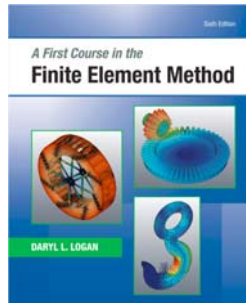


Chapter 3a – Development of Truss Equations



Learning Objectives

- To derive the stiffness matrix for a bar element.
- To illustrate how to solve a bar assemblage by the direct stiffness method.
- To introduce guidelines for selecting displacement functions.
- To describe the concept of transformation of vectors in two different coordinate systems in the plane.
- To derive the stiffness matrix for a bar arbitrarily oriented in the plane.
- To demonstrate how to compute stress for a bar in the plane.
- To show how to solve a plane truss problem.
- To develop the transformation matrix in three-dimensional space and show how to use it to derive the stiffness matrix for a bar arbitrarily oriented in space.
- To demonstrate the solution of space trusses.

Development of Truss Equations

Having set forth the foundation on which the direct stiffness method is based, we will now derive the stiffness matrix for a linear-elastic bar (or truss) element using the general steps outlined in Chapter 2.

We will include the introduction of both a local coordinate system, chosen with the element in mind, and a global or reference coordinate system, chosen to be convenient (for numerical purposes) with respect to the overall structure.

We will also discuss the transformation of a vector from the local coordinate system to the global coordinate system, using the concept of transformation matrices to express the stiffness matrix of an arbitrarily oriented bar element in terms of the global system.

Development of Truss Equations

Next we will describe how to handle inclined, or skewed, supports.

We will then extend the stiffness method to include space trusses.

We will develop the transformation matrix in three-dimensional space and analyze a space truss.

We will then use the principle of minimum potential energy and apply it to the bar element equations.

Finally, we will apply Galerkin's residual method to derive the bar element equations.

Development of Truss Equations



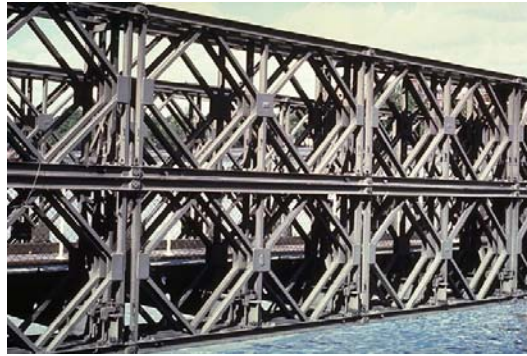
Development of Truss Equations



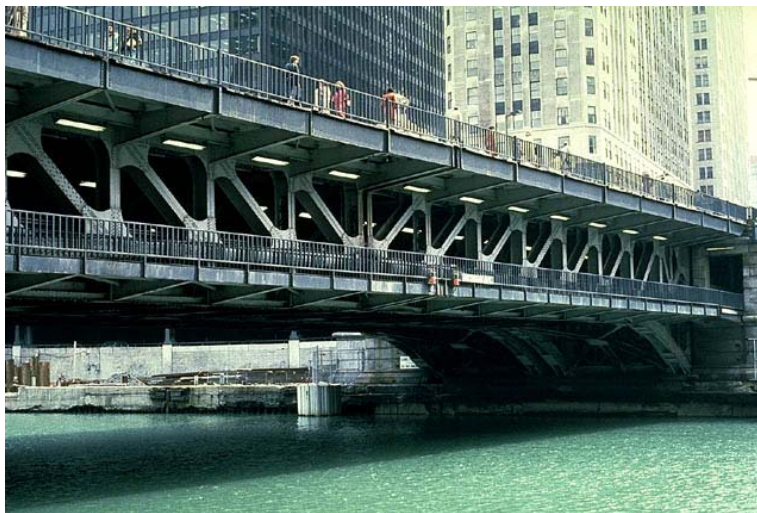
Development of Truss Equations



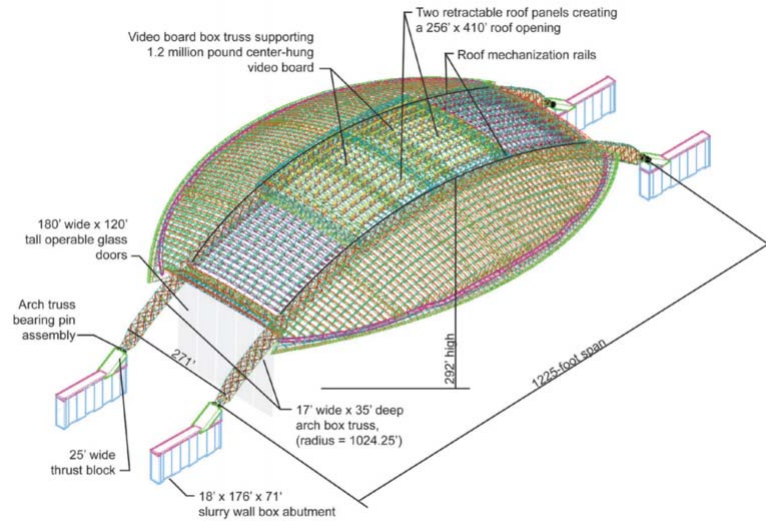
Development of Truss Equations



Development of Truss Equations



Development of Truss Equations



Development of Truss Equations

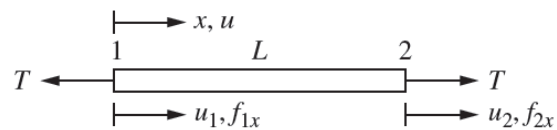


Development of Truss Equations



Stiffness Matrix for a Bar Element

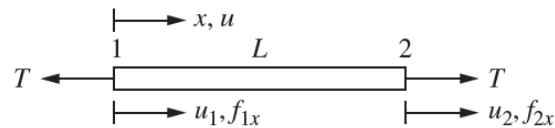
Consider the derivation of the stiffness matrix for the linear-elastic, constant cross-sectional area (prismatic) bar element shown below.



This application is directly applicable to the solution of pin-connected truss problems.

Stiffness Matrix for a Bar Element

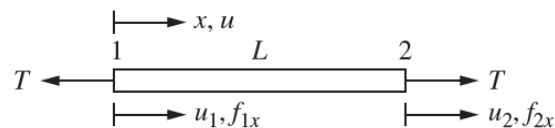
Consider the derivation of the stiffness matrix for the linear-elastic, constant cross-sectional area (prismatic) bar element show below.



where T is the tensile force directed along the axis at nodes 1 and 2, x is the ***local coordinate system*** directed along the length of the bar.

Stiffness Matrix for a Bar Element

Consider the derivation of the stiffness matrix for the linear-elastic, constant cross-sectional area (prismatic) bar element show below.



The bar element has a constant cross-section A , an initial length L , and modulus of elasticity E .

The nodal degrees of freedom are the local axial displacements u_1 and u_2 at the ends of the bar.

Stiffness Matrix for a Bar Element

The strain-displacement relationship is: $\varepsilon = \frac{du}{dx}$ $\sigma = E\varepsilon$

From equilibrium of forces, assuming no distributed loads acting on the bar, we get:

$$A\sigma_x = T = \text{constant}$$

Combining the above equations gives:

$$AE \frac{du}{dx} = T = \text{constant}$$

Taking the derivative of the above equation with respect to the local coordinate x gives:

$$\frac{d}{dx} \left(AE \frac{du}{dx} \right) = 0$$

Stiffness Matrix for a Bar Element

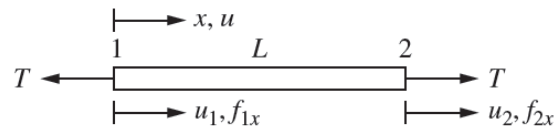
The following assumptions are considered in deriving the bar element stiffness matrix:

1. The bar cannot sustain shear force: $f_{1y} = f_{2y} = 0$
2. Any effect of transverse displacement is ignored.
3. Hooke's law applies; stress is related to strain: $\sigma_x = E\varepsilon_x$

Stiffness Matrix for a Bar Element

Step 1 - Select Element Type

We will consider the linear bar element shown below.



The bar element has a constant cross-section A , an initial length L , and modulus of elasticity E .

Stiffness Matrix for a Bar Element

Step 2 - Select a Displacement Function

A linear displacement function u is assumed: $u = a_1 + a_2 x$

The number of coefficients in the displacement function, a_i , is equal to the total number of degrees of freedom associated with the element.

Applying the boundary conditions and solving for the unknown coefficients gives:

$$u = \left(\frac{u_2 - u_1}{L} \right) x + u_1 \qquad u = \left[\left(1 - \frac{x}{L} \right) \quad \frac{x}{L} \right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Stiffness Matrix for a Bar Element

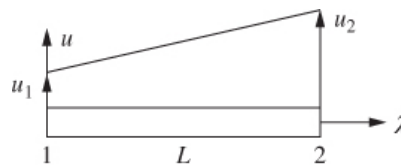
Step 2 - Select a Displacement Function

Or in another form: $u = [N_1 \quad N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

where N_1 and N_2 are the interpolation functions gives as:

$$N_1 = 1 - \frac{x}{L} \quad N_2 = \frac{x}{L}$$

The linear displacement function u plotted over the length of the bar element is shown here.



