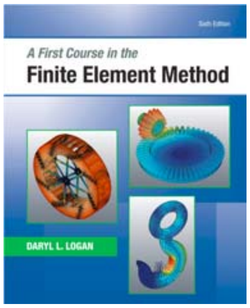
Chapter 5a – Plane Frame and Grid Equations



Learning Objectives

- To derive the two-dimensional arbitrarily oriented beam element stiffness matrix
- To demonstrate solutions of rigid plane frames by the direct stiffness method
- To describe how to handle inclined or skewed supports

Plane Frame and Grid Equations

Many structures, such as buildings and bridges, are composed of frames and/or grids.

This chapter develops the equations and methods for solution of plane frames and grids.

First, we will develop the stiffness matrix for a beam element arbitrarily oriented in a plane.

Plane Frame and Grid Equations

Many structures, such as buildings and bridges, are composed of frames and/or grids.

We will then include the axial nodal displacement degree of freedom in the local beam element stiffness matrix.

Plane Frame and Grid Equations

Many structures, such as buildings and bridges, are composed of frames and/or grids.

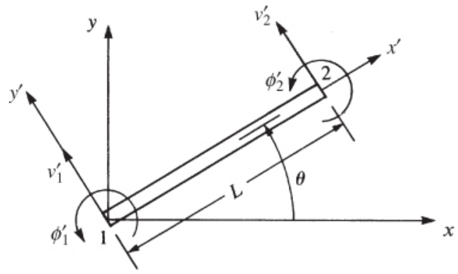
Then we will combine these results to develop the stiffness matrix, including axial deformation effects, for an arbitrarily oriented beam element.

We will also consider frames with inclined or skewed supports.

Plane Frame and Grid Equations

Two-Dimensional Arbitrarily Oriented Beam Element

We can derive the stiffness matrix for an arbitrarily oriented beam element, in a manner similar to that used for the bar element.



The local axes and are located along the beam element and transverse to the beam element, respectively, and the global axes *x'* and *y'* are located to be convenient for the total structure.

Plane Frame and Grid Equations

Two-Dimensional Arbitrarily Oriented Beam Element

The transformation from local displacements to global displacements is given in matrix form as:

$$\begin{Bmatrix} u' \\ v' \end{Bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \qquad \begin{matrix} C = \cos\theta \\ S = \sin\theta \end{matrix}$$

Using the second equation for the beam element, we can relate local nodal degrees of freedom to global degree of freedom:

$$\begin{Bmatrix} v'_1 \\ \phi'_1 \\ v'_2 \\ \phi'_2 \end{Bmatrix} = \left[\begin{array}{ccc|ccc} -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

$$v'_1 = -Su_1 + Cv_1$$

$$\mathbf{d}' = \mathbf{T}\mathbf{d}$$

Plane Frame and Grid Equations

Two-Dimensional Arbitrarily Oriented Beam Element

For a beam, we will define the following as the ***transformation matrix***:

$$\mathbf{T} = \left[\begin{array}{ccc|ccc} -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Notice that the rotations are not affected by the orientation of the beam.

Plane Frame and Grid Equations

Two-Dimensional Arbitrarily Oriented Beam Element

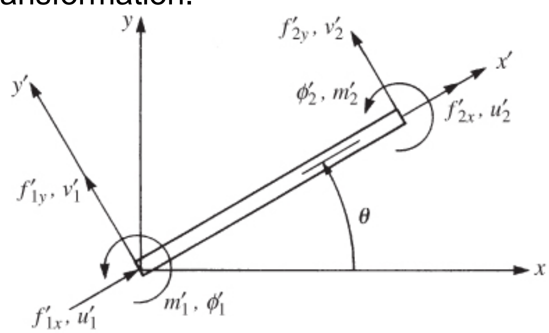
Substituting the above transformation into the general form of the stiffness matrix $\mathbf{T}^T\mathbf{k}'\mathbf{T}$ gives:

$$\begin{array}{cc} & \begin{array}{cccccc} \quad u_1 & \quad v_1 & \quad \phi_1 & \quad u_2 & \quad v_2 & \quad \phi_2 \end{array} \\ \mathbf{k} = \dfrac{EI}{L^3} & \left[\begin{array}{ccc|ccc} 12S^2 & -12SC & -6LS & -12S^2 & 12SC & -6LS \\ -12SC & 12C^2 & 6LC & 12SC & -12C^2 & 6LC \\ -6LS & 6LC & 4L^2 & 6LS & -6LC & 2L^2 \\ \hline -12S^2 & 12SC & 6LS & 12S^2 & -12SC & 6LS \\ 12SC & -12C^2 & -6LC & -12SC & 12C^2 & -6LC \\ -6LS & 6LC & 2L^2 & 6LS & -6LC & 4L^2 \end{array}\right] \end{array}$$

Plane Frame and Grid Equations

Two-Dimensional Arbitrarily Oriented Beam Element

Let's now consider the effects of an axial force in the general beam transformation.



Recall the simple axial deformation, define in the spring element:

$$\begin{Bmatrix} f'_{1x} \\ f'_{2x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix}$$

Plane Frame and Grid Equations

Two-Dimensional Arbitrarily Oriented Beam Element

Combining the axial effects with the shear force and bending moment effects, in local coordinates gives:

$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ m'_1 \\ f'_{2x} \\ f'_{2y} \\ m'_2 \end{Bmatrix} = \left[\begin{array}{ccc|ccc} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ \hline -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{array}\right]\begin{Bmatrix} u'_1 \\ v'_1 \\ \phi'_1 \\ u'_2 \\ v'_2 \\ \phi'_2 \end{Bmatrix}$$

$$C_1 = \frac{AE}{L} \qquad C_2 = \frac{EI}{L^3}$$

Plane Frame and Grid Equations

Two-Dimensional Arbitrarily Oriented Beam Element

Therefore:

$$\mathbf{k}' = \left[\begin{array}{ccc|ccc} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ \hline -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{array}\right]$$

The above stiffness matrix include the effects of axial force in the *x*' direction, shear force in the *y*' direction, and bending moment about the *z*' axis.

Plane Frame and Grid Equations

Two-Dimensional Arbitrarily Oriented Beam Element

The local degrees of freedom may be related to the global degrees of freedom by:

$$\left\{\begin{array}{c} u_1' \\ v_1' \\ \phi_1' \\ \hline u_2' \\ v_2' \\ \phi_2' \end{array}\right\} = \left[\begin{array}{ccc|ccc} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left\{\begin{array}{c} u_1 \\ v_1 \\ \phi_1 \\ \hline u_2 \\ v_2 \\ \phi_2 \end{array}\right\} \qquad \mathbf{d}' = \mathbf{T}\mathbf{d}$$

where **T** has been expanded to include axial effects

Plane Frame and Grid Equations

Two-Dimensional Arbitrarily Oriented Beam Element

Substituting the above transformation **T** into the general form of the stiffness matrix gives:

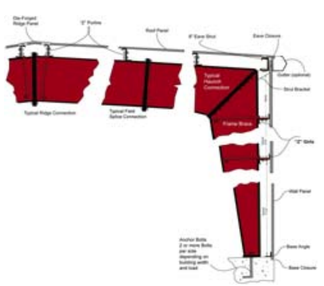
$$[k] = \frac{E}{L}\begin{bmatrix} AC^2 + \frac{12I}{L^2}S^2 & \left(A - \frac{12I}{L^2}\right)CS & -\frac{6I}{L}S & -\left(AC^2 + \frac{12I}{L^2}S^2\right) & -\left(A - \frac{12I}{L^2}\right)CS & -\frac{6I}{L}S \\ & AS^2 + \frac{12I}{L^2}C^2 & \frac{6I}{L}C & -\left(A - \frac{12I}{L^2}\right)CS & -\left(AS^2 + \frac{12I}{L^2}C^2\right) & \frac{6I}{L}C \\ & & 4I & \frac{6I}{L}S & -\frac{6I}{L}C & 2I \\ & & & AC^2 + \frac{12I}{L^2}S^2 & \left(A - \frac{12I}{L^2}\right)CS & \frac{6I}{L}S \\ & \textit{symmetric} & & & AS^2 + \frac{12I}{L^2}C^2 & -\frac{6I}{L}C \\ & & & & & 4I \end{bmatrix}$$

Plane Frame and Grid Equations

Two-Dimensional Arbitrarily Oriented Beam Element

The analysis of a rigid plane frame can be accomplished by applying stiffness matrix.

A ***rigid plane frame*** is: *a series of beam elements rigidly connected to each other*; that is, the original angles made between elements at their joints remain unchanged after the deformation.

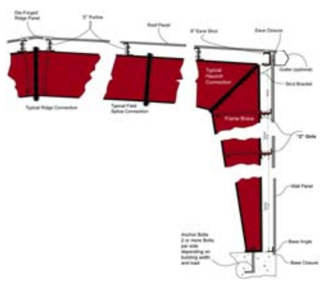


Plane Frame and Grid Equations

Two-Dimensional Arbitrarily Oriented Beam Element

Furthermore, moments are transmitted from one element to another at the joints.

Hence, moment continuity exists at the rigid joints.

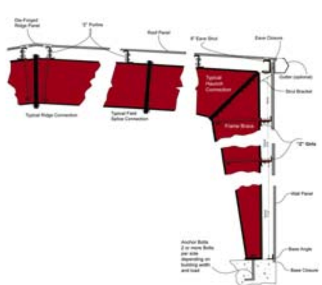


Plane Frame and Grid Equations

Two-Dimensional Arbitrarily Oriented Beam Element

In addition, the element centroids, as well as the applied loads, lie in a common plane.

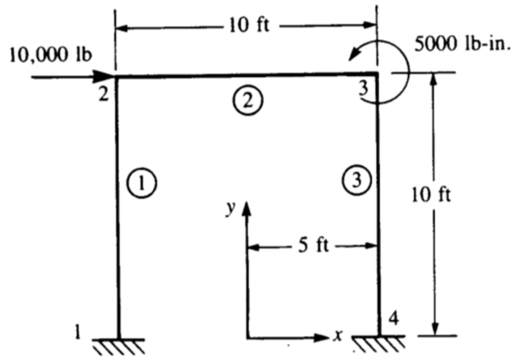
We observe that the element stiffnesses of a frame are functions of *E, A, L, I*, and the angle of orientation θ of the element with respect to the global-coordinate axes.



Plane Frame and Grid Equations

Rigid Plane Frame Example 1

Consider the frame shown in the figure below.

