Chapter 3a – Development of Truss Equations

Learning Objectives

- To derive the stiffness matrix for a bar element.
- To illustrate how to solve a bar assemblage by the direct stiffness method.
- To introduce guidelines for selecting displacement functions.
- To describe the concept of transformation of vectors in two different coordinate systems in the plane.
- To derive the stiffness matrix for a bar arbitrarily oriented in the plane.
- To demonstrate how to compute stress for a bar in the plane.
- To show how to solve a plane truss problem.
- To develop the transformation matrix in three-dimensional space and show how to use it to derive the stiffness matrix for a bar arbitrarily oriented in space.
- To demonstrate the solution of space trusses.

Chapter 3a – Development of Truss Equations

Development of Truss Equations

Having set forth the foundation on which the direct stiffness method is based, we will now derive the stiffness matrix for a linear-elastic bar (or truss) element using the general steps outlined in Chapter 2.

We will include the introduction of both a local coordinate system, chosen with the element in mind, and a global or reference coordinate system, chosen to be convenient (for numerical purposes) with respect to the overall structure.

We will also discuss the transformation of a vector from the local coordinate system to the global coordinate system, using the concept of transformation matrices to express the stiffness matrix of an arbitrarily oriented bar element in terms of the global system.
Development of Truss Equations

Next we will describe how to handle inclined, or skewed, supports.

We will then extend the stiffness method to include space trusses.

We will develop the transformation matrix in three-dimensional space and analyze a space truss.

We will then use the principle of minimum potential energy and apply it to the bar element equations.

Finally, we will apply Galerkin's residual method to derive the bar element equations.
Development of Truss Equations

Development of Truss Equations
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Development of Truss Equations

Consider the derivation of the stiffness matrix for the linear-elastic, constant cross-sectional area (prismatic) bar element shown below.

This application is directly applicable to the solution of pin-connected truss problems.

**Stiffness Matrix for a Bar Element**
Stiffness Matrix for a Bar Element

Consider the derivation of the stiffness matrix for the linear-elastic, constant cross-sectional area (prismatic) bar element shown below.

\[ T \]

\[ x, u \]

\[ 1 \]

\[ L \]

\[ 2 \]

\[ T \]

\[ u_1, f_{1x} \]

\[ u_2, f_{2x} \]

where \( T \) is the tensile force directed along the axis at nodes 1 and 2, \( x \) is the local coordinate system directed along the length of the bar.

The bar element has a constant cross-section \( A \), an initial length \( L \), and modulus of elasticity \( E \).

The nodal degrees of freedom are the local axial displacements \( u_1 \) and \( u_2 \) at the ends of the bar.
**Stiffness Matrix for a Bar Element**

The strain-displacement relationship is: \( \sigma = E \varepsilon \quad \varepsilon = \frac{du}{dx} \)

From equilibrium of forces, assuming no distributed loads acting on the bar, we get:

\[ A \sigma_x = T = \text{constant} \]

Combining the above equations gives:

\[ A E \frac{du}{dx} = T = \text{constant} \]

Taking the derivative of the above equation with respect to the local coordinate \( x \) gives:

\[ \frac{d}{dx} \left( A E \frac{du}{dx} \right) = 0 \]

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**Stiffness Matrix for a Bar Element**

The following assumptions are considered in deriving the bar element stiffness matrix:

1. The bar cannot sustain shear force: \( f_{1y} = f_{2y} = 0 \)
2. Any effect of transverse displacement is ignored.
3. Hooke’s law applies; stress is related to strain: \( \sigma_x = E \varepsilon_x \)
**Stiffness Matrix for a Bar Element**

**Step 1 - Select Element Type**

We will consider the linear bar element shown below.

![Linear Bar Element Diagram]

**Stiffness Matrix for a Bar Element**

**Step 2 - Select a Displacement Function**

A linear displacement function \( u \) is assumed: \( u = a_1 + a_2 x \)

The number of coefficients in the displacement function, \( a_i \), is equal to the total number of degrees of freedom associated with the element.

Applying the boundary conditions and solving for the unknown coefficients gives:

\[
\begin{align*}
    u &= \left( \frac{u_2 - u_1}{L} \right) x + u_1 \\
    u &= \left[ \begin{array}{c} 1 - \frac{x}{L} \\ \frac{x}{L} \end{array} \right] \{ u_1 \}
\end{align*}
\]
**Stiffness Matrix for a Bar Element**

**Step 2 - Select a Displacement Function**

Or in another form: \[ u = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \]

where \( N_1 \) and \( N_2 \) are the interpolation functions gives as:

\[ N_1 = 1 - \frac{x}{L} \quad N_2 = \frac{x}{L} \]

The linear displacement function \( u \) plotted over the length of the bar element is shown here.