Stiffness Matrix for a Bar Element

Step 3 - Define the Strain/Displacement and Stress/Strain Relationships

The stress-displacement relationship is: $\epsilon_x = \frac{du}{dx} = \frac{u_2 - u_1}{L}$

Step 4 - Derive the Element Stiffness Matrix and Equations

We can now derive the element stiffness matrix as follows:

$$T = A\sigma_x$$

Substituting the stress-displacement relationship into the above equation gives:

$$T = AE \left(\frac{u_2 - u_1}{L} \right)$$

Stiffness Matrix for a Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

The nodal force sign convention, defined in element figure, is:

$$f_{1x} = -T \quad f_{2x} = T$$

therefore,

$$f_{1x} = AE \left(\frac{u_1 - u_2}{L} \right) \quad f_{2x} = AE \left(\frac{u_2 - u_1}{L} \right)$$

Writing the above equations in matrix form gives:

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Notice that AE/L for a bar element is analogous to the spring constant k for a spring element.

Stiffness Matrix for a Bar Element

Step 5 - Assemble the Element Equations and Introduce Boundary Conditions

The ***global stiffness matrix*** and the ***global force vector*** are assembled using the nodal force equilibrium equations, and force/deformation and compatibility equations.

$$\mathbf{K} = [\mathbf{K}] = \sum_{e=1}^n \mathbf{k}^{(e)} \quad \mathbf{F} = \{\mathbf{F}\} = \sum_{e=1}^n \mathbf{f}^{(e)}$$

Where \mathbf{k} and \mathbf{f} are the element stiffness and force matrices expressed in global coordinates.

Stiffness Matrix for a Bar Element

Step 6 - Solve for the Nodal Displacements

Solve the displacements by imposing the boundary conditions and solving the following set of equations:

$$\mathbf{F} = \mathbf{K}\mathbf{u}$$

Step 7 - Solve for the Element Forces

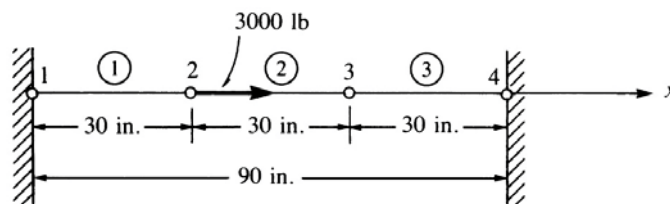
Once the displacements are found, the stress and strain in each element may be calculated from:

$$\varepsilon_x = \frac{du}{dx} = \frac{u_2 - u_1}{L} \qquad \sigma_x = E\varepsilon_x$$

Stiffness Matrix for a Bar Element

Example 1 - Bar Problem

Consider the following three-bar system shown below. Assume for elements 1 and 2: $A = 1 \text{ in}^2$ and $E = 30 (10^6) \text{ psi}$ and for element 3: $A = 2 \text{ in}^2$ and $E = 15 (10^6) \text{ psi}$.



Determine: (a) the global stiffness matrix, (b) the displacement of nodes 2 and 3, and (c) the reactions at nodes 1 and 4.

Stiffness Matrix for a Bar Element

Example 1 - Bar Problem

For elements 1 and 2:

$$\mathbf{k}^{(1)} = \mathbf{k}^{(2)} = \frac{(1)(30 \times 10^6)}{30} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{lb/in} = 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{lb/in}$$

1 2 node numbers for element 1
2 3 node numbers for element 2

For element 3:

$$\mathbf{k}^{(3)} = \frac{(2)(15 \times 10^6)}{30} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{lb/in} = 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{lb/in}$$

3 4 node numbers for element 3

As before, the numbers above the matrices indicate the displacements associated with the matrix.

Stiffness Matrix for a Bar Element

Example 1 - Bar Problem

Assembling the global stiffness matrix by the direct stiffness methods gives:

$$\mathbf{K} = 10^6 \begin{bmatrix} \overset{\text{E1}}{1} & \overset{\text{E1}}{-1} & 0 & 0 \\ \overset{\text{E1}}{-1} & \overset{\text{E1}}{2} & \overset{\text{E2}}{-1} & 0 \\ 0 & \overset{\text{E2}}{-1} & \overset{\text{E2}}{2} & \overset{\text{E3}}{-1} \\ 0 & 0 & \overset{\text{E3}}{-1} & \overset{\text{E3}}{1} \end{bmatrix}$$

Relating global nodal forces related to global nodal displacements gives:

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = 10^6 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Stiffness Matrix for a Bar Element

Example 1 - Bar Problem

The boundary conditions are: $u_1 = u_4 = 0$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = 10^6 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix}$$

Applying the boundary conditions and the known forces ($F_{2x} = 3,000 \text{ lb}$) gives:

$$\begin{Bmatrix} 3,000 \\ 0 \end{Bmatrix} = 10^6 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

Stiffness Matrix for a Bar Element

Example 1 - Bar Problem

Solving for u_2 and u_3 gives: $u_2 = 0.002 \text{ in}$
 $u_3 = 0.001 \text{ in}$

The global nodal forces are calculated as:

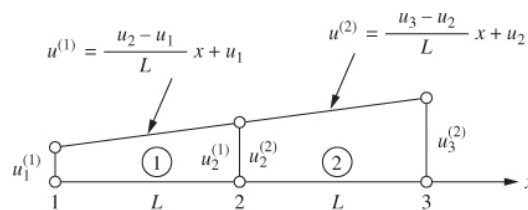
$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = 10^6 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.002 \\ 0.001 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -2,000 \\ 3,000 \\ 0 \\ -1,000 \end{Bmatrix} \text{ lb}$$

Stiffness Matrix for a Bar Element

Selecting Approximation Functions for Displacements

Consider the following guidelines, as they relate to the one-dimensional bar element, when selecting a displacement function.

1. Common approximation functions are usually polynomials.
2. The approximation function should be continuous within the bar element.

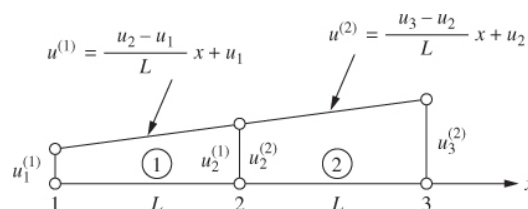


Stiffness Matrix for a Bar Element

Selecting Approximation Functions for Displacements

Consider the following guidelines, as they relate to the one-dimensional bar element, when selecting a displacement function.

3. The approximating function should provide interelement continuity for all degrees of freedom at each node for discrete line elements, and along common boundary lines and surfaces for two- and three-dimensional elements.

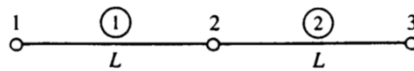


Stiffness Matrix for a Bar Element

Selecting Approximation Functions for Displacements

Consider the following guidelines, as they relate to the one-dimensional bar element, when selecting a displacement function.

For the bar element, we must ensure that nodes common to two or more elements remain common to these elements upon deformation and thus prevent overlaps or voids between elements.



The linear function is then called a **conforming** (or **compatible**) function for the bar element because it ensures both the satisfaction of continuity between adjacent elements and of continuity within the element.

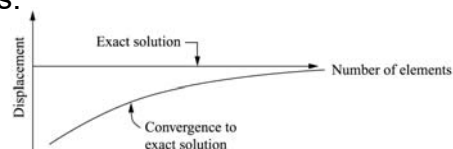
Stiffness Matrix for a Bar Element

Selecting Approximation Functions for Displacements

Consider the following guidelines, as they relate to the one-dimensional bar element, when selecting a displacement function.

4. The approximation function should allow for rigid-body displacement and for a state of constant strain within the element.

Completeness of a function is necessary for convergence to the exact answer, for instance, for displacements and stresses.



Stiffness Matrix for a Bar Element

Selecting Approximation Functions for Displacements

The interpolation function must allow for a rigid-body displacement, that means the function must be capable of yielding a constant value.

Consider the follow situation: $u = a_1$ $a_1 = u_1 = u_2$

Therefore: $u = N_1 u_1 + N_2 u_2 = (N_1 + N_2) a_1$

Since $u = a_1$ then: $u = a_1 = (N_1 + N_2) a_1$

This means that: $N_1 + N_2 = 1$

The displacement interpolation function must add to unity at every point within the element so the it will yield a constant value when a rigid-body displacement occurs.

Stiffness Matrix for a Bar Element

Transformation of Vectors in Two Dimensions

In many problems it is convenient to introduce both ***local*** and ***global*** (or reference) coordinates.

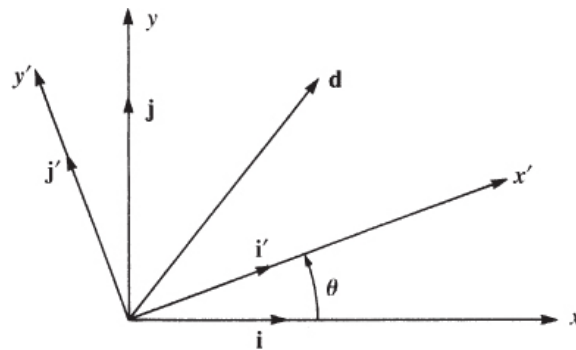
Local coordinates are always chosen to conveniently represent the individual element.