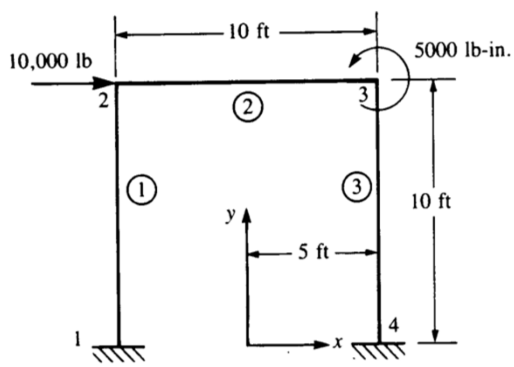


The frame is fixed at nodes 1 and 4 and subjected to a positive horizontal force of 10,000 *lb* applied at node 2 and to a positive moment of 5,000 *lb-in* applied at node 3.

Plane Frame and Grid Equations

Rigid Plane Frame Example 1

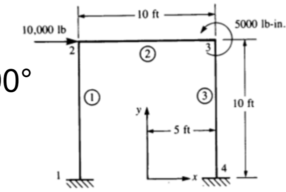
Consider the frame shown in the figure below.



Let $E = 30 \times 10^6$ *psi* and $A = 10$ in^2 for all elements, and let $I = 200$ in^4 for elements 1 and 3, and $I = 100$ in^4 for element 2.

Plane Frame and Grid Equations

Rigid Plane Frame Example 1



Element 1: The angle between *x* and *x'* is 90°

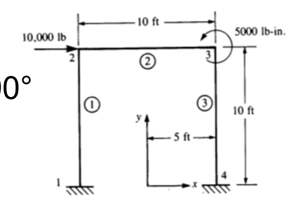
$$C = 0 \qquad S = 1$$

$$\frac{12I}{L^2} = \frac{12(200)}{(120)^2} = 0.167\ in^2 \qquad \frac{6I}{L} = \frac{6(200)}{120} = 10.0\ in^3$$

$$\frac{E}{L} = \frac{30 \times 10^6}{120} = 250{,}000\ {}^{lb}\!/_{in^3}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 1

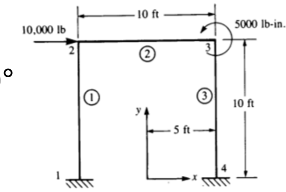


Element 1: The angle between *x* and *x'* is 90°

$$\mathbf{k}^{(1)} = 250{,}000 \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & \phi_1 & u_2 & v_2 & \phi_2 \end{array} \\ \left[\begin{array}{ccc|ccc} 0.167 & 0 & -10 & -0.167 & 0 & -10 \\ 0 & 10 & 0 & 0 & -10 & 0 \\ -10 & 0 & 800 & 10 & 0 & 400 \\ \hline -0.167 & 0 & 10 & 0.167 & 0 & 10 \\ 0 & -10 & 0 & 0 & 10 & 0 \\ -10 & 0 & 400 & 10 & 0 & 800 \end{array}\right] \end{array} {}^{lb}\!/_{in}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 1



Element 2: The angle between x and x' is 0°

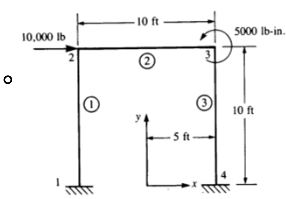
$$C = 1 \qquad S = 0$$

$$\frac{12I}{L^2} = \frac{12(100)}{(120)^2} = 0.0835\ in^2 \qquad \frac{6I}{L} = \frac{6(100)}{120} = 5.0\ in^3$$

$$\frac{E}{L} = \frac{30 \times 10^6}{120} = 250{,}000\ {}^{lb}\!/\!_{in^3}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 1

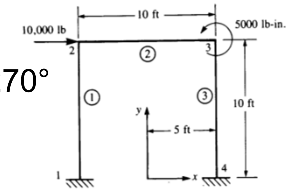


Element 2: The angle between x and x' is 0°

$$\begin{array}{cc} & \begin{array}{cccccc} u_2 & v_2 & \phi_2 & u_3 & v_3 & \phi_3 \end{array} \\ \mathbf{k}^{(2)} = 250{,}000 & \left[\begin{array}{ccc|ccc} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.0835 & 5 & 0 & 0.0835 & 5 \\ 0 & 5 & 400 & 0 & -5 & 200 \\ \hline -10 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0.0835 & -5 & 0 & 0.0835 & -5 \\ 0 & 5 & 200 & 0 & -5 & 400 \end{array}\right] {}^{lb}\!/\!_{in} \end{array}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 1



Element 3: The angle between x and x' is 270°

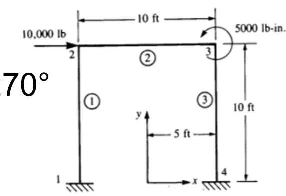
$$C = 0 \qquad S = -1$$

$$\frac{12I}{L^2} = \frac{12(200)}{(120)^2} = 0.167\ in^2 \qquad \frac{6I}{L} = \frac{6(200)}{120} = 10.0\ in^3$$

$$\frac{E}{L} = \frac{30 \times 10^6}{120} = 250{,}000\ {}^{lb}\!/\!_{in^3}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 1



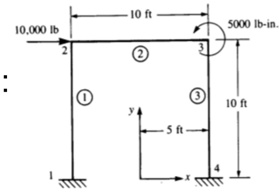
Element 3: The angle between x and x' is 270°

$$\mathbf{k}^{(3)} = 250{,}000 \begin{array}{c} \begin{array}{cccccc} u_3 & v_3 & \phi_3 & u_4 & v_4 & \phi_4 \end{array} \\ \left[\begin{array}{ccc|ccc} 0.167 & 0 & 10 & -0.167 & 0 & 10 \\ 0 & 10 & 0 & 0 & -10 & 0 \\ 10 & 0 & 800 & -10 & 0 & 400 \\ \hline -0.167 & 0 & -10 & 0.167 & 0 & -10 \\ 0 & -10 & 0 & 0 & 10 & 0 \\ 10 & 0 & 400 & -10 & 0 & 800 \end{array}\right] \end{array} {}^{lb}\!/\!_{in}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 1

The boundary conditions for this problem are:



$$u_1 = v_1 = \phi_1 = u_4 = v_4 = \phi_4 = 0$$

After applying the boundary conditions the global beam equations reduce to:

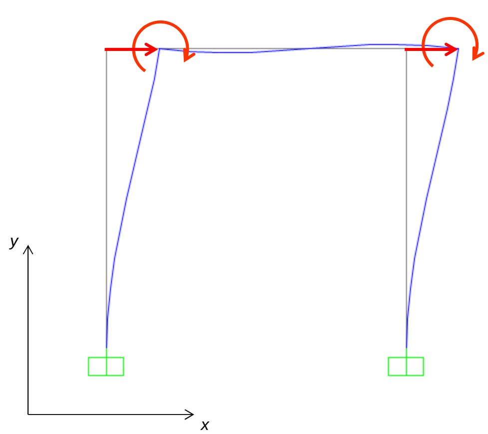
$$\begin{Bmatrix} 10{,}000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5{,}000 \end{Bmatrix} = 2.5\times 10^5 \begin{array}{c} \begin{array}{cccccc} u_2 & v_2 & \phi_2 & u_3 & v_3 & \phi_3 \end{array} \\ \left[\begin{array}{ccc|ccc} 10.167 & 0 & 10 & -10 & 0 & 0 \\ 0 & 10.0835 & 5 & 0 & -0.0835 & 5 \\ 10 & 5 & 1200 & 0 & -5 & 200 \\ \hline -10 & 0 & 0 & 10.167 & 0 & 10 \\ 0 & -0.0835 & -5 & 0 & 10.0835 & -5 \\ 0 & 5 & 200 & 10 & -5 & 1200 \end{array}\right] \end{array} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \\ u_3 \\ v_3 \\ \phi_3 \end{Bmatrix}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 1

Solving the above equations gives:

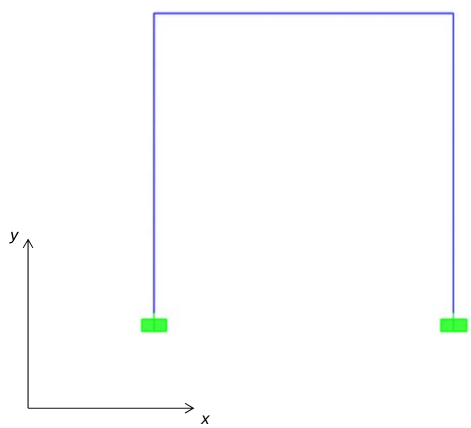
$$\begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \\ u_3 \\ v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0.211\,in \\ 0.00148\,in \\ -0.00153\,rad \\ 0.209\,in \\ -0.00148\,in \\ -0.00149\,rad \end{Bmatrix}$$



Plane Frame and Grid Equations

Rigid Plane Frame Example 1

Solving the above equations gives:



Plane Frame and Grid Equations

Rigid Plane Frame Example 1

Element 1: The element force-displacement equations can be obtained using **f' = k'Td**. Therefore, **Td** is:

$$T = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad C = 0 \qquad S = 1$$

$$\mathbf{Td} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ \phi_1 = 0 \\ u_2 = 0.211\,in \\ v_2 = 0.00148\,in \\ \phi_2 = -0.00153\,rad \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.00148\,in \\ -0.211\,in \\ -0.00153\,rad \end{Bmatrix}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 1

Element 1: Recall the elemental stiffness matrix is:

$$\mathbf{k}' = \left[\begin{array}{ccc|ccc} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ \hline -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{array}\right]$$

$$C_1 = \frac{AE}{L} = \frac{10\,in^2\left(30\times10^6\,psi\right)}{120\,in} = 2.5\times10^6\ {}^{lb}\!/_{in}$$

$$C_2 = \frac{EI}{L^3} = \frac{200\,in^4\left(30\times10^6\,psi\right)}{\left(120\,in\right)^3} = 3{,}472.22\ {}^{lb}\!/_{in}$$

The local force-displacement equations are:

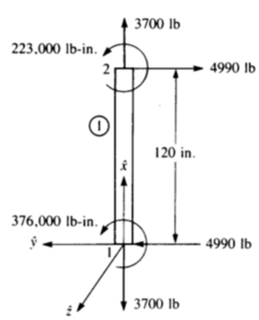
$$\mathbf{f}'^{(1)} = \mathbf{k}'\mathbf{T}\mathbf{d} = 2.5\times10^5 \left[\begin{array}{ccc|ccc} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.167 & 10 & 0 & -0.167 & 10 \\ 0 & 10 & 800 & 0 & -10 & 400 \\ \hline -10 & 0 & 0 & 10 & 0 & 10 \\ 0 & -0.167 & -10 & 0 & 0.167 & -10 \\ 0 & 10 & 400 & 0 & -10 & 800 \end{array}\right] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.00148\,in \\ -0.211\,in \\ -0.00153\,rad \end{Bmatrix}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 1

Element 1: Simplifying the above equations gives:

