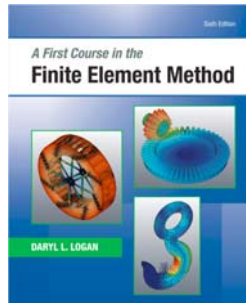### *Chapter 5b – Plane Frame and Grid Equations*

Learning Objectives

- To derive the stiffness matrix and equations for grid analysis
- To provide equations to determine torsional constants for various cross sections
- To illustrate the solution of grid structures
- To develop the stiffness matrix for a beam element arbitrarily oriented in space
- To present the solution of a space frame
- To introduce the concept of substructuring

Plane Frame and Grid Equations

Grid Equations

A **grid** is: *a structure on which the loads are applied perpendicular to the plane of the structure, as opposed to a plane frame where loads are applied in the plane of the structure.*
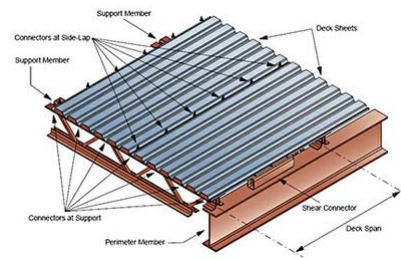
Both torsional and bending moment continuity are maintained at each node in a grid element.
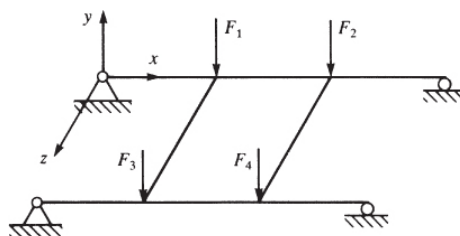
Plane Frame and Grid Equations
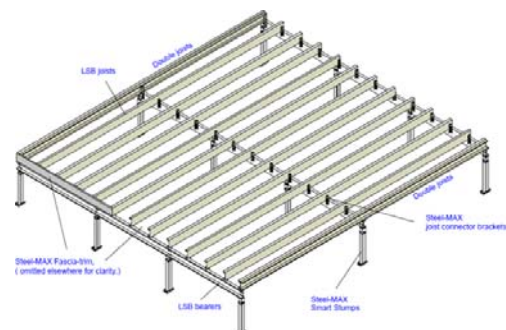
Grid Equations

Examples of a grid structure are floors and bridge deck systems.

Plane Frame and Grid Equations
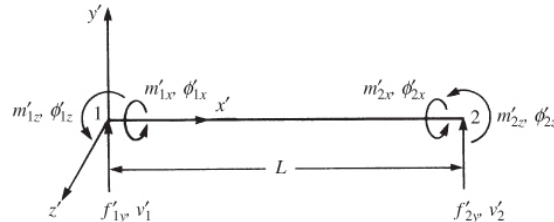
Grid Equations

Examples of a grid structure are floors and bridge deck systems.

Plane Frame and Grid Equations

Grid Equations

A representation of the grid element is shown below:

The degrees of freedom for a grid element are: a vertical displacement v'_i (normal to the grid), a torsional rotation ϕ'_{ix} about the x' axis, and a bending rotation ϕ'_{iz} about the z' axis.
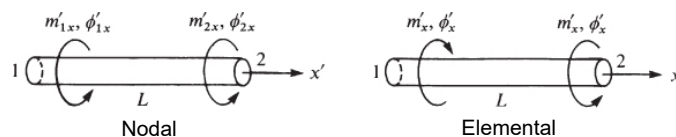
The nodal forces are: a transverse force f'_{iy}, a torsional m'_{ix} moment about the x' axis, and a bending moment m'_{iz} about the z' axis.

Plane Frame and Grid Equations

Grid Equations

Let's derive the torsional rotation components of the element stiffness matrix.

Consider the sign convention for nodal torque and angle of twist shown the figure below.

A linear displacement function is assumed. $\phi = a_1 + a_2 x'$

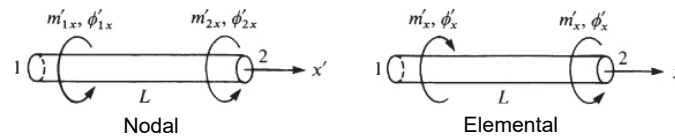Applying the boundary conditions and solving for the unknown coefficients gives:

$$\phi = \left(\frac{\phi'_{2x} - \phi'_{1x}}{L} \right) x' + \phi'_{1x}$$

Plane Frame and Grid Equations

Grid Equations

Let's derive the torsional rotation components of the element stiffness matrix.

Consider the sign convention for nodal torque and angle of twist shown the figure below.

Nodal   Elemental

Or in matrix form: $\phi' = \begin{bmatrix} N_1 & N_2 \end{bmatrix} = \begin{Bmatrix} \phi'_{1x} \\ \phi'_{2x} \end{Bmatrix}$

where: $N_1 = 1 - \dfrac{x'}{L}$ $N_2 = \dfrac{x'}{L}$

Plane Frame and Grid Equations

Grid Equations

To obtain the relationship between the shear strain γ and the angle of twist ϕ' consider the torsional deformation of the bar as shown below.
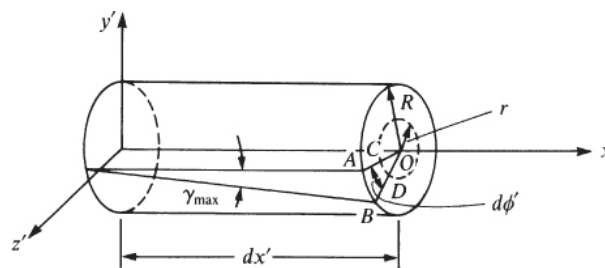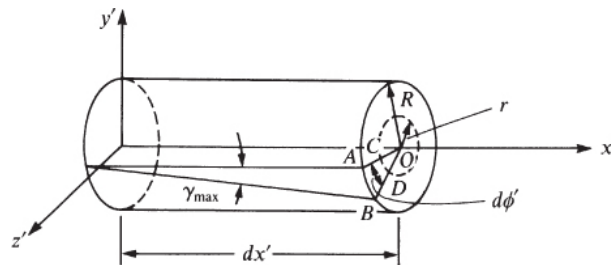

If we assume that all radial lines, such as **OA**, remain straight during twisting or torsional deformation, then the arc length $\overline{AB}$ is:

$$\overline{AB} = \gamma_{max}dx' = R\,d\phi' \qquad \Rightarrow \qquad \gamma_{max} = R\frac{d\phi'}{dx'}$$

Plane Frame and Grid Equations

Grid Equations

To obtain the relationship between the shear strain γ and the angle of twist ϕ' consider the torsional deformation of the bar as shown below.

At any radial position, r, we have, from similar triangles **OAB** and **OCD**:

$$\gamma = r\frac{d\phi'}{dx'} = \frac{r}{L}\left(\phi'_{2x} - \phi'_{1x}\right)$$

Plane Frame and Grid Equations

Grid Equations

To obtain the relationship between the shear strain γ and the angle of twist ϕ' consider the torsional deformation of the bar as shown below.

The relationship between shear stress and shear strain is:

$$\tau = G\gamma$$

where G is the **shear modulus** of the material.

Plane Frame and Grid Equations

Grid Equations

To obtain the relationship between the shear strain γ and the angle of twist ϕ' consider the torsional deformation of the bar as shown below.
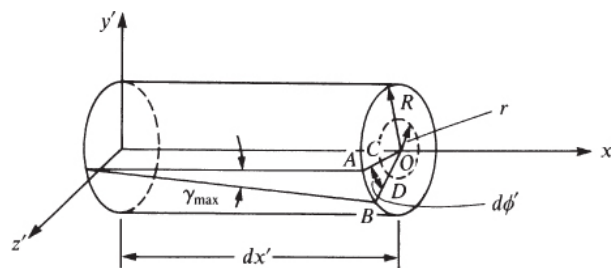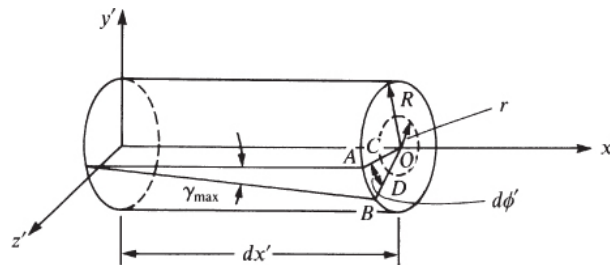

From elementary mechanics of materials, we get: $m'_x = \dfrac{\tau\, J}{R}$

Where J is the **polar moment of inertia** for a circular cross section or the **torsional constant** for non-circular cross sections.

Plane Frame and Grid Equations

Grid Equations

Cross Section	Torsional Constant
1. Channel	$J = \frac{t^3}{3}(h + 2b)$ $e = \frac{h^2 b^2 t}{4I}$
2. Angle	$J = \frac{1}{3}(b_1 t_1^3 + b_2 t_2^3)$
3. Z section	$J = \frac{t^3}{3}(2b + h)$

Cross Section	Torsional Constant
4. Wide-flanged beam with unequal flanges	$J = \frac{1}{3}(b_1 t_1^3 + b_2 t_2^3 + h t_w^3)$
5. Solid circular	$J = \frac{\pi}{2} r^4$
6. Closed hollow rectangular	$J = \frac{2 t t_1 (a - t)^2 (b - t_1)^2}{a t + b t_1 - t^2 - t_1^2}$

Plane Frame and Grid Equations

Grid Equations

Rewriting the above equation we get: $m'_x = \dfrac{GJ}{L}\left(\phi'_{2x} - \phi'_{1x}\right)$

The nodal torque sign convention gives: $\begin{aligned} m'_{1x} &= -m'_x \\ m'_{2x} &= m'_x \end{aligned}$

Therefore: $m'_{1x} = \dfrac{GJ}{L}\left(\phi'_{1x} - \phi'_{2x}\right)$ $m'_{2x} = \dfrac{GJ}{L}\left(\phi'_{2x} - \phi'_{1x}\right)$

In matrix form, the above equations are:

$$\begin{Bmatrix} m'_{1x} \\ m'_{2x} \end{Bmatrix} = \dfrac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \phi'_{1x} \\ \phi'_{2x} \end{Bmatrix}$$

Plane Frame and Grid Equations
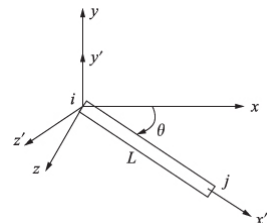
Grid Equations

Combining the torsional effects with shear and bending effects, we obtain the local stiffness matrix equations for a grid element.

$$\begin{Bmatrix} f'_{1y} \\ m'_{1x} \\ m'_{1z} \\ f'_{2y} \\ m'_{2x} \\ m'_{2z} \end{Bmatrix} = \begin{bmatrix} \frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} \\ 0 & \frac{GJ}{L} & 0 & 0 & -\frac{GJ}{L} & 0 \\ \frac{6EI}{L^2} & 0 & \frac{4EI}{L} & -\frac{6EI}{L^2} & 0 & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} \\ 0 & -\frac{GJ}{L} & 0 & 0 & \frac{GJ}{L} & 0 \\ \frac{6EI}{L^2} & 0 & \frac{2EI}{L} & -\frac{6EI}{L^2} & 0 & \frac{4EI}{L} \end{bmatrix}\begin{Bmatrix} v'_1 \\ \phi'_{1x} \\ \phi'_{1z} \\ v'_2 \\ \phi'_{2x} \\ \phi'_{2z} \end{Bmatrix}$$