Global coordinates are chosen to be convenient for the whole structure.

Stiffness Matrix for a Bar Element

Transformation of Vectors in Two Dimensions

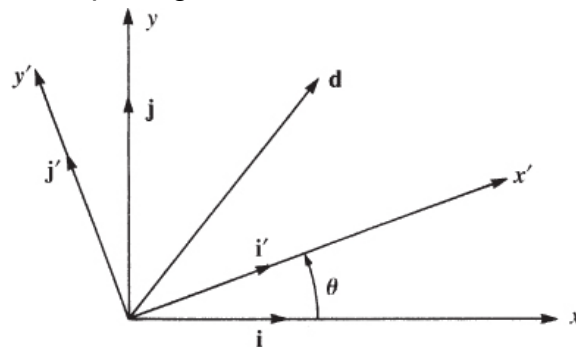
Given the nodal displacement of an element, represented by the vector \mathbf{d} in the figure below, we want to relate the components of this vector in one coordinate system to components in another.



Stiffness Matrix for a Bar Element

Transformation of Vectors in Two Dimensions

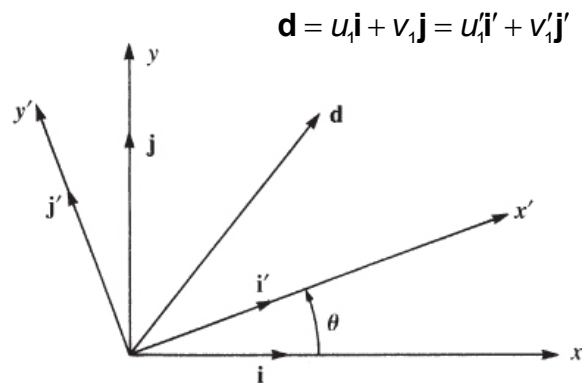
Let's consider that \mathbf{d} does not coincide with either the local or global axes. In this case, we want to relate global displacement components to local ones. In so doing, we will develop a ***transformation matrix*** that will subsequently be used to develop the global stiffness matrix for a bar element.



Stiffness Matrix for a Bar Element

Transformation of Vectors in Two Dimensions

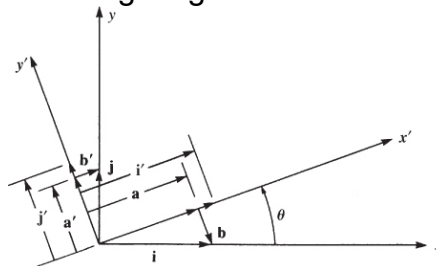
We define the angle θ to be positive when measured counterclockwise from x to x' . We can express vector displacement \mathbf{d} in both global and local coordinates by:



Stiffness Matrix for a Bar Element

Transformation of Vectors in Two Dimensions

Consider the following diagram:



Using vector addition: $\mathbf{a} + \mathbf{b} = \mathbf{i}$

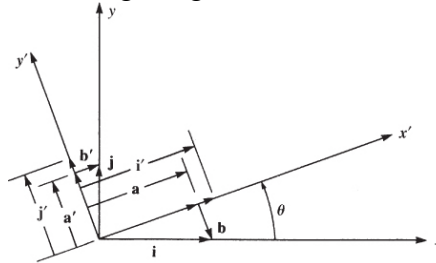
Using the law of cosines, we get: $|\mathbf{a}| = |\mathbf{i}| \cos \theta$ $|\mathbf{a}| = \cos \theta$

Similarly: $|\mathbf{b}| = |\mathbf{i}| \sin \theta$ $|\mathbf{b}| = \sin \theta$

Stiffness Matrix for a Bar Element

Transformation of Vectors in Two Dimensions

Consider the following diagram:



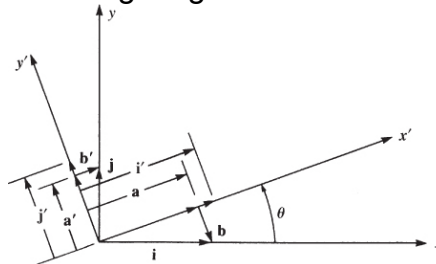
The vector **a** is in the **i'** direction and **b** is in the **j'** direction, therefore:

$$\mathbf{a} = |\mathbf{a}| \mathbf{i}' = (\cos \theta) \mathbf{i}' \quad \mathbf{b} = |\mathbf{b}| (-\mathbf{j}') = (\sin \theta) (-\mathbf{j}')$$

Stiffness Matrix for a Bar Element

Transformation of Vectors in Two Dimensions

Consider the following diagram:



The vector **i** can be rewritten as: $\mathbf{i} = \cos \theta \mathbf{i}' - \sin \theta \mathbf{j}'$

The vector **j** can be rewritten as: $\mathbf{j} = \sin \theta \mathbf{i}' + \cos \theta \mathbf{j}'$

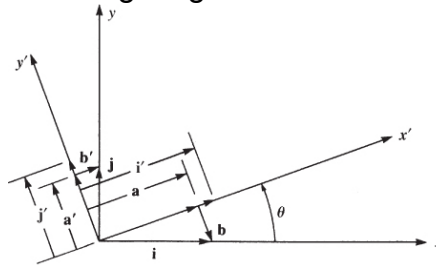
Therefore, the displacement vector is:

$$u_1 \mathbf{i} + v_1 \mathbf{j} = u_1 (\cos \theta \mathbf{i}' - \sin \theta \mathbf{j}') + v_1 (\sin \theta \mathbf{i}' + \cos \theta \mathbf{j}') = u_1' \mathbf{i}' + v_1' \mathbf{j}'$$

Stiffness Matrix for a Bar Element

Transformation of Vectors in Two Dimensions

Consider the following diagram:



Combining like coefficients of the local unit vectors gives:

$$\begin{aligned} u_1 \cos \theta + v_1 \sin \theta &= u'_1 \\ -u_1 \sin \theta + v_1 \cos \theta &= v'_1 \end{aligned} \quad \begin{aligned} C &= \cos \theta \\ S &= \sin \theta \end{aligned} \quad \begin{Bmatrix} u'_1 \\ v'_1 \end{Bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

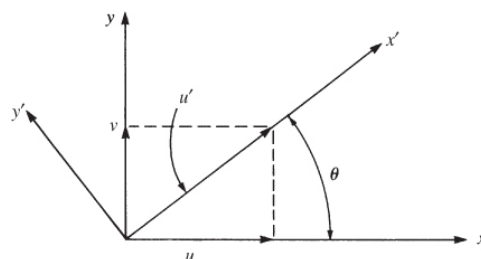
Stiffness Matrix for a Bar Element

Transformation of Vectors in Two Dimensions

The previous equation relates the global displacement \mathbf{d} to the \mathbf{d}' local displacements

The matrix is called the **transformation matrix**: $\begin{bmatrix} C & S \\ -S & C \end{bmatrix}$

The figure below shows u' expressed in terms of the global coordinates x and y .

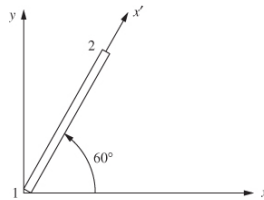


$$u' = Cu + Sv$$

Stiffness Matrix for a Bar Element

Example 2 - Bar Element Problem

The global nodal displacement at node 2 is $u_2 = 0.1$ in and $v_2 = 0.2$ in for the bar element shown below. Determine the local displacement.



Using the following expression we just derived, we get:

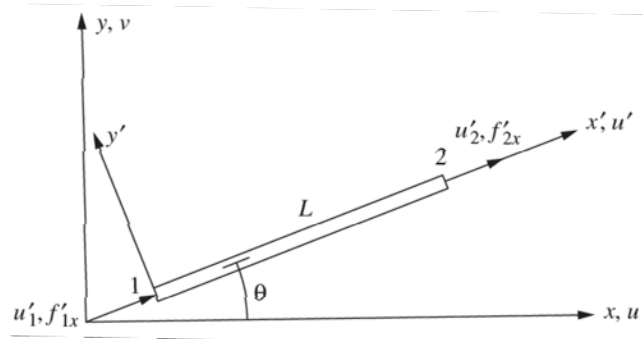
$$u' = Cu + Sv$$

$$u'_2 = \cos 60^\circ (0.1) + \sin 60^\circ (0.2) = 0.223 \text{ in}$$

Stiffness Matrix for a Bar Element

Global Stiffness Matrix

We will now use the transformation relationship developed above to obtain the global stiffness matrix for a bar element.



Stiffness Matrix for a Bar Element

Global Stiffness Matrix

We know that for a bar element in local coordinates we have:

$$\begin{Bmatrix} f'_{1x} \\ f'_{2x} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} \quad \mathbf{f}' = \mathbf{k}'\mathbf{d}'$$

We want to relate the global element forces \mathbf{f} to the global displacements \mathbf{d} for a bar element with an arbitrary orientation.

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = k \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \quad \mathbf{f} = k\mathbf{d}$$

Stiffness Matrix for a Bar Element

Global Stiffness Matrix

Using the relationship between local and global components, we can develop the global stiffness matrix.