$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ m'_1 \\ f'_{2x} \\ f'_{2y} \\ m'_2 \end{Bmatrix} = \begin{Bmatrix} -3{,}700\,lb \\ 4{,}990\,lb \\ 376\,k\cdot in \\ 3{,}700\,lb \\ -4{,}990\,lb \\ 223\,k\cdot in \end{Bmatrix}$$



Plane Frame and Grid Equations

Rigid Plane Frame Example 1

Element 2: The element force-displacement equations can be obtained using **f' = k'Td**. Therefore, **Td** is:

$$T = \left[\begin{array}{ccc|ccc} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \qquad C = 1 \qquad S = 0$$

$$\mathbf{Td} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_2 = 0.211\,in \\ v_2 = 0.00148\,in \\ \phi_2 = -0.00153\,rad \\ u_3 = 0.209\,in \\ v_3 = -0.00148\,in \\ \phi_3 = -0.00149\,rad \end{Bmatrix} = \begin{Bmatrix} 0.211\,in \\ 0.00148\,in \\ -0.00153\,rad \\ 0.209\,in \\ -0.00148\,in \\ -0.00149\,rad \end{Bmatrix}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 1

Element 2: The local force-displacement equations are:

$$\mathbf{k}' = \left[\begin{array}{ccc|ccc} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ \hline -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{array}\right]$$

$$C_1 = \frac{AE}{L} = \frac{10\,in^2\left(30\times10^6\,psi\right)}{120\,in} = 2.5\times10^6\,{}^{lb}\!/_{in}$$

$$C_2 = \frac{EI}{L^3} = \frac{100\,in^4\left(30\times10^6\,psi\right)}{\left(120\,in\right)^3} = 1{,}736.11\,{}^{lb}\!/_{in}$$

The local force-displacement equations are:

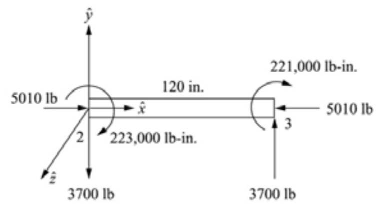
$$\mathbf{f}'^{(2)} = \mathbf{k}'\mathbf{Td} = 2.5\times10^5 \left[\begin{array}{ccc|ccc} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.0833 & 5 & 0 & -0.0833 & 5 \\ 0 & 5 & 400 & 0 & -5 & 200 \\ \hline -10 & 0 & 0 & 10 & 0 & 0 \\ 0 & -0.0833 & -5 & 0 & 0.0833 & -5 \\ 0 & 5 & 200 & 0 & -5 & 400 \end{array}\right] \begin{Bmatrix} 0.211\,in \\ 0.00148\,in \\ -0.00153\,rad \\ 0.209\,in \\ -0.00148\,in \\ -0.00149\,rad \end{Bmatrix}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 1

Element 2: Simplifying the above equations gives:

$$\begin{Bmatrix} f'_{2x} \\ f'_{2y} \\ m'_2 \\ f'_{3x} \\ f'_{3y} \\ m'_3 \end{Bmatrix} = \begin{Bmatrix} 5{,}010\ lb \\ -3{,}700\ lb \\ -223\ k \cdot in \\ -5{,}010\ lb \\ 3{,}700\ lb \\ -221\ k \cdot in \end{Bmatrix}$$



Plane Frame and Grid Equations

Rigid Plane Frame Example 1

Element 3: The element force-displacement equations can be obtained using **f' = k'Td**. Therefore, **Td** is:

$$T = \left[\begin{array}{ccc|ccc} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \qquad C = 0 \qquad S = -1$$

$$\mathbf{Td} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_3 = 0.209\ in \\ v_3 = -0.00148\ in \\ \phi_3 = -0.00149\ rad \\ u_4 = 0 \\ v_4 = 0 \\ \phi_4 = 0 \end{Bmatrix} = \begin{Bmatrix} 0.00148\ in \\ 0.209\ in \\ -0.00149\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 1

Element 3: The local force-displacement equations are:

$$\mathbf{k}' = \left[\begin{array}{ccc|ccc} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ \hline -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{array}\right]$$

$$C_1 = \frac{AE}{L} = \frac{10\,in^2\left(30\times10^6\,psi\right)}{120\,in} = 2.5\times10^6\ {}^{lb}\!/_{in}$$

$$C_2 = \frac{EI}{L^3} = \frac{200\,in^4\left(30\times10^6\,psi\right)}{\left(120\,in\right)^3} = 3{,}472.22\ {}^{lb}\!/_{in}$$

The local force-displacement equations are:

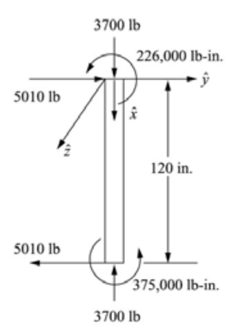
$$\mathbf{f}'^{(3)} = \mathbf{k}'\mathbf{T}\mathbf{d} = 2.5\times10^5 \left[\begin{array}{ccc|ccc} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.167 & 10 & 0 & -0.167 & 10 \\ 0 & 10 & 800 & 0 & -10 & 400 \\ \hline -10 & 0 & 0 & 10 & 0 & 10 \\ 0 & -0.167 & -10 & 0 & 0.167 & -10 \\ 0 & 10 & 400 & 0 & -10 & 800 \end{array}\right] \begin{Bmatrix} 0.00148\,in \\ 0.209\,in \\ -0.00149\,rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 1

Element 3: Simplifying the above equations gives:

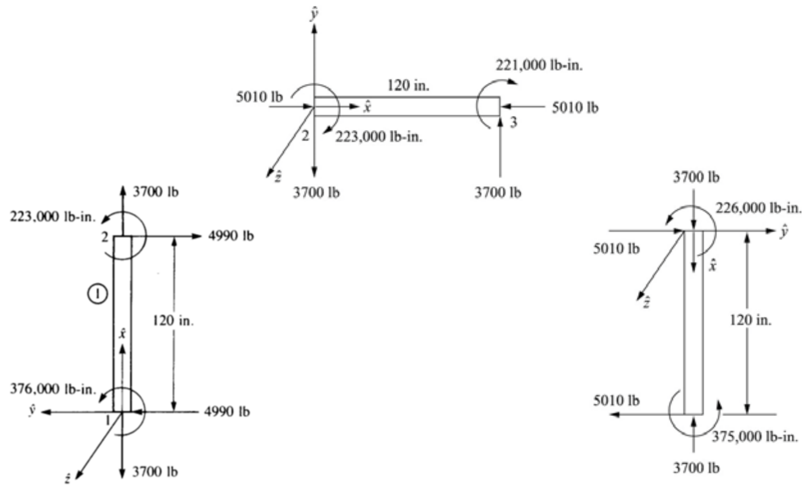
$$\begin{Bmatrix} f'_{3x} \\ f'_{3y} \\ m'_3 \\ f'_{4x} \\ f'_{4y} \\ m'_4 \end{Bmatrix} = \begin{Bmatrix} 3{,}700\,lb \\ 5{,}010\,lb \\ 226\,k\cdot in \\ -3{,}700\,lb \\ -5{,}010\,lb \\ 375\,k\cdot in \end{Bmatrix}$$



Plane Frame and Grid Equations

Rigid Plane Frame Example 1

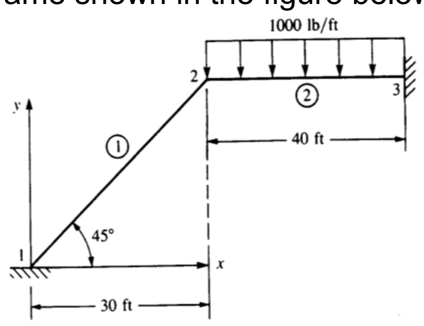
Check the equilibrium of all the elements:



Plane Frame and Grid Equations

Rigid Plane Frame Example 2

Consider the frame shown in the figure below.



The frame is fixed at nodes 1 and 3 and subjected to a positive distributed load of 1,000 *lb/ft* applied along element 2.

Let E = 30 x 10^6 *psi* and A = 100 in^2 for all elements, and let I = 1,000 in^4 for all elements.

Plane Frame and Grid Equations

Rigid Plane Frame Example 2

First we need to replace the distributed load with a set of equivalent nodal forces and moments acting at nodes 2 and 3.

For a beam with both end fixed, subjected to a uniform distributed load, *w*, the nodal forces and moments are:

$$f_{2y} = f_{3y} = -\frac{wL}{2} = -\frac{(1{,}000\,lb/ft)40\,ft}{2} = -20k$$

$$m_2 = -m_3 = -\frac{wL^2}{12} = -\frac{(1{,}000\,lb/ft)(40\,ft)^2}{12} = -133{,}333\ lb \cdot ft$$

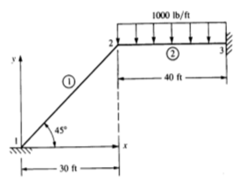
$$= 1{,}600\ k \cdot in$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 2

If we consider only the parts of the stiffness matrix associated with the three degrees of freedom at node 2, we get:

Element 1: The angle between *x* and *x'* is 45º



$$C = 0.707 \qquad S = 0.707$$

$$\frac{12I}{L^2} = \frac{12(1{,}000)}{\left(12 \times 30\sqrt{2}\right)^2} = 0.0463\ in^2 \qquad \frac{6I}{L} = \frac{6(1{,}000)}{12 \times 30\sqrt{2}} = 11.785\ in^3$$

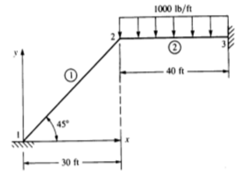
$$\frac{E}{L} = \frac{30 \times 10^6}{12 \times 30\sqrt{2}} = 58.93\ {}^{k}\!/_{in^3}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 2

If we consider only the parts of the stiffness matrix associated with the three degrees of freedom at node 2, we get:

Element 1: The angle between *x* and *x'* is 45º



$$\mathbf{k}^{(1)} = 58.93 \begin{array}{c} \begin{matrix} u_2 & \quad v_2 & \quad \phi_2 \end{matrix} \\ \begin{bmatrix} 50.02 & 49.98 & 8.33 \\ 49.98 & 50.02 & -8.33 \\ 8.33 & -8.33 & 4000 \end{bmatrix} \end{array} \, {}^{k}\!/_{in}$$

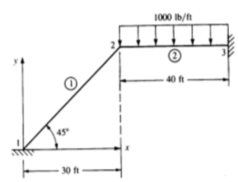
$$\mathbf{k}^{(1)} = \begin{array}{c} \begin{matrix} u_2 & \quad v_2 & \quad \phi_2 \end{matrix} \\ \begin{bmatrix} 2,948 & 2,945 & 491 \\ 2,945 & 2,948 & -491 \\ 491 & -491 & 235,700 \end{bmatrix} \end{array} \, {}^{k}\!/_{in}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 2

