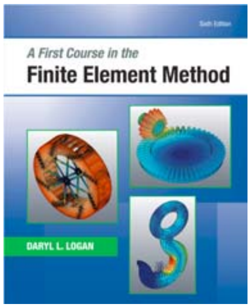
Chapter 4b – Development of Beam Equations



Learning Objectives

- To introduce the work-equivalence method for replacing distributed loading by a set of discrete loads
- To introduce the general formulation for solving beam problems with distributed loading acting on them
- To analyze beams with distributed loading acting on them
- To compare the finite element solution to an exact solution for a beam
- To derive the stiffness matrix for the beam element with nodal hinge
- To show how the potential energy method can be used to derive the beam element equations
- To apply Galerkin's residual method for deriving the beam element equations

Beam Stiffness

General Formulation

We can account for the distributed loads or concentrated loads acting on beam elements by considering the following formulation for a general structure:

$$\mathbf{F} = \mathbf{K}\mathbf{d} - \mathbf{F}_0$$

where $\mathbf{F}_0$ are the ***equivalent nodal forces***, expressed in terms of the global-coordinate components.

These forces would yield the same displacements as the original distributed load.

If we assume that the global nodal forces are not present ($\mathbf{F}$ = 0) then:

$$\mathbf{F}_0 = \mathbf{K}\mathbf{d}$$

Beam Stiffness

General Formulation

We now solve for the displacements, **d**, given the nodal forces $\mathbf{F}_0$.

Next, substitute the displacements and the equivalent nodal forces $\mathbf{F}_0$ back into the original expression and solve for the global nodal forces.

$$\mathbf{F} = \mathbf{Kd} - \mathbf{F}_0$$

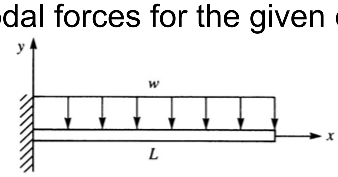
This concept can be applied on a local basis to obtain the local nodal forces in individual elements of structures as:

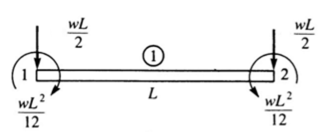
$$\mathbf{f} = \mathbf{kd} - \mathbf{f}_0$$

Beam Stiffness

Example 5 - Load Replacement

Consider the beam shown below; determine the equivalent nodal forces for the given distributed load.





The work equivalent nodal forces are shown above.

Using the beam stiffness equations:

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{wL}{2} \\ -\frac{wL^2}{12} \\ -\frac{wL}{2} \\ \frac{wL^2}{12} \end{Bmatrix}$$

Beam Stiffness

Example 5 - Load Replacement

Apply the boundary conditions: $v_1 = \phi_1 = 0$

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

We can solve for the displacements

$$\begin{Bmatrix} -\dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix} \qquad \begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{wL^4}{8EI} \\ -\dfrac{wL^3}{6EI} \end{Bmatrix}$$

Beam Stiffness

Example 5 - Load Replacement

In this case, the method of equivalent nodal forces gives the exact solution for the displacements and rotations.

To obtain the global nodal forces, we will first define the product of **Kd** to be $\mathbf{F^e}$, where $\mathbf{F^e}$ is called the ***effective global nodal forces***. Therefore:

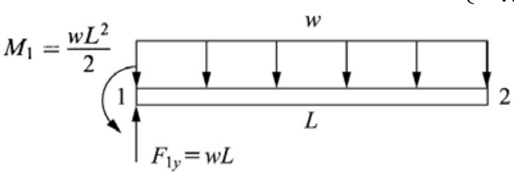
$$\begin{Bmatrix} F^e_{1y} \\ M^e_1 \\ F^e_{2y} \\ M^e_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -\frac{wL^4}{8EI} \\ -\frac{wL^3}{6EI} \end{Bmatrix} = \begin{Bmatrix} \dfrac{wL}{2} \\ \dfrac{5wL^2}{12} \\ -\dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix}$$

Beam Stiffness

Example 5 - Load Replacement

Using the above expression and the fix-end moments in:

$$\mathbf{F = Kd - F_0} \qquad \begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \end{Bmatrix} = \begin{Bmatrix} \frac{wL}{2} \\ \frac{5wL^2}{12} \\ -\frac{wL}{2} \\ \frac{wL^2}{12} \end{Bmatrix} - \begin{Bmatrix} -\frac{wL}{2} \\ -\frac{wL^2}{12} \\ -\frac{wL}{2} \\ \frac{wL^2}{12} \end{Bmatrix} = \begin{Bmatrix} wL \\ \frac{wL^2}{2} \\ 0 \\ 0 \end{Bmatrix}$$



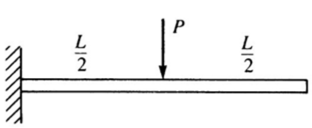
$$\sum F_y = 0 \quad = F_{1y} - wL$$

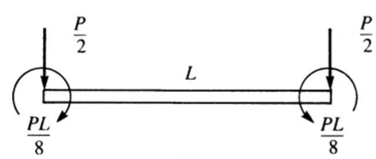
$$\circlearrowleft^{+} \sum M_1 = 0 \quad = \frac{wL^2}{2} - (wL)\frac{L}{2}$$

Beam Stiffness

Example 6 - Cantilever Beam

Consider the beam, shown below, determine the vertical displacement and rotation at the free-end and the nodal forces, including reactions. Assume ***EI*** is constant throughout the beam.





We will use one element and replace the concentrated load with the appropriate nodal forces.