Plane Frame and Grid Equations

Grid Equations

The **transformation matrix** relating local to global degrees of freedom for a grid is:

$$\mathbf{T}_G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C & S & 0 & 0 & 0 \\ 0 & -S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & S \\ 0 & 0 & 0 & 0 & -S & C \end{bmatrix}$$

where θ is now positive taken counterclockwise from x to x' in the x-z plane: therefore:

$$C = \cos\theta = \frac{x_j - x_i}{L} \qquad S = \sin\theta = \frac{z_j - z_i}{L}$$

Plane Frame and Grid Equations

Grid Equations

The **transformation matrix** relating local to global degrees of freedom for a grid is:

$$\mathbf{T}_G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C & S & 0 & 0 & 0 \\ 0 & -S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & S \\ 0 & 0 & 0 & 0 & -S & C \end{bmatrix}$$
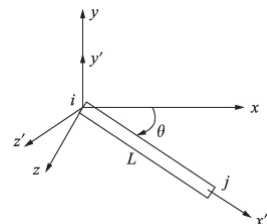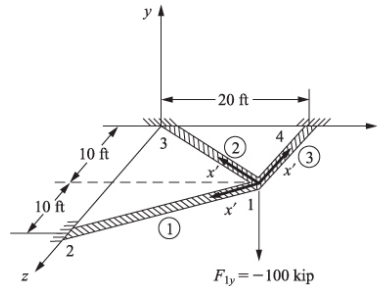
The global stiffness matrix for a grid element arbitrary oriented in the x-z plane is given by:

$$\mathbf{k}_G = \mathbf{T}_G{}^{\mathsf{T}} \mathbf{k}'_G \mathbf{T}_G$$

Plane Frame and Grid Equations

Grid Example 1

Consider the frame shown in the figure below.

$$F_{1y} = -100 \text{ kip}$$
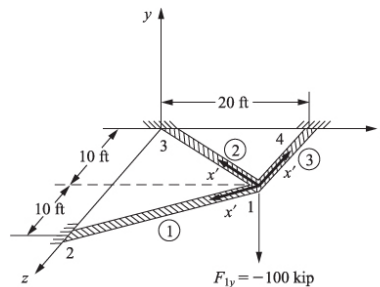
The frame is fixed at nodes 2, 3, and 4, and is subjected to a load of 100 *kips* applied at node 1.

Assume I = 400 in^4, J = 110 in^4, G = 12 x 10 ^{3} *ksi*, and E = 30 x 10 ^{3} *ksi* for all elements.

Plane Frame and Grid Equations

Grid Example 1

Consider the frame shown in the figure below.

$$F_{1y} = -100 \text{ kip}$$

To facilitate a timely solution, the boundary conditions at nodes 2, 3, and 4 are applied to the local stiffness matrices at the beginning of the solution.

$$v_2 = \phi_{2x} = \phi_{2z} = 0$$
$$v_3 = \phi_{3x} = \phi_{3z} = 0$$
$$v_4 = \phi_{4x} = \phi_{4z} = 0$$

Plane Frame and Grid Equations

Grid Example 1

Recall the general elemental stiffness matrix:

$$
\begin{Bmatrix} f'_{1y} \\ m'_{1x} \\ m'_{1z} \\ f'_{2y} \\ m'_{2x} \\ m'_{2z} \end{Bmatrix} =
\begin{bmatrix}
\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} \\
0 & \frac{GJ}{L} & 0 & 0 & -\frac{GJ}{L} & 0 \\
\frac{6EI}{L^2} & 0 & \frac{4EI}{L} & -\frac{6EI}{L^2} & 0 & \frac{2EI}{L} \\
-\frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} \\
0 & -\frac{GJ}{L} & 0 & 0 & \frac{GJ}{L} & 0 \\
\frac{6EI}{L^2} & 0 & \frac{2EI}{L} & -\frac{6EI}{L^2} & 0 & \frac{4EI}{L}
\end{bmatrix}
\begin{Bmatrix} v'_1 \\ \phi'_{1x} \\ \phi'_{1z} \\ v'_2 \\ \phi'_{2x} \\ \phi'_{2z} \end{Bmatrix}
$$

Plane Frame and Grid Equations

Grid Example 1

Recall the general transformation matrix:

$$
\mathbf{T}_G =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & C & S & 0 & 0 & 0 \\
0 & -S & C & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & C & S \\
0 & 0 & 0 & 0 & -S & C
\end{bmatrix}
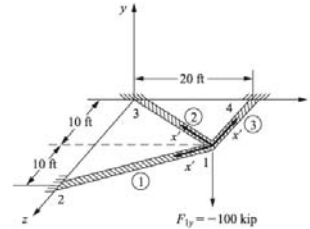\qquad
\mathbf{k}_G = \mathbf{T}_G^{\mathsf{T}} \mathbf{k}'_G \mathbf{T}_G
$$

Plane Frame and Grid Equations

Grid Example 1

Beam Element 1:

$$C = \cos\theta = \frac{x_2 - x_1}{L^{(1)}} = \frac{0 - 20}{22.36} = -0.894$$

$$S = \sin\theta = \frac{z_2 - z_1}{L^{(1)}} = \frac{20 - 10}{22.36} = 0.447$$

$$\frac{12EI}{L^3} = \frac{12(30\times10^3)(400)}{(22.36\times12)^3} = 7.45 \; k/_{in} \qquad \frac{6EI}{L^2} = \frac{6(30\times10^3)(400)}{(22.36\times12)^2} = 1,000 \; k$$

$$\frac{4EI}{L} = \frac{4(30\times10^3)(400)}{(22.36\times12)} = 179,000 \; k\cdot in \qquad \frac{GJ}{L} = \frac{(12\times10^3)(110)}{(22.36\times12)} = 4,920 \; k\cdot in$$

Plane Frame and Grid Equations

Grid Example 1

Beam Element 1:

The global stiffness matrix for element 1, considering only the parts associated with node 1, and the following relationship:

$$\mathbf{k}_G = \mathbf{T}_G{}^T \mathbf{k}'_G \mathbf{T}_G$$

$$\mathbf{T}_G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.894 & 0.447 \\ 0 & -0.447 & -0.894 \end{bmatrix} \qquad \mathbf{T}_G{}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.894 & -0.447 \\ 0 & 0.447 & -0.894 \end{bmatrix}$$

$$\mathbf{k}'^{(1)} = \begin{array}{ccc} v_1 & \phi_{1x} & \phi_{1z} \end{array} \\ \begin{bmatrix} 7.45 & 0 & 1,000 \\ 0 & 4,920 & 0 \\ 1,000 & 0 & 179,000 \end{bmatrix} k/_{in}$$

Plane Frame and Grid Equations

Grid Example 1

Beam Element 1:

The global stiffness matrix for element 1, considering only the parts associated with node 1, and the following relationship:

$$\mathbf{k}_G = \mathbf{T}_G{}^{\mathsf{T}}\mathbf{k}'_G\mathbf{T}_G$$

$$\mathbf{k}_G{}^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.894 & -0.447 \\ 0 & 0.447 & -0.894 \end{bmatrix} \begin{bmatrix} 7.45 & 0 & 1{,}000 \\ 0 & 4{,}920 & 0 \\ 1{,}000 & 0 & 179{,}000 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.894 & 0.447 \\ 0 & -0.447 & -0.894 \end{bmatrix}$$

$$\mathbf{k}_G{}^{(1)} = \begin{matrix} \quad v_1 \qquad\quad \phi_{1x} \qquad\quad \phi_{1z} \\ \begin{bmatrix} 7.45 & -447 & -894 \\ -447 & 39{,}700 & 69{,}600 \\ -894 & 69{,}600 & 144{,}000 \end{bmatrix} \end{matrix} k/_{in}$$

Plane Frame and Grid Equations

Grid Example 1

Beam Element 2:

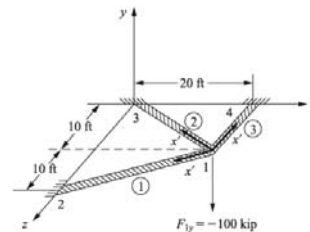$$C = \cos\theta = \frac{x_3 - x_1}{L^{(2)}} = \frac{0 - 20}{22.36} = -0.894$$

$$S = \sin\theta = \frac{z_3 - z_1}{L^{(2)}} = \frac{0 - 10}{22.36} = -0.447$$

$$\frac{12EI}{L^3} = \frac{12(30 \times 10^3)(400)}{(22.36 \times 12)^3} = 7.45 \ k/_{in} \qquad \frac{6EI}{L^2} = \frac{6(30 \times 10^3)(400)}{(22.36 \times 12)^2} = 1{,}000 \ k$$

$$\frac{4EI}{L} = \frac{4(30 \times 10^3)(400)}{(22.36 \times 12)} = 179{,}000 \ k \cdot in \qquad \frac{GJ}{L} = \frac{(12 \times 10^3)(110)}{(22.36 \times 12)} = 4{,}920 \ k \cdot in$$

Plane Frame and Grid Equations

Grid Example 1

Beam Element 2:

The global stiffness matrix for element 2, considering only the parts associated with node 1, and the following relationship:

$$\mathbf{k}_G = \mathbf{T}_G{}^T \mathbf{k}'_G \mathbf{T}_G$$

$$\mathbf{T}_G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.894 & -0.447 \\ 0 & 0.447 & -0.894 \end{bmatrix} \qquad \mathbf{T}_G{}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.894 & 0.447 \\ 0 & -0.447 & -0.894 \end{bmatrix}$$

$$
\begin{array}{ccc}
\quad v_1 & \phi_{1x} & \phi_{1z}
\end{array}
$$
$$\mathbf{k}'^{(2)} = \begin{bmatrix} 7.45 & 0 & 1{,}000 \\ 0 & 4{,}920 & 0 \\ 1{,}000 & 0 & 179{,}000 \end{bmatrix} k/_{in}$$

Plane Frame and Grid Equations

Grid Example 1

Beam Element 2:

The global stiffness matrix for element 2, considering only the parts associated with node 1, and the following relationship:

$$\mathbf{k}_G = \mathbf{T}_G{}^T \mathbf{k}'_G \mathbf{T}_G$$

$$\mathbf{k}_G{}^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.894 & 0.447 \\ 0 & -0.447 & -0.894 \end{bmatrix} \begin{bmatrix} 7.45 & 0 & 1{,}000 \\ 0 & 4{,}920 & 0 \\ 1{,}000 & 0 & 179{,}000 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.894 & -0.447 \\ 0 & 0.447 & -0.894 \end{bmatrix}$$

$$
\begin{array}{ccc}
\quad v_1 & \phi_{1x} & \phi_{1z}
\end{array}
$$
$$\mathbf{k}_G{}^{(2)} = \begin{bmatrix} 7.45 & 447 & -894 \\ 447 & 39{,}700 & -69{,}600 \\ -894 & -69{,}600 & 144{,}000 \end{bmatrix} k/_{in}$$
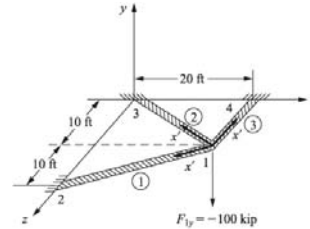
Plane Frame and Grid Equations

Grid Example 1

Beam Element 3:

$$C = \cos\theta = \frac{x_4 - x_1}{L^{(3)}} = \frac{20 - 20}{10} = 0$$

$$S = \sin\theta = \frac{z_4 - z_1}{L^{(3)}} = \frac{0 - 10}{10} = -1$$

$$\frac{12EI}{L^3} = \frac{12(30 \times 10^3)(400)}{(10 \times 12)^3} = 83.3 \ k \,/\, in \qquad \frac{6EI}{L^2} = \frac{6(30 \times 10^3)(400)}{(10 \times 12)^2} = 5,000 \ k$$

$$\frac{4EI}{L} = \frac{4(30 \times 10^3)(400)}{(10 \times 12)} = 400,000 \ k \cdot in \qquad \frac{GJ}{L} = \frac{(12 \times 10^3)(110)}{(10 \times 12)} = 11,000 \ k \cdot in$$