If we consider only the parts of the stiffness matrix associated with the three degrees of freedom at node 2, we get:

Element 2: The angle between *x* and *x'* is 0º



$$C = 1 \qquad S = 0$$

$$\frac{12I}{L^2} = \frac{12(1{,}000)}{(12 \times 40)^2} = 0.0521\, in^2 \qquad \frac{6I}{L} = \frac{6(1{,}000)}{12 \times 40} = 12.5\ in^3$$

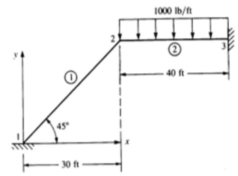
$$\frac{E}{L} = \frac{30 \times 10^6}{12 \times 40} = 52.5\ {}^{k}\!/_{in^3}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 2

If we consider only the parts of the stiffness matrix associated with the three degrees of freedom at node 2, we get:

Element 2: The angle between *x* and *x'* is 0º



$$\begin{array}{c} \quad\quad\quad\quad\quad u_2 \quad\quad v_2 \quad\quad\quad \phi_2 \\ \mathbf{k}^{(2)} = 62.50\begin{bmatrix} 100 & 0 & 0 \\ 0 & 0.052 & 12.5 \\ 0 & 12.5 & 4{,}000 \end{bmatrix} \tfrac{k}{in} \end{array}$$

$$\begin{array}{c} \quad\quad u_2 \quad\quad\quad v_2 \quad\quad\quad\quad \phi_2 \\ \mathbf{k}^{(2)} = \begin{bmatrix} 6{,}250 & 0 & 0 \\ 0 & 3.25 & 781.25 \\ 0 & 781.25 & 250{,}000 \end{bmatrix} \tfrac{k}{in} \end{array}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 2

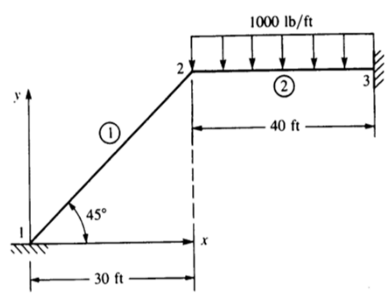
The global beam equations reduce to:

$$\begin{Bmatrix} 0 \\ -20\ k \\ -1{,}600\ k \cdot in \end{Bmatrix} = \begin{bmatrix} 9{,}198 & 2{,}945 & 491 \\ 2{,}945 & 2{,}951 & 290 \\ 491 & 290 & 485{,}700 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Nodal equivalent forces

Solving the above equations gives:

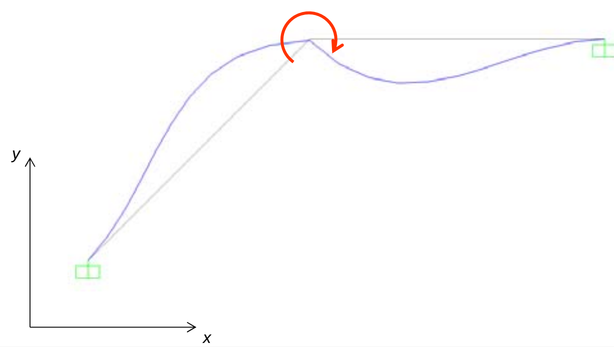
$$\begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0.0033\ in \\ -0.0097\ in \\ -0.0033\ rad \end{Bmatrix}$$



Plane Frame and Grid Equations

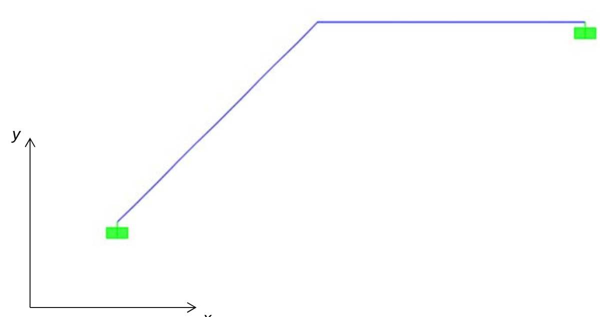
Rigid Plane Frame Example 2

$$\begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0.0033\ in \\ -0.0097\ in \\ -0.0033\ rad \end{Bmatrix}$$



Plane Frame and Grid Equations

Rigid Plane Frame Example 2



Plane Frame and Grid Equations

Rigid Plane Frame Example 2

Element 1: The element force-displacement equations can be obtained using **f**' = **k'Td**. Therefore, **Td** is:

$$\mathbf{Td} = \left[\begin{array}{ccc|ccc} 0.707 & 0.707 & 0 & 0 & 0 & 0 \\ -0.707 & 0.707 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0.707 & 0.707 & 0 \\ 0 & 0 & 0 & -0.707 & 0.707 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.0033\ in \\ -0.0097\ in \\ -0.0033\ rad \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00452\ in \\ -0.0092\ in \\ -0.0033\ rad \end{Bmatrix}$$

Recall the elemental stiffness matrix is a function of values C_1, C_2, and L

$$C_1 = \frac{AE}{L} = \frac{(100)30\times10^6}{12\times30\sqrt{2}} = 5{,}893\ {}^{k}\!/\!_{in} \qquad C_2 = \frac{EI}{L^3} = \frac{30\times10^6(1{,}000)}{\left(12\times30\sqrt{2}\right)^3} = 0.2273\ {}^{k}\!/\!_{in}$$

Plane Frame and Grid Equations

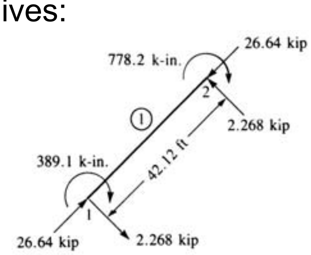
Rigid Plane Frame Example 2

Element 1: The local force-displacement equations are:

$$\mathbf{f}'_{(1)} = \mathbf{k}'\mathbf{Td} = \left[\begin{array}{ccc|ccc} 5{,}893 & 0 & 10 & -5{,}893 & 0 & 0 \\ 0 & 2.730 & 694.8 & 0 & -2.730 & 694.8 \\ 10 & 694.8 & 117{,}900 & 0 & -694.8 & 117{,}000 \\ \hline -5{,}893 & 0 & 0 & 5{,}983 & 0 & 0 \\ 0 & -2.730 & -694.8 & 0 & 2.730 & -694.8 \\ 0 & 694.8 & 117{,}000 & 0 & -694.8 & 235{,}800 \end{array}\right] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00452\ in \\ -0.0092\ in \\ -0.0033\ rad \end{Bmatrix}$$

Simplifying the above equations gives:

$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ m'_1 \\ f'_{2x} \\ f'_{2y} \\ m'_2 \end{Bmatrix} = \begin{Bmatrix} 26.64\ k \\ -2.268\ k \\ -389.1\ k\cdot in \\ -26.64\ k \\ 2.268\ k \\ -778.2\ k\cdot in \end{Bmatrix}$$



Plane Frame and Grid Equations

Rigid Plane Frame Example 2

Element 2: The element force-displacement equations can be obtained using **f' = k'Td**. Therefore, **Td** is:

$$\mathbf{Td} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} 0.0033\ in \\ -0.0097\ in \\ -0.0033\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.0033\ in \\ -0.0097\ in \\ -0.0033\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Recall the elemental stiffness matrix is a function of values C_1, C_2, and L

$$C_1 = \frac{AE}{L} = \frac{(100)30 \times 10^6}{12 \times 40} = 6{,}250\ {}^{k}\!/_{in} \qquad C_2 = \frac{EI}{L^3} = \frac{30 \times 10^6 (1{,}000)}{(12 \times 40)^3} = 0.2713\ {}^{k}\!/_{in}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 2

Element 2: The local force-displacement equations are:

$$\mathbf{f}'_{(2)} = \mathbf{k}'\mathbf{Td} = \left[\begin{array}{ccc|ccc} 6{,}250 & 0 & 0 & -6{,}250 & 0 & 0 \\ 0 & 3.25 & 781.1 & 0 & -3.25 & 781.1 \\ 0 & 781.1 & 250{,}000 & 0 & -781.1 & 125{,}000 \\ \hline -6{,}250 & 0 & 0 & 6{,}250 & 0 & 0 \\ 0 & -3.25 & -781.1 & 0 & 3.25 & -781.1 \\ 0 & 781.1 & 125{,}000 & 0 & -781.1 & 250{,}00 \end{array}\right] \begin{Bmatrix} -0.0033\ in \\ -0.0097\ in \\ -0.0033\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Simplifying the above equations gives:

$$\mathbf{k}'\mathbf{d}' = \begin{Bmatrix} 20.63\ k \\ -2.58\ k \\ -832.57\ k \cdot in \\ -20.63\ k \\ 2.58\ k \\ -412.50\ k \cdot in \end{Bmatrix}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 2

Element 2: To obtain the actual element local forces, we must subtract the equivalent nodal forces.

$$\mathbf{f} = \mathbf{kd} - \mathbf{f}_0$$

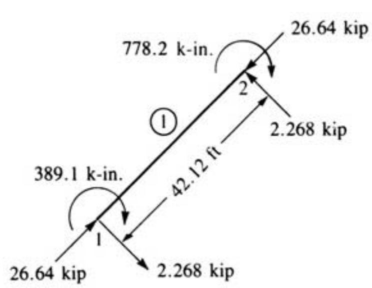
$$\begin{Bmatrix} f'_{2x} \\ f'_{2y} \\ m'_2 \\ f'_{3x} \\ f'_{3y} \\ m'_3 \end{Bmatrix} = \begin{Bmatrix} 20.63\ k \\ -2.58\ k \\ -832.57\ k \cdot in \\ -20.63\ k \\ 2.58\ k \\ -412.50\ k \cdot in \end{Bmatrix} - \begin{Bmatrix} 0 \\ -20\ k \\ -1600\ k \cdot in \\ 0 \\ -20\ k \\ 1600\ k \cdot in \end{Bmatrix} = \begin{Bmatrix} 20.63\ k \\ 17.42\ k \\ 767.4\ k \cdot in \\ -20.63\ k \\ 22.58\ k \\ -2{,}013\ k \cdot in \end{Bmatrix}$$

Plane Frame and Grid Equations

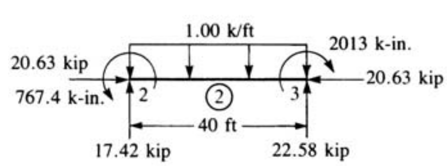
Rigid Plane Frame Example 2

The local forces in both elements are:

Element 1



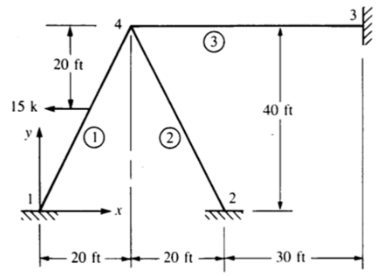
Element 2



Plane Frame and Grid Equations

Rigid Plane Frame Example 3

In this example, we will illustrate the equivalent joint force replacement method for a frame subjected to a load acting on an element instead of at one of the joints of the structure.