Beam Stiffness

Example 6 - Cantilever Beam

The beam stiffness equations are:

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{P}{2} \\ -\frac{PL}{8} \\ -\frac{P}{2} \\ \frac{PL}{8} \end{Bmatrix}$$

Apply the boundary conditions: $v_1 = \phi_1 = 0$

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Beam Stiffness

Example 6 - Cantilever Beam

The beam stiffness equations become:

$$\begin{Bmatrix} -\frac{P}{2} \\ \frac{PL}{8} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix} \qquad \begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{5PL^3}{48EI} \\ -\frac{PL^2}{8EI} \end{Bmatrix}$$

To obtain the global nodal forces, we begin by evaluating the effective nodal forces.

$$\begin{Bmatrix} F^e{}_{1y} \\ M^e{}_1 \\ F^e{}_{2y} \\ M^e{}_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -\frac{5PL^3}{48EI} \\ -\frac{PL^2}{8EI} \end{Bmatrix} = \begin{Bmatrix} \frac{P}{2} \\ \frac{3PL}{8} \\ -\frac{P}{2} \\ \frac{PL}{8} \end{Bmatrix}$$

Beam Stiffness

Example 6 - Cantilever Beam

Using the above expression in the following equation, gives:

$$\mathbf{F = Kd - F_0} \qquad \underbrace{\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \end{Bmatrix}}_{\mathbf{F}} = \underbrace{\begin{Bmatrix} \frac{P}{2} \\ \frac{3PL}{8} \\ -\frac{P}{2} \\ \frac{PL}{8} \end{Bmatrix}}_{\mathbf{Kd = F^e}} - \underbrace{\begin{Bmatrix} -\frac{P}{2} \\ -\frac{PL}{8} \\ -\frac{P}{2} \\ \frac{PL}{8} \end{Bmatrix}}_{\mathbf{F_0}} = \underbrace{\begin{Bmatrix} P \\ \frac{PL}{2} \\ 0 \\ 0 \end{Bmatrix}}_{\mathbf{F}}$$

Beam Stiffness

Example 6 - Cantilever Beam

In general, for any structure in which an equivalent nodal force replacement is made, the actual nodal forces acting on the structure are determined by first evaluating the effective nodal forces $\mathbf{F^e}$ for the structure and then subtracting off the equivalent nodal forces $\mathbf{F_0}$ for the structure.

Similarly, for any element of a structure in which equivalent nodal force replacement is made, the actual local nodal forces acting on the element are determined by first evaluating the effective local nodal forces for the element and then subtracting off the equivalent local nodal forces associated only with the element.

$$\mathbf{f = kd - f_0}$$

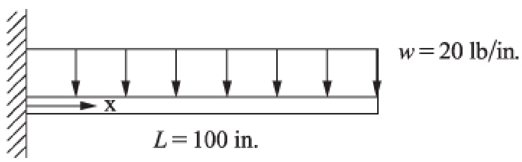
Beam Stiffness

Comparison of FE Solution to Exact Solution

We will now compare the finite element solution to the exact classical beam theory solution for the cantilever beam shown below.

Both one- and two-element finite element solutions will be presented and compared to the exact solution obtained by the direct double-integration method.

Let E = 30 x 10^6 *psi*, I = 100 in^4, L = 100 *in*, and uniform load w = 20 *lb/in*.



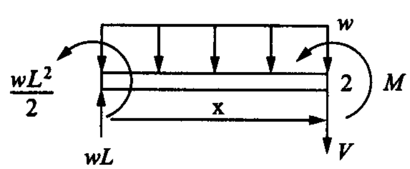
Beam Stiffness

Comparison of FE Solution to Exact Solution

To obtain the solution from classical beam theory, we use the double-integration method:

$$y'' = \left(\frac{M(x)}{EI}\right)$$

where the double prime superscript indicates differentiation twice with respect to x and M is expressed as a function of x by using a section of the beam as shown:



$$+\uparrow \sum F_y = 0 \quad \Rightarrow \quad V(x) = wL - wx$$

$$\circlearrowleft^{+} \sum M = 0 \Rightarrow \boxed{M(x) = -\frac{wx^2}{2} + wLx - \frac{wL^2}{2}}$$

Beam Stiffness

Comparison of FE Solution to Exact Solution

To obtain the solution from classical beam theory, we use the double-integration method:

$$y'' = \left(\frac{M(x)}{EI}\right)$$

$$y = \frac{w}{EI}\iint\left(-\frac{x^2}{2} + Lx - \frac{L^2}{2}\right)dx\,dx$$

$$y = \frac{w}{EI}\int\left(-\frac{x^3}{6} + \frac{Lx^2}{2} - \frac{xL^2}{2} + C_1\right)dx$$

Boundary Conditions

$$y'(0) = 0 \qquad y(0) = 0$$

$$y = \frac{w}{EI}\left(-\frac{x^4}{24} + \frac{Lx^3}{6} - \frac{x^2L^2}{4}\right) + C_1x + C_2$$

$$\boxed{y = \frac{w}{EI}\left(-\frac{x^4}{24} + \frac{Lx^3}{6} - \frac{x^2L^2}{4}\right)}$$

Beam Stiffness

Comparison of FE Solution to Exact Solution

Recall the one-element solution to the cantilever beam is:

$$\begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{wL^4}{8EI} \\ -\dfrac{wL^3}{6EI} \end{Bmatrix}$$

Using the numerical values for this problem we get:

$$\begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{20\,{}^{lb}\!/_{in}(100\,in)^4}{8\left(30\times10^6\,psi\right)100\,in^4} \\ -\dfrac{20\,{}^{lb}\!/_{in}(100\,in)^3}{6\left(30\times10^6\,psi\right)100\,in^4} \end{Bmatrix} = \begin{Bmatrix} -0.0833\,in \\ -0.00111\,rad \end{Bmatrix}$$

Beam Stiffness

Comparison of FE Solution to Exact Solution

The slope and displacement from the one-element FE solution identically match the beam theory values evaluated at $x = L$.

The reason why these nodal values from the FE solution are correct is that the element nodal forces were calculated on the basis of being energy or work equivalent to the distributed load based on the assumed cubic displacement field within each beam element.

Beam Stiffness

Comparison of FE Solution to Exact Solution

Values of displacement and slope at other locations along the beam for the FE are obtained by using the assumed cubic displacement function.

$$v(x) = \frac{1}{L^3}\left(-2x^3 + 3x^2L\right)v_2 + \frac{1}{L^3}\left(x^3L - x^2L^2\right)\phi_2$$

The value of the displacement at the midlength $v(x = 50\ in)$ is:

$$v(x = 50\,in) = -0.0278\,in$$

Using beam theory, the displacement at $v(x = 50\ in)$ is:

$$v(x = 50\,in) = -0.0295\,in$$

Beam Stiffness

Comparison of FE Solution to Exact Solution

In general, the displacements evaluated by the FE method using the cubic function for *v* are lower than by those of beam theory except at the nodes.

