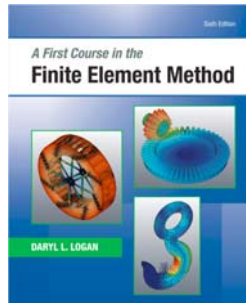## *Chapter 4a – Development of Beam Equations*

Learning Objectives

- To review the basic concepts of beam bending
- To derive the stiffness matrix for a beam element
- To demonstrate beam analysis using the direct stiffness method
- To illustrate the effects of shear deformation in shorter beams
- To introduce the work-equivalence method for replacing distributed loading by a set of discrete loads
- To introduce the general formulation for solving beam problems with distributed loading acting on them
- To analyze beams with distributed loading acting on them
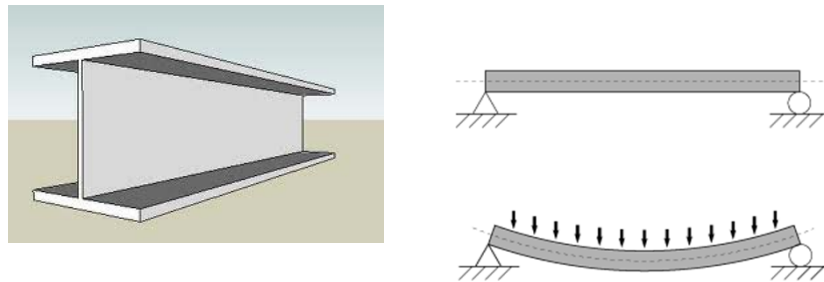
Chapter 4a – Development of Beam Equations
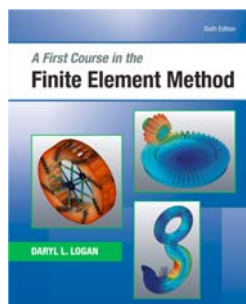
Learning Objectives

- To compare the finite element solution to an exact solution for a beam
- To derive the stiffness matrix for the beam element with nodal hinge
- To show how the potential energy method can be used to derive the beam element equations
- To apply Galerkin's residual method for deriving the beam element equations

Development of Beam Equations

In this section, we will develop the stiffness matrix for a beam element, the most common of all structural elements.
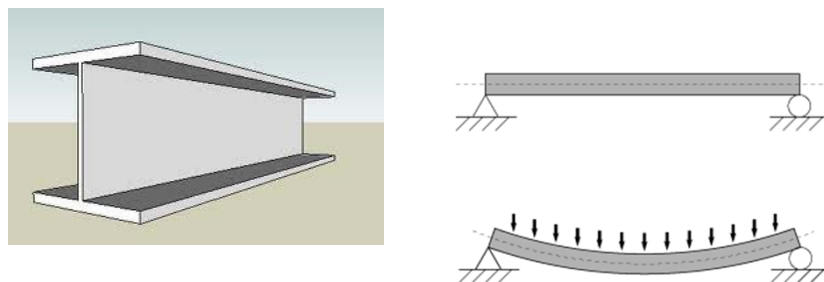
The **beam** element is considered to be straight and to have constant cross-sectional area.

Development of Beam Equations

We will derive the beam element stiffness matrix by using the principles of simple beam theory.

The degrees of freedom associated with a node of a beam element are a transverse displacement and a rotation.

Development of Beam Equations

We will discuss procedures for handling distributed loading and concentrated nodal loading.
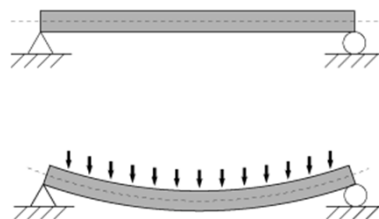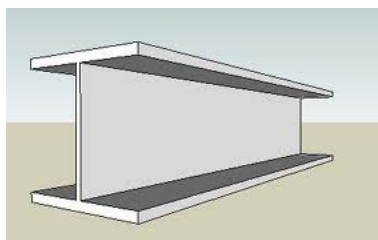
We will include the nodal shear forces and bending moments and the resulting shear force and bending moment diagrams as part of the total solution.
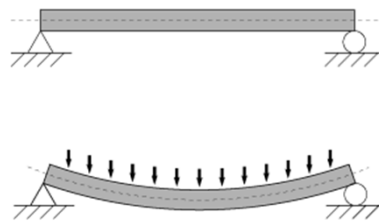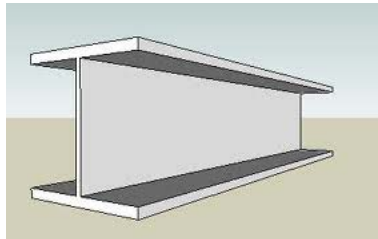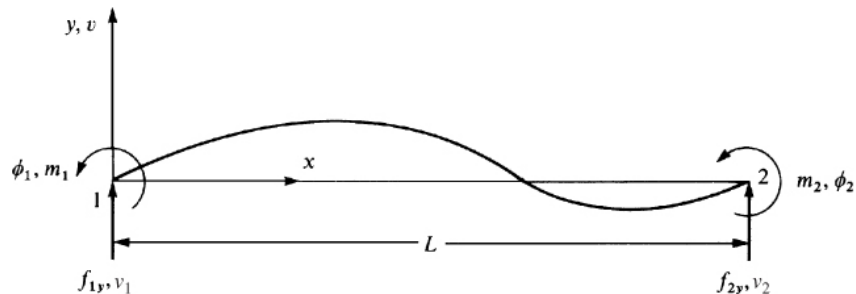
Development of Beam Equations

We will develop the beam bending element equations using the potential energy approach.

Finally, the Galerkin residual method is applied to derive the beam element equations

Beam Stiffness

Consider the beam element shown below.

The beam is of length L with axial local coordinate x and transverse local coordinate y.

The local transverse nodal displacements are given by v_i and the rotations by ϕ_i. The local nodal forces are given by f_{iy} and the bending moments by m_i.

Beam Stiffness

At all nodes, the following sign conventions are used on the element level:

1. **Moments** are positive in the counterclockwise direction.
2. **Rotations** are positive in the counterclockwise direction.
3. **Forces** are positive in the positive y direction.
4. **Displacements** are positive in the positive y direction.
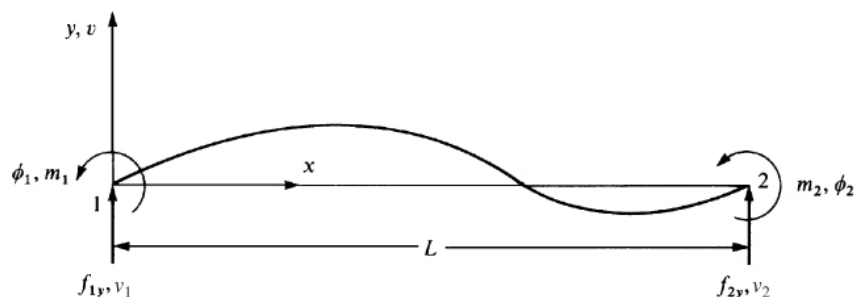
Beam Stiffness

At all nodes, the following sign conventions are used on the global level:

1. **Bending moments _m_** are positive if they cause the beam to bend concave up.