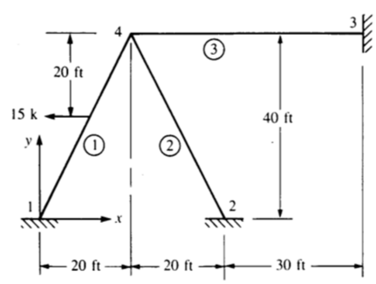


Since no distributed loads are present, the point of application of the concentrated load could be treated as an extra joint in the analysis.

Plane Frame and Grid Equations

Rigid Plane Frame Example 3

In this example, we will illustrate the equivalent joint force replacement method for a frame subjected to a load acting on an element instead of at one of the joints of the structure.

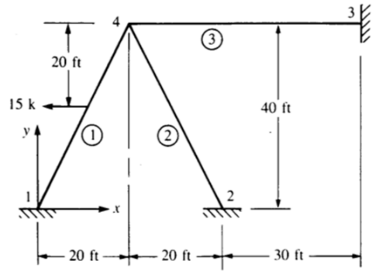


This approach has the disadvantage of increasing the total number of joints, as well as the size of the total structure stiffness matrix ***K***.

Plane Frame and Grid Equations

Rigid Plane Frame Example 3

In this example, we will illustrate the equivalent joint force replacement method for a frame subjected to a load acting on an element instead of at one of the joints of the structure.

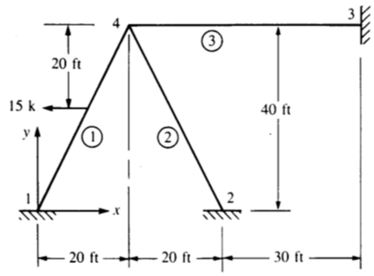


For small structures solved by computer, this does not pose a problem.

Plane Frame and Grid Equations

Rigid Plane Frame Example 3

In this example, we will illustrate the equivalent joint force replacement method for a frame subjected to a load acting on an element instead of at one of the joints of the structure.

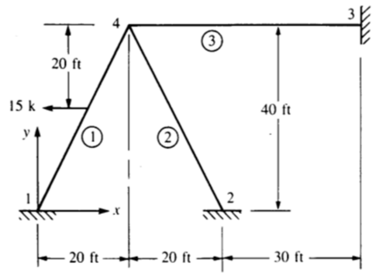


However, for very large structures, this might reduce the maximum size of the structure that could be analyzed.

Plane Frame and Grid Equations

Rigid Plane Frame Example 3

In this example, we will illustrate the equivalent joint force replacement method for a frame subjected to a load acting on an element instead of at one of the joints of the structure.

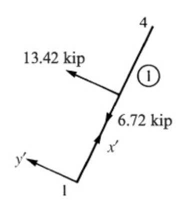


The frame is fixed at nodes 1, 2, and 3 and subjected to a concentrated load of 15 *k* applied at mid-length of element 1. Let E = 30 x 10^6 *psi*, A = 8 in^2, and let I = 800 in^4 for all elements.

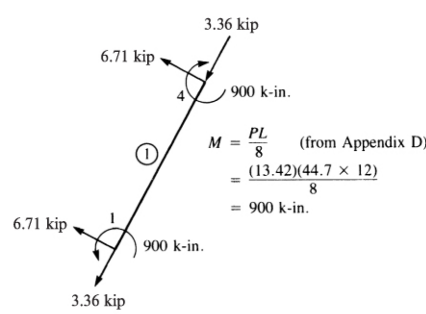
Plane Frame and Grid Equations

Rigid Plane Frame Example 3

1. Express the applied load in the element 1 local coordinate system (here x' is directed from node 1 to node 4).



2. Next, determine the equivalent joint forces at each end of element 1, using the Table D-1 in Appendix D.



Plane Frame and Grid Equations

Rigid Plane Frame Example 3

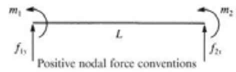
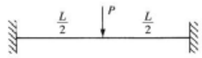
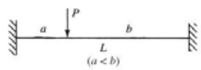
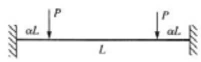
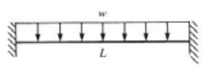
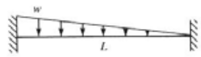
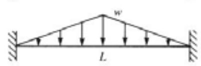


Table D–1 Single element equivalent joint forces f_0 for different types of loads

	f_{1y}	m_1	Loading case	f_{2y}	m_2
1.	$\frac{-P}{2}$	$\frac{-PL}{8}$		$\frac{-P}{2}$	$\frac{PL}{8}$
2.	$\frac{-Pb^2(L+2a)}{L^3}$	$\frac{-Pab^2}{L^2}$		$\frac{-Pa^2(L+2b)}{L^3}$	$\frac{Pa^2b}{L^2}$
3.	$-P$	$-\alpha(1-\alpha)PL$		$-P$	$\alpha(1-\alpha)PL$
4.	$\frac{-wL}{2}$	$\frac{-wL^2}{12}$		$\frac{-wL}{2}$	$\frac{wL^2}{12}$
5.	$\frac{-7wL}{20}$	$\frac{-wL^2}{20}$		$\frac{-3wL}{20}$	$\frac{wL^2}{30}$
6.	$\frac{-wL}{4}$	$\frac{-5wL^2}{96}$		$\frac{-wL}{4}$	$\frac{5wL^2}{96}$

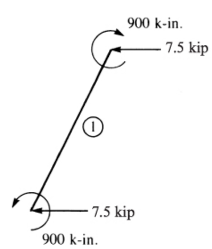
(*Continued*)

Plane Frame and Grid Equations

Rigid Plane Frame Example 3

3. Then transform the equivalent joint forces from the local coordinate system forces into the global coordinate system forces, using the equation: $\mathbf{f} = \mathbf{T}^T\mathbf{f}'$

   These global joint forces are:



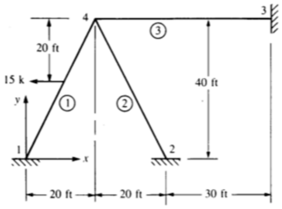
4. Then we analyze the structure, using the equivalent joint forces (plus actual joint forces, if any) in the usual manner.

5. The final internal forces developed at the ends of each element may be obtained by subtracting Step 2 joint forces from Step 4 joint forces.

Plane Frame and Grid Equations

Rigid Plane Frame Example 3

Element 1: The angle between *x* and *x'* is 63.43° (from nodes 1 to 4)



$C = 0.447 \qquad S = 0.895$

$$\frac{12I}{L^2} = \frac{12(800)}{(12\times 44.7)^2} = 0.0334\ in^2$$

$$\frac{6I}{L} = \frac{6(800)}{12\times 44.7} = 8.95\ in^3$$

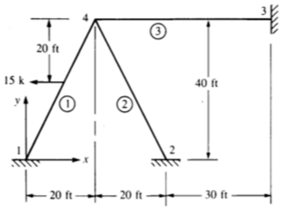
$$\frac{E}{L} = \frac{30\times 10^6}{12\times 44.7} = 55.9\ {}^{k}\!/\!{}_{in^3}$$

$$\begin{array}{cc} & \begin{array}{ccc} u_4 & v_4 & \phi_4 \end{array} \\ \mathbf{k}^{(1)} = & \begin{bmatrix} 90.0 & 178 & 448 \\ 178 & 359 & -244 \\ 448 & -244 & 179{,}000 \end{bmatrix} {}^{k}\!/\!{}_{in} \end{array}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 3

Element 2: The angle between *x* and *x'* is 116.57° (from nodes 2 to 4)



$C = -0.447 \qquad S = 0.895$

$$\frac{12I}{L^2} = \frac{12(800)}{(12\times 44.7)^2} = 0.0334\ in^2$$

$$\frac{6I}{L} = \frac{6(800)}{12\times 44.7} = 8.95\ in^3$$

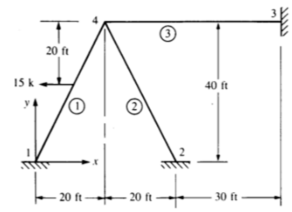
$$\frac{E}{L} = \frac{30\times 10^6}{12\times 44.7} = 55.9\ {}^{k}\!/\!{}_{in^3}$$

$$\begin{array}{cc} & \begin{array}{ccc} u_4 & v_4 & \phi_4 \end{array} \\ \mathbf{k}^{(2)} = & \begin{bmatrix} 90.0 & -178 & 448 \\ -178 & 359 & 244 \\ 448 & 244 & 179{,}000 \end{bmatrix} {}^{k}\!/\!{}_{in} \end{array}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 3

Element 3: The angle between x and x' is 0º (from nodes 4 to 3)



$$C = 1 \qquad S = 0$$

$$\frac{12I}{L^2} = \frac{12(800)}{(12 \times 50)^2} = 0.0267\ in^2$$

$$\frac{6I}{L} = \frac{6(800)}{12 \times 50} = 8.0\ in^3$$

$$\frac{E}{L} = \frac{30 \times 10^6}{12 \times 50} = 50.0\ {}^{k}\!/\!_{in^3}$$

$$\mathbf{k}^{(3)} = \begin{array}{c} \begin{array}{ccc} u_4 & v_4 & \phi_4 \end{array} \\ \begin{bmatrix} 400 & 0 & 0 \\ 0 & 1.334 & 400 \\ 0 & 400 & 160{,}000 \end{bmatrix} \end{array} {}^{k}\!/\!_{in}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 3

The global beam equations reduce to:

$$\begin{Bmatrix} -7.5\ k \\ 0 \\ -900\ k \cdot in \end{Bmatrix} = \begin{bmatrix} 582 & 0 & 896 \\ 0 & 719 & 400 \\ 896 & 400 & 518{,}000 \end{bmatrix} \begin{Bmatrix} u_4 \\ v_4 \\ \phi_4 \end{Bmatrix}$$

Solving the above equations gives:

$$\begin{Bmatrix} u_4 \\ v_4 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} -0.0103\ in \\ 0.000956\ in \\ -0.00172\ rad \end{Bmatrix}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 3