We already know the transformation relationships:

$$u'_1 = u_1 \cos \theta + v_1 \sin \theta \quad u'_2 = u_2 \cos \theta + v_2 \sin \theta$$

Combining both expressions for the two local degrees-of-freedom, in matrix form, we get:

$$\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \quad \mathbf{d}' = \mathbf{T}^* \mathbf{d}$$

$$\mathbf{T}^* = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}$$

Stiffness Matrix for a Bar Element

Global Stiffness Matrix

A similar expression for the force transformation can be developed.

$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ f'_{2x} \\ f'_{2y} \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix} \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} \quad \boxed{\mathbf{f}' = \mathbf{T}^* \mathbf{f}}$$

Substituting the global force expression into element force equation gives: $\mathbf{f}' = \mathbf{k}' \mathbf{d}' \Rightarrow \mathbf{T}^* \mathbf{f} = \mathbf{k}' \mathbf{d}'$

Substituting the transformation between local and global displacements gives: $\boxed{\mathbf{d}' = \mathbf{T} \mathbf{d}} \Rightarrow \boxed{\mathbf{T}^* \mathbf{f} = \mathbf{k}' \mathbf{T} \mathbf{d}}$

Stiffness Matrix for a Bar Element

Global Stiffness Matrix

The matrix \mathbf{T}^* is not a square matrix so we cannot invert it. Let's expand the relationship between local and global displacement.

$$\begin{Bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \quad \mathbf{d}' = \mathbf{T} \mathbf{d}$$

where \mathbf{T} is:

$$\mathbf{T} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix}$$

Stiffness Matrix for a Bar Element

Global Stiffness Matrix

We can write a similar expression for the relationship between local and global forces.

$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ f'_{2x} \\ f'_{2y} \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} \quad \mathbf{f}' = \mathbf{T}\mathbf{f}$$

Therefore our original local coordinate force-displacement expression

$$\begin{Bmatrix} f'_{1x} \\ f'_{2x} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} \quad \mathbf{f}' = \mathbf{k}'\mathbf{d}'$$

Stiffness Matrix for a Bar Element

Global Stiffness Matrix

May be expanded:

$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ f'_{2x} \\ f'_{2y} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{Bmatrix}$$

The global force-displacement equations are:

$$\mathbf{f}' = \mathbf{k}'\mathbf{d}' \quad \Rightarrow \quad \mathbf{T}\mathbf{f} = \mathbf{k}'\mathbf{T}\mathbf{d}$$

Multiply both side by \mathbf{T}^{-1} we get: $\mathbf{f} = \mathbf{T}^{-1}\mathbf{k}'\mathbf{T}\mathbf{d}$

where \mathbf{T}^{-1} is the **inverse** of \mathbf{T} . It can be shown that: $\mathbf{T}^{-1} = \mathbf{T}^T$

Stiffness Matrix for a Bar Element

Global Stiffness Matrix

The global force-displacement equations become: $\mathbf{f} = \mathbf{T}^T \mathbf{k}' \mathbf{T} \mathbf{d}$

Where the global stiffness matrix \mathbf{k} is: $\mathbf{k} = \mathbf{T}^T \mathbf{k}' \mathbf{T}$

Expanding the above transformation gives:

$$\mathbf{k} = \frac{AE}{L} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

We can assemble the total stiffness matrix by using the above element stiffness matrix and the direct stiffness method.

$$\mathbf{K} = [\mathbf{K}] = \sum_{e=1}^n \mathbf{k}^{(e)} \quad \mathbf{F} = \{\mathbf{F}\} = \sum_{e=1}^n \mathbf{f}^{(e)} \quad \mathbf{F} = \mathbf{K} \mathbf{d}$$

Stiffness Matrix for a Bar Element

Global Stiffness Matrix

Local forces can be computed as:

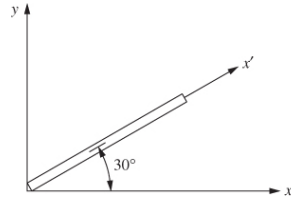
$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ f'_{2x} \\ f'_{2y} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ f'_{2x} \\ f'_{2y} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} Cu_1 + Sv_1 - Cu_2 - Sv_2 \\ 0 \\ -Cu_1 - Sv_1 + Cu_2 + Sv_2 \\ 0 \end{bmatrix}$$

Stiffness Matrix for a Bar Element

Example 3 - Bar Element Problem

For the bar element shown below, evaluate the global stiffness matrix. Assume the cross-sectional area is 2 in^2 , the length is 60 in , and the E is $30 \times 10^6 \text{ psi}$.



$$\mathbf{k} = \frac{AE}{L} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

Therefore: $C = \cos 30^\circ = \frac{\sqrt{3}}{2}$ $S = \sin 30^\circ = \frac{1}{2}$

Stiffness Matrix for a Bar Element

Example 3 - Bar Element Problem

The global elemental stiffness matrix is:

$$\mathbf{k} = \frac{(2 \text{ in}^2)(30 \times 10^6 \text{ psi})}{60 \text{ in}} \begin{bmatrix} 3/4 & \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 & -1/4 \\ -3/4 & -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 & 1/4 \end{bmatrix}$$

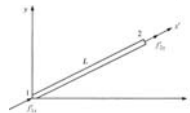
Simplifying the global elemental stiffness matrix is:

$$\mathbf{k} = 10^6 \begin{bmatrix} 0.750 & 0.433 & -0.750 & -0.433 \\ 0.433 & 0.250 & -0.433 & -0.250 \\ -0.750 & -0.433 & 0.750 & 0.433 \\ -0.433 & -0.250 & 0.433 & 0.250 \end{bmatrix} \text{ lb/in}$$

Stiffness Matrix for a Bar Element

Computation of Stress for a Bar in the x-y Plane

For a bar element the local forces are related to the local displacements by:



$$\begin{Bmatrix} f'_{1x} \\ f'_{2x} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix}$$

The force-displacement equation for f'_{2x} is:

$$f'_{2x} = \frac{AE}{L} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix}$$

The stress in terms of global displacement is:

$$\sigma = \frac{E}{L} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \frac{E}{L} [-Cu_1 - Sv_1 + Cu_2 + Sv_2]$$

Stiffness Matrix for a Bar Element

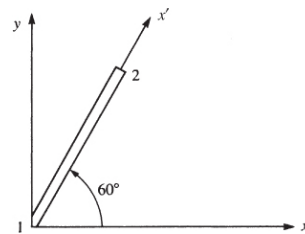
Example 4 - Bar Element Problem

For the bar element shown below, determine the axial stress.

Assume the cross-sectional area is $4 \times 10^{-4} \text{ m}^2$, the length is 2 m , and the E is 210 GPa .

The global displacements are known as

$u_1 = 0.25 \text{ mm}$, $v_1 = 0$, $u_2 = 0.5 \text{ mm}$,
and $v_2 = 0.75 \text{ mm}$.



$$\sigma = \frac{E}{L} [-Cu_1 - Sv_1 + Cu_2 + Sv_2]$$

$$\sigma = \frac{210 \times 10^6}{2} \left[-\frac{1}{2}(0.25) - \frac{\sqrt{3}}{4}(0) + \frac{1}{2}(0.5) + \frac{\sqrt{3}}{4}(0.75) \right] \text{ kN/m}^2$$

$$\sigma = 81.32 \times 10^3 \text{ kN/m}^2 = 81.32 \text{ MPa}$$

Stiffness Matrix for a Bar Element

Solution of a Plane Truss

We will now illustrate the use of equations developed above along with the direct stiffness method to solve the following plane truss example problems.

A plane truss is a structure composed of bar elements all lying in a common plane that are connected together by frictionless pins.

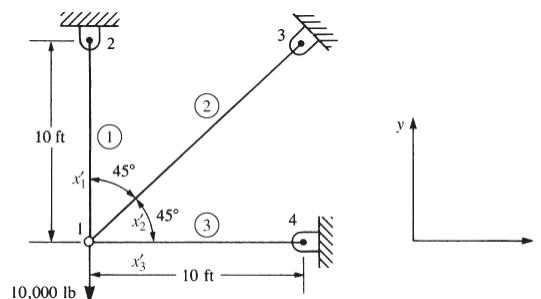
The plane truss also must have loads acting only in the common plane.

Stiffness Matrix for a Bar Element

Example 5 - Plane Truss Problem

The plane truss shown below is composed of three bars subjected to a downward force of 10 *kips* at node 1. Assume the cross-sectional area $A = 2 \text{ in}^2$ and E is $30 \times 10^6 \text{ psi}$ for all elements.

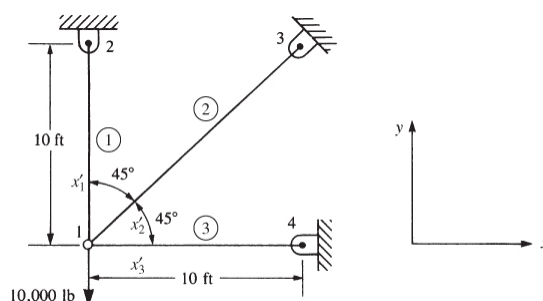
Determine the x and y displacement at node 1 and stresses in each element.