Plane Frame and Grid Equations

Grid Example 1

Beam Element 3:

The global stiffness matrix for element 3, considering only the parts associated with node 1, and the following relationship:

$$\mathbf{k}_G = \mathbf{T}_G{}^T \mathbf{k}'_G \mathbf{T}_G$$

$$\mathbf{T}_G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{T}_G{}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\mathbf{k}'^{(3)} = \begin{array}{ccc} v_1 & \phi_{1x} & \phi_{1z} \\ \begin{bmatrix} 83.3 & 0 & 5,000 \\ 0 & 11,000 & 0 \\ 5,000 & 0 & 400,000 \end{bmatrix} \end{array} k\!\!\Big/_{\!in}$$

Plane Frame and Grid Equations

Grid Example 1

Beam Element 3:

The global stiffness matrix for element 3, considering only the parts associated with node 1, and the following relationship:

$$\mathbf{k}_G = \mathbf{T}_G{}^T\mathbf{k}_G'\mathbf{T}_G$$

$$\mathbf{k}_G{}^{(3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 83.3 & 0 & 5{,}000 \\ 0 & 11{,}000 & 0 \\ 5{,}000 & 0 & 400{,}000 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\mathbf{k}_G{}^{(3)} = \begin{array}{ccc} v_1 & \phi_{1x} & \phi_{1z} \end{array}$$
$$\mathbf{k}_G{}^{(3)} = \begin{bmatrix} 83.3 & 5{,}000 & 0 \\ 5{,}000 & 400{,}000 & 0 \\ 0 & 0 & 11{,}000 \end{bmatrix} k\!\big/\!_{in}$$

Plane Frame and Grid Equations

Grid Example 1

Superimposing the three elemental stiffness matrices gives:

$$\begin{array}{ccc} v_1 & \phi_{1x} & \phi_{1z} \end{array}$$
$$\mathbf{K}_G = \begin{bmatrix} 98.2 & 5{,}000 & -1{,}790 \\ 5{,}000 & 479{,}000 & 0 \\ -1{,}790 & 0 & 299{,}000 \end{bmatrix}$$
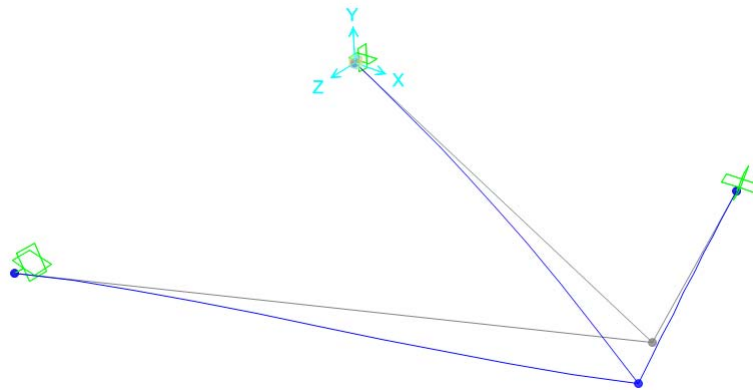
The global equations are:

$$\begin{Bmatrix} F_{1y} = -100\ k \\ M_{1x} = 0 \\ M_{1z} = 0 \end{Bmatrix} = \begin{bmatrix} 98.2 & 5{,}000 & -1{,}790 \\ 5{,}000 & 479{,}000 & 0 \\ -1{,}790 & 0 & 299{,}000 \end{bmatrix} \begin{Bmatrix} v_1 \\ \phi_{1x} \\ \phi_{1z} \end{Bmatrix}$$

Plane Frame and Grid Equations

Grid Example 1
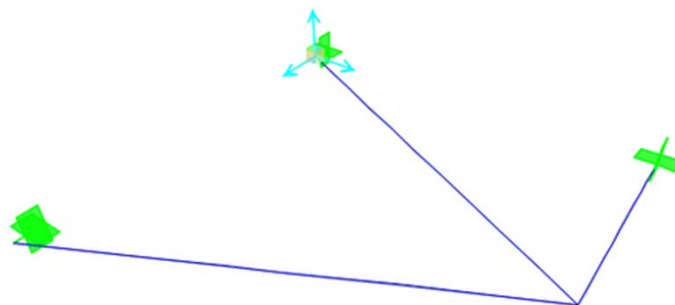
Solving the above equations gives:
$$\begin{Bmatrix} v_1 \\ \phi_{1x} \\ \phi_{1z} \end{Bmatrix} = \begin{Bmatrix} -2.83\ in \\ 0.0295\ rad \\ -0.0169\ rad \end{Bmatrix}$$

Plane Frame and Grid Equations

Grid Example 1

Solving the above equations gives:
$$\begin{Bmatrix} v_1 \\ \phi_{1x} \\ \phi_{1z} \end{Bmatrix} = \begin{Bmatrix} -2.83\ in \\ 0.0295\ rad \\ -0.0169\ rad \end{Bmatrix}$$

Plane Frame and Grid Equations

Grid Example 1

Solving the above equations gives:

$$\begin{Bmatrix} v_1 \\ \phi_{1x} \\ \phi_{1z} \end{Bmatrix} = \begin{Bmatrix} -2.83 \ in \\ 0.0295 \ rad \\ -0.0169 \ rad \end{Bmatrix}$$

The results indicate that the *y* displacement at node 1 is downward as indicated by the minus sign.

The rotation about the *x*-axis is positive.

The rotation about the *z*-axis is negative.

Based on the downward loading location with respect to the supports, these results are expected.
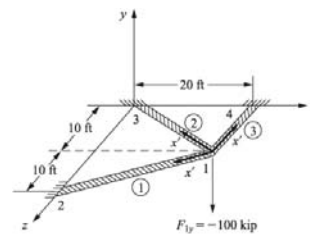
Plane Frame and Grid Equations

Grid Example 1

Beam Element 1: The grid element force-displacement equations can be obtained using $\mathbf{f}' = \mathbf{k}'_G \mathbf{T}_G \mathbf{d}$

$$C = \cos\theta = \frac{x_2 - x_1}{L^{(1)}} = \frac{0 - 20}{22.36} = -0.894$$

$$S = \sin\theta = \frac{z_2 - z_1}{L^{(1)}} = \frac{20 - 10}{22.36} = 0.447$$

$$\mathbf{T}_G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C & S & 0 & 0 & 0 \\ 0 & -S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & S \\ 0 & 0 & 0 & 0 & -S & C \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.894 & 0.447 & 0 & 0 & 0 \\ 0 & -0.447 & -0.894 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.894 & 0.447 \\ 0 & 0 & 0 & 0 & -0.447 & -0.894 \end{bmatrix}$$

Plane Frame and Grid Equations

Grid Example 1

Beam Element 1: The grid element force-displacement equations can be obtained using $f' = k'_G T_G d$

$$
T_G d = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.894 & 0.447 & 0 & 0 & 0 \\
0 & -0.447 & -0.894 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.894 & 0.447 \\
0 & 0 & 0 & 0 & -0.447 & -0.894
\end{bmatrix}
\begin{Bmatrix}
-2.83\ in \\
0.0295\ rad \\
-0.0169\ rad \\
0 \\
0 \\
0
\end{Bmatrix}
=
\begin{Bmatrix}
-2.83\ in \\
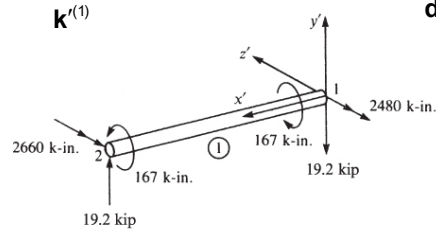-0.0339\ rad \\
0.00192\ rad \\
0 \\
0 \\
0
\end{Bmatrix}
$$

Plane Frame and Grid Equations

Grid Example 1

Beam Element 1: The grid element force-displacement equations can be obtained using $f' = k'_G T_G d$

$$
f'_{(1)} = k'T d = \begin{bmatrix}
7.45 & 0 & 1,000 & -7.45 & 0 & 1,000 \\
0 & 4,920 & 0 & 0 & -4,920 & 0 \\
1,000 & 0 & 179,000 & -1,000 & 0 & 89,500 \\
-7.45 & 0 & -1,000 & 7.45 & 0 & -1,000 \\
0 & -4,920 & 0 & 0 & 4,920 & 0 \\
1,000 & 0 & 89,500 & -1,000 & 0 & 179,000
\end{bmatrix}
\begin{Bmatrix}
-2.83\ in \\
-0.0339\ rad \\
0.00192\ rad \\
0 \\
0 \\
0
\end{Bmatrix}
$$

$$\underbrace{\hspace{5cm}}_{k'^{(1)}} \qquad \underbrace{\hspace{2cm}}_{d'}$$

$$
\begin{Bmatrix}
f'_{1y} \\
m'_{1x} \\
m'_{1z} \\
f'_{2y} \\
m'_{2x} \\
m'_{2z}
\end{Bmatrix}
=
\begin{Bmatrix}
-19.2\ k \\
-167\ k \cdot in \\
-2,480\ k \cdot in \\
19.2\ k \\
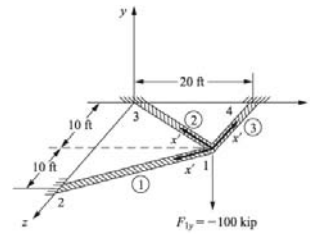167\ k \cdot in \\
-2,660\ k \cdot in
\end{Bmatrix}
$$

Plane Frame and Grid Equations

Grid Example 1

Beam Element 2: The grid element force-displacement
equations can be obtained using $\mathbf{f'} = \mathbf{k'}_G \mathbf{T}_G \mathbf{d}$

$$C = \cos\theta = \frac{x_3 - x_1}{L^{(2)}} = \frac{0-20}{22.36} = -0.894$$

$$S = \sin\theta = \frac{z_3 - z_1}{L^{(2)}} = \frac{0-10}{22.36} = -0.447$$

$$\mathbf{T}_G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C & S & 0 & 0 & 0 \\ 0 & -S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & S \\ 0 & 0 & 0 & 0 & -S & C \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.894 & -0.447 & 0 & 0 & 0 \\ 0 & 0.447 & -0.894 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.894 & -0.447 \\ 0 & 0 & 0 & 0 & 0.447 & -0.894 \end{bmatrix}$$

Plane Frame and Grid Equations

Grid Example 1

Beam Element 2: The grid element force-displacement
equations can be obtained using $\mathbf{f'} = \mathbf{k'}_G \mathbf{T}_G \mathbf{d}$

$$\mathbf{T}_G\mathbf{d} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.894 & -0.447 & 0 & 0 & 0 \\ 0 & 0.447 & -0.894 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.894 & -0.447 \\ 0 & 0 & 0 & 0 & 0.447 & -0.894 \end{bmatrix} \begin{Bmatrix} -2.83 \ in \\ 0.0295 \ rad \\ -0.0169 \ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -2.83 \ in \\ -0.0188 \ rad \\ 0.0283 \ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Plane Frame and Grid Equations