Element 1: The element force-displacement equations can be obtained using **f' = k'Td**

$$C = 0.447 \qquad S = 0.895 \qquad \mathbf{T} = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Td} = \begin{bmatrix} 0.447 & 0.895 & 0 & 0 & 0 & 0 \\ -0.895 & 0.447 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.447 & 0.895 & 0 \\ 0 & 0 & 0 & -0.895 & 0.447 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.0103\ in \\ 0.000956\ in \\ -0.00172\ rad \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00374\ in \\ 0.00963\ in \\ -0.00172\ rad \end{Bmatrix}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 3

Element 1: Recall the elemental stiffness matrix is:

$$\mathbf{k}' = \begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{bmatrix}$$

$$C_1 = \frac{AE}{L} = \frac{(8)30\times10^6}{12\times44.72} = 447.2\ {}^{k}\!/_{in} \qquad C_2 = \frac{EI}{L^3} = \frac{30\times10^6(800)}{(12\times44.72)^3} = 0.155\ {}^{k}\!/_{in}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 3

Element 1: The local force-displacement equations are:

$$\mathbf{f}'_{(1)} = \mathbf{k}'\mathbf{T}\mathbf{d} = \underbrace{\left[\begin{array}{ccc|ccc} 447 & 0 & 0 & -447 & 0 & 0 \\ 0 & 1.868 & 500.5 & 0 & -1.868 & 500.5 \\ 0 & 500.5 & 179{,}000 & 0 & -500.5 & 89{,}490 \\ \hline -447 & 0 & 0 & 447 & 0 & 0 \\ 0 & -1.868 & -500.5 & 0 & 1.868 & -500.5 \\ 0 & 500.5 & 89{,}490 & 0 & -500.5 & 179{,}000 \end{array}\right]}_{\mathbf{k}'^{(1)}} \underbrace{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00374\ in \\ 0.00963\ in \\ -0.00172\ rad \end{Bmatrix}}_{\mathbf{Td}}$$

$$\mathbf{f}'_{(1)} = \mathbf{k}'\mathbf{d}' = \begin{Bmatrix} 1.67\ k \\ -0.88\ k \\ -158\ k \cdot in \\ -1.67\ k \\ 0.88\ k \\ -311\ k \cdot in \end{Bmatrix}$$

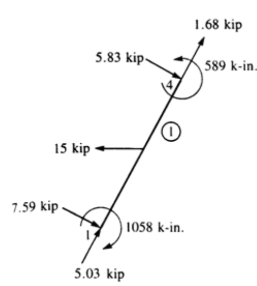
Plane Frame and Grid Equations

Rigid Plane Frame Example 3

Element 1: To obtain the actual element local forces, we must subtract the equivalent nodal forces.

$$\mathbf{f} = \mathbf{kd} - \mathbf{f}_0$$

$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ m'_1 \\ f'_{4x} \\ f'_{4y} \\ m'_4 \end{Bmatrix} = \begin{Bmatrix} 1.67\ k \\ -0.88\ k \\ -158\ k \cdot in \\ -1.67\ k \\ 0.88\ k \\ -311\ k \cdot in \end{Bmatrix} - \begin{Bmatrix} -3.36\ k \\ 6.71\ k \\ 900\ k \cdot in \\ -3.36\ k \\ 6.71\ k \\ -900\ k \cdot in \end{Bmatrix} = \begin{Bmatrix} 5.03\ k \\ -7.59\ k \\ -1{,}058\ k \cdot in \\ 1.68\ k \\ -5.83\ k \\ 589\ k \cdot in \end{Bmatrix}$$



Plane Frame and Grid Equations

Rigid Plane Frame Example 3

Element 2: The element force-displacement equations can be obtained using **f' = k'Td**

$$C = -0.447 \qquad S = 0.895$$

$$\mathbf{T} = \left[\begin{array}{ccc|ccc} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{Td} = \left[\begin{array}{ccc|ccc} -0.447 & 0.895 & 0 & 0 & 0 & 0 \\ -0.895 & -0.447 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -0.447 & 0.895 & 0 \\ 0 & 0 & 0 & -0.895 & -0.447 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.0103 \textit{ in} \\ 0.000956 \textit{ in} \\ -0.00172 \textit{ rad} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.00546 \textit{ in} \\ 0.00879 \textit{ in} \\ -0.00172 \textit{ rad} \end{Bmatrix}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 3

Element 2: Recall the elemental stiffness matrix is:

$$\mathbf{k}' = \left[\begin{array}{ccc|ccc} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ \hline -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{array}\right]$$

$$C_1 = \frac{AE}{L} = \frac{(8)30\times10^6}{12\times44.72} = 447.2 \,{}^{k}\!/_{in} \qquad C_2 = \frac{EI}{L^3} = \frac{30\times10^6(800)}{(12\times44.72)^3} = 0.155 \,{}^{k}\!/_{in}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 3

Element 2: The local force-displacement equations are:

$$\mathbf{f}'_{(2)} = \mathbf{k}'\mathbf{T}\mathbf{d} = \underbrace{\left[\begin{array}{ccc|ccc} 447 & 0 & 0 & -447 & 0 & 0 \\ 0 & 1.868 & 500.5 & 0 & -1.868 & 500.5 \\ 0 & 500.5 & 179{,}000 & 0 & -500.5 & 89{,}490 \\ \hline -447 & 0 & 0 & 447 & 0 & 0 \\ 0 & -1.868 & -500.5 & 0 & 1.868 & -500.5 \\ 0 & 500.5 & 89{,}490 & 0 & -500.5 & 179{,}000 \end{array}\right]}_{\mathbf{k}'^{(2)}} \underbrace{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.00546\ in \\ 0.00879\ in \\ -0.00172\ rad \end{Bmatrix}}_{\mathbf{Td}}$$

$$\mathbf{f}'_{(2)} = \mathbf{k}'\mathbf{d}' = \begin{Bmatrix} -2.44\ k \\ -0.877\ k \\ -158\ k \cdot in \\ 2.44\ k \\ 0.877\ k \\ -312\ k \cdot in \end{Bmatrix}$$

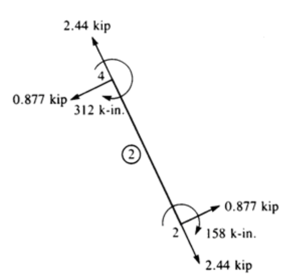
Plane Frame and Grid Equations

Rigid Plane Frame Example 3

Element 2: Since there are no applied loads on element 2, there are no equivalent nodal forces to account for.

Therefore, the above equations are the final local nodal forces

$$\mathbf{f}'_{(2)} = \mathbf{k}'\mathbf{d}' = \begin{Bmatrix} -2.44\ k \\ -0.877\ k \\ -158\ k \cdot in \\ 2.44\ k \\ 0.877\ k \\ -312\ k \cdot in \end{Bmatrix}$$



Plane Frame and Grid Equations

Rigid Plane Frame Example 3

Element 3: The element force-displacement equations can be obtained using **f' = k'Td**

$$C = 1 \qquad S = 0$$

$$\mathbf{T} = \left[\begin{array}{ccc|ccc} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{Td} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} -0.0103\ in \\ 0.000956\ in \\ -0.00172\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.0103\ in \\ 0.000956\ in \\ -0.00172\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 3

Element 3: Recall the elemental stiffness matrix is:

$$\mathbf{k}' = \left[\begin{array}{ccc|ccc} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ \hline -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{array}\right]$$

$$C_1 = \frac{AE}{L} = \frac{(8)30 \times 10^6}{12 \times 50} = 400\ {}^{k}\!/_{in} \qquad C_2 = \frac{EI}{L^3} = \frac{30 \times 10^6 (800)}{(12 \times 50)^3} = 0.111\ {}^{k}\!/_{in}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 3

Element 3: The local force-displacement equations are:

$$\mathbf{f}'_{(3)} = \mathbf{k}'\mathbf{T}\mathbf{d} = \underbrace{\begin{bmatrix} 400 & 0 & 0 & -400 & 0 & 0 \\ 0 & 1.335 & 400 & 0 & -1.335 & 400 \\ 0 & 400 & 160{,}000 & 0 & -400 & 80{,}000 \\ -400 & 0 & 0 & 400 & 0 & 0 \\ 0 & -1.335 & -400 & 0 & 1.335 & -400 \\ 0 & 400 & 80{,}000 & 0 & -400 & 160{,}000 \end{bmatrix}}_{\mathbf{k}'^{(3)}} \underbrace{\begin{Bmatrix} -0.0103\ in \\ 0.000956\ in \\ -0.00172\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}}_{\mathbf{Td}}$$

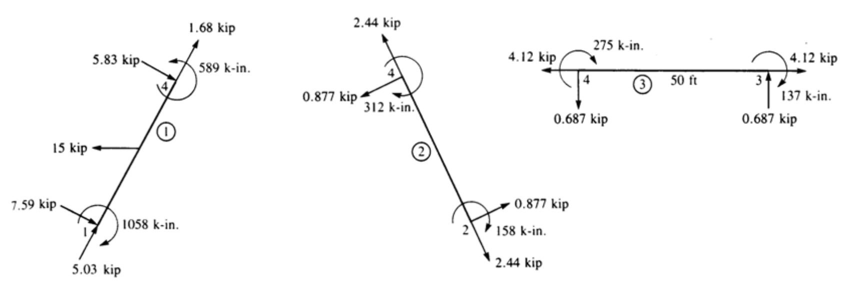
$$\mathbf{f}'_{(3)} = \mathbf{k}'\mathbf{d}' = \begin{Bmatrix} -4.12\ k \\ -0.687\ k \\ -275\ k \cdot in \\ 4.12\ k \\ 0.687\ k \\ -137\ k \cdot in \end{Bmatrix}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 3

Element 3: Since there are no applied loads on element 3, there are no equivalent nodal forces to account for.

Therefore, the above equations are the final local nodal forces

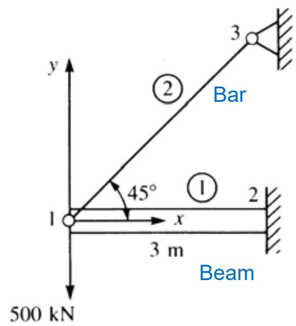


Plane Frame and Grid Equations

Rigid Plane Frame Example 4

The frame shown below is fixed at nodes 2 and 3 and subjected to a concentrated load of 500 *kN* applied at node 1.

For the bar, $A = 1 \times 10^{-3}\ m^2$, for the beam, $A = 2 \times 10^{-3}\ m^2$, $I = 5 \times 10^{-5}\ m^4$, and $L = 3\ m$. Let $E = 210\ GPa$ for both elements.