2. **Shear forces _V_** are positive is the cause the beam to rotate clockwise.
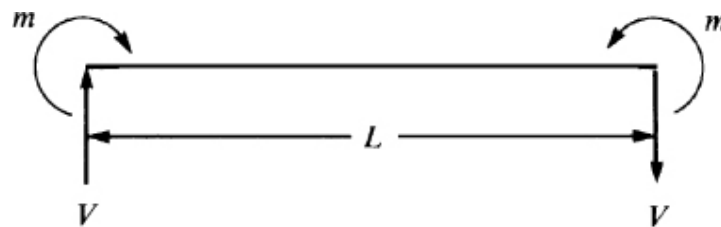
Beam Stiffness
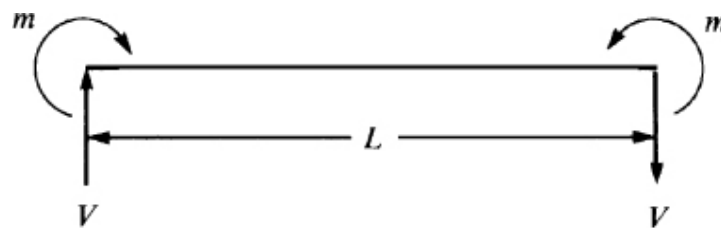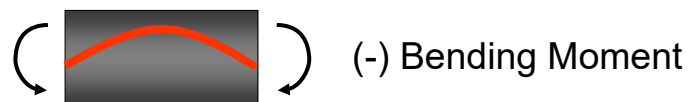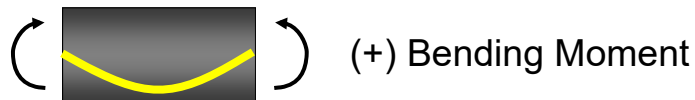
(+) Bending Moment
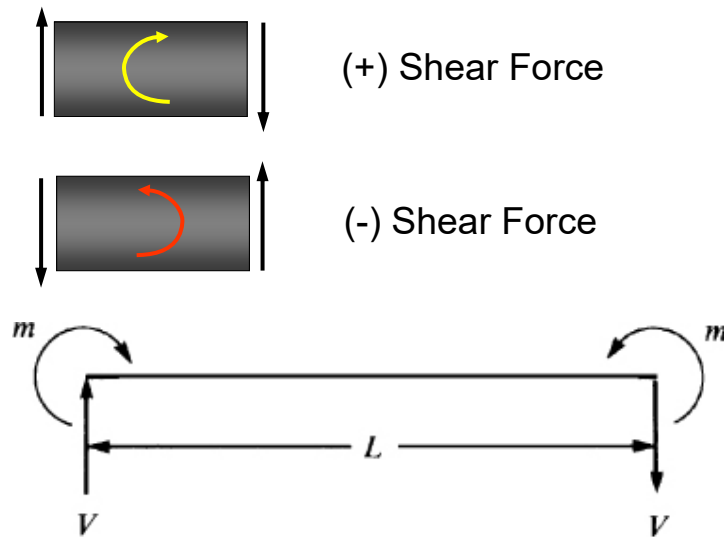
(-) Bending Moment

Beam Stiffness

(+) Shear Force

(-) Shear Force

Beam Stiffness

The differential equation governing simple linear-elastic beam behavior can be derived as follows. Consider the beam shown below.

(a) Undeformed beam under load $w(x)$

(b) Deformed beam due to applied loading
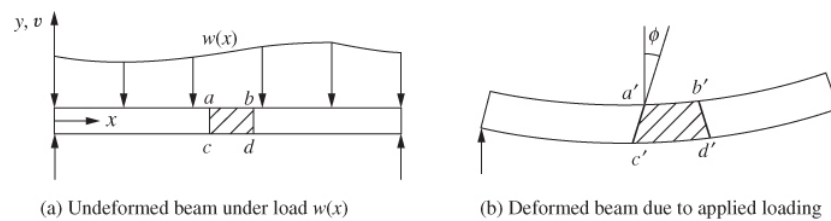
Beam Stiffness

The differential equation governing simple linear-elastic beam behavior can be derived as follows. Consider the beam shown below.
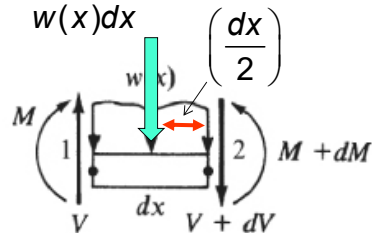
Write the equations of equilibrium for the differential element:

$$\circlearrowleft^+ \sum M_{right-side} = 0 \; = -\cancel{M} + (\cancel{M} + dM) - Vdx + w(x)dx\left(\frac{dx}{2}\right)$$

$$^+\uparrow \sum F_y = 0 \; = \cancel{V} - (\cancel{V} + dV) - w(x)dx$$

0

Beam Stiffness

From force and moment equilibrium of a differential beam element, we get:

$$\sum M_{right-side} = 0 \quad \Rightarrow \quad -Vdx + dM = 0 \quad \text{or} \quad V = \frac{dM}{dx}$$

$$\sum F_y = 0 \quad \Rightarrow \quad -wdx - dV = 0 \quad \text{or} \quad w = -\frac{dV}{dx}$$

$$w = -\frac{d}{dx}\left(\frac{dM}{dx}\right)$$

Beam Stiffness

The curvature κ of the beam is related to the moment by:

$$\kappa = \frac{1}{\rho} = \frac{M}{EI}$$

where ρ is the radius of the deflected curve, v is the transverse displacement function in the y direction, E is the modulus of elasticity, and I is the principle moment of inertia about y direction, as shown below.
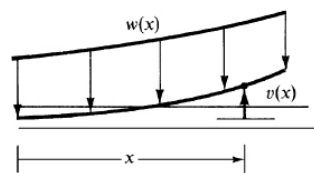
(a) Portion of deflected curve of beam   (b) Radius of deflected curve at $v(x)$
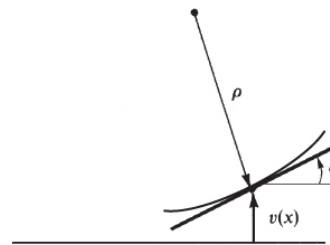
Beam Stiffness

The curvature, κ for small slopes $\phi = \dfrac{dv}{dx}$ is given as:

$$\kappa = \frac{d^2v}{dx^2}$$

Therefore: $\dfrac{d^2v}{dx^2} = \dfrac{M}{EI}$ $\Rightarrow$ $M = EI\dfrac{d^2v}{dx^2}$

Substituting the moment expression into the moment-load equations gives:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = -w(x)$$

For constant values of EI, the above equation reduces to:

$$EI\left(\frac{d^4v}{dx^4}\right) = -w(x)$$

Beam Stiffness

Step 1 - Select Element Type

We will consider the linear-elastic beam element shown below.
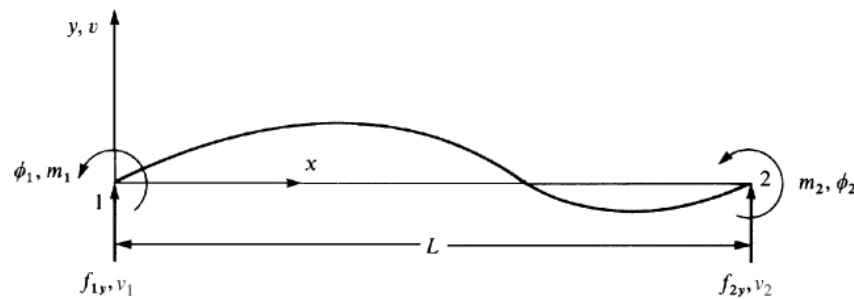
Beam Stiffness