This is always true for beams subjected to some form of distributed load that are modeled using the cubic displacement function.

The exception to this result is at the nodes, where the beam theory and FE results are identical because of the work-equivalence concept used to replace the distributed load by work-equivalent discrete loads at the nodes.

Beam Stiffness

Comparison of FE Solution to Exact Solution

The beam theory solution predicts a quartic (fourth-order) polynomial expression for a beam subjected to uniformly distributed loading, while the FE solution *v*(*x*) assumes a cubic (third-order) displacement behavior in each beam all load conditions.

The FE solution predicts a stiffer structure than the actual one.

This is expected, as the FE model forces the beam into specific modes of displacement and effectively yields a stiffer model than the actual structure.

However, as more elements are used in the model, the FE solution converges to the beam theory solution.

Beam Stiffness

Comparison of FE Solution to Exact Solution

For the special case of a beam subjected to only nodal concentrated loads, the beam theory predicts a cubic displacement behavior.

The FE solution for displacement matches the beam theory solution for all locations along the beam length, as both $v(x)$ and $y(x)$ are cubic functions.

Beam Stiffness

Comparison of FE Solution to Exact Solution

Under uniformly distributed loading, the beam theory solution predicts a quadratic moment and a linear shear force in the beam.

However, the FE solution using the cubic displacement function predicts a linear bending moment and a constant shear force within each beam element used in the model.

Beam Stiffness

Comparison of FE Solution to Exact Solution

We will now determine the bending moment and shear force in the present problem based on the FE method.

$$M(x) = EIy'' = EI\frac{d^2(\mathbf{Nd})}{dx^2} = EI\frac{(d^2\mathbf{N})\mathbf{d}}{dx^2}$$

$$M(x) = EI[\mathbf{B}]\{\mathbf{d}\}$$

$$[\mathbf{B}] = \left[\left(-\frac{6}{L^2}+\frac{12x}{L^3}\right) \quad \left(-\frac{4}{L}+\frac{6x}{L^2}\right) \quad \left(\frac{6}{L^2}-\frac{12x}{L^3}\right) \quad \left(-\frac{2}{L}+\frac{6x}{L^2}\right)\right]$$

$$M = EI\left[\left(-\frac{6}{L^2}+\frac{12x}{L^3}\right)v_1 + \left(-\frac{4}{L}+\frac{6x}{L^2}\right)\phi_1 + \left(\frac{6}{L^2}-\frac{12x}{L^3}\right)v_2 + \left(-\frac{2}{L}+\frac{6x}{L^2}\right)\phi_2\right]$$

Beam Stiffness

Comparison of FE Solution to Exact Solution

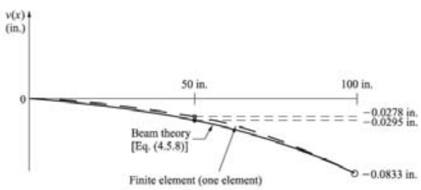
We will now determine the bending moment and shear force in the present problem based on the FE method.

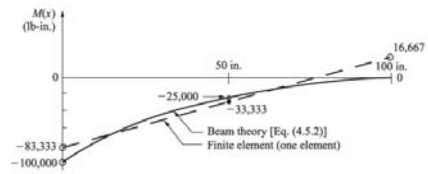
Position	M_{FE}	M_{exact}
X = 0	-83,333 *lb-in*	-100,000 *lb-in*
X = 50 *in*	-33,333 *lb-in*	-25,000 *lb-in*
X = 100 *in*	16,667 *lb-in*	0

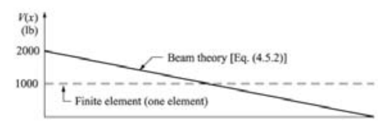
Beam Stiffness

Comparison of FE Solution to Exact Solution

The plots below show the displacement, bending moment , and shear force over the beam using beam theory and the one-element FE solutions.



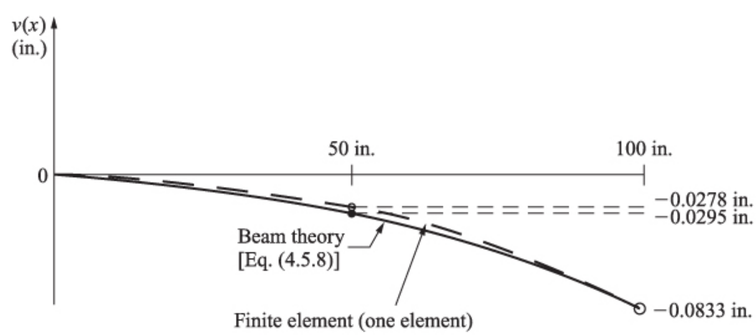




Beam Stiffness

Comparison of FE Solution to Exact Solution

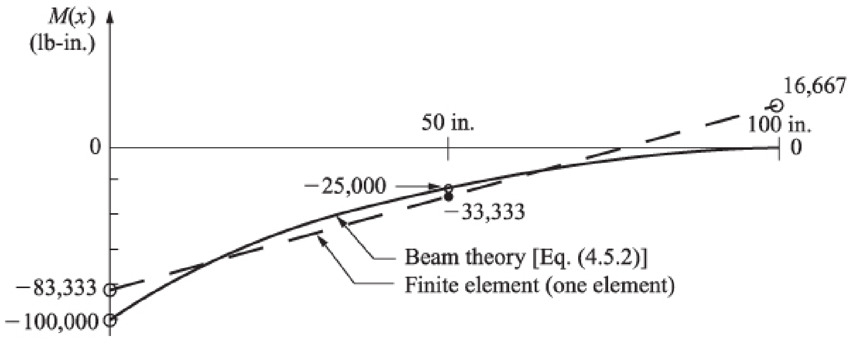
The FE solution for displacement matches the beam theory solution at the nodes but predicts smaller displacements (less deflection) at other locations along the beam length.