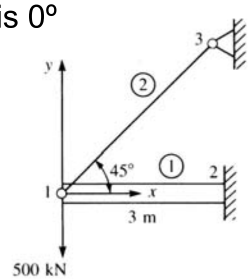


Plane Frame and Grid Equations

Rigid Plane Frame Example 4

Element 1: The angle between *x* and *x'* is 0°

$$C = 1 \qquad S = 0$$



$$\frac{12I}{L^2} = \frac{12(5\times10^{-5})}{(3)^2} = 6.67\times10^{-5}\ m^2$$

$$\frac{6I}{L} = \frac{6(5\times10^{-5})}{3} = 10^{-4}\ m^3$$

$$\frac{E}{L} = \frac{210\times10^6}{3} = 70\times10^6\ kN/m^3$$

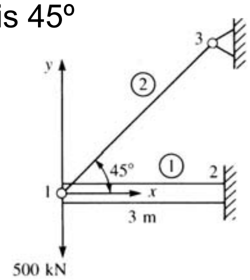
$$\mathbf{k}^{(1)} = 70\times10^3 \begin{array}{c} \begin{array}{ccc} u_1 & v_1 & \phi_1 \end{array} \\ \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.067 & 0.10 \\ 0 & 0.10 & 0.20 \end{bmatrix} \end{array} \ {kN}/{m}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 4

Element 2: The angle between *x* and *x'* is 45º

$C = 0.707 \qquad S = 0.707$



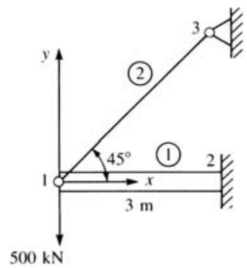
$$\mathbf{k}^{(2)} = \frac{10^{-3}m^2\left(210\times10^6\ {}^{kN}\!/\!_{m^2}\right)}{4.24\ m}\begin{array}{c} \begin{array}{cc} u_1 & v_1 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \end{array} {}^{kN}\!/\!_{m}$$

$$\mathbf{k}^{(2)} = 70\times10^3 \begin{array}{c} \begin{array}{cc} u_1 & v_1 \end{array} \\ \begin{bmatrix} 0.354 & 0.354 \\ 0.354 & 0.354 \end{bmatrix} \end{array} {}^{kN}\!/\!_{m}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 4

Assembling the elemental stiffness matrices we obtain the global stiffness matrix:



$$\mathbf{K} = 70\times10^3 \begin{bmatrix} 2.354 & 0.354 & 0 \\ 0.354 & 0.421 & 0.10 \\ 0 & 0.10 & 0.20 \end{bmatrix} {}^{kN}\!/\!_{m}$$

The global equations are:

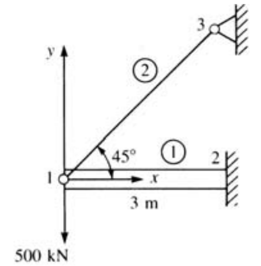
$$\begin{Bmatrix} 0 \\ -500\ kN \\ 0 \end{Bmatrix} = 70\times10^3\ {}^{kN}\!/\!_{m} \begin{bmatrix} 2.354 & 0.354 & 0 \\ 0.354 & 0.421 & 0.10 \\ 0 & 0.10 & 0.20 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \phi_1 \end{Bmatrix}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 4

Solving the above equations gives:

$$\begin{Bmatrix} u_1 \\ v_1 \\ \phi_1 \end{Bmatrix} = \begin{Bmatrix} 0.00388\ m \\ -0.0225\ m \\ 0.0113\ rad \end{Bmatrix}$$



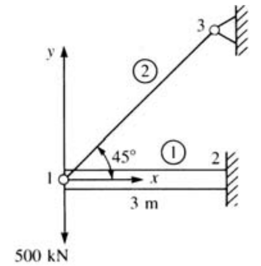
Plane Frame and Grid Equations

Rigid Plane Frame Example 4

Bar Element: The bar element force-displacement equations can be obtained using **f’** = **k’Td**

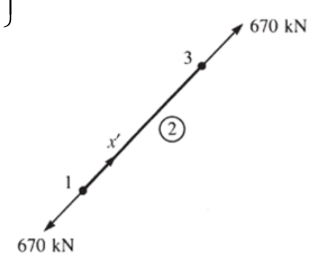
$$C = 0.707 \qquad S = 0.707$$

$$\begin{Bmatrix} f'_{1x} \\ f'_{3x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{Bmatrix}$$



$$f'_{1x} = \frac{AE}{L}\left(Cu_1 + Sv_1\right) = -670\ kN$$

$$f'_{3x} = -\frac{AE}{L}\left(Cu_1 + Sv_1\right) = 670\ kN$$

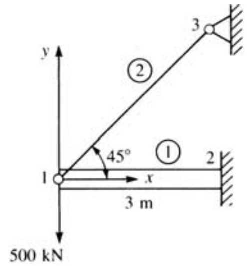


Plane Frame and Grid Equations

Rigid Plane Frame Example 4

Beam Element: The bar element force-displacement equations can be obtained using **f' = k'Td**

$$\mathbf{T} = \left[\begin{array}{ccc|ccc} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} C = 1 \\ S = 0 \end{array}$$

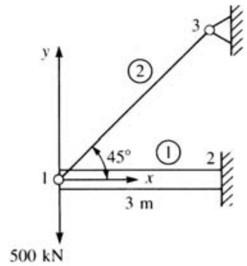


$$\mathbf{d}' = \mathbf{Td} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} 0.00388\ m \\ -0.0225\ m \\ 0.0113\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.00388\ m \\ -0.0225\ m \\ 0.0113\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 4

Beam Element: The bar element force-displacement equations can be obtained using **f' = k'Td**



$$\mathbf{k}' = \left[\begin{array}{ccc|ccc} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ \hline -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{array}\right]$$

$$C_1 = \frac{AE}{L} = \frac{(0.002)210\times10^6}{3} = 140\times10^3\ {}^{kN}\!/\!_{m}$$

$$C_2 = \frac{EI}{L^3} = \frac{210\times10^6(5\times10^{-5})}{(3)^3} = 388.89\ {}^{kN}\!/\!_{m}$$

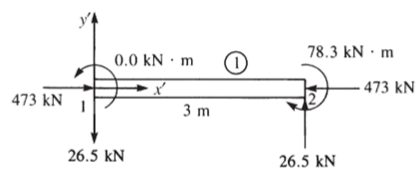
Plane Frame and Grid Equations

Rigid Plane Frame Example 4

Beam Element: The bar element force-displacement equations can be obtained using **f' = k'Td**

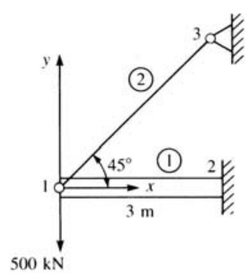
$$\mathbf{f}'_{(1)} = \mathbf{k}'\mathbf{d}' = 70 \times 10^3 \underbrace{\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0.067 & 0.10 & 0 & -0.067 & 0.10 \\ 0 & 0.10 & 0.20 & 0 & -0.10 & 0.10 \\ \hline -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & -0.067 & -0.10 & 0 & 0.067 & -0.10 \\ 0 & 0.10 & 0.10 & 0 & -0.10 & 0.20 \end{array}\right]}_{\mathbf{k}'^{(1)}} \underbrace{\begin{Bmatrix} 0.00388\ m \\ -0.0225\ m \\ 0.0113\ kN \cdot m \\ 0 \\ 0 \\ 0 \end{Bmatrix}}_{\mathbf{d}'}$$

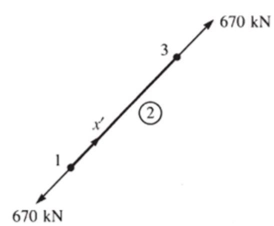
$$\mathbf{f}'_{(1)} = \begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ m'_1 \\ f'_{2x} \\ f'_{2y} \\ m'_2 \end{Bmatrix} = \begin{Bmatrix} 473\ kN \\ -26.5\ kN \\ 0.0 \\ -473\ kN \\ 26.5\ kN \\ -78.3\ kN \cdot m \end{Bmatrix}$$

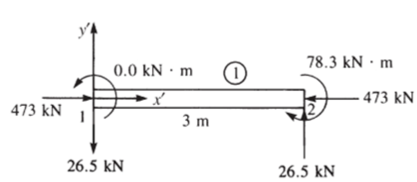


Plane Frame and Grid Equations

Rigid Plane Frame Example 4





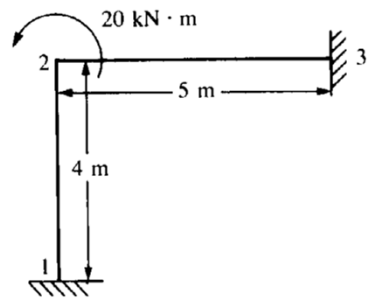


Plane Frame and Grid Equations

Rigid Plane Frame Example 5

The frame is fixed at nodes 1 and 3 and subjected to a moment of 20 *kN-m* applied at node 2

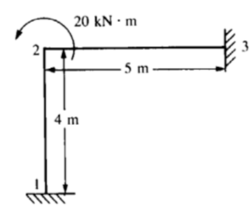
Assume $A = 2 \times 10^{-2}\ m^2$, $I = 2 \times 10^{-4}\ m^4$, and $E = 210\ GPa$ for all elements.



Plane Frame and Grid Equations

Rigid Plane Frame Example 5

Element 1: The angle between *x* and *x'* is 90°



$$C = 0 \qquad S = 1$$

$$\frac{12I}{L^2} = \frac{12(2\times10^{-4})}{(4)^2} = 1.5\times10^{-4}\,m^2$$

$$\frac{6I}{L} = \frac{6(2\times10^{-4})}{4} = 3\times10^{-4}\,m^3$$

$$\frac{E}{L} = \frac{210\times10^6}{4} = 5.25\times10^7\ {}^{kN}\!/_{m^3}$$

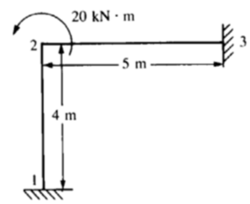
The parts of **k** associated with node 2 are:

$$\mathbf{k}^{(1)} = 5.25\times10^5 \begin{array}{c} \begin{array}{ccc} u_2 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 0.015 & 0 & 0.03 \\ 0 & 2 & 0 \\ 0.03 & 0 & 0.08 \end{bmatrix} \end{array} {}^{kN}\!/_{m}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 5

Element 2: The angle between *x* and *x'* is 0º



$$C = 1 \qquad S = 0$$

$$\frac{12I}{L^2} = \frac{12(2\times10^{-4})}{(5)^2} = 9.6\times10^{-4}\, m^2$$

$$\frac{6I}{L} = \frac{6(2\times10^{-4})}{5} = 2.4\times10^{-4}\, m^3$$

$$\frac{E}{L} = \frac{210\times10^6}{5} = 4.2\times10^7\ {}^{kN}\!/_{m^3}$$

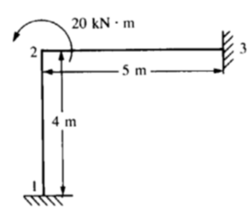
The parts of **k** associated with node 2 are:

$$\begin{array}{c} \\ \mathbf{k}^{(2)} = 4.2\times10^5 \end{array} \begin{array}{c} \begin{array}{ccc} u_2 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.0096 & 0.024 \\ 0 & 0.024 & 0.08 \end{bmatrix} \end{array} {}^{kN}\!/_{m}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 5