Step 2 - Select a Displacement Function

Assume the transverse displacement function v is:

$$v = a_1 x^3 + a_2 x^2 + a_3 x + a_4$$

The number of coefficients in the displacement function a_i is equal to the total number of degrees of freedom associated with the element (displacement and rotation at each node). The boundary conditions are:

$$v(x = 0) = v_1 \qquad v(x = L) = v_2$$

$$\left.\frac{dv}{dx}\right|_{x=0} = \phi_1 \qquad \left.\frac{dv}{dx}\right|_{x=L} = \phi_2$$

Beam Stiffness

Step 2 - Select a Displacement Function

Applying the boundary conditions and solving for the unknown coefficients gives:

$$v(0) = v_1 = a_4 \qquad\qquad v(L) = v_2 = a_1 L^3 + a_2 L^2 + a_3 L + a_4$$

$$\frac{dv(0)}{dx} = \phi_1 = a_3 \qquad\qquad \frac{dv(L)}{dx} = \phi_2 = 3a_1 L^2 + 2a_2 L + a_3$$

Solving these equations for a_1, a_2, a_3, and a_4 gives:

$$v = \left[\frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\phi_1 + \phi_2) \right] x^3$$

$$+ \left[-\frac{3}{L^2}(v_1 - v_2) - \frac{1}{L}(2\phi_1 + \phi_2) \right] x^2 + \phi_1 x + v_1$$

Beam Stiffness

Step 2 - Select a Displacement Function

In matrix form the above equations are: $v = [N]\{d\}$

$$\{d\} = \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix} \qquad [N] = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix}$$

where

$$N_1 = \frac{1}{L^3}\left(2x^3 - 3x^2 L + L^3 \right) \qquad\qquad N_2 = \frac{1}{L^3}\left(x^3 L - 2x^2 L^2 + xL^3 \right)$$

$$N_3 = \frac{1}{L^3}\left(-2x^3 + 3x^2 L \right) \qquad\qquad N_4 = \frac{1}{L^3}\left(x^3 L - x^2 L^2 \right)$$

Beam Stiffness

Step 2 - Select a Displacement Function

N_1, N_2, N_3, and N_4 are called the ***interpolation functions*** for a beam element.
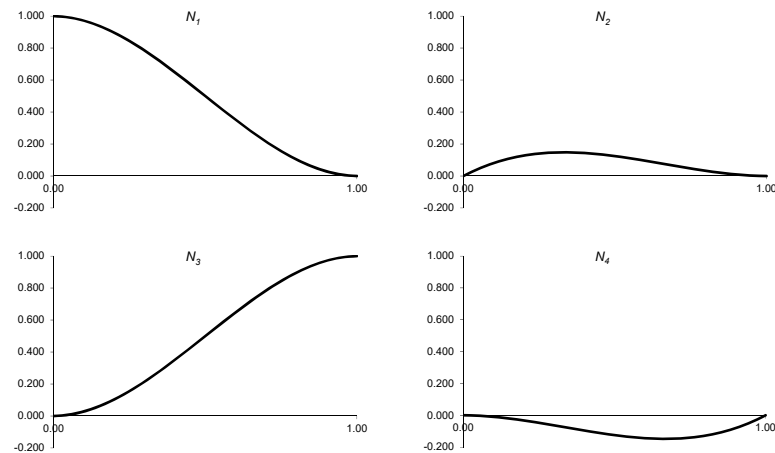
Beam Stiffness

Step 3 - Define the Strain/Displacement
and Stress/Strain Relationships

The stress-displacement relationship is: $\varepsilon_x(x,y) = \dfrac{du}{dx}$

where u is the axial displacement function.

We can relate the axial displacement to the transverse displacement by considering the beam element shown below:

Beam Stiffness

Step 3 - Define the Strain/Displacement
and Stress/Strain Relationships
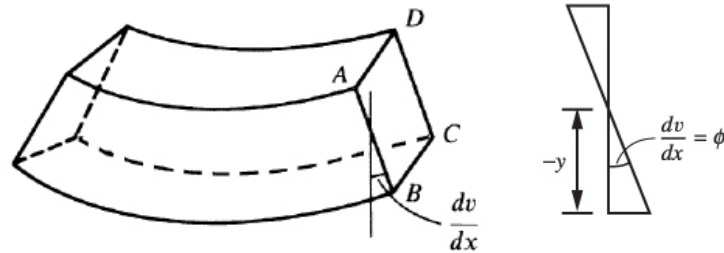
$$u = -y\frac{dv}{dx}$$

Beam Stiffness

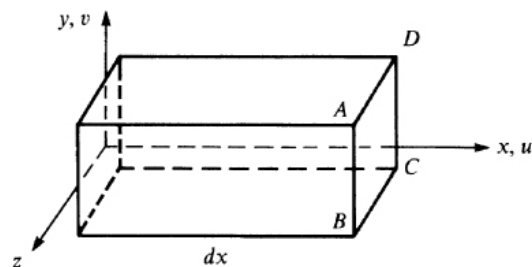### Step 3 - Define the Strain/Displacement
and Stress/Strain Relationships

One of the basic assumptions in simple beam theory is that
planes remain planar after deformation, therefore:

$$\varepsilon_x(x,y) = \frac{du}{dx} = -y\left(\frac{d^2v}{dx^2}\right)$$

Moments and shears are related to the transverse
displacement as:

$$m(x) = EI\left(\frac{d^2v}{dx^2}\right) \qquad\qquad V(x) = EI\left(\frac{d^3v}{dx^3}\right)$$

Beam Stiffness

Step 4 - Derive the Element Stiffness Matrix and Equations

Use beam theory sign convention for shear force and bending moment.

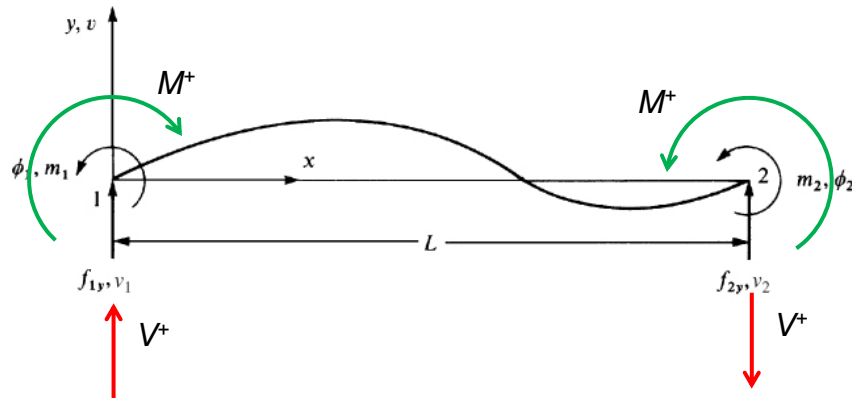# Beam Stiffness

Step 4 - Derive the Element Stiffness Matrix and Equations

Using beam theory sign convention for shear force and bending moment we obtain the following equations:

$$f_{1y} = V = EI\frac{d^3v}{dx^3}\bigg|_{x=0} = \frac{EI}{L^3}\left(12v_1 + 6L\phi_1 - 12v_2 + 6L\phi_2\right)$$

$$f_{2y} = -V = -EI\frac{d^3v}{dx^3}\bigg|_{x=L} = \frac{EI}{L^3}\left(-12v_1 - 6L\phi_1 + 12v_2 - 6L\phi_2\right)$$

$$m_1 = -m = -EI\frac{d^2v}{dx^2}\bigg|_{x=0} = \frac{EI}{L^3}\left(6Lv_1 + 4L^2\phi_1 - 6Lv_2 + 2L^2\phi_2\right)$$

$$m_2 = m = EI\frac{d^2v}{dx^2}\bigg|_{x=L} = \frac{EI}{L^3}\left(6Lv_1 + 2L^2\phi_1 - 6Lv_2 + 4L^2\phi_2\right)$$