Grid Example 1

Beam Element 2: The grid element force-displacement equations can be obtained using $\mathbf{f'} = \mathbf{k'}_G \mathbf{T}_G \mathbf{d}$

$$\mathbf{f'}_{(2)} = \mathbf{k'Td} = \begin{bmatrix} 7.45 & 0 & 1{,}000 & -7.45 & 0 & 1{,}000 \\ 0 & 4{,}920 & 0 & 0 & -4{,}920 & 0 \\ 1{,}000 & 0 & 179{,}000 & -1{,}000 & 0 & 89{,}500 \\ -7.45 & 0 & -1{,}000 & 7.45 & 0 & -1{,}000 \\ 0 & -4{,}920 & 0 & 0 & 4{,}920 & 0 \\ 1{,}000 & 0 & 89{,}500 & -1{,}000 & 0 & 179{,}000 \end{bmatrix} \begin{Bmatrix} -2.83 \ in \\ -0.0188 \ rad \\ 0.0283 \ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\underbrace{}_{\mathbf{k'}^{(2)}} \qquad \underbrace{}_{\mathbf{d'}}$$

$$\begin{Bmatrix} f'_{1y} \\ m'_{1x} \\ m'_{1z} \\ f'_{3y} \\ m'_{3x} \\ m'_{3z} \end{Bmatrix} = \begin{Bmatrix} 7.23 \ k \\ -92.5 \ k \cdot in \\ -2{,}240 \ k \cdot in \\ -7.23 \ k \\ 92.5 \ k \cdot in \\ -295 \ k \cdot in \end{Bmatrix}$$
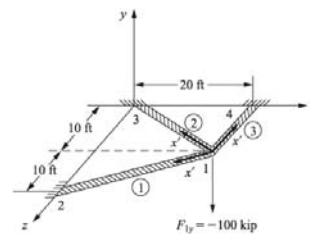

Plane Frame and Grid Equations

Grid Example 1

Beam Element 3: The grid element force-displacement equations can be obtained using $\mathbf{f'} = \mathbf{k'}_G \mathbf{T}_G \mathbf{d}$

$$C = \cos\theta = \frac{x_4 - x_1}{L^{(3)}} = \frac{20 - 20}{10} = 0$$

$$S = \sin\theta = \frac{z_4 - z_1}{L^{(3)}} = \frac{0 - 10}{10} = -1$$

$$\mathbf{T}_G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C & S & 0 & 0 & 0 \\ 0 & -S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & S \\ 0 & 0 & 0 & 0 & -S & C \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Plane Frame and Grid Equations

Grid Example 1

Beam Element 3: The grid element force-displacement equations can be obtained using $\mathbf{f}' = \mathbf{k}'_G \mathbf{T}_G \mathbf{d}$

$$\mathbf{T}_G \mathbf{d} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} -2.83\ in \\ 0.0295\ rad \\ -0.0169\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -2.83\ in \\ 0.0169\ rad \\ 0.0295\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$
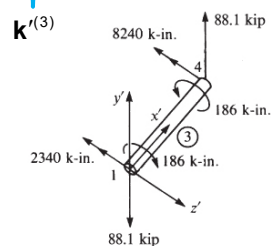
Plane Frame and Grid Equations

Grid Example 1

Beam Element 3: The grid element force-displacement equations can be obtained using $\mathbf{f}' = \mathbf{k}'_G \mathbf{T}_G \mathbf{d}$
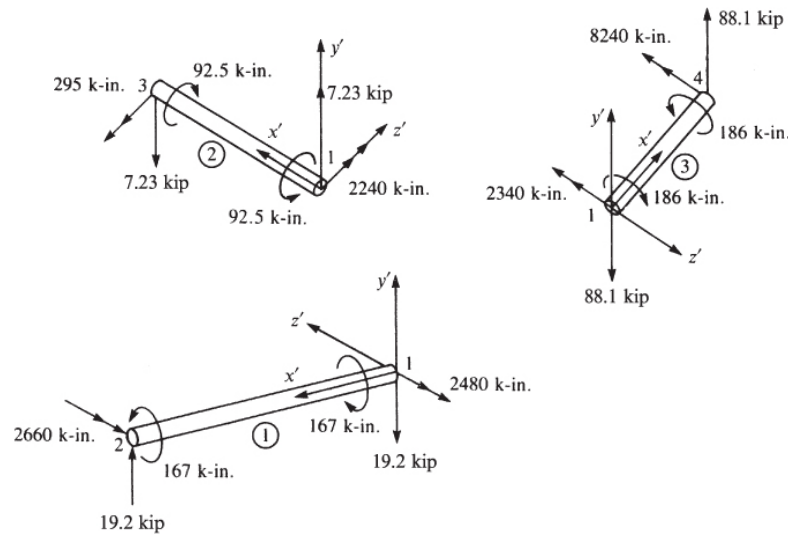
$$\mathbf{f}'_{(3)} = \mathbf{k}'\mathbf{T}\mathbf{d} = \begin{bmatrix} 83.3 & 0 & 5{,}000 & -83.3 & 0 & 5{,}000 \\ 0 & 11{,}000 & 0 & 0 & -11{,}000 & 0 \\ 5{,}000 & 0 & 400{,}000 & -5{,}000 & 0 & 200{,}000 \\ -83.3 & 0 & -5{,}000 & 83.3 & 0 & -5{,}000 \\ 0 & -11{,}000 & 0 & 0 & 11{,}000 & 0 \\ 5{,}000 & 0 & 200{,}000 & -5{,}000 & 0 & 400{,}000 \end{bmatrix} \begin{Bmatrix} -2.83\ in \\ 0.0169\ rad \\ 0.0295\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\underbrace{}_{\mathbf{k}'^{(3)}} \underbrace{}_{\mathbf{d}'}$$

$$\begin{Bmatrix} f'_{1y} \\ m'_{1x} \\ m'_{1z} \\ f'_{4y} \\ m'_{4x} \\ m'_{4z} \end{Bmatrix} = \begin{Bmatrix} -88.1\ k \\ 186\ k \cdot in \\ -2{,}340\ k \cdot in \\ 88.1\ k \\ -186\ k \cdot in \\ -8{,}240\ k \cdot in \end{Bmatrix}$$
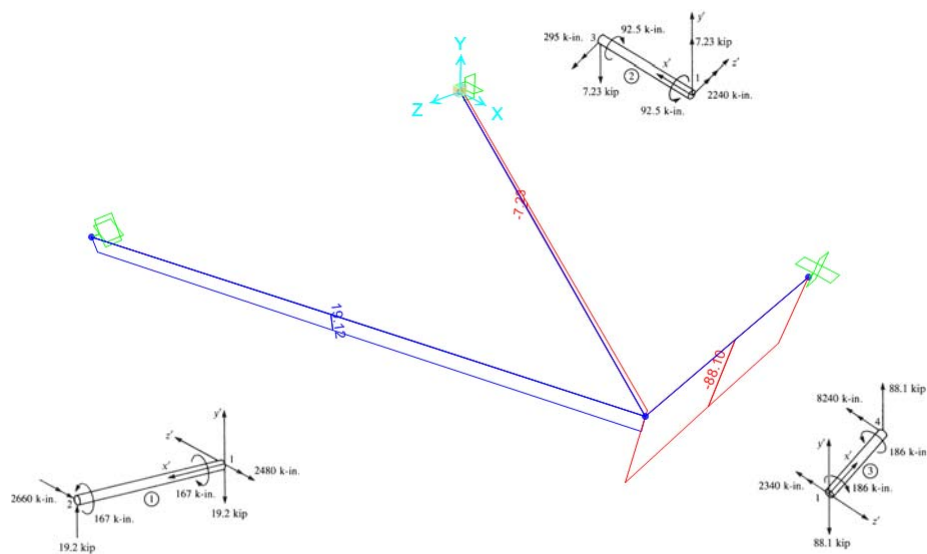
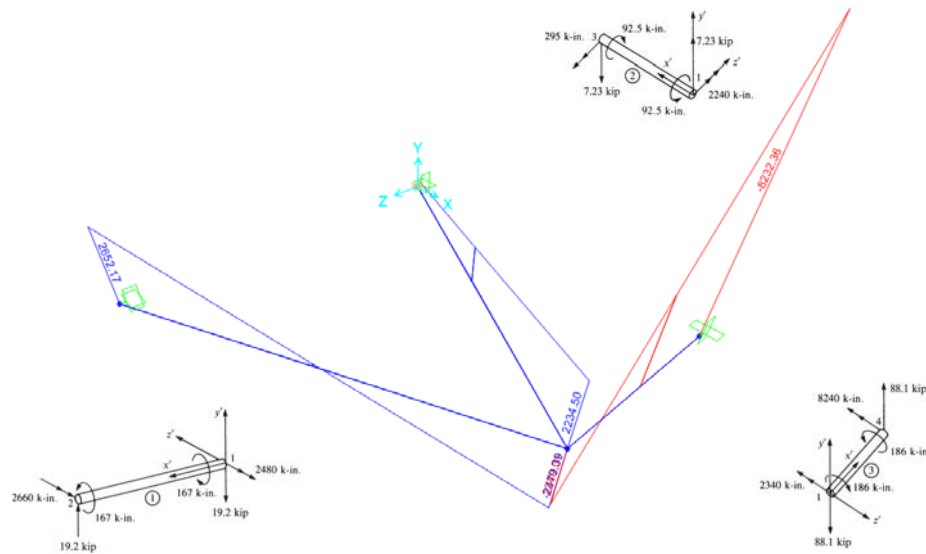# Plane Frame and Grid Equations

Grid Example 1

Plane Frame and Grid Equations

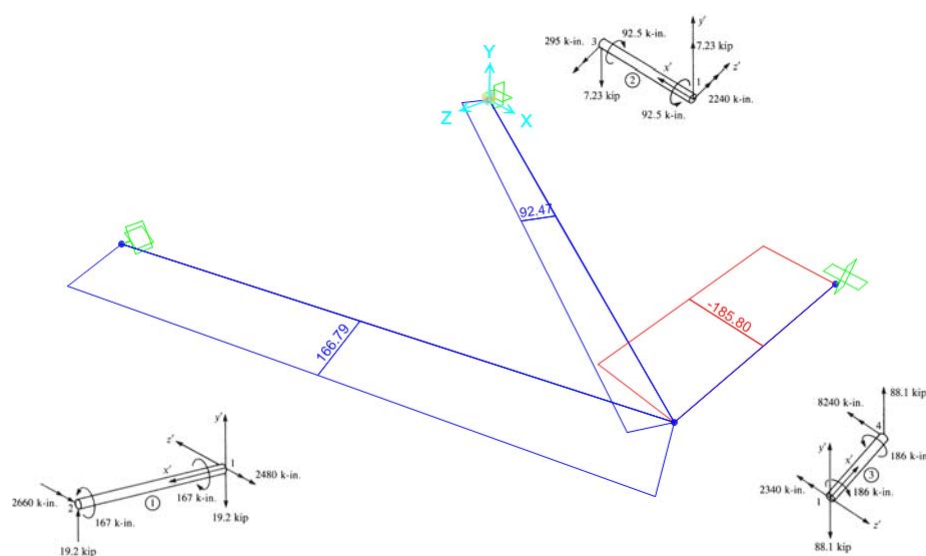## Grid Example 1 – Forces in the y-direction

Plane Frame and Grid Equations
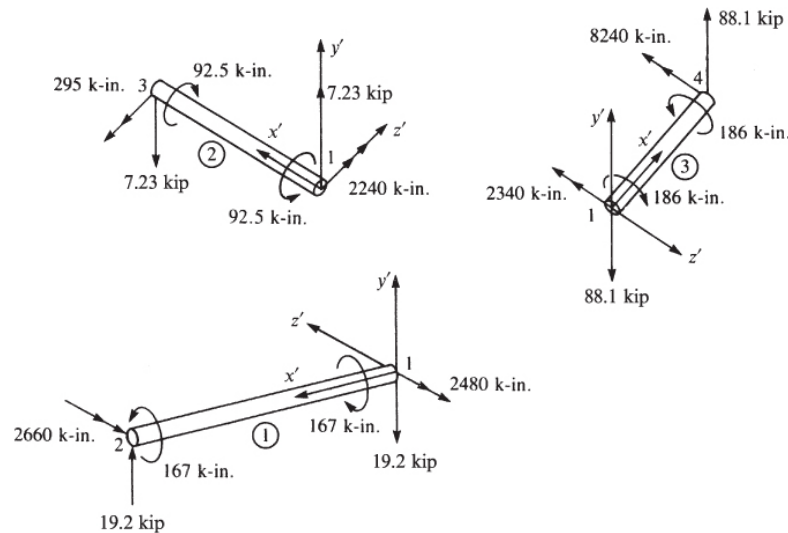
Grid Example 1 - Moment about the y' axis