Stiffness Matrix for a Bar Element

Example 5 - Plane Truss Problem

Element	Node 1	Node 2	θ	C	S
1	1	2	90°	0	1
2	1	3	45°	0.707	0.707
3	1	4	0°	1	0



Stiffness Matrix for a Bar Element

Example 5 - Plane Truss Problem

The global elemental stiffness matrix are:

$$\mathbf{k} = \frac{AE}{L} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

element 1: $C=0$ $S=1$ $\Rightarrow \mathbf{k}^{(1)} = \frac{(2 \text{ in}^2)(30 \times 10^6 \text{ psi})}{120 \text{ in}} \begin{bmatrix} u_1 & v_1 & u_2 & v_2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \text{ lb/in}$

element 2: $C = \frac{\sqrt{2}}{2}$ $S = \frac{\sqrt{2}}{2}$ $\Rightarrow \mathbf{k}^{(2)} = \frac{(2 \text{ in}^2)(30 \times 10^6 \text{ psi})}{240\sqrt{2} \text{ in}} \begin{bmatrix} u_1 & v_1 & u_3 & v_3 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \text{ lb/in}$

element 3: $C=1$ $S=0$ $\Rightarrow \mathbf{k}^{(3)} = \frac{(2 \text{ in}^2)(30 \times 10^6 \text{ psi})}{120 \text{ in}} \begin{bmatrix} u_1 & v_1 & u_4 & v_4 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ lb/in}$

Stiffness Matrix for a Bar Element

Example 5 - Plane Truss Problem

The total global stiffness matrix is:

$$\mathbf{K} = 5 \times 10^5 \begin{bmatrix} 1.354 & 0.354 & 0 & 0 & -0.354 & -0.354 & -1 & 0 \\ 0.354 & 1.354 & 0 & -1 & -0.354 & -0.354 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ lb/in}$$

element 1
element 2
element 3

The total global force-displacement equations are:

$$\begin{Bmatrix} 0 \\ -10,000 \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{Bmatrix} = 5 \times 10^5 \begin{bmatrix} 1.354 & 0.354 & 0 & 0 & -0.354 & -0.354 & -1 & 0 \\ 0.354 & 1.354 & 0 & -1 & -0.354 & -0.354 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Stiffness Matrix for a Bar Element

Example 5 - Plane Truss Problem

Applying the boundary conditions for the truss, the above equations reduce to:

$$\begin{Bmatrix} 0 \\ -10,000 \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{Bmatrix} = 5 \times 10^5 \begin{bmatrix} 1.354 & 0.354 & 0 & 0 & -0.354 & -0.354 & -1 & 0 \\ 0.354 & 1.354 & 0 & -1 & -0.354 & -0.354 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Stiffness Matrix for a Bar Element

Example 5 - Plane Truss Problem

Applying the boundary conditions for the truss, the above equations reduce to:

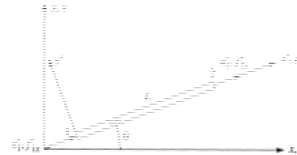
$$\begin{Bmatrix} 0 \\ -10,000 \end{Bmatrix} = 5 \times 10^5 \begin{bmatrix} 1.354 & 0.354 \\ 0.354 & 1.354 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

Solving the equations gives: $u_1 = 0.414 \times 10^{-2} \text{ in}$

$$v_1 = -1.59 \times 10^{-2} \text{ in}$$

The stress in an element is: $\sigma = \frac{E}{L} [-Cu_{\boxed{1}} - Sv_{\boxed{1}} + Cu_{\boxed{2}} + Sv_{\boxed{2}}]$

where \boxed{i} is the local node number



Stiffness Matrix for a Bar Element

Example 5 - Plane Truss Problem

Element	Node 1	Node 2	θ	C	S
1	1	2	90°	0	1
2	1	3	45°	0.707	0.707
3	1	4	0°	1	0

$$\sigma = \frac{E}{L} [-Cu_{\boxed{1}} - Sv_{\boxed{1}} + Cu_{\boxed{2}} + Sv_{\boxed{2}}]$$

element 1 $\sigma^{(1)} = \frac{30 \times 10^6}{120} [-v_1] = 3,965 \text{ psi}$

element 2 $\sigma^{(2)} = -\frac{30 \times 10^6}{120} [(0.707)u_1 + (0.707)v_1] = 1,471 \text{ psi}$

Stiffness Matrix for a Bar Element

Example 5 - Plane Truss Problem

Element	Node 1	Node 2	θ	C	S
1	1	2	90°	0	1
2	1	3	45°	0.707	0.707
3	1	4	0°	1	0

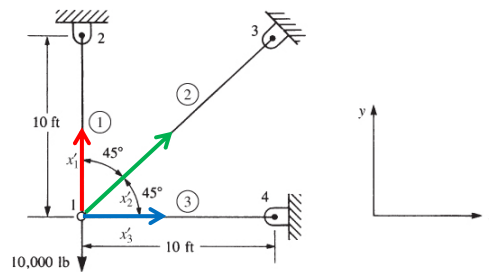
$$\sigma = \frac{E}{L} [-Cu_{\boxed{1}} - Sv_{\boxed{1}} + Cu_{\boxed{2}} + Sv_{\boxed{2}}]$$

element 3 $\sigma^{(3)} = \frac{30 \times 10^6}{120} [-u_1] = -1,035 \text{ psi}$

Stiffness Matrix for a Bar Element

Example 5 - Plane Truss Problem

Let's check equilibrium at node 1:



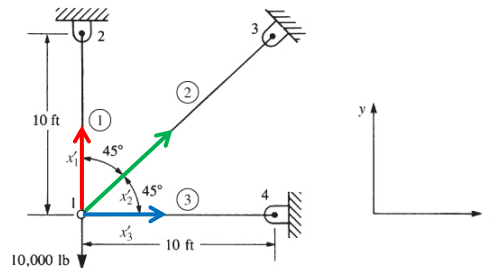
$$\sum F_x = \underbrace{f_{1x}^{(2)} \cos(45^\circ)}_{\text{green}} + \underbrace{f_{1x}^{(3)}}_{\text{blue}}$$

$$\sum F_y = \underbrace{f_{1x}^{(2)} \sin(45^\circ)}_{\text{green}} + \underbrace{f_{1x}^{(1)}}_{\text{red}} - 10,000 \text{ lb}$$

Stiffness Matrix for a Bar Element

Example 5 - Plane Truss Problem

Let's check equilibrium at node 1:



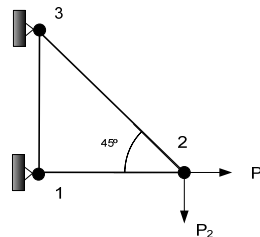
$$\sum F_x = (1,471 \text{ psi})(2 \text{ in}^2)(0.707) - (1,035 \text{ psi})(2 \text{ in}^2) = 0$$

$$\sum F_y = (3,965 \text{ psi})(2 \text{ in}^2) + (1,471 \text{ psi})(2 \text{ in}^2)(0.707) - 10,000 = 0$$

Stiffness Matrix for a Bar Element

Example 6 - Plane Truss Problem

Develop the element stiffness matrices and system equations for the plane truss below.

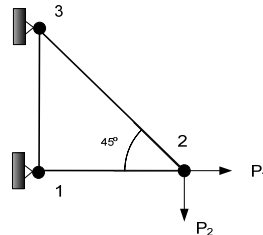


Assume the stiffness of each element is constant. Use the numbering scheme indicated. Solve the equations for the displacements and compute the member forces. All elements have a constant value of AE/L

Stiffness Matrix for a Bar Element

Example 6 - Plane Truss Problem

Develop the element stiffness matrices and system equations for the plane truss below.



Member	Node 1	Node 2	Elemental Stiffness	θ
1	1	2	k	0
2	2	3	k	$3\pi/4$
3	1	3	k	$\pi/2$

Stiffness Matrix for a Bar Element

Example 6 - Plane Truss Problem

Compute the elemental stiffness matrix for each element. The general form of the matrix is:

$$k = \frac{AE}{L} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

Member	Node 1	Node 2	Elemental Stiffness	θ
1	1	2	k	0
2	2	3	k	$3\pi/4$
3	1	3	k	$\pi/2$

Stiffness Matrix for a Bar Element

Example 6 - Plane Truss Problem

For element 1:

$$\mathbf{k}^{(1)} = k \begin{matrix} & \begin{matrix} u_1 & v_1 & u_2 & v_2 \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{matrix} \end{matrix}$$

Member	Node 1	Node 2	Elemental Stiffness	θ
1	1	2	k	0
2	2	3	k	$3\pi/4$
3	1	3	k	$\pi/2$

Stiffness Matrix for a Bar Element

Example 6 - Plane Truss Problem

For element 2:

$$\mathbf{k}^{(2)} = \frac{k}{2} \begin{matrix} & \begin{matrix} u_2 & v_2 & u_3 & v_3 \end{matrix} \\ \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} & \begin{matrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{matrix} \end{matrix}$$

Member	Node 1	Node 2	Elemental Stiffness	θ
1	1	2	k	0
2	2	3	k	$3\pi/4$
3	1	3	k	$\pi/2$

Stiffness Matrix for a Bar Element

Example 6 - Plane Truss Problem

For element 3:

$$\mathbf{k}^{(3)} = k \begin{bmatrix} u_1 & v_1 & u_3 & v_3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{matrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{matrix}$$