Beam Stiffness

Step 4 - Derive the Element Stiffness Matrix and Equations

In matrix form the above equations are:

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} \boxed{12} & \boxed{6L} & -12 & 6L \\ 6L & \boxed{4L^2} & -6L & \boxed{2L^2} \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix} \qquad \begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = k \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

where the stiffness matrix is:

$$\mathbf{k} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Beam Stiffness

Step 4 - Derive the Element Stiffness Matrix and Equations

Beam stiffness based on Timoshenko Beam Theory
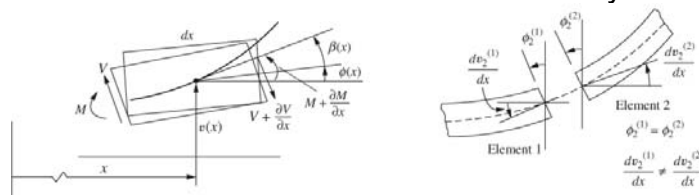
The total deflection of the beam at a point *x* consists of two parts, one caused by bending and one by shear force. The slope of the deflected curve at a point *x* is:

$$\frac{dv}{dx} = \phi(x) + \beta(x)$$

Beam Stiffness

Step 4 - Derive the Element Stiffness Matrix
and Equations

Beam stiffness based on Timoshenko Beam Theory

The relationship between bending moment and bending deformation is:

$$M(x) = EI\frac{d\phi(x)}{dx}$$

Beam Stiffness

Step 4 - Derive the Element Stiffness Matrix
and Equations

Beam stiffness based on Timoshenko Beam Theory

The relationship between shear force and shear deformation is:

$$V(x) = k_s AG\beta(x)$$

where $k_s A$ is the shear area.

Beam Stiffness

Step 4 - Derive the Element Stiffness Matrix
and Equations
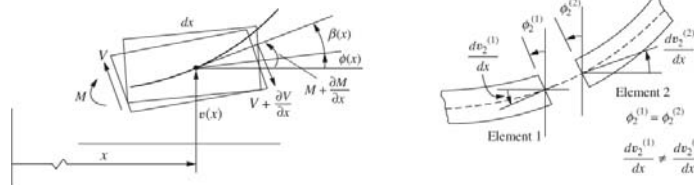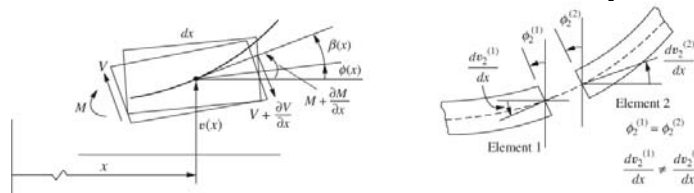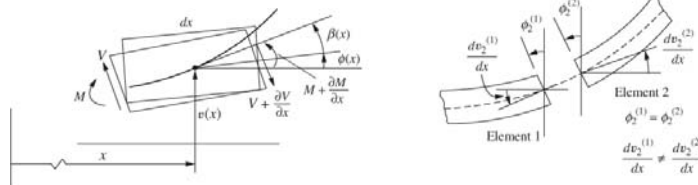
Beam stiffness based on Timoshenko Beam Theory

You can review the details in your book, but by including the
effects of shear deformations into the relationship between
forces and nodal displacements a modified elemental
stiffness can be developed.

Beam Stiffness

Step 4 - Derive the Element Stiffness Matrix
and Equations

Beam stiffness based on Timoshenko Beam Theory
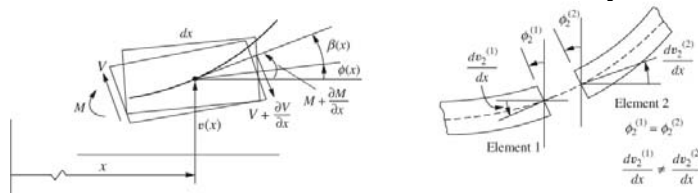
$$\mathbf{k} = \frac{EI}{L^3\left(1+\varphi\right)}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & \left(4+\varphi\right)L^2 & -6L & \left(2-\varphi\right)L^2 \\ -12 & -6L & 12 & -6L \\ 6L & \left(2-\varphi\right)L^2 & -6L & \left(4+\varphi\right)L^2 \end{bmatrix} \qquad \varphi = \frac{12EI}{k_s AGL^2}$$

Beam Stiffness

Step 5 - Assemble the Element Equations
and Introduce Boundary Conditions

Consider a beam modeled by two beam elements (do not include shear deformations):
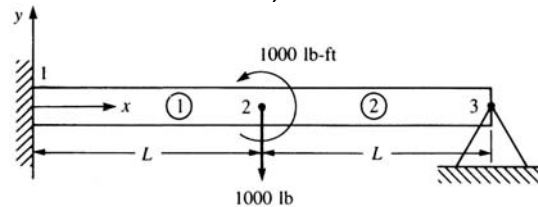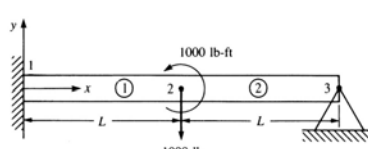
Assume the *EI* to be constant throughout the beam. A force of 1,000 *lb* and moment of 1,000 *lb-ft* are applied to the mid-point of the beam.

Beam Stiffness

Step 5 - Assemble the Element Equations
and Introduce Boundary Conditions

The beam element stiffness matrices are:

$$\mathbf{k}^{(1)} = \frac{EI}{L^3} \begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \end{array} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$\mathbf{k}^{(2)} = \frac{EI}{L^3} \begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \end{array} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Beam Stiffness

Step 5 - Assemble the Element Equations and Introduce Boundary Conditions

In this example, the local coordinates coincide with the global coordinates of the whole beam (therefore there is no transformation required for this problem).

The total stiffness matrix can be assembled as:

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12+12 & -6L+6L & -12 & 6L \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix}$$

Element 1   Element 2

Beam Stiffness

Step 5 - Assemble the Element Equations and Introduce Boundary Conditions

The boundary conditions are: $\quad v_1 = \phi_1 = v_3 = 0$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12+12 & -6L+6L & -12 & 6L \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ v_2 \\ \phi_2 \\ 0 \\ \phi_3 \end{Bmatrix}$$

Beam Stiffness

Step 5 - Assemble the Element Equations and Introduce Boundary Conditions

By applying the boundary conditions the beam equations reduce to:

$$\begin{Bmatrix} -1{,}000 \; lb \\ 1{,}000 \; lb\,ft \\ 0 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 24 & 0 & 6L \\ 0 & 8L^2 & 2L^2 \\ 6L & 2L^2 & 4L^2 \end{bmatrix} \begin{Bmatrix} v_2 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

Beam Stiffness

Step 6 - Solve for the Unknown Degrees of Freedom

Solving the above equations gives:

$$v_2 = \frac{-875L^3 + 375L^2}{12EI} \; in \qquad \phi_2 = \frac{-125L^2 + 625L}{4EI} \; rad \qquad \phi_3 = \frac{125L^2 - 125L}{EI} \; rad$$

Step 7 - Solve for the Element Strains and Stresses

$$m(x) = EI\left(\frac{d^2v}{dx^2}\right) = EI\left(\frac{d^2N}{dx^2}\right) \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

The second derivative of N is linear; therefore $m(x)$ is linear.

