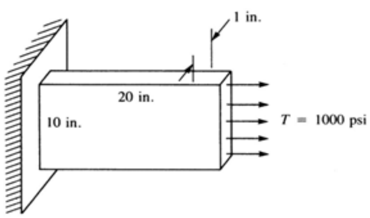


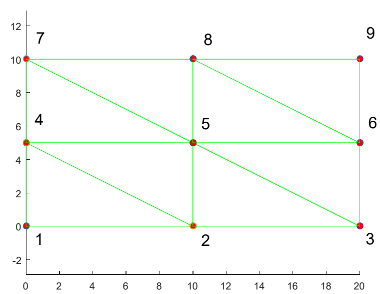

```
Nodal Displacements:
Node      Coords          u1        u2
1     0.0000    0.0000  -0.0000  -0.0000
2     0.0000   10.0000  -0.0000   0.0000
3    20.0000   10.0000   0.6096   0.0042
4    20.0000    0.0000   0.6637   0.1041

Strains and Stresses
Element; 1
int pt      Coords           e_11       e_22       e_12       s_11       s_22       s_12
   1  6.6667  6.6667       0.0305     0.0000     0.0001     1.0048     0.3014     0.0024
Element; 2
int pt      Coords           e_11       e_22       e_12       s_11       s_22       s_12
   1 13.3333  3.3333       0.0332    -0.0100    -0.0001     0.9952    -0.0012    -0.0024
```

Plane Stress and Plane Strain Equations

FEM_2Dor3D_linelast_standard – 8 elements



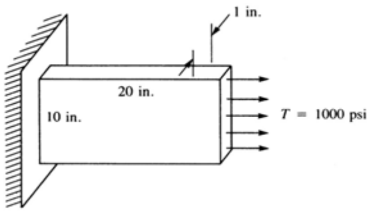


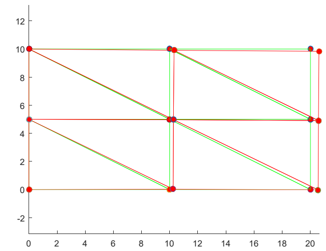
Logan_p364_8_element.txt

```
No._material_props:    3
    Shear_modulus:   11.5385
    Poissons_ratio:  0.3
    Plane_strain/stress: 1
No._coords_per_node:   2
No._DOF_per_node:      2
No._nodes:             9
Nodal_coords:
    0.0   0.0
   10.0   0.0
   20.0   0.0
    0.0   5.0
   10.0   5.0
   20.0   5.0
    0.0  10.0
   10.0  10.0
   20.0  10.0
No._elements:                       8
Max_no._nodes_on_any_one_element:   3
element_identifier; no._nodes_on_element; connectivity:
    1  3  1 2 4
    2  3  2 5 4
    3  3  2 3 5
    4  3  3 6 5
    5  3  4 5 7
    6  3  5 8 7
    7  3  5 6 8
    8  3  6 9 8
No._nodes_with_prescribed_DOFs:  6
Node_#, DOF#, Value:
   1 1 0.0
   1 2 0.0
   4 1 0.0
   4 2 0.0
   7 1 0.0
   7 2 0.0
No._elements_with_prescribed_loads: 2
Element_#, Face_#, Traction_components
  4 1 1.0 0.0
  8 1 1.0 0.0
```

Plane Stress and Plane Strain Equations

FEM_2Dor3D_linelast_standard – 8 elements

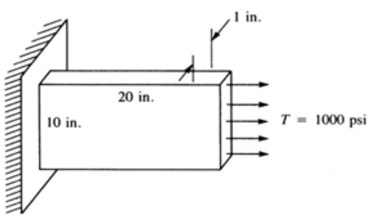


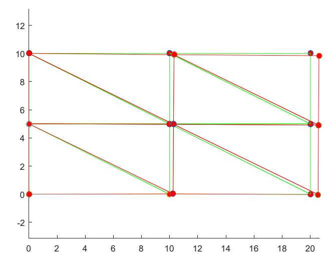


```
Nodal Displacements:
Node      Coords          u1        u2
1  0.0000   0.0000   0.0000  -0.0000
2 10.0000   0.0000   0.2489   0.0317
3 20.0000   0.0000   0.5461  -0.0231
4  0.0000   5.0000  -0.0000   0.0000
5 10.0000   5.0000   0.2706  -0.0276
6 20.0000   5.0000   0.5758  -0.0893
7  0.0000  10.0000   0.0000  -0.0000
8 10.0000  10.0000   0.3068  -0.0923
9 20.0000  10.0000   0.6082  -0.1541
```

Plane Stress and Plane Strain Equations

FEM_2Dor3D_linelast_standard – 8 elements

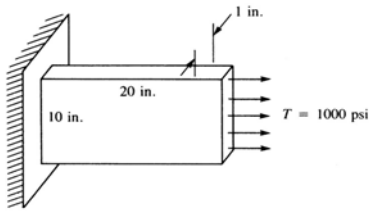


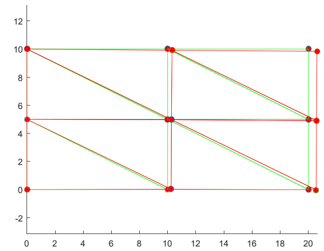


```
Strains and Stresses
Element; 1
int pt    Coords          e_11      e_22      e_12      s_11      s_22      s_12
   1  3.3333  1.6667    0.0249    0.0000    0.0016    1.0051    0.4308    0.0366
Element; 2
int pt    Coords          e_11      e_22      e_12      s_11      s_22      s_12
   1  6.6667  3.3333    0.0271   -0.0119    0.0008    0.8875   -0.0106    0.0183
Element; 3
int pt    Coords          e_11      e_22      e_12      s_11      s_22      s_12
   1 13.3333  1.6667    0.0297   -0.0119   -0.0006    0.9949    0.0355   -0.0132
Element; 4
int pt    Coords          e_11      e_22      e_12      s_11      s_22      s_12
   1 16.6667  3.3333    0.0305   -0.0132   -0.0001    1.0035   -0.0066   -0.0026
```