Member	Node 1	Node 2	Elemental Stiffness	θ
1	1	2	k	0
2	2	3	k	$3\pi/4$
3	1	3	k	$\pi/2$

Stiffness Matrix for a Bar Element

Example 6 - Plane Truss Problem

Assemble the global stiffness matrix by superimposing the elemental global matrices.

$$\mathbf{K} = \frac{k}{2} \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \\ \boxed{2} & \boxed{0} & \boxed{-2} & \boxed{0} & \boxed{0} & \boxed{0} & u_1 \\ \boxed{0} & \boxed{2} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{-2} & v_1 \\ \boxed{-2} & \boxed{0} & \boxed{3} & \boxed{-1} & \boxed{-1} & \boxed{1} & u_2 \\ \boxed{0} & \boxed{0} & \boxed{-1} & \boxed{1} & \boxed{1} & \boxed{-1} & v_2 \\ \boxed{0} & \boxed{0} & \boxed{-1} & \boxed{1} & \boxed{1} & \boxed{-1} & u_3 \\ \boxed{0} & \boxed{-2} & \boxed{1} & \boxed{-1} & \boxed{-1} & \boxed{3} & v_3 \end{bmatrix}$$

element 1
element 2
element 3

Stiffness Matrix for a Bar Element

Example 6 - Plane Truss Problem

The unconstrained (no boundary conditions satisfied) equations are:

$$\frac{k}{2} \begin{bmatrix} 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -2 \\ -2 & 0 & 3 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & -2 & 1 & -1 & -1 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ P_1 \\ -P_2 \\ F_{3x} \\ F_{3y} \end{Bmatrix}$$

The displacement at nodes 1 and 3 are zero in both directions.
Applying these conditions to the system equations gives:

$$\frac{k}{2} \begin{bmatrix} 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -2 \\ -2 & 0 & 3 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & -2 & 1 & -1 & -1 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ P_1 \\ -P_2 \\ F_{3x} \\ F_{3y} \end{Bmatrix}$$

Stiffness Matrix for a Bar Element

Example 6 - Plane Truss Problem

Applying the boundary conditions to the system equations gives:

$$\frac{k}{2} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ -P_2 \end{Bmatrix}$$

Solving this set of equations is fairly easy. The solution is:

$$u_2 = \frac{P_1 - P_2}{k} \quad v_2 = \frac{P_1 - 3P_2}{k}$$

Stiffness Matrix for a Bar Element

Example 6 - Plane Truss Problem

Using the force-displacement relationship the force in each member may be computed.

Member (element) 1:

$$C = 1 \quad S = 0 \quad \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = k \begin{Bmatrix} Cu_1 + Sv_1 - Cu_2 - Sv_2 \\ 0 \\ -Cu_1 - Sv_1 + Cu_2 + Sv_2 \\ 0 \end{Bmatrix}$$

$$f_{1x} = k(-Cu_2) = k\left(-\frac{P_1 - P_2}{k}\right) = -(P_1 - P_2) \quad f_{1y} = 0$$

$$f_{2x} = k(Cu_2) = k\left(\frac{P_1 - P_2}{k}\right) = P_1 - P_2 \quad f_{2y} = 0$$

Stiffness Matrix for a Bar Element

Example 6 - Plane Truss Problem

Using the force-displacement relationship the force in each member may be computed.

Member (element) 2:

$$C = -\frac{1}{\sqrt{2}} \quad S = \frac{1}{\sqrt{2}} \quad \begin{Bmatrix} f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = k \begin{Bmatrix} Cu_2 + Sv_2 - Cu_3 - Sv_3 \\ 0 \\ -Cu_2 - Sv_2 + Cu_3 + Sv_3 \\ 0 \end{Bmatrix}$$

$$f_{2x} = k(Cu_2 + Sv_2)$$

$$= k\left[\left(\frac{P_1 - P_2}{k}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{P_1 - 3P_2}{k}\right)\left(\frac{1}{\sqrt{2}}\right)\right] = -\sqrt{2}P_2$$

$$f_{3x} = k(-Cu_2 - Sv_2)$$

$$= k\left[\left(\frac{P_1 - P_2}{k}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{P_1 - 3P_2}{k}\right)\left(-\frac{1}{\sqrt{2}}\right)\right] = \sqrt{2}P_2$$

Stiffness Matrix for a Bar Element

Example 6 - Plane Truss Problem

Using the force-displacement relationship the force in each member may be computed.

Member (element) 3:

$$C=0 \quad S=1 \quad \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = k \begin{bmatrix} Cu_1 + Sv_1 - Cu_3 - Sv_3 \\ 0 \\ -Cu_1 - Sv_1 + Cu_3 + Sv_3 \\ 0 \end{bmatrix}$$

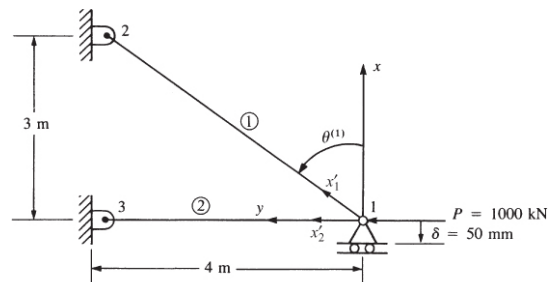
$$\begin{aligned} f_{1x} &= 0 & f_{1y} &= 0 \\ f_{3x} &= 0 & f_{3y} &= 0 \end{aligned}$$

The solution to this simple problem can be readily checked by using simple static equilibrium equations.

Stiffness Matrix for a Bar Element

Example 7 - Plane Truss Problem

Consider the two bar truss shown below.



Determine the displacement in the y direction of node 1 and the axial force in each element.

Assume $E = 210 \text{ GPa}$ and $A = 6 \times 10^{-4} \text{ m}^2$

Stiffness Matrix for a Bar Element

Example 7 - Plane Truss Problem

The global elemental stiffness matrix for **element 1** is:

$$\begin{aligned} \cos \theta^{(1)} &= \frac{3}{5} = 0.6 \\ \sin \theta^{(1)} &= \frac{4}{5} = 0.8 \end{aligned} \quad \mathbf{k}^{(1)} = \frac{210 \times 10^6 (6 \times 10^{-4})}{5} \begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 \\ 0.48 & 0.64 & -0.48 & -0.64 \\ -0.36 & -0.48 & 0.36 & 0.48 \\ -0.48 & -0.64 & 0.48 & 0.64 \end{bmatrix}$$

Simplifying the above expression gives:

$$\mathbf{k}^{(1)} = 25,200 \begin{bmatrix} u_1 & v_1 & u_2 & v_2 \\ 0.36 & 0.48 & -0.36 & -0.48 \\ 0.48 & 0.64 & -0.48 & -0.64 \\ -0.36 & -0.48 & 0.36 & 0.48 \\ -0.48 & -0.64 & 0.48 & 0.64 \end{bmatrix}$$

Stiffness Matrix for a Bar Element

Example 7 - Plane Truss Problem

The global elemental stiffness matrix for **element 2** is:

$$\begin{aligned} \cos \theta^{(2)} &= 0 \\ \sin \theta^{(2)} &= 1 \end{aligned} \quad \mathbf{k}^{(2)} = \frac{(210 \times 10^6)(6 \times 10^{-4})}{4} \begin{bmatrix} 0 & 0 & 0 & -0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Simplifying the above expression gives:

$$\mathbf{k}^{(2)} = 25,200 \begin{bmatrix} u_1 & v_1 & u_3 & v_3 \\ 0 & 0 & 0 & 0 \\ 0 & 1.25 & 0 & -1.25 \\ 0 & 0 & 0 & 0 \\ 0 & -1.25 & 0 & 1.25 \end{bmatrix}$$

Stiffness Matrix for a Bar Element

Example 7 - Plane Truss Problem

The total global equations are:

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = 25,200 \begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 & 0 & 0 \\ 0.48 & 1.89 & -0.48 & -0.64 & 0 & -1.25 \\ -0.36 & -0.48 & 0.36 & 0.48 & 0 & 0 \\ -0.48 & -0.64 & 0.48 & 0.64 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.25 & 0 & 0 & 0 & 1.25 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

element 1

element 2

The displacement boundary conditions are:

$$u_1 = \delta \quad u_2 = v_2 = u_3 = v_3 = 0$$

Stiffness Matrix for a Bar Element

Example 7 - Plane Truss Problem

The total global equations are:

$$\begin{Bmatrix} F_{1x} \\ P \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = 25,200 \begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 & 0 & 0 \\ 0.48 & 1.89 & -0.48 & -0.64 & 0 & -1.25 \\ -0.36 & -0.48 & 0.36 & 0.48 & 0 & 0 \\ -0.48 & -0.64 & 0.48 & 0.64 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.25 & 0 & 0 & 0 & 1.25 \end{bmatrix} \begin{Bmatrix} \delta \\ v_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

By applying the boundary conditions the force-displacement equations reduce to:

$$P = 25,200(0.48\delta + 1.89v_1)$$

Stiffness Matrix for a Bar Element

Example 7 - Plane Truss Problem

Solving the equation gives: $v_1 = (2.1 \times 10^{-5})P - 0.25\delta$

By substituting $P = 1,000 \text{ kN}$ and $\delta = -0.05 \text{ m}$ in the above equation gives:

$$v_1 = 0.0337 \text{ m}$$

The local element forces for element 1 are:

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 25,200 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 \end{bmatrix} \begin{Bmatrix} u_1 = -0.05 \\ v_1 = 0.0337 \\ u_2 \\ v_2 \end{Bmatrix}$$

The element forces are: $f_{1x} = -76.6 \text{ kN}$ $f_{2x} = 76.7 \text{ kN}$

Tension

Stiffness Matrix for a Bar Element

Example 7 - Plane Truss Problem

The local element forces for element 2 are:

$$\begin{Bmatrix} f_{1x} \\ f_{3x} \end{Bmatrix} = 31,500 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = -0.05 \\ v_1 = 0.0337 \\ u_3 \\ v_3 \end{Bmatrix}$$

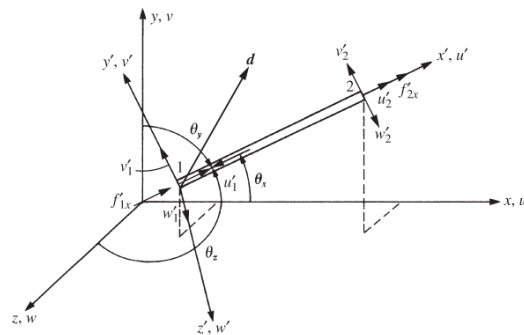
The element forces are: $f_{1x} = 1,061 \text{ kN}$ $f_{3x} = -1,061 \text{ kN}$

Compression

Stiffness Matrix for a Bar Element

Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space

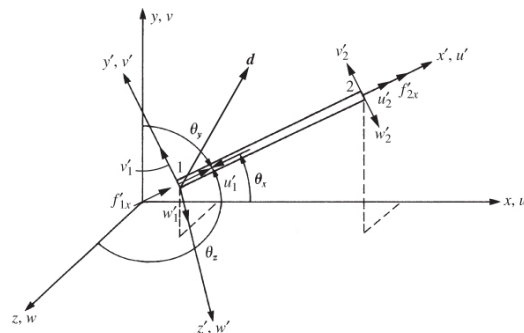
Let's derive the transformation matrix for the stiffness matrix for a bar element in three-dimensional space as shown below:



Stiffness Matrix for a Bar Element

Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space

The coordinates at node 1 are x_1 , y_1 , and z_1 , and the coordinates of node 2 are x_2 , y_2 , and z_2 . Also, let θ_x , θ_y , and θ_z be the angles measured from the global x , y , and z axes, respectively, to the local x' axis.

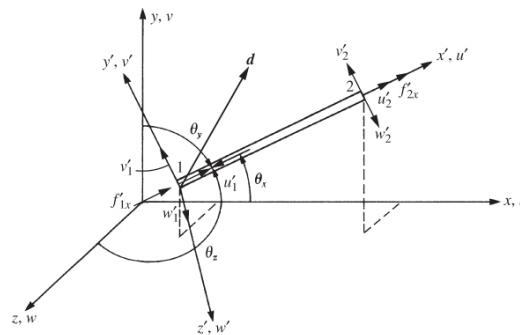


Stiffness Matrix for a Bar Element

Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space

The three-dimensional vector representing the bar element is given as:

$$\mathbf{d} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = u'\mathbf{i}' + v'\mathbf{j}' + w'\mathbf{k}'$$



Stiffness Matrix for a Bar Element

Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space

Taking the dot product of the above equation with \mathbf{i}' gives:

$$u(\mathbf{i} \cdot \mathbf{i}') + v(\mathbf{j} \cdot \mathbf{i}') + w(\mathbf{k} \cdot \mathbf{i}') = u'$$

By the definition of the dot product we get:

$$\mathbf{i} \cdot \mathbf{i}' = \frac{x_2 - x_1}{L} = C_x \quad \mathbf{j} \cdot \mathbf{i}' = \frac{y_2 - y_1}{L} = C_y \quad \mathbf{k} \cdot \mathbf{i}' = \frac{z_2 - z_1}{L} = C_z$$

$$\text{where } L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$C_x = \cos \theta_x \quad C_y = \cos \theta_y \quad C_z = \cos \theta_z$$

where C_x , C_y , and C_z are projections of \mathbf{i}' on to \mathbf{i} , \mathbf{j} , and \mathbf{k} , respectively.

Stiffness Matrix for a Bar Element

Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space

Therefore: $u' = C_x u + C_y v + C_z w$

The transformation between local and global displacements is:

$$\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} = \begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix} \quad \mathbf{d}' = \mathbf{T}^* \mathbf{d}$$

$$\mathbf{T}^* = \begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix}$$

Stiffness Matrix for a Bar Element

Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space

The transformation from the local to the global stiffness matrix is:

$$\mathbf{k} = \mathbf{T}^T \mathbf{k}' \mathbf{T} \quad \mathbf{k} = \begin{bmatrix} C_x & 0 \\ C_y & 0 \\ C_z & 0 \\ 0 & C_x \\ 0 & C_y \\ 0 & C_z \end{bmatrix} \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix}$$

$$\mathbf{k} = \frac{AE}{L} \begin{bmatrix} C_x^2 & C_x C_y & C_x C_z & -C_x^2 & -C_x C_y & -C_x C_z \\ C_x C_y & C_y^2 & C_y C_z & -C_x C_y & -C_y^2 & -C_y C_z \\ C_x C_z & C_y C_z & C_z^2 & -C_x C_z & -C_y C_z & -C_z^2 \\ -C_x^2 & -C_x C_y & -C_x C_z & C_x^2 & C_x C_y & C_x C_z \\ -C_x C_y & -C_y^2 & -C_y C_z & C_x C_y & C_y^2 & C_y C_z \\ -C_x C_z & -C_y C_z & -C_z^2 & C_x C_z & C_y C_z & C_z^2 \end{bmatrix}$$

Stiffness Matrix for a Bar Element

Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space

The global stiffness matrix can be written in a more convenient form as:

$$\mathbf{k} = \frac{AE}{L} \begin{bmatrix} \lambda & -\lambda \\ -\lambda & \lambda \end{bmatrix} \quad \lambda = \begin{bmatrix} C_x^2 & C_x C_y & C_x C_z \\ C_x C_y & C_y^2 & C_y C_z \\ C_x C_z & C_y C_z & C_z^2 \end{bmatrix}$$

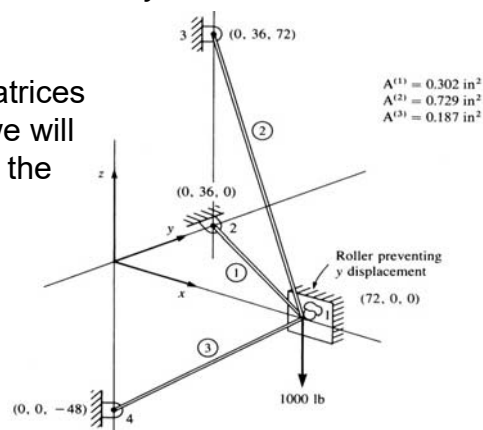
Stiffness Matrix for a Bar Element

Example 8 – Space Truss Problem

Consider the space truss shown below. The modulus of elasticity, $E = 1.2 \times 10^6 \text{ psi}$ for all elements. Node 1 is constrained from movement in the y direction.

To simplify the stiffness matrices for the three elements, we will express each element in the following form:

$$\mathbf{k} = \frac{AE}{L} \begin{bmatrix} \lambda & -\lambda \\ -\lambda & \lambda \end{bmatrix}$$



Stiffness Matrix for a Bar Element

Example 8 – Space Truss Problem

Consider element 1: $L^{(1)} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

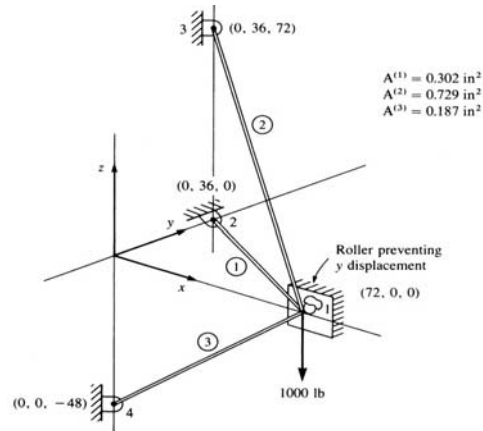
$$L^{(1)} = \sqrt{(-72)^2 + (36)^2} = 80.5 \text{ in}$$

$$C_x = \frac{-72}{80.5} = -0.894$$

$$C_y = \frac{36}{80.5} = 0.447$$

$$C_z = 0$$

$$\lambda = \begin{bmatrix} 0.80 & -0.40 & 0 \\ -0.40 & 0.20 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