Plane Frame and Grid Equations

Grid Example 1 - Torsional Moment about the x' axis

Plane Frame and Grid Equations

Grid Example 1

Plane Frame and Grid Equations

Grid Example 1

To check the equilibrium of node 1, the local forces and moments for each element need to be transformed to global coordinates. Recall, that:

$$\mathbf{f}' = \mathbf{T}\mathbf{f} \quad \Rightarrow \quad \mathbf{f} = \mathbf{T}^{\mathsf{T}}\mathbf{f}' \qquad \mathbf{T}^{\mathsf{T}} = \mathbf{T}^{-1}$$

Since we are only checking the forces and moments at node 1, we need only the upper-left-hand portion of the transformation matrix $\mathbf{T}_G$

Element 1:

$$
\begin{Bmatrix} f_{1y} \\ m_{1x} \\ m_{1z} \end{Bmatrix} =
\begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.894 & -0.447 \\ 0 & 0.447 & -0.894 \end{bmatrix}
\begin{Bmatrix} -19.2\,k \\ -167\,k \cdot in \\ -2{,}480\,k \cdot in \end{Bmatrix} =
\begin{Bmatrix} -19.2\,k \\ 1{,}260\,k \cdot in \\ 2{,}150\,k \cdot in \end{Bmatrix}
$$

Plane Frame and Grid Equations

Grid Example 1

To check the equilibrium of node 1, the local forces and moments for each element need to be transformed to global coordinates. Recall, that:

$$\mathbf{f'} = \mathbf{Tf} \quad \Rightarrow \quad \mathbf{f} = \mathbf{T^T f'} \qquad \mathbf{T^T} = \mathbf{T^{-1}}$$

Element 2:

$$\begin{Bmatrix} f_{1y} \\ m_{1x} \\ m_{1z} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.894 & 0.447 \\ 0 & -0.447 & -0.894 \end{bmatrix} \begin{Bmatrix} -7.23\ k \\ -92.5\ k \cdot in \\ -2,240\ k \cdot in \end{Bmatrix} = \begin{Bmatrix} 7.23\ k \\ 1,080\ k \cdot in \\ -1,960\ k \cdot in \end{Bmatrix}$$
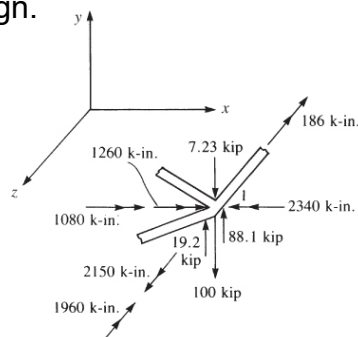
Element 3:

$$\begin{Bmatrix} f_{1y} \\ m_{1x} \\ m_{1z} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} -88.1\ k \\ -2,340\ k \cdot in \\ -186\ k \cdot in \end{Bmatrix} = \begin{Bmatrix} -88.1\ k \\ -2,340\ k \cdot in \\ -186\ k \cdot in \end{Bmatrix}$$

Plane Frame and Grid Equations

Grid Example 1

Check the equilibrium of node 1.

Remember that forces and moments from each element are equal in magnitude but opposite in sign.

Element 1:

$$\begin{Bmatrix} f_{1y} \\ m_{1x} \\ m_{1z} \end{Bmatrix} = \begin{Bmatrix} -19.2\ k \\ 1,260\ k \cdot in \\ 2,150\ k \cdot in \end{Bmatrix}$$

Element 2:

$$\begin{Bmatrix} f_{1y} \\ m_{1x} \\ m_{1z} \end{Bmatrix} = \begin{Bmatrix} 7.23\ k \\ 1,080\ k \cdot in \\ -1,960\ k \cdot in \end{Bmatrix}$$

Element 3:

$$\begin{Bmatrix} f_{1y} \\ m_{1x} \\ m_{1z} \end{Bmatrix} = \begin{Bmatrix} -88.1\ k \\ -2,340\ k \cdot in \\ -186\ k \cdot in \end{Bmatrix}$$
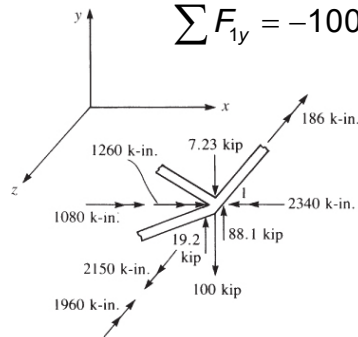
Plane Frame and Grid Equations

Grid Example 1

Check the equilibrium of node 1.

$$\sum M_{1x} = -1{,}260 - 1{,}080 + 2{,}340 = 0.0 \; k \cdot in$$

$$\sum M_{1z} = -2{,}150 + 1{,}060 + 186 = -4.0 \; k \cdot in$$
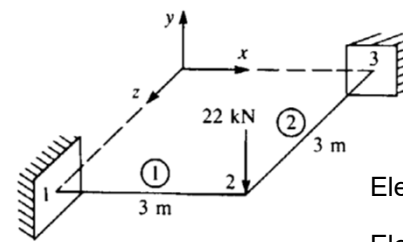
$$\sum F_{1y} = -100 - 7.23 + 19.2 + 88.1 = 0.07 \; k$$

Plane Frame and Grid Equations

Grid Example 2

Consider the frame shown in the figure below.

Element 1: nodes 1 to 2

Element 2: nodes 3 to 2

The frame is fixed at nodes 1 and 3, and is subjected to a load of 22 *kN* applied at node 2.

Assume *I* = 16.6 x 10^{-5} m^4, *J* = 4.6 x 10^{-5} m^4, *G* = 84 *GPa*, and *E* = 210 *GPa* for all elements.

Plane Frame and Grid Equations

Grid Example 2

Consider the frame shown in the figure below.
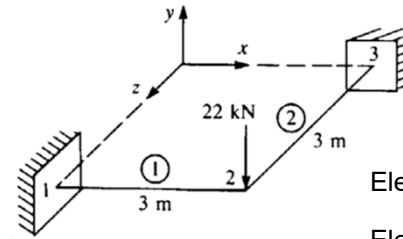

Element 1: nodes 1 to 2

Element 2: nodes 3 to 2

To facilitate a timely solution, the boundary conditions at nodes 1 and 3 are applied to the local stiffness matrices at the beginning of the solution.

$$v_1 = \phi_{1x} = \phi_{1z} = 0$$

$$v_3 = \phi_{3x} = \phi_{3z} = 0$$

Plane Frame and Grid Equations

Grid Example 2

Recall the general elemental stiffness matrix:

$$
\begin{Bmatrix} f'_{1y} \\ m'_{1x} \\ m'_{1z} \\ f'_{2y} \\ m'_{2x} \\ m'_{2z} \end{Bmatrix} =
\begin{bmatrix}
\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} \\
0 & \frac{GJ}{L} & 0 & 0 & -\frac{GJ}{L} & 0 \\
\frac{6EI}{L^2} & 0 & \frac{4EI}{L} & -\frac{6EI}{L^2} & 0 & \frac{2EI}{L} \\
-\frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} \\
0 & -\frac{GJ}{L} & 0 & 0 & \frac{GJ}{L} & 0 \\
\frac{6EI}{L^2} & 0 & \frac{2EI}{L} & -\frac{6EI}{L^2} & 0 & \frac{4EI}{L}
\end{bmatrix}
\begin{Bmatrix} v'_1 \\ \phi'_{1x} \\ \phi'_{1z} \\ v'_2 \\ \phi'_{2x} \\ \phi'_{2z} \end{Bmatrix}
$$

Plane Frame and Grid Equations

Grid Example 2

Recall the general transformation matrix:

$$
\mathbf{T}_G = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & C & S & 0 & 0 & 0 \\
0 & -S & C & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & C & S \\
0 & 0 & 0 & 0 & -S & C
\end{bmatrix}
\qquad
\mathbf{k}_G = \mathbf{T}_G{}^{\mathsf{T}}\mathbf{k}_G'\mathbf{T}_G
$$

Plane Frame and Grid Equations

Grid Example 2

Beam Element 1: from nodes 1 to 2

$$
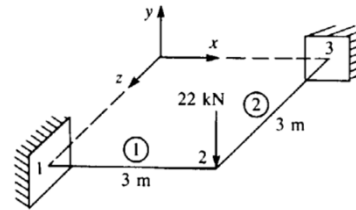C = \cos\theta = \frac{x_2 - x_1}{L^{(1)}} = \frac{3}{3} = 1
$$

$$
S = \sin\theta = \frac{z_2 - z_1}{L^{(1)}} = \frac{0}{3} = 0
$$

$$
\frac{12EI}{L^3} = \frac{12(210 \times 10^6)(16.6 \times 10^{-5})}{(3)^3} = 1.55 \times 10^4 \; kN/m
$$

$$
\frac{6EI}{L^2} = \frac{6(210 \times 10^6)(16.6 \times 10^{-5})}{(3)^2} = 2.32 \times 10^4 \; kN
$$

$$
\frac{4EI}{L} = \frac{4(210 \times 10^6)(16.6 \times 10^{-5})}{3} = 4.65 \times 10^4 \; kN{\cdot}m
$$

$$
\frac{GJ}{L} = \frac{(84 \times 10^6)(4.6 \times 10^{-5})}{3} = 0.128 \times 10^4 \; kN{\cdot}m
$$

Plane Frame and Grid Equations

Grid Example 2

Beam Element 1:

The global stiffness matrix for element 1, considering only the parts associated with node 2, and the following relationship:

$$\mathbf{k}_G = \mathbf{T}_G{}^T\mathbf{k}'_G\mathbf{T}_G$$

$$\mathbf{T}_G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad\qquad \mathbf{T}_G{}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{k}'^{(1)} = 10^4 \begin{array}{ccc} v_2 & \phi_{2x} & \phi_{2z} \\ \begin{bmatrix} 1.55 & 0 & -2.32 \\ 0 & 0.128 & 0 \\ -2.32 & 0 & 4.65 \end{bmatrix} \end{array} KN\!\big/\!_m$$

Plane Frame and Grid Equations

Grid Example 2

Beam Element 1:

The global stiffness matrix for element 1, considering only the parts associated with node 2, and the following relationship:

$$\mathbf{k}_G = \mathbf{T}_G{}^T\mathbf{k}'_G\mathbf{T}_G$$

$$\mathbf{k}_G{}^{(1)} = 10^4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.55 & 0 & -2.32 \\ 0 & 0.128 & 0 \\ -2.32 & 0 & 4.65 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} kN\!\big/\!_m$$

$$\mathbf{k}_G{}^{(1)} = 10^4 \begin{array}{ccc} v_2 & \phi_{2x} & \phi_{2z} \\ \begin{bmatrix} 1.55 & 0 & -2.32 \\ 0 & 0.128 & 0 \\ -2.32 & 0 & 4.65 \end{bmatrix} \end{array} kN\!\big/\!_m$$
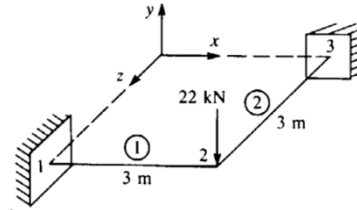
Plane Frame and Grid Equations