Beam Stiffness

Comparison of FE Solution to Exact Solution

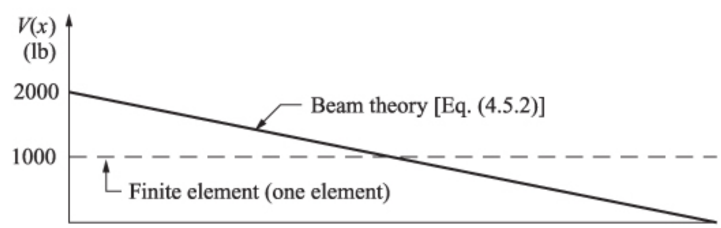
The bending moment is derived by taking two derivatives on the displacement function. It then takes more elements to model the second derivative of the displacement function.



Beam Stiffness

Comparison of FE Solution to Exact Solution

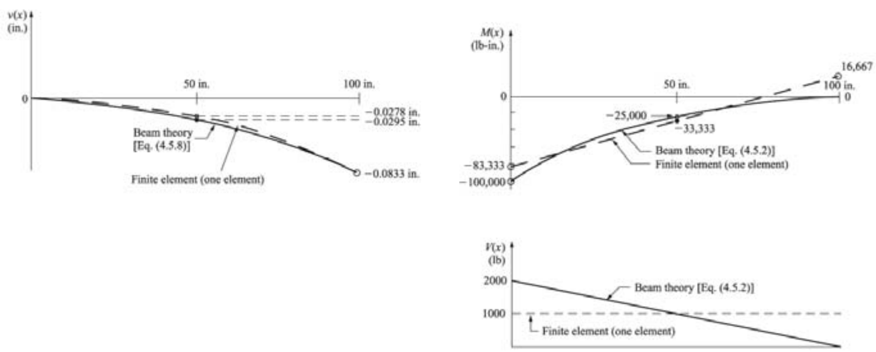
The shear force is derived by taking three derivatives on the displacement function. For the uniformly loaded beam, the shear force is a constant throughout the single-element model.



Beam Stiffness

Comparison of FE Solution to Exact Solution

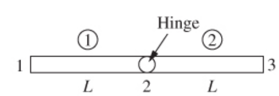
To improve the FE solution we need to use more elements in the model (refine the mesh) or use a higher-order element, such as a fifth-order approximation for the displacement function.

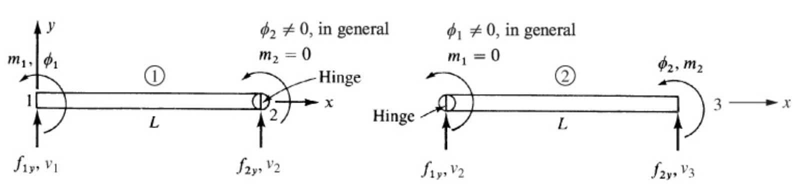


Beam Stiffness

Beam Element with Nodal Hinge

Consider the beam, shown below, with an internal hinge. An internal hinge causes a discontinuity in the slope of the deflection curve at the hinge and the bending moment is zero at the hinge.





Beam Stiffness

Beam Element with Nodal Hinge

For a beam with a hinge on the **right** end:

$[K_{11}]$ $[K_{12}]$

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\left[\begin{array}{ccc|c} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ \hline 6L & 2L^2 & -6L & 4L^2 \end{array}\right]\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

$[K_{21}]$ $[K_{22}]$

The moment m_2 is zero and we can partition the matrix to eliminate the degree of freedom associated with ϕ_2.

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right]\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} \qquad \begin{aligned} f_1 &= K_{11}d_1 + K_{12}d_2 \\ f_2 &= K_{21}d_1 + K_{22}d_2 \end{aligned}$$

Beam Stiffness

Beam Element with Nodal Hinge

For a beam with a hinge on the **right** end:

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\left[\begin{array}{ccc|c} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ \hline 6L & 2L^2 & -6L & 4L^2 \end{array}\right]\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

$$\{d_1\} = \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \end{Bmatrix} \qquad \{f_1\} = \begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \end{Bmatrix}$$

$$\{d_2\} = \{\phi_2\} \qquad \{f_2\} = \{m_2\}$$

$$d_2 = K_{22}^{-1}\left(f_2 - K_{21}d_1\right)$$

$$f_1 = K_{11}d_1 + K_{12}d_2 \quad = K_{11}d_1 + K_{12}\left[K_{22}^{-1}\left(f_2 - K_{21}d_1\right)\right]$$

$$f_c = k_c d_1 \qquad f_c = f_1 - K_{12}K_{22}^{-1}\overset{m_2 = 0}{f_2} \qquad k_c = K_{11} - K_{12}K_{22}^{-1}K_{21}$$

Beam Stiffness

Beam Element with Nodal Hinge

We can condense out the degree of freedom by using the partitioning method discussed earlier.

$$k_c = K_{11} - K_{12}K_{22}^{-1}K_{21}$$

$$k_c = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 \\ 6L & 4L^2 & -6L \\ -12 & -6L & 12 \end{bmatrix} - \frac{EI}{L^3}\begin{Bmatrix} 6L \\ 2L^2 \\ -6L \end{Bmatrix}\frac{1}{4L^2}\begin{bmatrix} 6L & 2L^2 & -6L \end{bmatrix}$$

Therefore, the condensed stiffness matrix is:

$$k_c = \frac{3EI}{L^3}\begin{bmatrix} 1 & L & -1 \\ L & L^2 & -L \\ -1 & -L & 1 \end{bmatrix}$$

Beam Stiffness

Beam Element with Nodal Hinge

The element force-displacement equations are:

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \end{Bmatrix} = \frac{3EI}{L^3}\begin{bmatrix} 1 & L & -1 \\ L & L^2 & -L \\ -1 & -L & 1 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \end{Bmatrix}$$

Expanding the element force-displacement equations and maintaining $m_2 = 0$ gives:

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{3EI}{L^3}\begin{bmatrix} 1 & L & -1 & 0 \\ L & L^2 & -L & 0 \\ -1 & -L & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Beam Stiffness

Beam Element with Nodal Hinge

Once the displacements $\boldsymbol{d}_1$ are found, $\boldsymbol{d}_2$ may be computed:

$$d_2 = K_{22}^{-1}\left(f_2 - K_{21}d_1\right) \qquad \{\phi_2\} = K_{22}^{-1}\left(\overset{m_2 = 0}{\{m_2\}} - K_{21}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \end{Bmatrix}\right)$$

In this case, $\boldsymbol{d}_2$ is ϕ_2 (the rotation at right side of the element at the hinge.