Plane Stress and Plane Strain Equations

FEM_2Dor3D_linelast_standard – 8 elements

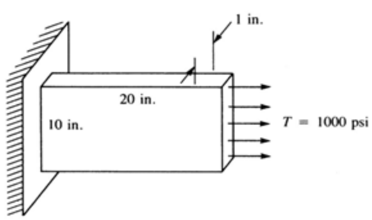


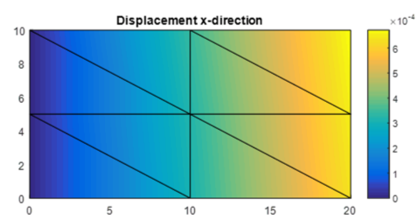


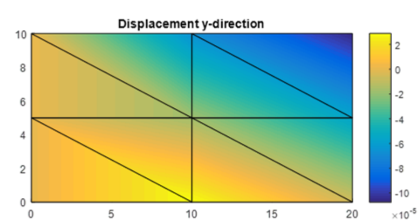
```
Strains and Stresses
Element; 5
int pt     Coords          e_11      e_22      e_12      s_11       s_22      s_12
   1  3.3333  6.6667     0.0271   -0.0000   -0.0014    1.0927    0.4683   -0.0318
Element; 6
int pt     Coords          e_11      e_22      e_12      s_11       s_22      s_12
   1  6.6667  8.3333     0.0307   -0.0129   -0.0010    1.0147    0.0079   -0.0230
Element; 7
int pt     Coords          e_11      e_22      e_12      s_11       s_22      s_12
   1 13.3333  6.6667     0.0305   -0.0129    0.0005    1.0086    0.0053    0.0122
Element; 8
int pt     Coords          e_11      e_22      e_12      s_11       s_22      s_12
   1 16.6667  8.3333     0.0301   -0.0130    0.0001    0.9931   -0.0017    0.0035
```

Plane Stress and Plane Strain Equations

FEM_2Dor3D_linelast_standard – 8 elements

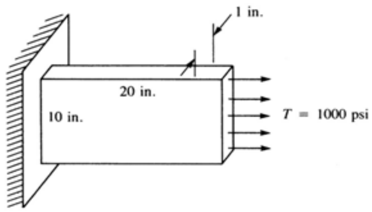


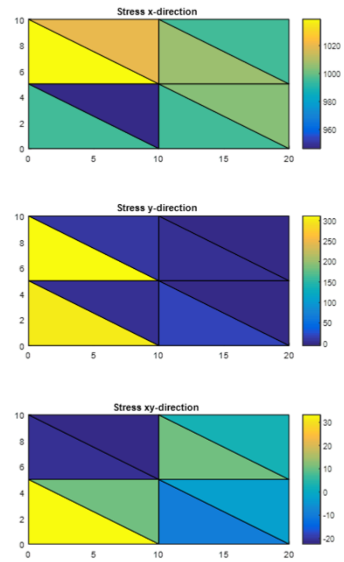




Plane Stress and Plane Strain Equations

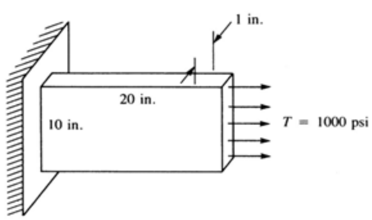
FEM_2Dor3D_linelast_standard – 8 elements

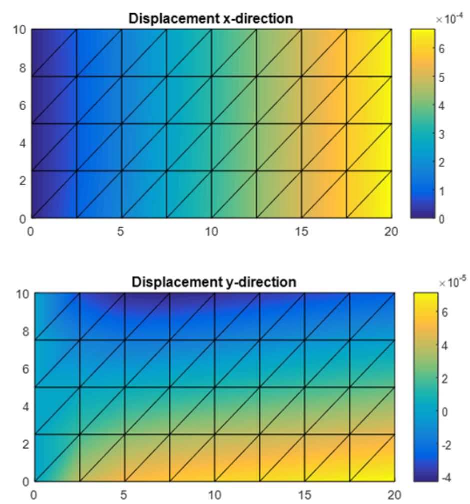




Plane Stress and Plane Strain Equations

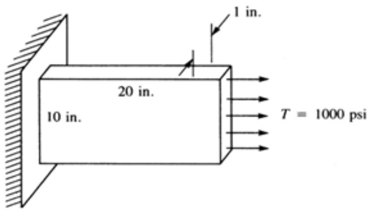
FEM_2Dor3D_linelast_standard – 64 elements

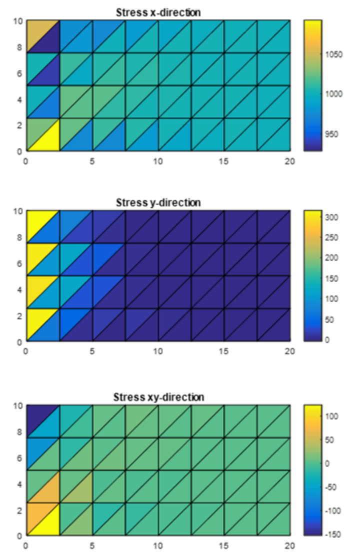




Plane Stress and Plane Strain Equations

FEM_2Dor3D_linelast_standard – 64 elements





End of Chapter 6a