**Stiffness Matrix for a Bar Element**

**Step 3 - Define the Strain/Displacement and Stress/Strain Relationships**

The stress-displacement relationship is:

\[ \varepsilon_x = \frac{du}{dx} = \frac{u_2 - u_1}{L} \]

**Step 4 - Derive the Element Stiffness Matrix and Equations**

We can now derive the element stiffness matrix as follows:

\[ T = A\sigma_x \]

Substituting the stress-displacement relationship into the above equation gives:

\[ T = AE \left( \frac{u_2 - u_1}{L} \right) \]
**Stiffness Matrix for a Bar Element**

**Step 4 - Derive the Element Stiffness Matrix and Equations**

The nodal force sign convention, defined in element figure, is:

\[ f_{1x} = -T \quad f_{2x} = T \]

therefore,

\[ f_{1x} = AE \left( \frac{u_{1} - u_{2}}{L} \right) \quad f_{2x} = AE \left( \frac{u_{2} - u_{1}}{L} \right) \]

Writing the above equations in matrix form gives:

\[
\begin{bmatrix}
  f_{1x} \\
  f_{2x}
\end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_{1} \\
  u_{2}\end{bmatrix}
\]

Notice that \( \frac{AE}{L} \) for a bar element is analogous to the spring constant \( k \) for a spring element.

**Stiffness Matrix for a Bar Element**

**Step 5 - Assemble the Element Equations and Introduce Boundary Conditions**

The *global stiffness matrix* and the *global force vector* are assembled using the nodal force equilibrium equations, and force/deformation and compatibility equations.

\[
K = \sum_{e=1}^{n} k^{(e)} \\
F = \sum_{e=1}^{n} f^{(e)}
\]

Where \( k \) and \( f \) are the element stiffness and force matrices expressed in global coordinates.
**Stiffness Matrix for a Bar Element**

**Step 6 - Solve for the Nodal Displacements**

Solve the displacements by imposing the boundary conditions and solving the following set of equations:

\[ \mathbf{F} = \mathbf{Ku} \]

**Step 7 - Solve for the Element Forces**

Once the displacements are found, the stress and strain in each element may be calculated from:

\[ \varepsilon_x = \frac{d}{dx} = \frac{u_2 - u_1}{L} \quad \sigma_x = E\varepsilon_x \]

---

**Stiffness Matrix for a Bar Element**

**Example 1 - Bar Problem**

Consider the following three-bar system shown below. Assume for elements 1 and 2: \( A = 1 \text{ in}^2 \) and \( E = 30 \times 10^6 \text{ psi} \) and for element 3: \( A = 2 \text{ in}^2 \) and \( E = 15 \times 10^6 \text{ psi} \).

Determine: (a) the global stiffness matrix, (b) the displacement of nodes 2 and 3, and (c) the reactions at nodes 1 and 4.
**Stiffness Matrix for a Bar Element**

**Example 1 - Bar Problem**

For elements 1 and 2:

\[
k^{(1)} = k^{(2)} = \frac{(1)(30 \times 10^6)}{30} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\]

For element 3:

\[
k^{(3)} = \frac{(2)(15 \times 10^6)}{30} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\]

As before, the numbers above the matrices indicate the displacements associated with the matrix.

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**Stiffness Matrix for a Bar Element**

**Example 1 - Bar Problem**

Assembling the global stiffness matrix by the direct stiffness methods gives:

\[
K = 10^6 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}
\]

Relating global nodal forces related to global nodal displacements gives:

\[
\begin{bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{bmatrix} = 10^6 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}
\]
**Stiffness Matrix for a Bar Element**

**Example 1 – Bar Problem**

The boundary conditions are: \( u_1 = u_4 = 0 \)

\[
\begin{bmatrix}
  F_{1x} \\
  F_{2x} \\
  F_{3x} \\
  F_{4x}
\end{bmatrix} = 10^6 \begin{bmatrix}
  1 & -1 & 0 & 0 \\
  -1 & 2 & -1 & 0 \\
  0 & -1 & 2 & -1 \\
  0 & 0 & -1 & 1
\end{bmatrix} \begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3 \\
  u_4
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix}
\]

Applying the boundary conditions and the known forces \((F_{2x} = 3,000 \text{ lb})\) gives:

\[
\begin{bmatrix}
  3,000 \\
  0
\end{bmatrix} = 10^6 \begin{bmatrix}
  2 & -1 \\
  -1 & 2
\end{bmatrix} \begin{bmatrix}
  u_2 \\
  u_3
\end{bmatrix}
\]

**Stiffness Matrix for a Bar Element**

**Example 1 – Bar Problem**

Solving for \( u_2 \) and \( u_3 \) gives: \( u_2 = 0.002 \text{ in} \)

\( u_3 = 0.001 \text{ in} \)