Beam Stiffness

Beam Element with Nodal Hinge

For a beam with a hinge on the **left** end:

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\left[\begin{array}{cc:cc} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ \hdashline -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{array}\right]\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Rewriting the equations to move ϕ_1 and m_1 to the first row:

$$\begin{Bmatrix} m_1 \\ f_{1y} \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\left[\begin{array}{cc:cc} 4L^2 & 6L & -6L & 2L^2 \\ 6L & 12 & -12 & 6L \\ \hdashline -6L & -12 & 12 & -6L \\ 2L^2 & 6L & -6L & 4L^2 \end{array}\right]\begin{Bmatrix} \phi_1 \\ v_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Beam Stiffness

Beam Element with Nodal Hinge

For a beam with a hinge on the **left** end:

$[K_{11}]$ $[K_{12}]$

$$\begin{Bmatrix} m_1 \\ f_{1y} \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\left[\begin{array}{c|ccc} 4L^2 & 6L & -6L & 2L^2 \\ \hline 6L & 12 & -12 & 6L \\ -6L & -12 & 12 & -6L \\ 2L^2 & 6L & -6L & 4L^2 \end{array}\right]\begin{Bmatrix} \phi_1 \\ v_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

$[K_{21}]$ $[K_{22}]$

The moment m_1 is zero and we can partition the matrix to eliminate the degree of freedom associated with ϕ_1.

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right]\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} \qquad \begin{aligned} f_1 &= K_{11}d_1 + K_{12}d_2 \\ f_2 &= K_{21}d_1 + K_{22}d_2 \end{aligned}$$

Beam Stiffness

Beam Element with Nodal Hinge

For a beam with a hinge on the right end:

$$\begin{Bmatrix} m_1 \\ f_1 \\ f_2 \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\left[\begin{array}{c|ccc} 4L^2 & 6L & -6L & 2L^2 \\ \hline 6L & 12 & -12 & 6L \\ -6L & -12 & 12 & -6L \\ 2L^2 & 6L & -6L & 4L^2 \end{array}\right]\begin{Bmatrix} \phi_1 \\ v_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

$$\{d_1\} = \{\phi_1\} \qquad \{f_1\} = \{m_1\}$$

$$\{d_2\} = \begin{Bmatrix} v_1 \\ v_2 \\ \phi_2 \end{Bmatrix} \qquad \{f_2\} = \begin{Bmatrix} f_{1y} \\ f_{2y} \\ m_2 \end{Bmatrix}$$

$$d_1 = K_{11}^{-1}\left(f_1 - K_{12}d_2\right)$$

$$f_2 = K_{21}d_1 + K_{22}d_2 = K_{22}d_2 + K_{21}\left[K_{11}^{-1}\left(f_1 - K_{12}d_2\right)\right]$$

$$f_c = k_c d_2 \qquad f_c = f_2 - K_{21}K_{11}^{-1}\overset{m_1 = 0}{f_1} \qquad k_c = K_{22} - K_{21}K_{11}^{-1}K_{12}$$

Beam Stiffness

Beam Element with Nodal Hinge

We can condense out the degree of freedom by using the partitioning method discussed earlier.

$$k_c = K_{22} - K_{21} K_{11}^{-1} K_{12}$$

$$k_c = \frac{EI}{L^3}\begin{bmatrix} 12 & -12 & 6L \\ -12 & 12 & -6L \\ 6L & -6L & 4L^2 \end{bmatrix} - \frac{EI}{L^3}\begin{Bmatrix} 6L \\ -6L \\ 2L^2 \end{Bmatrix} \frac{1}{4L^2}\begin{bmatrix} 6L & -6L & 2L^2 \end{bmatrix}$$

Therefore, the condensed stiffness matrix is:

$$k_c = \frac{3EI}{L^3}\begin{bmatrix} 1 & -1 & L \\ -1 & 1 & -L \\ L & -L & L^2 \end{bmatrix}$$

Beam Stiffness

Beam Element with Nodal Hinge

The element force-displacement equations are:

$$\begin{Bmatrix} f_{1y} \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{3EI}{L^3}\begin{bmatrix} 1 & -1 & L \\ -1 & 1 & -L \\ L & -L & L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \\ \phi_1 \end{Bmatrix}$$

Expanding the element force-displacement equations and maintaining $m_1 = 0$ gives:

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{3EI}{L^3}\begin{bmatrix} 1 & 0 & L & -1 \\ 0 & 0 & 0 & 0 \\ L & 0 & L & -L \\ -1 & 0 & -L & 1 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Beam Stiffness

Beam Element with Nodal Hinge

Once the displacements $\boldsymbol{d}_2$ are found, $\boldsymbol{d}_1$ may be computed:

$$d_1 = K_{11}^{-1}\left(f_1 - K_{12}d_2\right) \qquad \{\phi_1\} = K_{11}^{-1}\left(\{\cancelto{m_1 = 0}{m_1}\} - K_{12}\begin{Bmatrix} v_1 \\ v_2 \\ \phi_1 \end{Bmatrix}\right)$$

In this case, $\boldsymbol{d}_1$ is ϕ_1 (the rotation at right side of the element at the hinge.