Beam Stiffness

Step 6 - Solve for the Unknown Degrees of Freedom

Solving the above equations gives:

$$v_2 = -\frac{875L^3 - 375L^2}{12EI}\ in \qquad \phi_2 = -\frac{125L^2 - 625L}{4EI}\ rad \qquad \phi_3 = -\frac{125L^2 - 125L}{EI}\ rad$$

Step 7 - Solve for the Element Strains and Stresses

$$V(x) = EI\left(\frac{d^3v}{dx^3}\right) = EI\left(\frac{d^3N}{dx^2}\right)\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

The third derivative of N is a constant; therefore $V(x)$ is constant.

Beam Stiffness

Step 7 - Solve for the Element Strains and Stresses

Assume L = 120 in, E = 29x10^6 psi, and I = 100 in^4:

$$v_2 = -0.0433\ in \qquad \phi_2 = -7.758 \times 10^{-5}\ rad \qquad \phi_3 = 5.586 \times 10^{-4}\ rad$$

Element #1:

$$m(x) = EI\left(\frac{d^2v}{dx^2}\right) = EI\left(\frac{d^2N}{dx^2}\right)\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

$$m_1 = \frac{EI}{L^3}\left(6Lv_1 + 4L^2\phi_1 - 6Lv_2 + 2L^2\phi_2\right) = 3,875\ lb \cdot ft$$

$$m_2 = \frac{EI}{L^3}\left(6Lv_1 + 2L^2\phi_1 - 6Lv_2 + 4L^2\phi_2\right) = 3,562.5\ lb \cdot ft$$

Beam Stiffness

Step 7 - Solve for the Element Strains and Stresses

Assume L = 120 in, E = 29x10^6 psi, and I = 100 in^4:

$$v_2 = -0.0433 \ in \qquad \phi_2 = -7.758 \times 10^{-5} \ rad \qquad \phi_3 = 5.586 \times 10^{-4} \ rad$$

Element #2:

$$m(x) = EI\left(\frac{d^2v}{dx^2}\right) = EI\left(\frac{d^2N}{dx^2}\right)\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

$$m_2 = \frac{EI}{L^3}\left(6Lv_2 + 4L^2\phi_2 - 6Lv_3 + 2L^2\phi_3\right) = -2,562.5 \, lb \cdot ft$$

$$m_3 = \frac{EI}{L^3}\left(6Lv_2 + 2L^2\phi_2 - 6Lv_3 + 4L^2\phi_3\right) = 0$$

Beam Stiffness

Step 7 - Solve for the Element Strains and Stresses

Assume L = 120 in, E = 29x10^6 psi, and I = 100 in^4:

$$v_2 = -0.0433 \ in \qquad \phi_2 = -7.758 \times 10^{-5} \ rad \qquad \phi_3 = 5.586 \times 10^{-4} \ rad$$

Element #1:

$$V(x) = EI\left(\frac{d^3v}{dx^3}\right) = EI\left(\frac{d^3N}{dx^2}\right)\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

$$f_{1y} = \frac{EI}{L^3}\left(12v_1 + 6L\phi_1 - 12v_2 + 6L\phi_2\right) = 743.75 \, lb$$

$$f_{2y} = \frac{EI}{L^3}\left(-12v_1 - 6L\phi_1 + 12v_2 - 6L\phi_2\right) = -743.75 \, lb$$

Beam Stiffness

Step 7 - Solve for the Element Strains and Stresses

Assume L = 120 in, E = 29x10^6 psi, and I = 100 in^4:

$$v_2 = -0.0433 \text{ } in \qquad \phi_2 = -7.758 \times 10^{-5} \text{ } rad \qquad \phi_3 = 5.586 \times 10^{-4} \text{ } rad$$

Element #2:

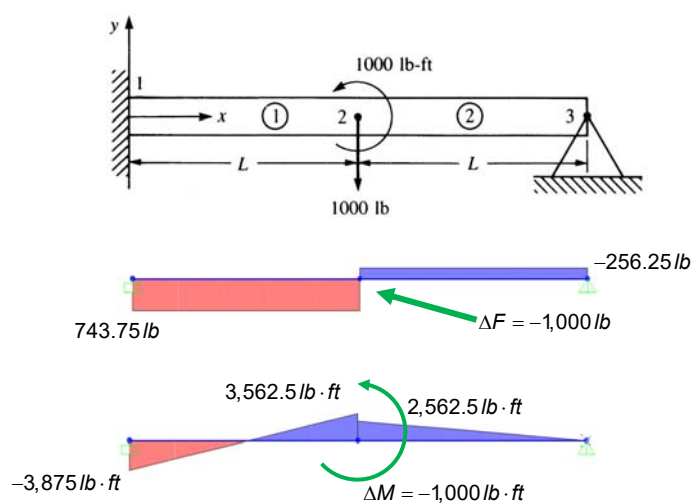$$V(x) = EI\left(\frac{d^3v}{dx^3}\right) = EI\left(\frac{d^3N}{dx^2}\right)\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

$$f_{2y} = \frac{EI}{L^3}\left(12v_2 + 6L\phi_2 - 12v_3 + 6L\phi_3\right) = -256.25\,lb$$

$$f_{3y} = \frac{EI}{L^3}\left(-12v_2 - 6L\phi_2 + 12v_3 - 6L\phi_3\right) = 256.25\,lb$$
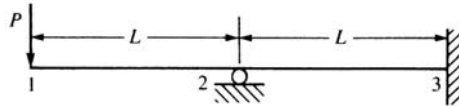
Beam Stiffness

Step 7 - Solve for the Element Strains and Stresses

Beam Stiffness

Example 1 - Beam Problem

Consider the beam shown below. Assume that EI is constant and the length is $2L$ (no shear deformation).

The beam element stiffness matrices are:

$$\mathbf{k}^{(1)} = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

$$\mathbf{k}^{(2)} = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

Beam Stiffness

Example 1 - Beam Problem

The local coordinates coincide with the global coordinates of the whole beam (therefore there is no transformation required for this problem).

The total stiffness matrix can be assembled as:

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix}$$
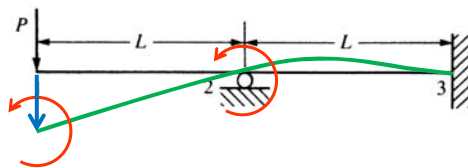
Element 1   Element 2

Beam Stiffness

Example 1 - Beam Problem

The boundary conditions are:　　$v_2 = v_3 = \phi_3 = 0$

$$
\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3}
\begin{bmatrix}
12 & 6L & -12 & 6L & 0 & 0 \\
6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\
-12 & -6L & 24 & 0 & -12 & 6L \\
6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\
0 & 0 & -12 & -6L & 12 & -6L \\
0 & 0 & 6L & 2L^2 & -6L & 4L^2
\end{bmatrix}
\begin{Bmatrix} v_1 \\ \phi_1 \\ 0 \\ \phi_2 \\ 0 \\ 0 \end{Bmatrix}
$$

Beam Stiffness

By applying the boundary conditions the beam equations reduce to:

$$
\begin{Bmatrix} -P \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{L^3}
\begin{bmatrix}
12 & 6L & 6L \\
6L & 4L^2 & 2L^2 \\
6L & 2L^2 & 8L^2
\end{bmatrix}
\begin{Bmatrix} v_1 \\ \phi_1 \\ \phi_2 \end{Bmatrix}
$$

Solving the above equations gives:

$$
\begin{Bmatrix} v_1 \\ \phi_1 \\ \phi_2 \end{Bmatrix} = \frac{PL^2}{4EI}
\begin{Bmatrix} -\dfrac{7L}{3} \\ 3 \\ 1 \end{Bmatrix}
$$

Beam Stiffness

Example 1 - Beam Problem

The positive signs for the rotations indicate that both are in the counterclockwise direction.