Grid Example 2

Beam Element 2: from nodes 3 to 2

$$C = \cos\theta = \frac{x_3 - x_2}{L^{(2)}} = \frac{0}{3} = 0$$

$$S = \sin\theta = \frac{z_3 - z_2}{L^{(1)}} = \frac{3}{3} = 1$$

$$\frac{12EI}{L^3} = \frac{12(210\times10^6)(16.6\times10^{-5})}{(3)^3} = 1.55\times10^4\,kN/m$$

$$\frac{6EI}{L^2} = \frac{6(210\times10^6)(16.6\times10^{-5})}{(3)^2} = 2.32\times10^4\,kN$$

$$\frac{4EI}{L} = \frac{4(210\times10^6)(16.6\times10^{-5})}{3} = 4.65\times10^4\,kN{\cdot}m$$

$$\frac{GJ}{L} = \frac{(84\times10^6)(4.6\times10^{-5})}{3} = 0.128\times10^4\,kN{\cdot}m$$

Plane Frame and Grid Equations

Grid Example 2

Beam Element 2:

The global stiffness matrix for element 2, considering only the parts associated with node 2, and the following relationship:

$$\mathbf{k}_G = \mathbf{T}_G{}^T \mathbf{k}_G' \mathbf{T}_G$$

$$\mathbf{T}_G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \qquad \mathbf{T}_G{}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{k}'^{(2)} = 10^4 \begin{array}{ccc} v_2 & \phi_{2x} & \phi_{2z} \\ \begin{bmatrix} 1.55 & 0 & -2.32 \\ 0 & 0.128 & 0 \\ -2.32 & 0 & 4.65 \end{bmatrix} \end{array} {}^{KN}\!\!/_m$$

Plane Frame and Grid Equations

Grid Example 2

Beam Element 2:

The global stiffness matrix for element 2, considering only the parts associated with node 2, and the following relationship:

$$\mathbf{k}_G = \mathbf{T}_G{}^T\mathbf{k}_G'\mathbf{T}_G$$

$$\mathbf{k}_G^{(2)} = 10^4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1.55 & 0 & -2.32 \\ 0 & 0.128 & 0 \\ -2.32 & 0 & 4.65 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} kN\!\!\Big/\!\!m$$

$$\begin{array}{ccc} v_2 & \phi_{2x} & \phi_{2z} \end{array}$$
$$\mathbf{k}_G^{(2)} = 10^4 \begin{bmatrix} 1.55 & 2.32 & 0 \\ 2.32 & 4.65 & 0 \\ 0 & 0 & 0.128 \end{bmatrix} kN\!\!\Big/\!\!m$$

Plane Frame and Grid Equations

Grid Example 2

Superimposing the two elemental stiffness matrices gives:

$$\begin{array}{ccc} v_2 & \phi_{2x} & \phi_{2z} \end{array}$$
$$\mathbf{K}_G = 10^4 \begin{bmatrix} 3.10 & 2.32 & -2.32 \\ 2.32 & 4.78 & 0 \\ -2.32 & 0 & 4.78 \end{bmatrix} kN\!\!\Big/\!\!m$$
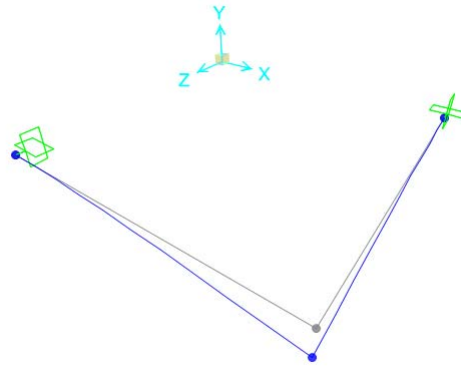
The global equations are:

$$\begin{Bmatrix} F_{2y} = -22\ kN \\ M_{2x} = 0 \\ M_{2z} = 0 \end{Bmatrix} = 10^4 \begin{bmatrix} 3.10 & 2.32 & -2.32 \\ 2.32 & 4.78 & 0 \\ -2.32 & 0 & 4.78 \end{bmatrix} \begin{Bmatrix} v_2 \\ \phi_{2x} \\ \phi_{2z} \end{Bmatrix}$$

Plane Frame and Grid Equations

Grid Example 2

Solving the above equations gives:
$$\begin{Bmatrix} v_2 \\ \phi_{2x} \\ \phi_{2z} \end{Bmatrix} = \begin{Bmatrix} -0.00259 \; m \\ 0.00126 \; rad \\ -0.00126 \; rad \end{Bmatrix}$$
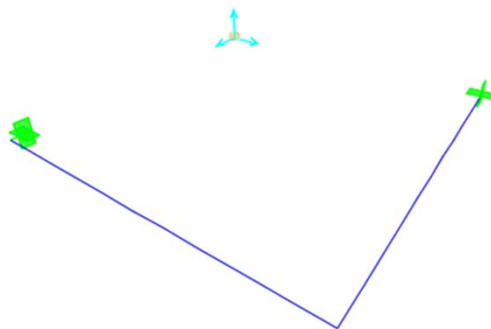

Plane Frame and Grid Equations

Grid Example 2

Solving the above equations gives:
$$\begin{Bmatrix} v_2 \\ \phi_{2x} \\ \phi_{2z} \end{Bmatrix} = \begin{Bmatrix} -0.00259 \; m \\ 0.00126 \; rad \\ -0.00126 \; rad \end{Bmatrix}$$

Plane Frame and Grid Equations

Grid Example 2

Solving the above equations gives:
$$\begin{Bmatrix} v_2 \\ \phi_{2x} \\ \phi_{2z} \end{Bmatrix} = \begin{Bmatrix} -0.00259 \ m \\ 0.00126 \ rad \\ -0.00126 \ rad \end{Bmatrix}$$

The results indicate that the *y* displacement at node 1 is downward as indicated by the minus sign.

The rotation about the *x*-axis is positive.

The rotation about the *z*-axis is negative.

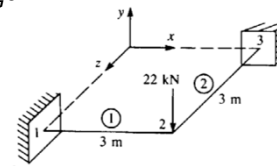Based on the downward loading location with respect to the supports, these results are expected.

Plane Frame and Grid Equations

Grid Example 2

Beam Element 1: The grid element force-displacement equations can be obtained using $\mathbf{f'} = \mathbf{k'}_G \mathbf{T}_G \mathbf{d}$

$$C = \cos\theta = \frac{x_2 - x_1}{L^{(1)}} = \frac{3}{3} = 1$$

$$S = \sin\theta = \frac{z_2 - z_1}{L^{(1)}} = \frac{0}{3} = 0$$

$$\mathbf{T}_G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C & S & 0 & 0 & 0 \\ 0 & -S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & S \\ 0 & 0 & 0 & 0 & -S & C \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Plane Frame and Grid Equations

Grid Example 2

Beam Element 1: The grid element force-displacement equations can be obtained using $\mathbf{f}' = \mathbf{k}'_G \mathbf{T}_G \mathbf{d}$

$$
\mathbf{T}_G\mathbf{d} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{Bmatrix}
0 \\
0 \\
0 \\
-0.00259\ m \\
0.00126\ rad \\
-0.00126\ rad
\end{Bmatrix}
=
\begin{Bmatrix}
0 \\
0 \\
0 \\
-0.00259\ m \\
0.00126\ rad \\
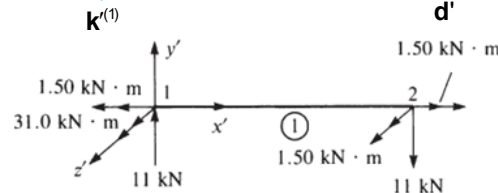-0.00126\ rad
\end{Bmatrix}
$$

Plane Frame and Grid Equations

Grid Example 2

Beam Element 1: The grid element force-displacement equations can be obtained using $\mathbf{f}' = \mathbf{k}'_G \mathbf{T}_G \mathbf{d}$

$$
\mathbf{f}'_{(1)} = \mathbf{k}'\mathbf{T}\mathbf{d} = 10^4
\begin{bmatrix}
1.55 & 0 & 2.32 & -1.55 & 0 & 2.32 \\
0 & 0.128 & 0 & 0 & -0.128 & 0 \\
2.32 & 0 & 4.65 & -2.32 & 0 & 2.33 \\
-1.55 & 0 & -2.32 & 1.55 & 0 & -2.32 \\
0 & -0.128 & 0 & 0 & 0.128 & 0 \\
2.32 & 0 & 2.33 & -2.32 & 0 & 4.65
\end{bmatrix}
\begin{Bmatrix}
0 \\
0 \\
0 \\
-0.00259\ m \\
0.00126\ rad \\
-0.00126\ rad
\end{Bmatrix}
$$

$$
\underbrace{\phantom{\begin{bmatrix} 1.55 & 0 & 2.32 & -1.55 & 0 & 2.32 \end{bmatrix}}}_{\mathbf{k}'^{(1)}}
\quad
\underbrace{\phantom{\begin{Bmatrix} 0 \end{Bmatrix}}}_{\mathbf{d}'}
$$

$$
\begin{Bmatrix}
f'_{1y} \\
m'_{1x} \\
m'_{1z} \\
f'_{2y} \\
m'_{2x} \\
m'_{2z}
\end{Bmatrix}
=
\begin{Bmatrix}
11.0\ kN \\
-1.50\ kN \cdot m \\
31.0\ kN \cdot m \\
-11.0\ kN \\
1.50\ kN \cdot m \\
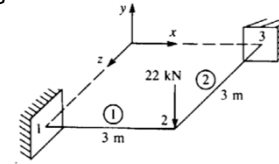1.50\ kN \cdot m
\end{Bmatrix}
$$

Plane Frame and Grid Equations

Grid Example 2

Beam Element 2: The grid element force-displacement equations can be obtained using $\mathbf{f}' = \mathbf{k}'_G \mathbf{T}_G \mathbf{d}$

$$C = \cos\theta = \frac{x_3 - x_2}{L^{(2)}} = \frac{0}{3} = 0$$

$$S = \sin\theta = \frac{z_3 - z_2}{L^{(1)}} = \frac{3}{3} = 1$$

$$
\mathbf{T}_G = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & C & S & 0 & 0 & 0 \\
0 & -S & C & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & C & S \\
0 & 0 & 0 & 0 & -S & C
\end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix}
$$

Plane Frame and Grid Equations

Grid Example 2

Beam Element 2: The grid element force-displacement equations can be obtained using $\mathbf{f}' = \mathbf{k}'_G \mathbf{T}_G \mathbf{d}$

$$
\mathbf{T}_G \mathbf{d} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix}
\begin{Bmatrix}
0 \\
0 \\
0 \\
-0.00259 \; m \\
0.00126 \; rad \\
-0.00126 \; rad
\end{Bmatrix}
=
\begin{Bmatrix}
0 \\
0 \\
0 \\
-0.00259 \; m \\
-0.00126 \; rad \\
-0.00126 \; rad
\end{Bmatrix}
$$

Plane Frame and Grid Equations

Grid Example 2

Beam Element 2: The grid element force-displacement equations can be obtained using $\mathbf{f'} = \mathbf{k'}_G \mathbf{T}_G \mathbf{d}$

$$\mathbf{f'}_{(2)} = \mathbf{k'Td} = 10^4 \begin{bmatrix} 1.55 & 0 & 2.32 & -1.55 & 0 & 2.32 \\ 0 & 0.128 & 0 & 0 & -0.128 & 0 \\ 2.32 & 0 & 4.65 & -2.32 & 0 & 2.33 \\ -1.55 & 0 & -2.32 & 1.55 & 0 & -2.32 \\ 0 & -0.128 & 0 & 0 & 0.128 & 0 \\ 2.32 & 0 & 2.33 & -2.32 & 0 & 4.65 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00259 \; m \\ -0.00126 \; rad \\ -0.00126 \; rad \end{Bmatrix}$$