Beam Stiffness

Beam Element with Nodal Hinge

For a beam element with a hinge at its left end, the element force-displacement equations are:

$$\begin{Bmatrix} f_{1y} \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{3EI}{L^3}\begin{bmatrix} 1 & -1 & L \\ -1 & 1 & -L \\ L & -L & L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Expanding the element force-displacement equations and maintaining $m_1 = 0$ gives:

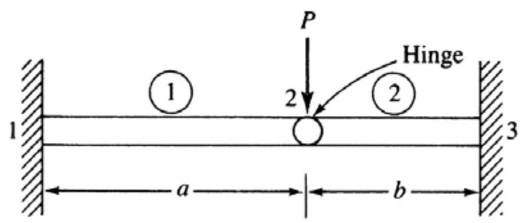
$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{3EI}{L^3}\begin{bmatrix} 1 & 0 & -1 & L \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -L \\ L & 0 & -L & L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Beam Stiffness

Example 7 - Beam With Hinge

In the following beam, shown below, determine the vertical displacement and rotation at node 2 and the element forces for the uniform beam with an internal hinge at node 2.

Assume ***EI*** is constant throughout the beam.



Beam Stiffness

Example 7 - Beam With Hinge

The stiffness matrix for element 1 (with hinge on right) is:

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{3EI}{L^3} \begin{bmatrix} 1 & L & -1 & 0 \\ L & L^2 & -L & 0 \\ -1 & -L & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

$$\begin{array}{cc} & \begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \end{array} \\ \begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{3EI}{a^3} & \begin{bmatrix} 1 & a & -1 & 0 \\ a & a^2 & -a & 0 \\ -1 & -a & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix} \end{array}$$

Beam Stiffness

Example 7 - Beam With Hinge

The stiffness matrix for element 2 (with no hinge) is:

$$\begin{Bmatrix} f_2 \\ m_2 \\ f_3 \\ m_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix}$$

$$\begin{Bmatrix} f_2 \\ m_2 \\ f_3 \\ m_3 \end{Bmatrix} = \frac{EI}{b^3}\begin{array}{c} \begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ \begin{bmatrix} 12 & 6b & -12 & 6b \\ 6b & 4b^2 & -6b & 2b^2 \\ -12 & -6b & 12 & -6b \\ 6b & 2b^2 & -6b & 4b^2 \end{bmatrix} \end{array}\begin{Bmatrix} v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix}$$

Beam Stiffness

Example 7 - Beam With Hinge

The assembled equations are:

Element 1

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = EI\begin{bmatrix} \frac{3}{a^3} & \frac{3}{a^2} & -\frac{3}{a^3} & 0 & 0 & 0 \\ \frac{3}{a^2} & \frac{3}{a} & -\frac{3}{a^2} & 0 & 0 & 0 \\ -\frac{3}{a^3} & -\frac{3}{a^2} & \frac{12}{b^3}+\frac{3}{a^3} & \frac{6}{b^2} & -\frac{12}{b^3} & \frac{6}{b^2} \\ 0 & 0 & \frac{6}{b^2} & \frac{4}{b} & -\frac{6}{b^2} & \frac{2}{b} \\ 0 & 0 & -\frac{12}{b^3} & -\frac{6}{b^2} & \frac{12}{b^3} & -\frac{6}{b^2} \\ 0 & 0 & \frac{6}{b^2} & \frac{2}{b} & -\frac{6}{b^2} & \frac{4}{b} \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix}$$

Element 2

Beam Stiffness

Example 7 - Beam With Hinge

The boundary conditions are: $v_1 = v_3 = \phi_1 = \phi_3 = 0$

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = EI \begin{bmatrix} \frac{3}{a^3} & \frac{3}{a^2} & -\frac{3}{a^3} & 0 & 0 & 0 \\ \frac{3}{a^2} & \frac{3}{a} & -\frac{3}{a^2} & 0 & 0 & 0 \\ -\frac{3}{a^3} & -\frac{3}{a^2} & \frac{3}{a^3}+\frac{12}{b^3} & \frac{6}{b^2} & -\frac{12}{b^3} & \frac{6}{b^2} \\ 0 & 0 & \frac{6}{b^2} & \frac{4}{b} & -\frac{6}{b^2} & \frac{2}{b} \\ 0 & 0 & -\frac{12}{b^3} & -\frac{6}{b^2} & \frac{12}{b^3} & -\frac{6}{b^2} \\ 0 & 0 & \frac{6}{b^2} & \frac{2}{b} & -\frac{6}{b^2} & \frac{4}{b} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ v_2 \\ \phi_2 \\ 0 \\ 0 \end{Bmatrix}$$

Beam Stiffness

Example 7 - Beam With Hinge

After applying the boundary conditions the global beam equations reduce to:

$$EI \begin{bmatrix} \frac{3}{a^3}+\frac{12}{b^3} & \frac{6}{b^2} \\ \frac{6}{b^2} & \frac{4}{b} \end{bmatrix} \begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -P \\ 0 \end{Bmatrix} \qquad \begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{a^3 b^3 P}{3\left(b^3 + a^3\right)EI} \\ \dfrac{a^3 b^2 P}{2\left(b^3 + a^3\right)EI} \end{Bmatrix}$$

Beam Stiffness

Example 7 - Beam With Hinge

The element force-displacement equations for element 1 are:

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \end{Bmatrix} = \frac{3EI}{a^3}\begin{bmatrix} 1 & a & -1 \\ a & a^2 & -a \\ -1 & -a & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -\dfrac{a^3 b^3 P}{3\left(b^3 + a^3\right)EI} \end{Bmatrix} \qquad \begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \end{Bmatrix} = \begin{Bmatrix} \dfrac{b^3 P}{b^3 + a^3} \\ \dfrac{ab^3 P}{b^3 + a^3} \\ -\dfrac{b^3 P}{b^3 + a^3} \end{Bmatrix}$$