The global nodal forces are calculated as:

\[
\begin{bmatrix}
  F_{1x} \\
  F_{2x} \\
  F_{3x} \\
  F_{4x}
\end{bmatrix} = 10^6 \begin{bmatrix}
  1 & -1 & 0 & 0 \\
  -1 & 2 & -1 & 0 \\
  0 & -1 & 2 & -1 \\
  0 & 0 & -1 & 1
\end{bmatrix} \begin{bmatrix}
  0 \\
  0.002 \\
  0.001 \\
  0
\end{bmatrix} = \begin{bmatrix}
  -2,000 \\
  3,000 \\
  0 \\
  -1,000
\end{bmatrix} \text{ lb}
\]
Stiffness Matrix for a Bar Element

Selecting Approximation Functions for Displacements

Consider the following guidelines, as they relate to the one-dimensional bar element, when selecting a displacement function.

1. Common approximation functions are usually polynomials.
2. The approximation function should be continuous within the bar element.

\[ u^{(1)} = \frac{u_2 - u_1}{L}(x + u_1) \quad u^{(2)} = \frac{u_3 - u_2}{L}(x + u_2) \]

Stiffness Matrix for a Bar Element

Selecting Approximation Functions for Displacements

Consider the following guidelines, as they relate to the one-dimensional bar element, when selecting a displacement function.

3. The approximating function should provide interelement continuity for all degrees of freedom at each node for discrete line elements, and along common boundary lines and surfaces for two- and three-dimensional elements.

\[ u^{(1)} = \frac{u_2 - u_1}{L}(x + u_1) \quad u^{(2)} = \frac{u_3 - u_2}{L}(x + u_2) \]
**Stiffness Matrix for a Bar Element**

**Selecting Approximation Functions for Displacements**

Consider the following guidelines, as they relate to the one-dimensional bar element, when selecting a displacement function.

For the bar element, we must ensure that nodes common to two or more elements remain common to these elements upon deformation and thus prevent overlaps or voids between elements.

![Diagram of bar elements with nodes 1, 2, and 3, and lengths marked as L.]

The linear function is then called a **conforming** (or **compatible**) function for the bar element because it ensures both the satisfaction of continuity between adjacent elements and of continuity within the element.

**Stiffness Matrix for a Bar Element**

**Selecting Approximation Functions for Displacements**

Consider the following guidelines, as they relate to the one-dimensional bar element, when selecting a displacement function.

4. The approximation function should allow for rigid-body displacement and for a state of constant strain within the element.

Completeness of a function is necessary for convergence to the exact answer, for instance, for displacements and stresses.
**Stiffness Matrix for a Bar Element**

**Selecting Approximation Functions for Displacements**

The interpolation function must allow for a rigid-body displacement, that means the function must be capable of yielding a constant value.

Consider the follow situation: \[ u = a_i \quad a_i = u_1 = u_2 \]

Therefore: \[ u = N_1 a_i + N_2 u_2 = (N_1 + N_2) a_i \]

Since \( u = a_1 \) then: \[ u = a_i = (N_1 + N_2) a_i \]

This means that: \( N_1 + N_2 = 1 \)

The displacement interpolation function must add to unity at every point within the element so the it will yield a constant value when a rigid-body displacement occurs.

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**Stiffness Matrix for a Bar Element**

**Transformation of Vectors in Two Dimensions**

In many problems it is convenient to introduce both **local** and **global** (or reference) coordinates.

Local coordinates are always chosen to conveniently represent the individual element.

Global coordinates are chosen to be convenient for the whole structure.
Stiffness Matrix for a Bar Element

Transformation of Vectors in Two Dimensions

Given the nodal displacement of an element, represented by the vector $\mathbf{d}$ in the figure below, we want to relate the components of this vector in one coordinate system to components in another.

Let's consider that $\mathbf{d}$ does not coincide with either the local or global axes. In this case, we want to relate global displacement components to local ones. In so doing, we will develop a transformation matrix that will subsequently be used to develop the global stiffness matrix for a bar element.
**Stiffness Matrix for a Bar Element**

**Transformation of Vectors in Two Dimensions**

We define the angle $\theta$ to be positive when measured counterclockwise from $x$ to $x'$. We can express vector displacement $\mathbf{d}$ in both global and local coordinates by:

$$
\mathbf{d} = u_i \mathbf{i} + v_j = u'_i \mathbf{i}' + v'_j \mathbf{j}'
$$

**Stiffness Matrix for a Bar Element**

**Transformation of Vectors in Two Dimensions**

Consider the following diagram:

Using vector addition: $\mathbf{a} + \mathbf{b} = \mathbf{i}$

Using the law of cosines, we get: $|\mathbf{a}| = |\mathbf{i}| \cos \theta$  \hspace{1cm} $|\mathbf{a}| = \cos \theta$

Similarly: $|\mathbf{b}| = |\mathbf{i}| \sin \theta$  \hspace{1cm} $|\mathbf{b}| = \sin \theta$
**Stiffness Matrix for a Bar Element**