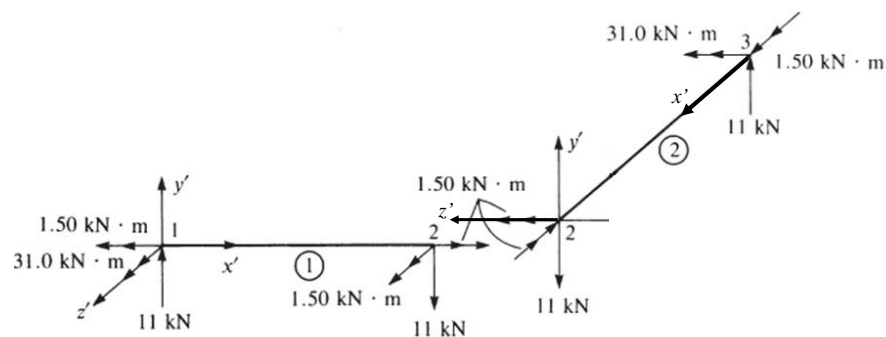$$\begin{Bmatrix} f'_{2y} \\ m'_{2x} \\ m'_{2z} \\ f'_{3y} \\ m'_{3x} \\ m'_{3z} \end{Bmatrix} = \begin{Bmatrix} -11.0 \; kN \\ -1.50 \; kN \cdot m \\ 1.50 \; kN \cdot m \\ 11.0 \; kN \\ 1.50 \; kN \cdot m \\ 31.0 \; kN \cdot m \end{Bmatrix}$$
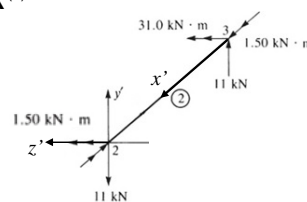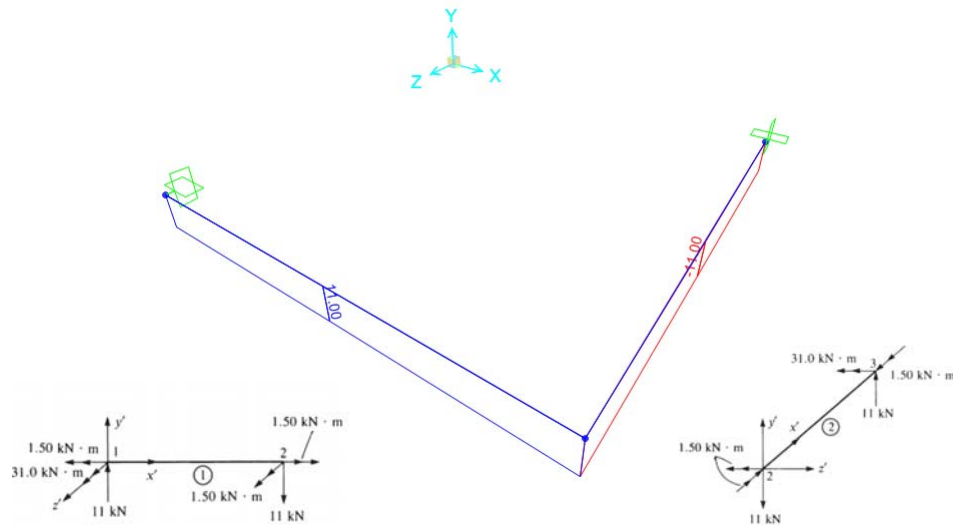

Plane Frame and Grid Equations

Grid Example 2
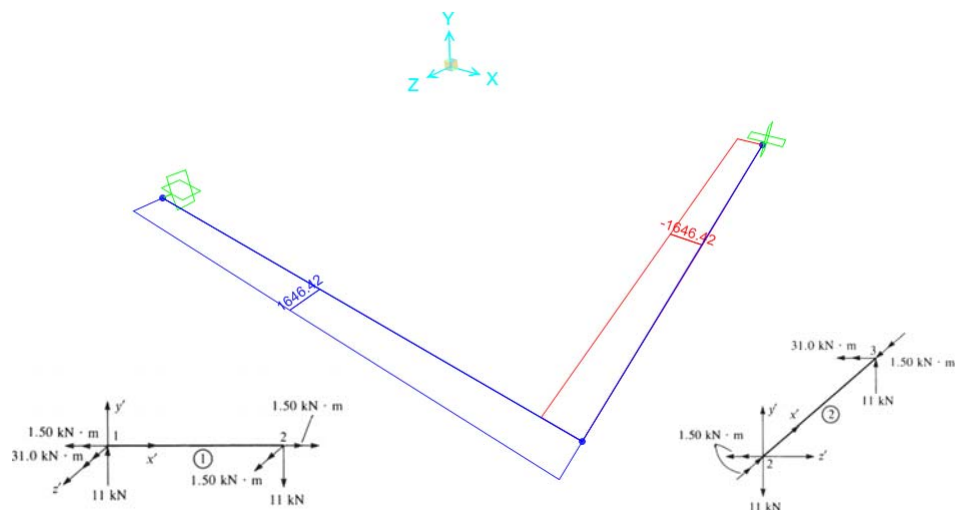
The resulting free-body diagrams:

Plane Frame and Grid Equations

Grid Example 2 – Forces in the y-direction

Plane Frame and Grid Equations

Grid Example 2 – Torsional Moment about the x' axis

Plane Frame and Grid Equations

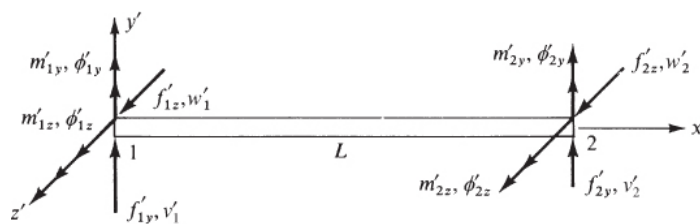*Grid Example 2* – Moment about the z' axis
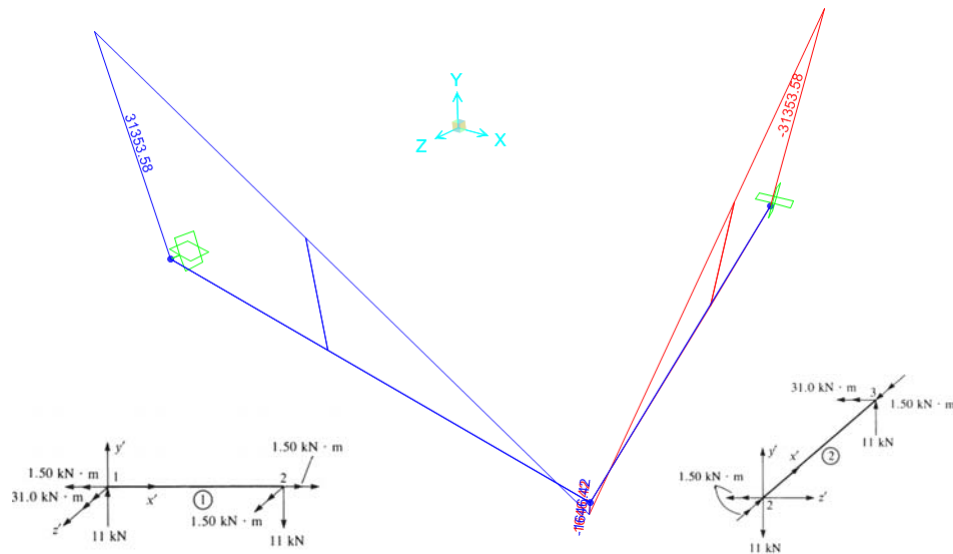
Plane Frame and Grid Equations

Beam Element Arbitrarily Oriented in Space

In this section, we will develop a beam element that is arbitrarily oriented in three-dimensions.

This element can be used to analyze three-dimensional frames.

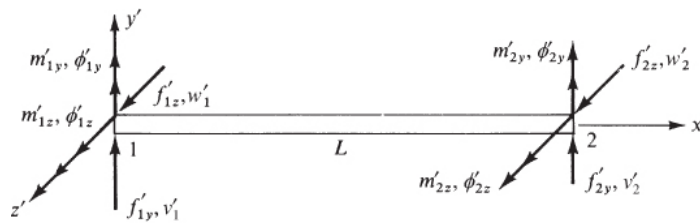Let consider bending about axes, as shown below.

Plane Frame and Grid Equations

Beam Element Arbitrarily Oriented in Space

The y' axis is the principle axis for which the moment of inertia is minimum, I_y

The right-hand rule is used to establish the z' axis and the maximum moment of inertia, I_z

Plane Frame and Grid Equations

Beam Element Arbitrarily Oriented in Space

Bending in the x'-z' plane: The bending in the x'-z' plane is defined by m'_y

The stiffness matrix for bending the in the x'-z' plane is:

$$\mathbf{k}'_y = \frac{EI_y}{L^4} \begin{bmatrix} 12L & 6L^2 & -12L & 6L^2 \\ 6L^2 & 4L^3 & -6L^2 & 2L^3 \\ -12L & -6L^2 & 12L & -6L^2 \\ 6L^2 & 2L^3 & -6L^2 & 4L^3 \end{bmatrix}$$

where I_y is the moment of inertia about the y' axis (the weak axis), therefore: $I_y < I_z$

Plane Frame and Grid Equations

Beam Element Arbitrarily Oriented in Space

Bending in the *x'-y'* plane: The bending in the *x'-y'* plane is defined by m'_z

The stiffness matrix for bending the in the *x'-z'* plane is:

$$\mathbf{k}'_z = \frac{EI_z}{L^4}\begin{bmatrix} 12L & 6L^2 & -12L & 6L^2 \\ 6L^2 & 4L^3 & -6L^2 & 2L^3 \\ -12L & -6L^2 & 12L & -6L^2 \\ 6L^2 & 2L^3 & -6L^2 & 4L^3 \end{bmatrix}$$

where I_z is the moment of inertia about the *z'* axis (the strong axis).

Plane Frame and Grid Equations

Beam Element Arbitrarily Oriented in Space

Direct superposition of the bending stiffness matrices with the effects of axial forces and torsional rotation give:

$$\mathbf{k}' = \begin{bmatrix}
\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \\
-\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 & \frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\
0 & 0 & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{bmatrix}$$
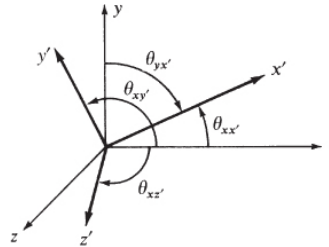
Plane Frame and Grid Equations

Beam Element Arbitrarily Oriented in Space

The global stiffness matrix may be obtained using: $\mathbf{k} = \mathbf{T}^T \mathbf{k}' \mathbf{T}$

where:
$$T = \begin{bmatrix} \lambda_{3 \times 3} & & & \\ & \lambda_{3 \times 3} & & \\ & & \lambda_{3 \times 3} & \\ & & & \lambda_{3 \times 3} \end{bmatrix} \qquad \lambda_{3 \times 3} = \begin{bmatrix} C_{xx'} & C_{yx'} & C_{zx'} \\ C_{xy'} & C_{yy'} & C_{zy'} \\ C_{xz'} & C_{yz'} & C_{zz'} \end{bmatrix}$$

the direction cosines, C_{ij}, are defined as shown below:

Plane Frame and Grid Equations

Beam Element Arbitrarily Oriented in Space
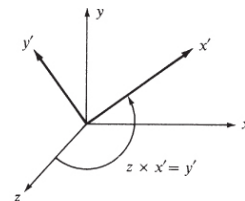
The direction cosines of the x' axis are:

$$x' = \cos\theta_{xx'}\mathbf{i} + \cos\theta_{yx'}\mathbf{j} + \cos\theta_{zx'}\mathbf{k}$$

$$\cos\theta_{xx'} = \frac{x_2 - x_1}{L} = l \qquad \cos\theta_{yx'} = \frac{y_2 - y_1}{L} = m \qquad \cos\theta_{zx'} = \frac{z_2 - z_1}{L} = n$$