Beam Stiffness

Example 7 - Beam With Hinge

The slope ϕ_2 on element 1 may be found using the condensed matrix:

$$d_2 = K_{22}^{-1}\left(f_2 - K_{21} d_1\right) \qquad \left\{\phi_2\right\}_{\text{element1}} = K_{22}^{-1}\left(\left\{\overset{m_2 = 0}{\cancel{m_2}}\right\} - K_{21}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \end{Bmatrix}\right)$$

$$\left\{\phi_2\right\}_1 = \frac{a}{4EI}\left(0 - \frac{EI}{a^3}\begin{bmatrix} 6a & 2a^2 & -6a \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -\dfrac{a^3 b^3 P}{3\left(b^3 + a^3\right)EI} \end{Bmatrix}\right)$$

$$\left\{\phi_2\right\}_1 = -\frac{a^2 b^3 P}{2\left(b^3 + a^3\right)EI}$$

Beam Stiffness

Example 7 - Beam With Hinge

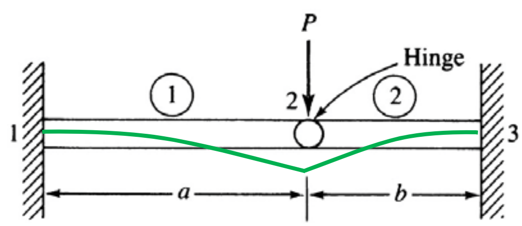
The element force-displacement equations for element 2 are:

$$\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = \frac{EI}{b^3}\begin{bmatrix} 12 & 6b & -12 & 6b \\ 6b & 4b^2 & -6b & 2b^2 \\ -12 & -6b & 12 & -6b \\ 6b & 2b^2 & -6b & 4b^2 \end{bmatrix}\begin{Bmatrix} -\dfrac{a^3b^3P}{3\left(b^3+a^3\right)EI} \\ -\dfrac{a^3b^2P}{2\left(b^3+a^3\right)EI} \\ 0 \\ 0 \end{Bmatrix} \qquad \begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{a^3P}{b^3+a^3} \\ 0 \\ \dfrac{a^3P}{b^3+a^3} \\ \dfrac{ba^3P}{b^3+a^3} \end{Bmatrix}$$

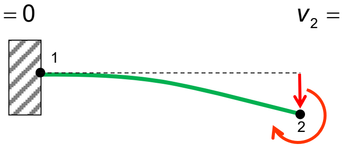
Beam Stiffness

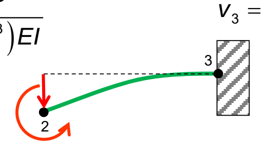
Example 7 - Beam With Hinge

Displacements and rotations on each element.



$v_1 = \phi_1 = 0$ $\qquad v_2 = -\dfrac{a^3b^3P}{3\left(b^3+a^3\right)EI}$ $\qquad v_3 = \phi_3 = 0$





$$\{\phi_2\}_1 = -\frac{a^2b^3P}{2\left(b^3+a^3\right)EI} \qquad \{\phi_2\}_2 = \frac{a^3b^2P}{2\left(b^3+a^3\right)EI}$$

Beam Stiffness

Potential Energy Approach to Derive Beam Element Equations

Let’s derive the equations for a beam element using the principle of minimum potential energy.

The procedure for applying the principle of minimum potential energy is similar to that used for the bar element.

The total potential energy π_p is defined as the sum of the internal strain energy ***U*** and the potential energy of the external forces Ω:

$$\pi_p = U + \Omega$$

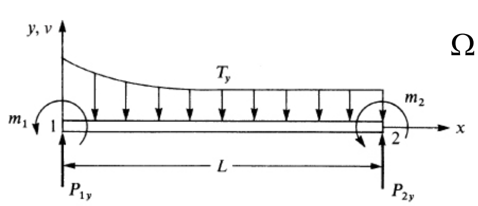
Beam Stiffness

Potential Energy Approach to Derive Beam Element Equations

The differential internal work (strain energy) *dU* in a one-dimensional beam element is:

$$U = \int_V \tfrac{1}{2}\sigma_x \varepsilon_x \, dV$$

For a single beam element, shown below, subjected to both distributed and concentrated nodal forces, the potential energy due to forces (or the work done these forces) is:



$$\Omega = -\int_S T_y v \, dS - \sum_{i=1}^{2} P_{iy} v_i - \sum_{i=1}^{2} m_i \phi_i$$

Beam Stiffness

Potential Energy Approach to Derive Beam Element Equations

If the beam element has a constant cross-sectional area *A*, then the differential volume of the beam is given as:

$$dV = dA\, dx$$

The differential element where the surface loading acts is given as: $dS = b\, dx$ (where *b* is the width of the beam element).

Therefore the total potential energy is:

$$\pi_p = \int_x \int_A \tfrac{1}{2}\sigma_x \varepsilon_x \, dA\, dx - \int_x T_y v b\, dx - \sum_{i=1}^{2}\left(P_{iy} v_i - m_i \phi_i\right)$$

Beam Stiffness

Potential Energy Approach to Derive Beam Element Equations

The strain-displacement relationship is: $\varepsilon_x = -y \dfrac{d^2 v}{dx^2}$

We can express the strain in terms of nodal displacements and rotations as:

$$\varepsilon_x = -y \left[\frac{12x - 6L}{L^3} \quad \frac{6xL - 4L^2}{L^3} \quad \frac{-12x + 6L}{L^3} \quad \frac{6xL - 2L^2}{L^3} \right] \{d\}$$

$$\{\varepsilon_x\} = -y[B]\{d\}$$

$$[B] = \left[\frac{12x - 6L}{L^3} \quad \frac{6xL - 4L^2}{L^3} \quad \frac{-12x + 6L}{L^3} \quad \frac{6xL - 2L^2}{L^3} \right]$$

Beam Stiffness

Potential Energy Approach to Derive Beam Element Equations

The stress-strain relationship in one-dimension is:

$$\{\sigma_x\} = [E]\{\varepsilon_x\}$$

where ***E*** is the modulus of elasticity. Therefore:

$$\{\sigma_x\} = -y[E][B]\{d\} \qquad \{\varepsilon_x\} = -y[B]\{d\}$$

The total potential energy can be written in matrix form as:

$$\pi_p = \int_x \int_A \tfrac{1}{2}\{\sigma_x\}^T \{\varepsilon_x\}\, dA\, dx - \int_x bT_y \{v\}^T\, dx - \{d\}^T \{P\}$$