Transformation of Vectors in Two Dimensions

Consider the following diagram:

The vector $\mathbf{a}$ is in the $i'$ direction and $\mathbf{b}$ is in the $j'$ direction, therefore:

$$
\mathbf{a} = |\mathbf{a}| i' = (\cos \theta) i' \quad \quad \mathbf{b} = |\mathbf{b}| (-j') = (\sin \theta)(-j')
$$

---

**Stiffness Matrix for a Bar Element**

Transformation of Vectors in Two Dimensions

Consider the following diagram:

The vector $\mathbf{i}$ can be rewritten as: $\mathbf{i} = \cos \theta i' - \sin \theta j'$

The vector $\mathbf{j}$ can be rewritten as: $\mathbf{j} = \sin \theta i' + \cos \theta j'$

Therefore, the displacement vector is:

$$
u_i + v_j = u_i (\cos \theta i' - \sin \theta j') + v_i (\sin \theta i' + \cos \theta j') = u'_{ij} + v'_{ij}
$$
Stiffness Matrix for a Bar Element

Transformation of Vectors in Two Dimensions

Consider the following diagram:

Combining like coefficients of the local unit vectors gives:

\[
\begin{align*}
    u_i \cos \theta + v_i \sin \theta &= u'_i \\
    -u_i \sin \theta + v_i \cos \theta &= v'_i
\end{align*}
\]

\[
\begin{pmatrix}
    u'_i \\
    v'_i
\end{pmatrix} =
\begin{bmatrix}
    C & S \\
    -S & C
\end{bmatrix}
\begin{pmatrix}
    u_i \\
    v_i
\end{pmatrix}
\]

The matrix is called the transformation matrix.

The figure below shows \( u' \) expressed in terms of the global coordinates \( x \) and \( y \).

\[ u' = Cu + Sv \]
**Stiffness Matrix for a Bar Element**

Example 2 - Bar Element Problem

The global nodal displacement at node 2 is $u_2 = 0.1 \text{ in}$ and $v_2 = 0.2 \text{ in}$ for the bar element shown below. Determine the local displacement.

Using the following expression we just derived, we get:

$$u' = Cu + Sv$$

$$u'_2 = \cos 60^\circ (0.1) + \sin 60^\circ (0.2) = 0.223 \text{ in}$$

**Stiffness Matrix for a Bar Element**

Global Stiffness Matrix

We will now use the transformation relationship developed above to obtain the global stiffness matrix for a bar element.
**Stiffness Matrix for a Bar Element**

**Global Stiffness Matrix**

We know that for a bar element in local coordinates we have:

\[
\begin{pmatrix}
    f'_{1x} \\
    f'_{2x}
\end{pmatrix} = \frac{AE}{L} \begin{pmatrix}
    1 & -1 \\
    -1 & 1
\end{pmatrix} \begin{pmatrix}
    u'_1 \\
    u'_2
\end{pmatrix}
\]

\[ f' = k'd' \]

We want to relate the global element forces \( f \) to the global displacements \( d \) for a bar element with an arbitrary orientation.

\[
\begin{pmatrix}
    f_{1x} \\
    f_{1y} \\
    f_{2x} \\
    f_{2y}
\end{pmatrix} = k \begin{pmatrix}
    u_1 \\
    v_1 \\
    u_2 \\
    v_2
\end{pmatrix}
\]

\[ f = kd \]

**Stiffness Matrix for a Bar Element**

**Global Stiffness Matrix**

Using the relationship between local and global components, we can develop the global stiffness matrix.

We already know the transformation relationships:

\[ u'_1 = u_1 \cos \theta + v_1 \sin \theta \quad \quad u'_2 = u_2 \cos \theta + v_2 \sin \theta \]

Combining both expressions for the two local degrees-of-freedom, in matrix form, we get:

\[
\begin{pmatrix}
    u'_1 \\
    u'_2
\end{pmatrix} = \begin{pmatrix}
    C & S & 0 & 0 \\
    0 & 0 & C & S
\end{pmatrix} \begin{pmatrix}
    u_1 \\
    v_1 \\
    u_2 \\
    v_2
\end{pmatrix}
\]

\[ d' = T'd \]

\[
T' = \begin{pmatrix}
    C & S & 0 & 0 \\
    0 & 0 & C & S
\end{pmatrix}
\]
**Stiffness Matrix for a Bar Element**

**Global Stiffness Matrix**

A similar expression for the force transformation can be developed.

\[
\begin{bmatrix}
  f'_{1x} \\
  f'_{2x}
\end{bmatrix} =
\begin{bmatrix}
  C & S & 0 & 0 \\
  0 & 0 & C & S
\end{bmatrix}
\begin{bmatrix}
  f_{1x} \\
  f_{1y} \\
  f_{2x} \\
  f_{2y}
\end{bmatrix}
\]

\[f' = T'f\]