The negative sign on the displacement indicates a deformation in the -y direction.

$$
\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{P}{4L}
\begin{bmatrix}
12 & 6L & -12 & 6L & 0 & 0 \\
6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\
-12 & -6L & 24 & 0 & -12 & 6L \\
6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\
0 & 0 & -12 & -6L & 12 & -6L \\
0 & 0 & 6L & 2L^2 & -6L & 4L^2
\end{bmatrix}
\begin{Bmatrix} -7L/3 \\ 3 \\ 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}
$$

Beam Stiffness

Example 1 - Beam Problem

The local nodal forces for element 1:

$$
\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{P}{4L}
\begin{bmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2
\end{bmatrix}
\begin{Bmatrix} -7L/3 \\ 3 \\ 0 \\ 1 \end{Bmatrix}
=
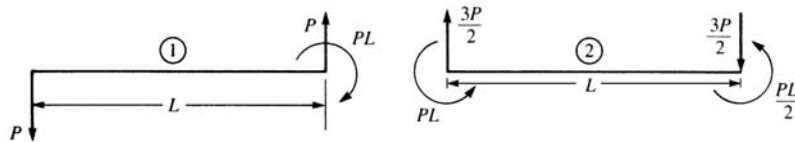\begin{Bmatrix} -P \\ 0 \\ P \\ -PL \end{Bmatrix}
$$

The local nodal forces for element 2:

$$
\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = \frac{P}{4L}
\begin{bmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2
\end{bmatrix}
\begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}
=
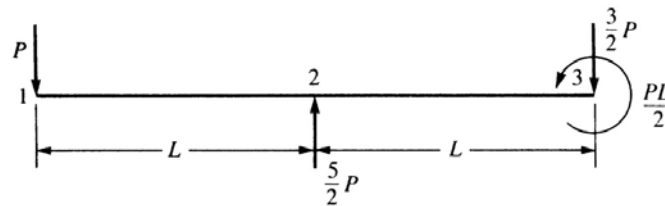\begin{Bmatrix} 1.5P \\ PL \\ -1.5P \\ 0.5PL \end{Bmatrix}
$$

Beam Stiffness

Example 1 - Beam Problem

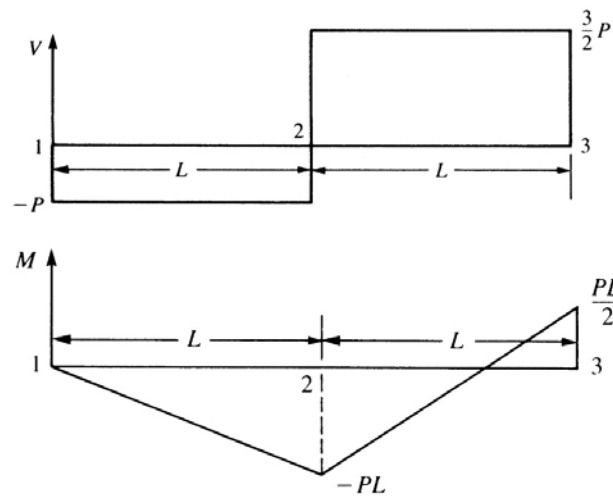The free-body diagrams for the each element are shown
below.

Combining the elements gives the forces and moments for the
original beam.

Beam Stiffness

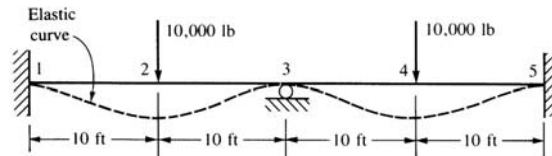### Example 1 - Beam Problem

Therefore, the shear force and bending moment diagrams are:

Beam Stiffness

Example 2 - Beam Problem

Consider the beam shown below. Assume E = 30 x 10^6 *psi* and I = 500 *in*^{4} are constant throughout the beam. Use four elements of equal length to model the beam.

The beam element stiffness matrices are:

$$\mathbf{k}^{(i)} = \frac{EI}{L^3} \begin{array}{c c c c c} & v_i & \phi_i & v_{(i+1)} & \phi_{(i+1)} \\ \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} & \begin{matrix} 12 \\ 6L \\ -12 \\ 6L \end{matrix} & \begin{matrix} 6L \\ 4L^2 \\ -6L \\ 2L^2 \end{matrix} & \begin{matrix} -12 \\ -6L \\ 12 \\ -6L \end{matrix} & \begin{matrix} 6L \\ 2L^2 \\ -6L \\ 4L^2 \end{matrix} \end{array}$$

Beam Stiffness

Example 2 - Beam Problem

Using the direct stiffness method, the four beam element stiffness matrices are superimposed to produce the global stiffness matrix.

Beam Stiffness

Example 2 - Beam Problem

The boundary conditions for this problem are:

$$v_1 = \phi_1 = v_3 = v_5 = \phi_5 = 0$$

Beam Stiffness

Example 2 - Beam Problem

The boundary conditions for this problem are:

$$v_1 = \phi_1 = v_3 = v_5 = \phi_5 = 0$$

After applying the boundary conditions the global beam equations reduce to:

$$\frac{EI}{L^3}\begin{bmatrix} 24 & 0 & 6L & 0 & 0 \\ 0 & 8L^2 & 2L^2 & 0 & 0 \\ 6L & 2L^2 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -6L & 24 & 0 \\ 0 & 0 & 2L^2 & 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \phi_2 \\ \phi_3 \\ v_4 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} -10{,}000\ lb \\ 0 \\ 0 \\ -10{,}000\ lb \\ 0 \end{Bmatrix}$$

Beam Stiffness

Example 2 - Beam Problem

Substituting $L = 120$ in, $E = 30 \times 10^6$ psi, and $I = 500$ in^4 into the above equations and solving for the unknowns gives:

$$v_2 = v_4 = -0.048 \ in \qquad \phi_2 = \phi_3 = \phi_4 = 0$$

The global forces and moments can be determined as:

$$F_{1y} = \quad 5 \ kips \qquad M_1 = \quad 25 \ kips \cdot ft$$
$$F_{2y} = -10 \ kips \qquad M_2 = 0$$
$$F_{3y} = \quad 10 \ kips \qquad M_3 = 0$$
$$F_{4y} = -10 \ kips \qquad M_4 = 0$$
$$F_{5y} = \quad 5 \ kips \qquad M_5 = -25 \ kips \cdot ft$$

Beam Stiffness

Example 2 - Beam Problem

The local nodal forces for element 1:

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -0.048 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 5 \ kips \\ 25 \ k \cdot ft \\ -5 \ kips \\ 25 \ k \cdot ft \end{Bmatrix}$$

The local nodal forces for element 2:

$$\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -0.048 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -5 \ kips \\ -25 \ k \cdot ft \\ 5 \ kips \\ -25 \ k \cdot ft \end{Bmatrix}$$