The y' axis is selected to be perpendicular to the x' and the z axes is such a way that the cross product of global z with x' results in the y' axis as shown in the figure.

$$z \times x' = y' = \frac{1}{D} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ l & m & n \end{vmatrix} = -\frac{m}{D}\mathbf{i} + \frac{l}{D}\mathbf{j} \qquad D = \sqrt{l^2 + m^2}$$

Plane Frame and Grid Equations

Beam Element Arbitrarily Oriented in Space

The z' axis is determined by the condition that $z' = x' \times y'$

$$z' = x' \times y' = \frac{1}{D} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ l & m & n \\ -m & l & 0 \end{vmatrix} = -\frac{ln}{D}\mathbf{i} - \frac{mn}{D}\mathbf{j} + D\mathbf{k}$$

Therefore, the transformation matrix becomes:

$$\lambda_{3 \times 3} = \begin{bmatrix} C_{xx'} & C_{yx'} & C_{zx'} \\ C_{xy'} & C_{yy'} & C_{zy'} \\ C_{xz'} & C_{yz'} & C_{zz'} \end{bmatrix} = \begin{bmatrix} l & m & n \\ -\dfrac{m}{D} & \dfrac{l}{D} & 0 \\ -\dfrac{ln}{D} & -\dfrac{mn}{D} & D \end{bmatrix}$$

Plane Frame and Grid Equations

Beam Element Arbitrarily Oriented in Space

There are two exceptions that arise when using the above expressions for mapping the local coordinates to the global system:

(1) when the positive x' coincides with z

For the this case, it is assumed that y' is y.

$$\lambda = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$
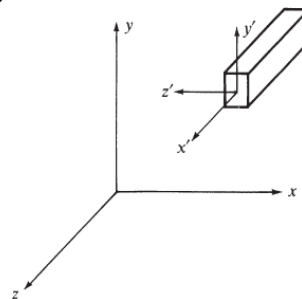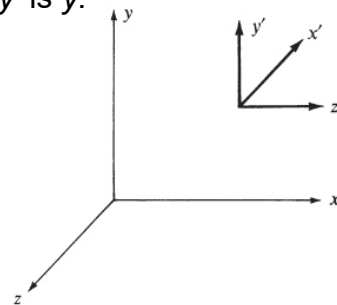
Plane Frame and Grid Equations

Beam Element Arbitrarily Oriented in Space

There are two exceptions that arise when using the above expressions for mapping the local coordinates to the global system:

(2) when the positive *x'* is in the opposite direction as *z*

For the second case, it is assumed that *y'* is *y*.

$$\lambda = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Plane Frame and Grid Equations

Beam Element Arbitrarily Oriented in Space

If the effects of axial force, both shear forces, twisting moment, and both bending moments are considered, the stiffness matrix for a frame element is:

$$\mathbf{k'} = \begin{bmatrix}
\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3(1+\Phi_y)} & 0 & 0 & 0 & \frac{6EI_z}{L^2(1+\Phi_y)} & 0 & \frac{-12EI_z}{L^3(1+\Phi_y)} & 0 & 0 & 0 & \frac{6EI_z}{L^2(1+\Phi_y)} \\
0 & 0 & \frac{12EI_y}{L^3(1+\Phi_z)} & 0 & \frac{-6EI_y}{L^2(1+\Phi_z)} & 0 & 0 & 0 & \frac{-12EI_y}{L^3(1+\Phi_z)} & 0 & \frac{-6EI_y}{L^2(1+\Phi_z)} & 0 \\
0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 \\
0 & 0 & \frac{-6EI_y}{L^2(1+\Phi_z)} & 0 & \frac{(4+\Phi_z)EI_y}{L(1+\Phi_z)} & 0 & 0 & 0 & \frac{6EI_y}{L^2(1+\Phi_z)} & 0 & \frac{(2-\Phi_z)EI_y}{L(1+\Phi_z)} & 0 \\
0 & \frac{6EI_z}{L^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(4+\Phi_y)EI_z}{L(1+\Phi_y)} & 0 & \frac{-6EI_z}{L^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(2-\Phi_y)EI_z}{L(1+\Phi_y)} \\
-\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 & \frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-12EI_z}{L^3(1+\Phi_y)} & 0 & 0 & 0 & \frac{-6EI_z}{L^2(1+\Phi_y)} & 0 & \frac{12EI_z}{L^3(1+\Phi_y)} & 0 & 0 & 0 & \frac{-6EI_z}{L^2(1+\Phi_y)} \\
0 & 0 & \frac{-12EI_y}{L^3(1+\Phi_z)} & 0 & \frac{6EI_y}{L^2(1+\Phi_z)} & 0 & 0 & 0 & \frac{12EI_y}{L^3(1+\Phi_z)} & 0 & \frac{6EI_y}{L^2(1+\Phi_z)} & 0 \\
0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\
0 & 0 & \frac{-6EI_y}{L^2(1+\Phi_z)} & 0 & \frac{(2-\Phi_z)EI_y}{L(1+\Phi_z)} & 0 & 0 & 0 & \frac{6EI_y}{L^2(1+\Phi_z)} & 0 & \frac{(4+\Phi_z)EI_y}{L(1+\Phi_z)} & 0 \\
0 & \frac{6EI_z}{L^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(2-\Phi_y)EI_z}{L(1+\Phi_y)} & 0 & \frac{-6EI_z}{L^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(4+\Phi_y)EI_z}{L(1+\Phi_y)}
\end{bmatrix}$$

Plane Frame and Grid Equations

Beam Element Arbitrarily Oriented in Space

In this case the symbol Φ are: $\Phi_y = \dfrac{12EI_y}{GA_sL^2}$ $\Phi_z = \dfrac{12EI_z}{GA_sL^2}$

Where A_s is the effective beam cross-section in shear.

$$k' = \begin{bmatrix}
\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3(1+\Phi_y)} & 0 & 0 & 0 & \frac{6EI_z}{L^2(1+\Phi_y)} & 0 & \frac{-12EI_z}{L^3(1+\Phi_y)} & 0 & 0 & 0 & \frac{6EI_z}{L^2(1+\Phi_y)} \\
0 & 0 & \frac{12EI_y}{L^3(1+\Phi_z)} & 0 & \frac{-6EI_y}{L^2(1+\Phi_z)} & 0 & 0 & 0 & \frac{-12EI_y}{L^3(1+\Phi_z)} & 0 & \frac{-6EI_y}{L^2(1+\Phi_z)} & 0 \\
0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 \\
0 & 0 & \frac{-6EI_y}{L^2(1+\Phi_z)} & 0 & \frac{(4+\Phi_z)EI_y}{L(1+\Phi_z)} & 0 & 0 & 0 & \frac{6EI_y}{L^2(1+\Phi_z)} & 0 & \frac{(2-\Phi_z)EI_y}{L(1+\Phi_z)} & 0 \\
0 & \frac{6EI_z}{L^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(4+\Phi_y)EI_z}{L(1+\Phi_y)} & 0 & \frac{-6EI_z}{L^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(2-\Phi_y)EI_z}{L(1+\Phi_y)} \\
-\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 & \frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-12EI_z}{L^3(1+\Phi_y)} & 0 & 0 & 0 & \frac{-6EI_z}{L^2(1+\Phi_y)} & 0 & \frac{12EI_z}{L^3(1+\Phi_y)} & 0 & 0 & 0 & \frac{-6EI_z}{L^2(1+\Phi_y)} \\
0 & 0 & \frac{-12EI_y}{L^3(1+\Phi_z)} & 0 & \frac{6EI_y}{L^2(1+\Phi_z)} & 0 & 0 & 0 & \frac{12EI_y}{L^3(1+\Phi_z)} & 0 & \frac{6EI_y}{L^2(1+\Phi_z)} & 0 \\
0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\
0 & 0 & \frac{-6EI_y}{L^2(1+\Phi_z)} & 0 & \frac{(2-\Phi_z)EI_y}{L(1+\Phi_z)} & 0 & 0 & 0 & \frac{6EI_y}{L^2(1+\Phi_z)} & 0 & \frac{(4+\Phi_z)EI_y}{L(1+\Phi_z)} & 0 \\
0 & \frac{6EI_z}{L^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(2-\Phi_y)EI_z}{L(1+\Phi_y)} & 0 & \frac{-6EI_z}{L^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(4+\Phi_y)EI_z}{L(1+\Phi_y)}
\end{bmatrix}$$

Plane Frame and Grid Equations

Beam Element Arbitrarily Oriented in Space

Recall the shear modulus of elasticity or the modulus of rigidity, G, is related to the modulus of elasticity and the Poisson's ratio, v as:

$$G = \frac{E}{2(1+v)}$$

This is the form of the stiffness matrix used by SAP2000 for its frame element.

Plane Frame and Grid Equations
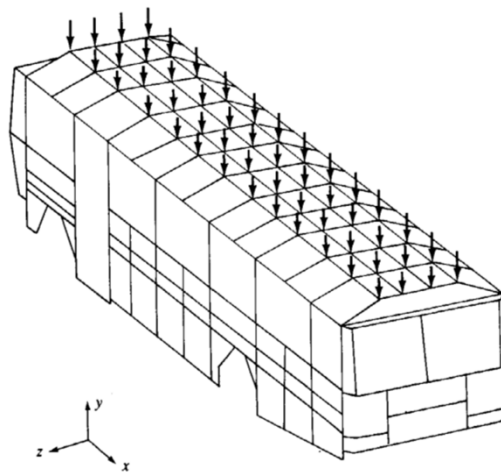
Beam Element Arbitrarily Oriented in Space

If Φ_y and Φ_z are set to zero, the stiffness matrix reduces to:

$$
k' = \begin{bmatrix}
\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} & 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \\
-\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 & \frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-12EI_z}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2} \\
0 & 0 & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{bmatrix}
$$

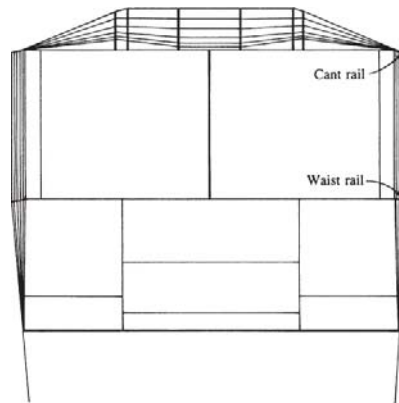## *Plane Frame and Grid Equations*

Example Frame Application

A bus subjected to a static roof-crush analysis. In this model 599 frame elements and 357 nodes are used.

Plane Frame and Grid Equations

Example Frame Application

A bus subjected to a static roof-crush analysis. In this model 599 frame elements and 357 nodes are used.

Plane Frame and Grid Equations

Concept of Substructure Analysis

Sometimes structures are too large to be analyzed as a single system or treated as a whole; that is, the final stiffness matrix and equations for solution exceed the memory capacity of the computer.

A procedure to overcome this problem is to separate the whole structure into smaller units called *substructures*.

Plane Frame and Grid Equations

Concept of Substructure Analysis

For example, the space frame of an airplane, as shown below, may require thousands of nodes and elements to completely model and describe the response of the whole structure.

If we separate the aircraft into substructures, such as parts of the fuselage or body, wing sections, etc., as shown below, then we can solve the problem more readily and on computers with limited memory.

Beam Stiffness

Problems:

10. Do problems **5.1, 5.7, 5.15, 5.28,** and **5.51** in your textbook.

11. Do problems **5.20, 5.23, 5.25, 5.35**, and **5.53** on pages 308 - 321 in your textbook. You may use the **SAP2000** to do frame analysis.

End of Chapter 5b