Substituting the global force expression into element force equation gives:

\[f' = k'd' \implies T'f = k'd'\]

Substituting the transformation between local and global displacements gives:

\[d' = T'd \implies T'f = k'T'd\]

---

**Stiffness Matrix for a Bar Element**

**Global Stiffness Matrix**

The matrix \(T^*\) is not a square matrix so we cannot invert it. Let’s expand the relationship between local and global displacement.

\[
\begin{bmatrix}
  u'_1 \\
  v'_1 \\
  u'_2 \\
  v'_2
\end{bmatrix} =
\begin{bmatrix}
  C & S & 0 & 0 \\
  -S & C & 0 & 0 \\
  0 & 0 & C & S \\
  0 & 0 & -S & C
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  v_1 \\
  u_2 \\
  v_2
\end{bmatrix}
\]

\[d' = Td\]

where \(T\) is:

\[
T =
\begin{bmatrix}
  C & S & 0 & 0 \\
  -S & C & 0 & 0 \\
  0 & 0 & C & S \\
  0 & 0 & -S & C
\end{bmatrix}
\]
**Stiffness Matrix for a Bar Element**

Global Stiffness Matrix

We can write a similar expression for the relationship between local and global forces.

\[
\begin{bmatrix}
  f'_{1x} \\
  f'_{1y} \\
  f'_{2x} \\
  f'_{2y}
\end{bmatrix} =
\begin{bmatrix}
  C & S & 0 & 0 \\
  -S & C & 0 & 0 \\
  0 & 0 & C & S \\
  0 & 0 & -S & C
\end{bmatrix}
\begin{bmatrix}
  f_{1x} \\
  f_{1y} \\
  f_{2x} \\
  f_{2y}
\end{bmatrix}
\]

\[f' = T f\]

Therefore our original local coordinate force-displacement expression

\[
\begin{bmatrix}
  f'_{1x} \\
  f'_{2x}
\end{bmatrix} = \frac{AE}{L} \begin{bmatrix}
  1 & -1 \\
  -1 & 1
\end{bmatrix}
\begin{bmatrix}
  u'_{1} \\
  u'_{2}
\end{bmatrix}
\]

\[f' = k'd'\]

**Stiffness Matrix for a Bar Element**

Global Stiffness Matrix

May be expanded:

\[
\begin{bmatrix}
  f'_{1x} \\
  f'_{1y} \\
  f'_{2x} \\
  f'_{2y}
\end{bmatrix} = \frac{AE}{L} \begin{bmatrix}
  1 & 0 & -1 & 0 \\
  0 & 0 & 0 & 0 \\
  -1 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  u'_{1} \\
  v'_{1} \\
  u'_{2} \\
  v'_{2}
\end{bmatrix}
\]

The global force-displacement equations are:

\[f' = k'd' \Rightarrow Tf = k'Td\]

Multiply both side by \(T^{-1}\) we get: \(f = T^T k'Td\)

where \(T^{-1}\) is the inverse of \(T\). It can be shown that: \(T^{-1} = T^T\)
**Stiffness Matrix for a Bar Element**

**Global Stiffness Matrix**

The global force-displacement equations become: \( \mathbf{f} = \mathbf{T}^T \mathbf{k} \mathbf{T} \mathbf{d} \)

Where the global stiffness matrix \( \mathbf{k} \) is: \( \mathbf{k} = \mathbf{T}^T \mathbf{k} \mathbf{T} \)

Expanding the above transformation gives:

\[
\mathbf{k} = \frac{AE}{L} \begin{bmatrix}
  C^2 & CS & -C^2 & -CS \\
  CS & S^2 & -CS & -S^2 \\
  -C^2 & -CS & C^2 & CS \\
  -CS & -S^2 & CS & S^2 \\
\end{bmatrix}
\]

We can assemble the total stiffness matrix by using the above element stiffness matrix and the direct stiffness method.

\[
\mathbf{K} = \left[ \mathbf{K} \right] = \sum_{e=1}^{n} \mathbf{k}^{(e)} \\
\mathbf{F} = \left\{ \mathbf{F} \right\} = \sum_{e=1}^{n} \mathbf{f}^{(e)} \\
\mathbf{F} = \mathbf{Kd}
\]