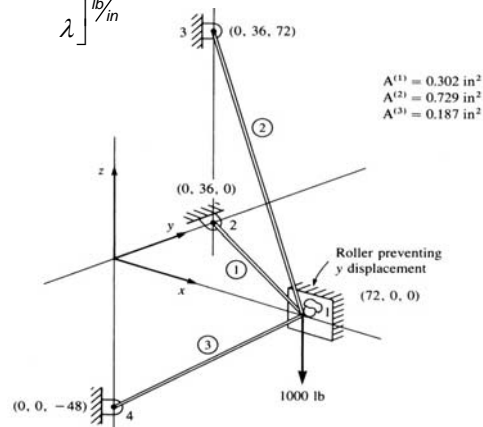


Stiffness Matrix for a Bar Element

Example 8 – Space Truss Problem

Consider element 1:

$$\mathbf{k} = \frac{(0.302 \text{ in}^2)(1.2 \times 10^6 \text{ psi})}{80.5 \text{ in}} \begin{bmatrix} \overset{u_1 \ v_1 \ w_1}{\lambda} & \overset{u_2 \ v_2 \ w_2}{-\lambda} \\ -\lambda & \lambda \end{bmatrix} \text{ lb/in}$$



Stiffness Matrix for a Bar Element

Example 8 – Space Truss Problem

Consider element 2: $L^{(2)} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2}$

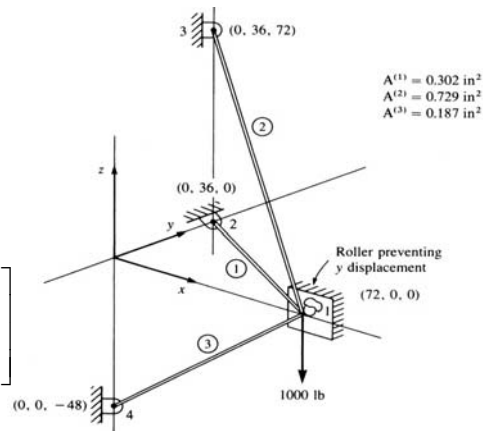
$$L^{(2)} = \sqrt{(-72)^2 + (36)^2 + (72)^2} = 108 \text{ in}$$

$$C_x = \frac{-72}{108} = -0.667$$

$$C_y = \frac{36}{108} = 0.333$$

$$C_z = \frac{72}{108} = 0.667$$

$$\lambda = \begin{bmatrix} 0.444 & -0.222 & -0.444 \\ -0.222 & 0.111 & 0.444 \\ -0.444 & 0.444 & 0.444 \end{bmatrix}$$

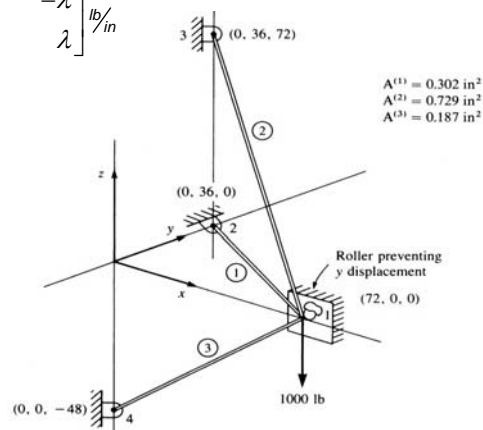


Stiffness Matrix for a Bar Element

Example 8 – Space Truss Problem

Consider element 2:

$$\mathbf{k} = \frac{(0.729 \text{ in}^2)(1.2 \times 10^6 \text{ psi})}{108 \text{ in}} \begin{bmatrix} \lambda & -\lambda \\ -\lambda & \lambda \end{bmatrix} \text{ lb/in}$$



Stiffness Matrix for a Bar Element

Example 8 – Space Truss Problem

Consider element 3: $L^{(3)} = \sqrt{(x_4 - x_1)^2 + (y_4 - y_1)^2 + (z_4 - z_1)^2}$

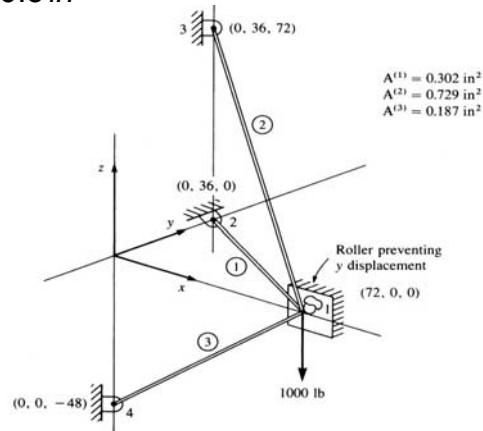
$$L^{(3)} = \sqrt{(-72)^2 + (-48)^2} = 86.5 \text{ in}$$

$$C_x = \frac{-72}{86.5} = -0.832$$

$$C_y = 0$$

$$C_z = \frac{-48}{86.5} = -0.555$$

$$\lambda = \begin{bmatrix} 0.692 & 0 & 0.462 \\ 0 & 0 & 0 \\ 0.462 & 0 & 0.308 \end{bmatrix}$$



Stiffness Matrix for a Bar Element

Example 8 – Space Truss Problem

Consider element 3:

$$\mathbf{k} = \frac{(0.187 \text{ in}^2)(1.2 \times 10^6 \text{ psi})}{86.5 \text{ in}} \begin{bmatrix} \lambda & -\lambda \\ -\lambda & \lambda \end{bmatrix} \text{ lb/in}$$

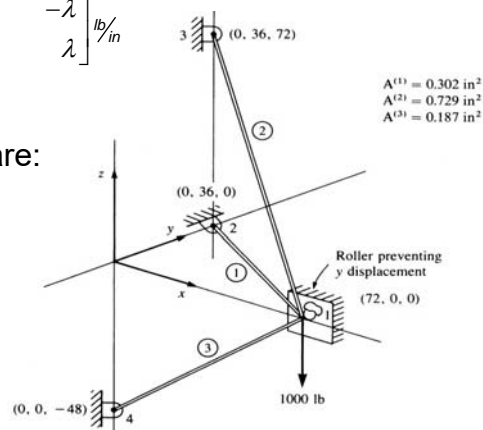
The boundary conditions are:

$$u_2 = v_2 = w_2 = 0$$

$$u_3 = v_3 = w_3 = 0$$

$$u_4 = v_4 = w_4 = 0$$

$$v_1 = 0$$



Stiffness Matrix for a Bar Element

Example 8 – Space Truss Problem

Canceling the rows and the columns associated with the boundary conditions reduces the global stiffness matrix to:

$$\mathbf{K} = \begin{matrix} & \begin{matrix} u_1 & w_1 \end{matrix} \\ \begin{bmatrix} 8,997 & -2,403 \\ -2,403 & 4,398 \end{bmatrix} \end{matrix}$$

The global force-displacement equations are:

$$\begin{bmatrix} 8,997 & -2,403 \\ -2,403 & 4,398 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1,000 \end{Bmatrix}$$

Solving the equation gives:

$$u_1 = -0.0711 \text{ in} \qquad w_1 = -0.2662 \text{ in}$$

Stiffness Matrix for a Bar Element

Example 8 – Space Truss Problem

It can be shown, that the local forces in an element are:

$$\begin{Bmatrix} f'_{ix} \\ f'_{jx} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} -C_x & -C_y & -C_z & C_x & C_y & C_z \\ C_x & C_y & C_z & -C_x & -C_y & -C_z \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \\ u_j \\ v_j \\ w_j \end{Bmatrix}$$

The stress in an element is:

$$\sigma = \frac{E}{L} \begin{bmatrix} -C_x & -C_y & -C_z & C_x & C_y & C_z \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \\ u_j \\ v_j \\ w_j \end{Bmatrix}$$

Stiffness Matrix for a Bar Element

Example 8 – Space Truss Problem

The stress in element 1 is:

$$\sigma^{(1)} = \frac{1.2 \times 10^6}{80.5} [0.894 \quad -0.447 \quad 0 \quad -0.894 \quad 0.447 \quad 0] \begin{Bmatrix} -0.0711 \\ 0 \\ -0.2662 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma^{(1)} = -948 \text{ psi}$$

The stress in element 2 is:

$$\sigma^{(2)} = \frac{1.2 \times 10^6}{108} [0.667 \quad -0.333 \quad -0.667 \quad -0.667 \quad 0.333 \quad 0.667] \begin{Bmatrix} -0.0711 \\ 0 \\ -0.2662 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma^{(2)} = 1,445 \text{ psi}$$

Stiffness Matrix for a Bar Element

Example 8 – Space Truss Problem

The stress in element 3 is:

$$\sigma^{(3)} = \frac{1.2 \times 10^6}{86.5} [0.832 \quad 0 \quad 0.555 \quad -0.832 \quad 0 \quad -0.555] \begin{Bmatrix} -0.0711 \\ 0 \\ -0.2662 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma^{(3)} = 2,843 \text{ psi}$$

Stiffness Matrix for a Bar Element**Problems:**

2. Verify the global stiffness matrix for a three-dimensional bar. **Hint:** First, expand \mathbf{T}^* to a 6 x 6 square matrix, then expand \mathbf{k} to 6 x 6 square matrix by adding the appropriate rows and columns of zeros, and finally, perform the matrix triple product $\mathbf{k} = \mathbf{T}^T \mathbf{k}' \mathbf{T}$.
- 3a. Do problems **3.4**, **3.10**, **3.12**, **3.15a,b**, **3.18**, **3.23**, **3.37**, **3.43**, and **3.48** on pages 146 - 165 in your textbook "A First Course in the Finite Element Method" by D. Logan.

End of Chapter 3a