Beam Stiffness

Example 2 - Beam Problem

The local nodal forces for element 3:

$$\begin{Bmatrix} f_{3y} \\ m_3 \\ f_{4y} \\ m_4 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -0.048 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 5 \; kips \\ 25 \; k{\cdot}ft \\ -5 \; kips \\ 25 \; k{\cdot}ft \end{Bmatrix}$$
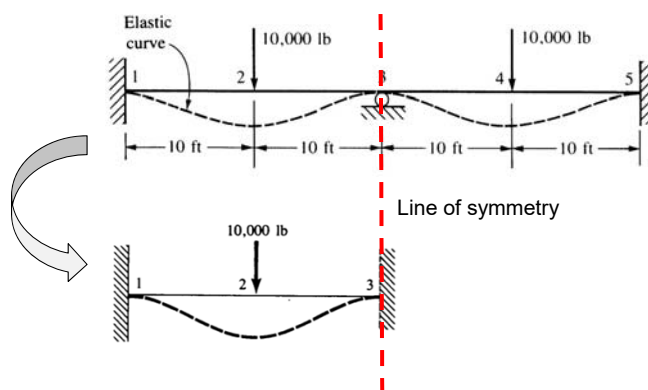
The local nodal forces for element 4:

$$\begin{Bmatrix} f_{4y} \\ m_4 \\ f_{5y} \\ m_5 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -0.048 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -5 \; kips \\ -25 \; k{\cdot}ft \\ 5 \; kips \\ -25 \; k{\cdot}ft \end{Bmatrix}$$

Beam Stiffness
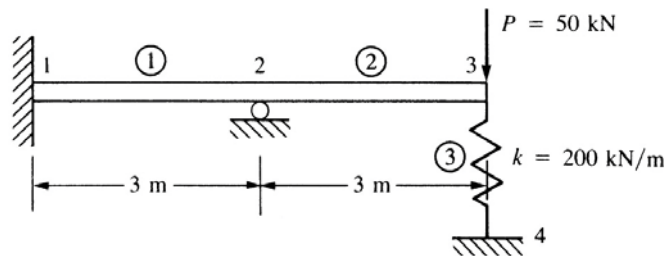
Example 2 - Beam Problem

Note: Due to symmetry about the vertical plane at node 3, we could have worked just half the beam, as shown below.

Beam Stiffness

Example 3 - Beam Problem

Consider the beam shown below. Assume E = 210 *GPa* and $I = 2 \times 10^{-4}$ m^4 are constant throughout the beam and the spring constant k = 200 *kN/m*. Use two beam elements of equal length and one spring element to model the structure.

Beam Stiffness

Example 3 - Beam Problem

The beam element stiffness matrices are:

$$\mathbf{k}^{(1)} = \frac{EI}{L^3}\begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array} \qquad \mathbf{k}^{(2)} = \frac{EI}{L^3}\begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

The spring element stiffness matrix is:

$$\mathbf{k}^{(3)} = \begin{array}{cc} v_3 & v_4 \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{array} \qquad \Rightarrow \qquad \mathbf{k}^{(3)} = \begin{array}{ccc} v_3 & \phi_3 & v_4 \\ \begin{bmatrix} k & 0 & -k \\ 0 & 0 & 0 \\ -k & 0 & k \end{bmatrix} \end{array}$$

Beam Stiffness

Example 3 - Beam Problem

Using the direct stiffness method and superposition gives the global beam equations.

$$
\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \\ F_{4y} \end{Bmatrix} = \frac{EI}{L^3}
\begin{bmatrix}
12 & 6L & -12 & 6L & 0 & 0 & 0 \\
6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\
-12 & -6L & 24 & 0 & -12 & 6L & 0 \\
6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 & 0 \\
0 & 0 & -12 & -6L & 12+k' & -6L & -k' \\
0 & 0 & 6L & 2L^2 & -6L & 4L^2 & 0 \\
0 & 0 & 0 & 0 & -k' & 0 & k'
\end{bmatrix}
\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \\ v_4 \end{Bmatrix}
\qquad k' = \frac{kL^3}{EI}
$$

Element 1   Element 2   Element 3

Beam Stiffness

Example 3 - Beam Problem

The boundary conditions for this problem are: $v_1 = \phi_1 = v_2 = v_4 = 0$

$$
\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \\ F_{4y} \end{Bmatrix} = \frac{EI}{L^3}
\begin{bmatrix}
12 & 6L & -12 & 6L & 0 & 0 & 0 \\
6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\
-12 & -6L & 24 & 0 & -12 & 6L & 0 \\
6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 & 0 \\
0 & 0 & -12 & -6L & 12+k' & -6L & -k' \\
0 & 0 & 6L & 2L^2 & -6L & 4L^2 & 0 \\
0 & 0 & 0 & 0 & -k' & 0 & k'
\end{bmatrix}
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \phi_2 \\ v_3 \\ \phi_3 \\ 0 \end{Bmatrix}
\qquad k' = \frac{kL^3}{EI}
$$

Beam Stiffness

Example 3 - Beam Problem

After applying the boundary conditions the global beam
equations reduce to:

$$\begin{Bmatrix} M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 8L^2 & -6L & 2L^2 \\ -6L & 12+k' & -6L \\ 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix}$$

Solving the above
equations gives:

$$\begin{Bmatrix} \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{3PL^2}{EI}\left(\dfrac{1}{12+7k'}\right) \\ -\dfrac{7PL^3}{EI}\left(\dfrac{1}{12+7k'}\right) \\ -\dfrac{9PL^2}{EI}\left(\dfrac{1}{12+7k'}\right) \end{Bmatrix} \qquad k' = \frac{kL^3}{EI}$$

Beam Stiffness

Example 3 - Beam Problem

Substituting $L = 3\ m$, $E = 210\ GPa$, $I = 2 \times 10^{-4}\ m^4$, and
$k = 200\ kN/m$ in the above equations gives:

$$v_3 = -0.0174\ m$$
$$\phi_2 = -0.00249\ rad$$
$$\phi_3 = -0.00747\ rad$$

Substituting the solution back into the global equations gives:

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \\ F_{4y} \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L & 0 \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 & 0 \\ 0 & 0 & -12 & -6L & 12+k' & -6L & -k' \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 & 0 \\ 0 & 0 & 0 & 0 & -k' & 0 & k' \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00249\ rad \\ -0.0174\ m \\ -0.00747\ rad \\ 0 \end{Bmatrix}$$

Beam Stiffness

Example 3 - Beam Problem

Substituting $L = 3\ m$, $E = 210\ GPa$, $I = 2 \times 10^{-4}\ m^4$, and $k = 200\ kN/m$ in the above equations gives:

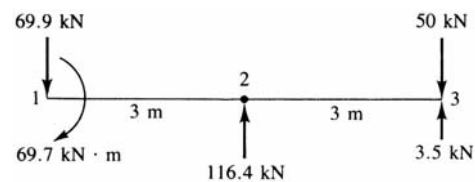$$v_3 = -0.0174\ m$$
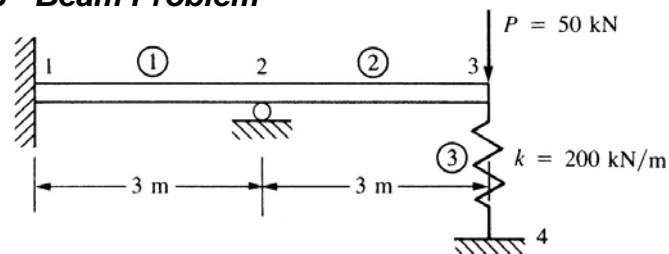$$\phi_2 = -0.00249\ rad$$
$$\phi_3 = -0.00747\ rad$$