**Local forces can be computed as:**

\[
\begin{align*}
\begin{bmatrix}
  f_{x_1}' \\
  f_{y_1}' \\
  f_{x_2}' \\
  f_{y_2}'
\end{bmatrix}
&= \frac{AE}{L} \begin{bmatrix}
  1 & 0 & -1 & 0 \\
  0 & 0 & 0 & 0 \\
  -1 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  u_1' \\
  v_1' \\
  u_2' \\
  v_2'
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
  f_{x_1}'' \\
  f_{y_1}'' \\
  f_{x_2}'' \\
  f_{y_2}''
\end{bmatrix}
&= \frac{AE}{L} \begin{bmatrix}
  C & S & 0 & 0 \\
  -S & C & 0 & 0 \\
  0 & 0 & C & S \\
  0 & 0 & -S & C
\end{bmatrix}
\begin{bmatrix}
  u_1'' \\
  v_1'' \\
  u_2'' \\
  v_2''
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
  f_{x_1}'' \\
  f_{y_1}'' \\
  f_{x_2}'' \\
  f_{y_2}''
\end{bmatrix}
&= \begin{bmatrix}
  Cu_1 + Sv_1 & -Cu_2 - Sv_2 \\
  0 \\
  -Cu_1 - Sv_1 + Cu_2 + Sv_2 \\
  0
\end{bmatrix}
\end{align*}
\]
### Stiffness Matrix for a Bar Element

**Example 3 - Bar Element Problem**

For the bar element shown below, evaluate the global stiffness matrix. Assume the cross-sectional area is 2 in\(^2\), the length is 60 in, and the \(E\) is 30 \(x\) 10\(^6\) psi.

Therefore:

\[ C = \cos 30^\circ = \frac{\sqrt{3}}{2} \]
\[ S = \sin 30^\circ = \frac{1}{2} \]

\[ k = \frac{AE}{L} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix} \]

Simplifying the global elemental stiffness matrix is:

The global elemental stiffness matrix is:

\[ k = \frac{(2 \text{ in}^2)(30 \times 10^6 \text{ psi})}{60 \text{ in}} \begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \]

Simplifying the global elemental stiffness matrix is:

\[ k = 10^6 \begin{bmatrix} 0.750 & 0.433 & -0.750 & -0.433 \\ 0.433 & 0.250 & -0.433 & -0.250 \\ -0.750 & -0.433 & 0.750 & 0.433 \\ -0.433 & -0.250 & 0.433 & 0.250 \end{bmatrix} \]
**Stiffness Matrix for a Bar Element**

Computation of Stress for a Bar in the x-y Plane

For a bar element the local forces are related to the local displacements by:

\[
\begin{bmatrix}
  f_{1x}' \\
  f_{2x}'
\end{bmatrix} = \begin{bmatrix}
  AE & 1 & -1 \\
  L & -1 & 1
\end{bmatrix}
\begin{bmatrix}
  u'_1 \\
  u'_2
\end{bmatrix}
\]

The force-displacement equation for \( f'_{2x} \) is:

\[
f'_{2x} = \frac{AE}{L} [-1 \quad 1] \begin{bmatrix}
  u'_1 \\
  u'_2
\end{bmatrix}
\]

The stress in terms of global displacement is:

\[
\sigma = \frac{E}{L} \begin{bmatrix}
  C & S & 0 & 0 \\
  0 & C & S & 0
\end{bmatrix} \begin{bmatrix}
  u_1 \\
  v_1 \\
  u_2 \\
  v_2
\end{bmatrix} = \frac{E}{L} [-Cu_1 - Sv_1 + Cu_2 + Sv_2]
\]

**Example 4 - Bar Element Problem**

For the bar element shown below, determine the axial stress. Assume the cross-sectional area is \( 4 \times 10^{-4} \, m^2 \), the length is \( 2 \, m \), and the \( E \) is 210 GPa.

The global displacements are known as \( u_1 = 0.25 \, mm \), \( v_1 = 0 \), \( u_2 = 0.5 \, mm \), and \( v_2 = 0.75 \, mm \).

\[
\sigma = \frac{E}{L} [-Cu_1 - Sv_1 + Cu_2 + Sv_2]
\]

\[
\sigma = \frac{210 \times 10^6}{2} \left[ -\frac{1}{2} (0.25) - \frac{\sqrt{3}}{4} (0) + \frac{1}{2} (0.5) + \frac{\sqrt{3}}{4} (0.75) \right] \left[ \text{KN/m} \right]
\]

\[
\sigma = 81.32 \times 10^3 \, \text{KN/m}^2 = 81.32 \, \text{MPa}
\]
**Stiffness Matrix for a Bar Element**

**Solution of a Plane Truss**

We will now illustrate the use of equations developed above along with the direct stiffness method to solve the following plane truss example problems.

A plane truss is a structure composed of bar elements all lying in a common plane that are connected together by frictionless pins.

The plane truss also must have loads acting only in the common plane.

**Stiffness Matrix for a Bar Element**

**Example 5 - Plane Truss Problem**

The plane truss shown below is composed of three bars subjected to a downward force of 10 kips at node 1. Assume the cross-sectional area $A = 2 \text{ in}^2$ and $E$ is $30 \times 10^6 \text{ psi}$ for all elements.