Beam Stiffness

Potential Energy Approach to Derive Beam Element Equations

If we define, $w = bT_y$ as a line load (load per unit length) in the y direction and the substitute the definitions of σ_x and ε_x the total potential energy can be written in matrix form as:

$$\pi_p = \int_0^L \int_A \frac{E}{2} y^2 \{d\}^T [B]^T [B]\{d\}\, dA\, dx - \int_0^L w\{d\}^T [N]^T dx - \{d\}^T \{P\}$$

Use the following definition for moment of inertia: $I = \int_A y^2 dA$

Then the total potential energy expression becomes:

$$\pi_p = \int_0^L \frac{EI}{2} \{d\}^T [B]^T [B]\{d\}\, dx - \int_0^L w\{d\}^T [N]^T dx - \{d\}^T \{P\}$$

Beam Stiffness

Potential Energy Approach to Derive Beam Element Equations

Differentiating the total potential energy with respect to the displacement and rotations (v_1, v_2, ϕ_1 and ϕ_2) and equating each term to zero gives:

$$EI\int_0^L [B]^T[B]\,dx\,\{d\} - \int_0^L w[N]^T\,dx - \{P\} = 0$$

The nodal forces vector is: $\{f\} = \int_0^L w[N]^T\,dx + \{P\}$

The elemental stiffness matrix is: $[k] = EI\int_0^L [B]^T[B]\,dx$

Beam Stiffness

Potential Energy Approach to Derive Beam Element Equations

Integrating the previous matrix expression gives:

$$[k] = EI\int_0^L [B]^T[B]\,dx$$

$$[k] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Beam Stiffness

Galerkin's Method to Derive Beam Element Equations

The governing differential equation for a one-dimensional beam is:

$$EI\left(\frac{d^4v}{dx^4}\right)+w=0$$

We can define the residual ***R*** as:

$$\int_0^L\left(EI\left(\frac{d^4v}{dx^4}\right)+w\right)N_i\,dx=0 \qquad i=1, 2, 3, \text{and } 4$$

If we apply integration by parts twice to the first term we get:

$$\int_0^L EI\left(v_{xxxx}\right)N_i dx=\int_0^L EI\left(v_{xx}\right)\left(N_{i,xx}\right)dx+EI\left[N_i\left(v_{xxx}\right)-\left(N_{i,x}\right)\left(v_{xx}\right)\right]_0^L$$

where the subscript *x* indicates a derivative with respect to *x*

Beam Stiffness

Galerkin's Method to Derive Beam Element Equations

Since $v=[N]\{d\}$, then the second derivative of *v* with respect to *x* is:

$$v_{xx}=\left[\frac{12x-6L}{L^3}\quad\frac{6xL-4L^2}{L^3}\quad\frac{-12x+6L}{L^3}\quad\frac{6xL-2L^2}{L^3}\right]\{d\}$$

or

$$v_{xx}=[B]\{d\}$$

Therefore the integration by parts becomes:

$$\int_0^L\left(N_{i,xx}\right)EI[B]\,dx\,\{d\}+\int_0^L N_i w\,dx+\left[N_iV-\left(N_{i,x}\right)m\right]\{d\}\Big|_0^L=0$$

$$i=1, 2, 3, \text{and } 4$$

Beam Stiffness

Galerkin's Method to Derive Beam Element Equations

The above expression is really four equations (one for each N_i) and can be written in matrix form as:

$$\int_0^L [B]^T EI[B]\, dx \{d\} = -\int_0^L [N]^T w\, dx + \Big|\, [N]_x{}^T m - [N]^T V \,\Big|_0^L$$

The interpolation function in the last two terms can be evaluated:

$$[N]_x\left(x=0\right) = [0 \quad 1 \quad 0 \quad 0] \qquad [N]_x\left(x=L\right) = [0 \quad 0 \quad 0 \quad 1]$$

$$[N]\left(x=0\right) = [1 \quad 0 \quad 0 \quad 0] \qquad [N]\left(x=L\right) = [0 \quad 0 \quad 1 \quad 0]$$

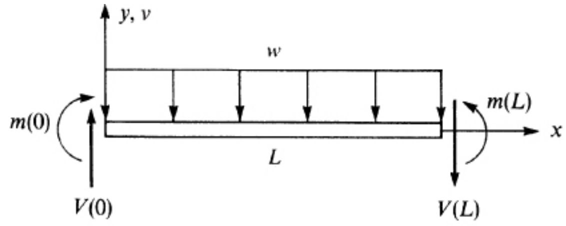
Beam Stiffness

Galerkin's Method to Derive Beam Element Equations

Therefore, the last two terms of the matrix form of the Galerkin formulation become (see the figure below):

$$i = 1 \quad \Rightarrow \quad V(0) \qquad i = 2 \quad \Rightarrow \quad m(0)$$

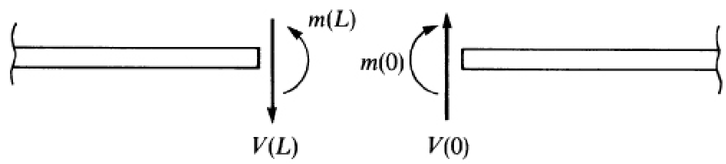
$$i = 3 \quad \Rightarrow \quad V(L) \qquad i = 4 \quad \Rightarrow \quad m(L)$$



Beam Stiffness

Galerkin's Method to Derive Beam Element Equations

When beam elements are assembled, as shown below:



Two shear forces and two moments form adjoining elements contribute to the concentrated force and the concentrated moment at the node common to both elements.

Beam Stiffness

Problems:

7. Verify the four beam element equations are contained in the following matrix expression.

$$EI\int_0^L [B]^T [B]\, dx\, \{d\} - \int_0^L w[N]^T dx - \{P\} = 0$$

8. Do problems ***4.10, 4.12, 4.18, 4.22, 4.40*** and ***4.47*** in your textbook.

Beam Stiffness

Problems:

9. Work problem ***4.36*** in your using the SAP2000.

   Attempt to select the lightest standard W section to support the loads for the beam.

   The bending stress must not exceed 160 MPa and the allowable deflection must not exceed (L = 6 m)/360

End of Chapter 4b