Substituting the solution back into the global equations gives:

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \\ F_{4y} \end{Bmatrix} = \begin{Bmatrix} -69.9\ kN \\ -69.7\ kN \cdot m \\ 116.4\ kN \\ 0 \\ -50\ kN \\ 0 \\ 3.5\ kN \end{Bmatrix}$$
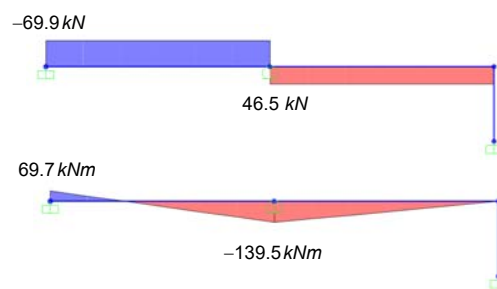

Beam Stiffness

Example 3 - Beam Problem

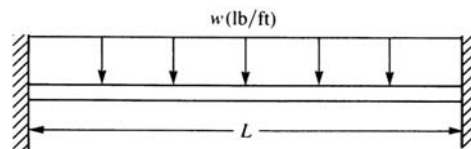The variation of shear force and bending moment is:

Beam Stiffness

Distributed Loadings

Beam members can support distributed loading as well as
concentrated nodal loading.

Therefore, we must be able to account for distributed loading.

Consider the fixed-fixed beam subjected to a uniformly
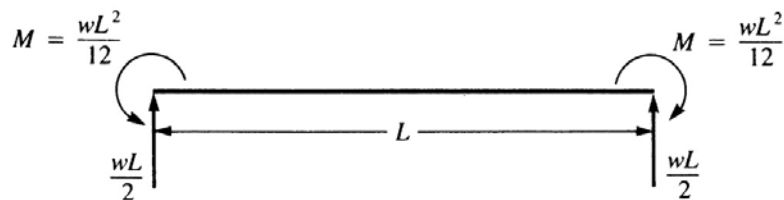distributed loading w shown the figure below.

The reactions, determined from structural analysis theory, are
called fixed-end reactions.

Beam Stiffness

Distributed Loadings

In general, fixed-end reactions are those reactions at the ends
of an element if the ends of the element are assumed to be
fixed (displacements and rotations are zero).

Therefore, guided by the results from structural analysis for the
case of a uniformly distributed load, we replace the load by
concentrated nodal forces and moments tending to have the
same effect on the beam as the actual distributed load.

Beam Stiffness

Distributed Loadings

The figures below illustrates the idea of equivalent nodal loads for a general beam. We can replace the effects of a uniform load by a set of nodal forces and moments.
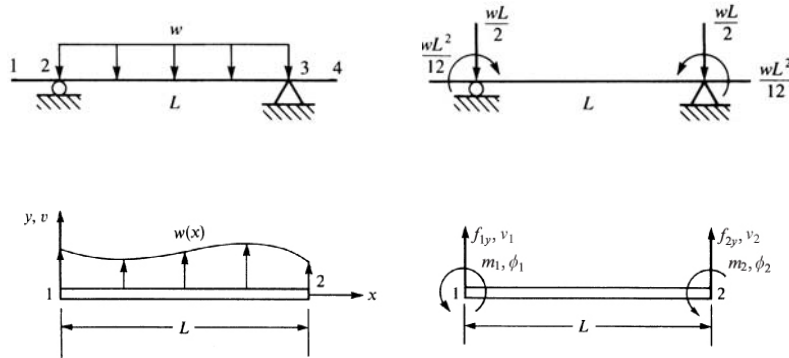
Beam Stiffness

Work Equivalence Method

This method is based on the concept that the work done by the distributed load is equal to the work done by the discrete nodal loads. The work done by the distributed load is:

$$W_{distributed} = \int_0^L w(x)\,v(x)\,dx$$

where $v(x)$ is the transverse displacement. The work done by the discrete nodal forces is:

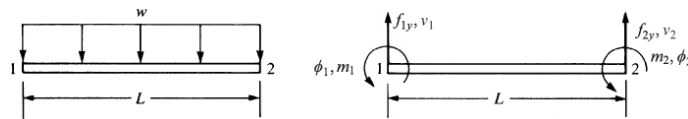$$W_{nodes} = m_1\phi_1 + m_2\phi_2 + f_{1y}v_1 + f_{2y}v_2$$

The nodal forces can be determined by setting $W_{distributed} = W_{nodes}$ for arbitrary displacements and rotations.

Beam Stiffness

Example 4 - Load Replacement

Consider the beam, shown below, and determine the
  equivalent nodal forces for the given distributed load.

Using the work equivalence method or: $W_{distributed} = W_{nodes}$

$$\int_0^L w(x)\, v(x)\, dx = m_1\phi_1 + m_2\phi_2 + f_{1y}v_1 + f_{2y}v_2$$

Beam Stiffness

Example 4 - Load Replacement

Evaluating the left-hand-side of the above expression with:

$$w(x) = -w$$

$$v(x) = \left[\frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\phi_1 + \phi_2)\right]x^3$$

$$+ \left[-\frac{3}{L^2}(v_1 - v_2) - \frac{1}{L}(2\phi_1 + \phi_2)\right]x^2 + \phi_1 x + v_1$$

gives:

$$\int_0^L w\, v(x)\, dx = \frac{Lw}{2}(v_1 - v_2) - \frac{L^2w}{4}(\phi_1 + \phi_2) - Lw(v_2 - v_1)$$

$$+ \frac{L^2w}{3}(2\phi_1 + \phi_2) - \frac{L^2w}{2}\phi_1 - wLv_1$$

Beam Stiffness

Example 4 - Load Replacement

Using a set of arbitrary nodal displacements, such as:

$$v_1 = v_2 = \phi_2 = 0 \qquad \phi_1 = 1$$

The resulting nodal equivalent force or moment is:

$$m_1\phi_1 + m_2\phi_2 + f_{1y}v_1 + f_{2y}v_2 = \int_0^L w(x)\, v(x)\, dx$$

$$m_1 = -\left(\frac{wL^2}{4} - \frac{2}{3}L^2 w + \frac{L^2}{2}w \right) = -\frac{wL^2}{12}$$

Beam Stiffness

Example 4 - Load Replacement

Using a set of arbitrary nodal displacements, such as:

$$v_1 = v_2 = \phi_1 = 0 \qquad \phi_2 = 1$$

The resulting nodal equivalent force or moment is:

$$m_1\phi_1 + m_2\phi_2 + f_{1y}v_1 + f_{2y}v_2 = \int_0^L w(x)\, v(x)\, dx$$

$$m_2 = -\left(\frac{wL^2}{4} - \frac{wL^2}{3} \right) = \frac{wL^2}{12}$$
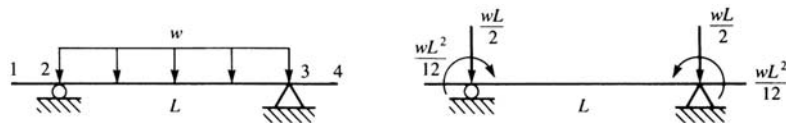
Beam Stiffness

Example 4 - Load Replacement

Setting the nodal rotations equal zero except for the nodal displacements gives:

$$f_{1y} = -\frac{LW}{2} + Lw - Lw = -\frac{Lw}{2}$$

$$f_{2y} = \frac{LW}{2} - Lw = -\frac{Lw}{2}$$

Summarizing, the equivalent nodal forces and moments are:

End of Chapter 4a