Determine the $x$ and $y$ displacement at node 1 and stresses in each element.
## Stiffness Matrix for a Bar Element

**Example 5 - Plane Truss Problem**

<table>
<thead>
<tr>
<th>Element</th>
<th>Node 1</th>
<th>Node 2</th>
<th>$\theta$ (deg)</th>
<th>$C$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>90</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>45</td>
<td>0.707</td>
<td>0.707</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

The global elemental stiffness matrix are:

\[
\begin{bmatrix}
C & CS & -C & -CS \\
CS & S^2 & -CS & -S^2 \\
-C & -CS & C^2 & CS \\
-CS & -S^2 & CS & S^2
\end{bmatrix}
\]

\[
\begin{align*}
&\text{element 1: } C = 0, S = 1 \Rightarrow k^{(1)} = \frac{(2 \text{in}^2)(30 \times 10^6 \text{ psi})}{120 \text{in}} \\
&\text{element 2: } C = \frac{5}{4}, S = \frac{5}{4} \Rightarrow k^{(2)} = \frac{(2 \text{in}^2)(30 \times 10^6 \text{ psi})}{240 \sqrt{2} \text{ in}} \\
&\text{element 3: } C = 1, S = 0 \Rightarrow k^{(3)} = \frac{(2 \text{in}^2)(30 \times 10^6 \text{ psi})}{120 \text{in}}
\end{align*}
\]
Stiffness Matrix for a Bar Element

Example 5 - Plane Truss Problem

The total global stiffness matrix is:

\[
K = 5 \times 10^5
\]

\[
\begin{bmatrix}
1.354 & 0.354 & 0 & 0 & -0.354 & -0.354 & -1 & 0 \\
0.354 & 1.354 & 0 & -1 & -0.354 & -0.354 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\
-0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The total global force-displacement equations are:

\[
\begin{bmatrix}
0 \\
-10,000 \\
F_{2x} \\
F_{2y} \\
F_{3x} \\
F_{3y} \\
F_{4x} \\
F_{4y}
\end{bmatrix}
- 5 \times 10^6
\left[
\begin{array}{ccccc}
1.354 & 0.354 & 0 & 0 & -0.354 & -0.354 & -1 & 0 \\
0.354 & 1.354 & 0 & -1 & -0.354 & -0.354 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\
-0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\right]
\begin{bmatrix}
u_1 \\
v_1
\end{bmatrix}
\]

\[
F_x \quad F_y
\]

Stiffness Matrix for a Bar Element

Example 5 - Plane Truss Problem

Applying the boundary conditions for the truss, the above equations reduce to:

\[
\begin{bmatrix}
0 \\
-10,000 \\
F_{2x} \\
F_{2y} \\
F_{3x} \\
F_{3y} \\
F_{4x} \\
F_{4y}
\end{bmatrix}
- 5 \times 10^6
\left[
\begin{array}{ccccc}
1.354 & 0.354 & 0 & 0 & -0.354 & -0.354 & -1 & 0 \\
0.354 & 1.354 & 0 & -1 & -0.354 & -0.354 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\
-0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\right]
\begin{bmatrix}
u_1 \\
v_1
\end{bmatrix}
\]
**Stiffness Matrix for a Bar Element**

**Example 5 - Plane Truss Problem**

Applying the boundary conditions for the truss, the above equations reduce to:

\[
\begin{bmatrix}
0 & 1.354 & 0.354 \\
-10,000 & 0.354 & 1.354
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
= 5 \times 10^6
\begin{bmatrix}
10,000 \\
10,000
\end{bmatrix}
\]

Solving the equations gives:

\[u_1 = 0.414 \times 10^{-2} \text{in} \]
\[v_1 = -1.59 \times 10^{-2} \text{in} \]

The stress in an element is:

\[\sigma = \frac{E}{L} \left[ -C_u \theta + S_v \theta + C_u + S_v \theta \right] \]

where \(\theta\) is the local node number

---

**Stiffness Matrix for a Bar Element**

**Example 5 - Plane Truss Problem**

<table>
<thead>
<tr>
<th>Element</th>
<th>Node 1</th>
<th>Node 2</th>
<th>(\theta)</th>
<th>C</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>90°</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>45°</td>
<td>0.707</td>
<td>0.707</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>4</td>
<td>0°</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

\[\sigma = \frac{E}{L} \left[ -C_u \theta + S_v \theta + C_u + S_v \theta \right] \]

**Element 1**

\[\sigma^{(1)} = \frac{30 \times 10^6}{120} [-v_1] = 3,965 \text{ psi} \]

**Element 2**

\[\sigma^{(2)} = \frac{-30 \times 10^6}{120} [(0.707)u_1 + (0.707)v_1] = 1,471 \text{ psi} \]
**Stiffness Matrix for a Bar Element**