Assembling the elemental stiffness matrices we obtain the global stiffness matrix:



$$\mathbf{K} = 10^6 \begin{bmatrix} 0.8480 & 0 & 0.0158 \\ 0 & 1.0500 & 0.0101 \\ 0.0158 & 0.0101 & 0.0756 \end{bmatrix} {}^{kN}\!/_{m}$$

The global equations are:

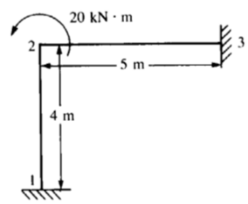
$$\begin{Bmatrix} 0 \\ 0 \\ 20\ kN\cdot m \end{Bmatrix} = 10^6 \begin{bmatrix} 0.8480 & 0 & 0.0158 \\ 0 & 1.0500 & 0.0101 \\ 0.0158 & 0.0101 & 0.0756 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 5

Solving the above equations gives:

$$\begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -4.95 \times 10^{-6}\,m \\ -2.56 \times 10^{-6}\,m \\ 2.66 \times 10^{-4}\,rad \end{Bmatrix}$$



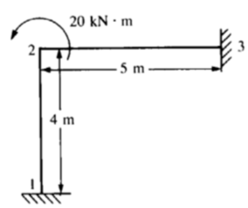
Plane Frame and Grid Equations

Rigid Plane Frame Example 5

Element 1: The element force-displacement equations can be obtained using **f'** = **k'Td**

$$\mathbf{T} = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = 0 \qquad S = 1$$

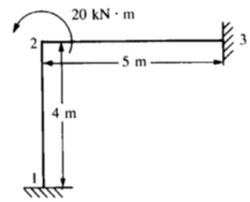


$$\mathbf{d}' = \mathbf{Td} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -4.95 \times 10^{-6}\,m \\ -2.56 \times 10^{-6}\,m \\ 2.66 \times 10^{-4}\,rad \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -2.56 \times 10^{-6}\,m \\ 4.95 \times 10^{-6}\,m \\ 2.66 \times 10^{-4}\,rad \end{Bmatrix}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 5

Element 1: The element force-displacement equations can be obtained using **f' = k'Td**



$$\mathbf{k}' = \left[\begin{array}{ccc|ccc} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ \hline -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{array}\right]$$

$$C_1 = \frac{AE}{L} = \frac{(2\times10^{-2})210\times10^{6}}{4} = 1.05\times10^{6}\ {}^{kN}\!/_{m}$$

$$C_2 = \frac{EI}{L^3} = \frac{210\times10^{6}(5\times10^{-5})}{(3)^3} = 388.89\ {}^{kN}\!/_{m}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 5

Element 1: The element force-displacement equations can be obtained using **f' = k'Td**

$$\mathbf{f}'_{(1)} = \mathbf{k}'\mathbf{d}' = 5.25\times10^{3} \underbrace{\left[\begin{array}{ccc|ccc} 200 & 0 & 0 & -200 & 0 & 0 \\ 0 & 1.5 & 3 & 0 & -1.5 & 3 \\ 0 & 3 & 8 & 0 & -3 & 4 \\ \hline -200 & 0 & 0 & 200 & 0 & 0 \\ 0 & -1.5 & -3 & 0 & 1.5 & -3 \\ 0 & 3 & 4 & 0 & -3 & 8 \end{array}\right]}_{\mathbf{k}^{(1)}} \underbrace{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -2.56\times10^{-6}\,m \\ 4.95\times10^{-6}\,m \\ 2.66\times10^{-4}\,rad \end{Bmatrix}}_{\mathbf{d}'}$$

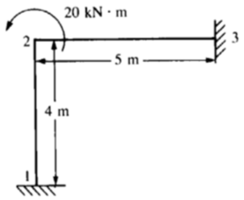
$$\mathbf{f}'_{(1)} = \begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ m'_1 \\ f'_{2x} \\ f'_{2y} \\ m'_2 \end{Bmatrix} = \begin{Bmatrix} 2.69\ kN \\ 4.2\ kN \\ 5.59\ kN\cdot m \\ -2.69\ kN \\ -4.2\ kN \\ 11.17\ kN\cdot m \end{Bmatrix}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 5

Element 2: The element force-displacement equations can be obtained using **f'** = **k'Td**

$$\mathbf{T} = \left[\begin{array}{ccc|ccc} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \qquad C = 1 \qquad S = 0$$

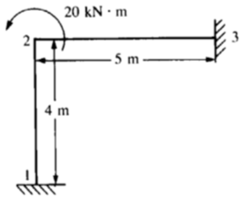


$$\mathbf{d}' = \mathbf{Td} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} -4.95\times10^{-6}\,m \\ -2.56\times10^{-6}\,m \\ 2.66\times10^{-4}\,rad \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -4.95\times10^{-6}\,m \\ -2.56\times10^{-6}\,m \\ 2.66\times10^{-4}\,rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 5

Element 2: The element force-displacement equations can be obtained using **f'** = **k'Td**



$$\mathbf{k}' = \left[\begin{array}{ccc|ccc} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ \hline -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{array}\right]$$

$$C_1 = \frac{AE}{L} = \frac{(2\times10^{-2})210\times10^6}{5} = 0.84\times10^6 \ {}^{kN}\!/_{m}$$

$$C_2 = \frac{EI}{L^3} = \frac{210\times10^6(2\times10^{-4})}{(5)^3} = 336 \ {}^{kN}\!/_{m}$$

Plane Frame and Grid Equations

Rigid Plane Frame Example 5

Element 2: The element force-displacement equations can be obtained using **f'** = **k'Td**

$$\mathbf{f}'_{(2)} = \mathbf{k}'\mathbf{d}' = 4.2\times10^{3}\underbrace{\begin{bmatrix} 200 & 0 & 0 & -200 & 0 & 0 \\ 0 & 0.96 & 2.40 & 0 & -0.96 & 2.40 \\ 0 & 2.40 & 8 & 0 & -2.40 & 4 \\ -200 & 0 & 0 & 200 & 0 & 0 \\ 0 & -0.96 & -2.40 & 0 & 0.96 & -2.40 \\ 0 & 2.40 & 4 & 0 & -2.40 & 8 \end{bmatrix}}_{\mathbf{k}'^{(2)}}\underbrace{\begin{Bmatrix} -4.95\times10^{-6}\,m \\ -2.56\times10^{-6}\,m \\ 2.66\times10^{-4}\,rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}}_{\mathbf{d}'}$$

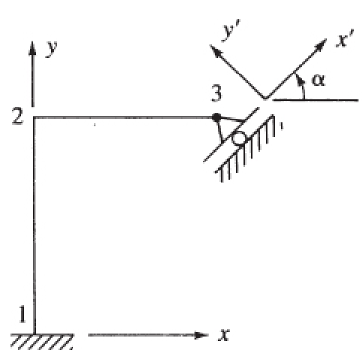
$$\mathbf{f}'_{(2)} = \begin{Bmatrix} f'_{2x} \\ f'_{2y} \\ m'_{2} \\ f'_{3x} \\ f'_{3y} \\ m'_{3} \end{Bmatrix} = \begin{Bmatrix} 4.16\,kN \\ -2.69\,kN \\ 8.92\,kN\cdot m \\ -4.16\,kN \\ 2.69\,kN \\ 4.47\,kN\cdot m \end{Bmatrix}$$

Plane Frame and Grid Equations

Inclined or Skewed Supports

If a support is inclined, or skewed, at some angle α for the global *x* axis, as shown below.

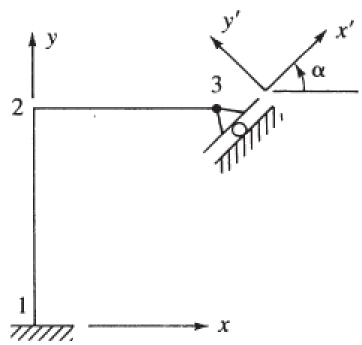
The boundary conditions on the displacements are not in the global *x*-*y* directions but in the *x'*-*y'* directions.



Plane Frame and Grid Equations

Inclined or Skewed Supports

We must transform the local boundary condition of $v'_3 = 0$ (in local coordinates) into the global *x*-*y* system.



Plane Frame and Grid Equations

Inclined or Skewed Supports

Therefore, the relationship between of the components of the displacement in the local and the global coordinate systems at node 3 is:

$$\begin{Bmatrix} u'_3 \\ v'_3 \\ \phi'_3 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ \phi_3 \end{Bmatrix}$$

We can rewrite the above expression as:

$$\{d'_3\} = [t_3]\{d_3\} \qquad [t_3] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Plane Frame and Grid Equations

Inclined or Skewed Supports

We can apply this sort of transformation to the entire displacement vector as:

$$\{d'\} = [T_i]\{d\} \qquad \text{or} \qquad \{d\} = [T_i]^T\{d'\}$$

where the matrix **[*T_i*]** is:

$$[T_i] = \begin{bmatrix} [I] & [0] & [0] \\ [0] & [I] & [0] \\ [0] & [0] & [t_3] \end{bmatrix}$$

Both the identity matrix **[*I*]** and the matrix **[*t_3*]** are 3 x 3 matrices.

Plane Frame and Grid Equations

Inclined or Skewed Supports

The force vector can be transformed by using the same transformation.

$$\{f'\} = [T_i]\{f\}$$

In global coordinates, the force-displacement equations are:

$$\{f\} = [K]\{d\}$$

Applying the skewed support transformation to both sides of the force-displacement equation gives:

$$[T_i]\{f\} = [T_i][K]\{d\}$$

By using the relationship between the local and the global displacements, the force-displacement equations become:

$$[T_i]\{f\} = [T_i][K][T_i]^T\{d'\} \quad \Rightarrow \quad \{f'\} = [T_i][K][T_i]^T\{d'\}$$

Plane Frame and Grid Equations

Inclined or Skewed Supports

Therefore the global equations become:

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ M_1 \\ F_{2x} \\ F_{2y} \\ M_2 \\ F'_{3x} \\ F'_{3y} \\ M_3 \end{Bmatrix} = [T_i][K][T_i]^T \begin{Bmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \\ u'_3 \\ v'_3 \\ \phi_3 \end{Bmatrix}$$

Elemental coordinates

End of Chapter 5a