Example 5 - Plane Truss Problem

<table>
<thead>
<tr>
<th>Element</th>
<th>Node 1</th>
<th>Node 2</th>
<th>$\theta$</th>
<th>C</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>90$^\circ$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>45$^\circ$</td>
<td>0.707</td>
<td>0.707</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>4</td>
<td>0$^\circ$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

$$
\sigma = \frac{E}{L} \left[ -Cu_{12} - Sv_{12} + Cu_{21} + Sv_{21} \right]
$$

**element 3**

$$
\sigma^{(3)} = \frac{30 \times 10^6}{120} [-u_1] = -1,035 \text{ psi}
$$

---

**Stiffness Matrix for a Bar Element**

Example 5 - Plane Truss Problem

Let's check equilibrium at node 1:

$$
\sum F_x = f_{x1}^{(2)} \cos(45^\circ) + f_{x1}^{(3)}
$$

$$
\sum F_y = f_{y1}^{(2)} \sin(45^\circ) + f_{y1}^{(1)} - 10,000 \text{ lb}
$$
Stiffness Matrix for a Bar Element

Example 5 - Plane Truss Problem

Let's check equilibrium at node 1:

\[ \sum F_x = (1,471 \text{ psi})(2 \text{ in}^2)(0.707) - (1,035 \text{ psi})(2 \text{ in}^2) = 0 \]

\[ \sum F_y = (3,965 \text{ psi})(2 \text{ in}^2) + (1,471 \text{ psi})(2 \text{ in}^2)(0.707) - 10,000 = 0 \]

Stiffness Matrix for a Bar Element

Example 6 - Plane Truss Problem

Develop the element stiffness matrices and system equations for the plane truss below.

Assume the stiffness of each element is constant. Use the numbering scheme indicated. Solve the equations for the displacements and compute the member forces. All elements have a constant value of \( AE/L \).
Stiffness Matrix for a Bar Element

Example 6 - Plane Truss Problem

Develop the element stiffness matrices and system equations for the plane truss below.

Compute the elemental stiffness matrix for each element. The general form of the matrix is:

\[
\begin{bmatrix}
  C^2 & CS & -C^2 & -CS \\
  CS & S^2 & -CS & -S^2 \\
  -C^2 & -CS & C^2 & CS \\
  -CS & -S^2 & CS & S^2 \\
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>Member</th>
<th>Node 1</th>
<th>Node 2</th>
<th>Elemental Stiffness</th>
<th>(\theta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>(k)</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>(k)</td>
<td>(3\pi/4)</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
<td>(k)</td>
<td>(\pi/2)</td>
</tr>
</tbody>
</table>
**Stiffness Matrix for a Bar Element**

**Example 6 - Plane Truss Problem**

For element 1:

$$k^{(1)} = k \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

<table>
<thead>
<tr>
<th>Member</th>
<th>Node 1</th>
<th>Node 2</th>
<th>Elemental Stiffness</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>$k$</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>$k$</td>
<td>$3\pi/4$</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
<td>$k$</td>
<td>$\pi/2$</td>
</tr>
</tbody>
</table>

For element 2:

$$k^{(2)} = \frac{k}{2} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

<table>
<thead>
<tr>
<th>Member</th>
<th>Node 1</th>
<th>Node 2</th>
<th>Elemental Stiffness</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>$k$</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>$k$</td>
<td>$3\pi/4$</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
<td>$k$</td>
<td>$\pi/2$</td>
</tr>
</tbody>
</table>
**Stiffness Matrix for a Bar Element**

**Example 6 - Plane Truss Problem**

For element 3:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
u_1 \\
v_1 \\
u_3 \\
v_3 \\
\end{bmatrix}
= k
\]

<table>
<thead>
<tr>
<th>Member</th>
<th>Node 1</th>
<th>Node 2</th>
<th>Elemental Stiffness</th>
<th>( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>( k )</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>( k )</td>
<td>( 3\pi/4 )</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
<td>( k )</td>
<td>( \pi/2 )</td>
</tr>
</tbody>
</table>

\[
K = \frac{k}{2}
\]

Assemble the global stiffness matrix by superimposing the elemental global matrices.
### Stiffness Matrix for a Bar Element

**Example 6 - Plane Truss Problem**

The unconstrained (no boundary conditions satisfied) equations are:

\[
\begin{bmatrix}
2 & 0 & -2 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & -2 \\
k & -2 & 0 & 3 & -1 & 1 \\
0 & 0 & -1 & 1 & 1 & 1 \\
0 & 0 & -1 & 1 & 1 & 1 \\
0 & -2 & 1 & -1 & -1 & 3
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_6
\end{bmatrix}
= 
\begin{bmatrix}
F_{u_1} \\
F_{u_2} \\
F_{u_3} \\
F_{u_4} \\
P_1 \\
P_2
\end{bmatrix}
\]

The displacement at nodes 1 and 3 are zero in both directions. Applying these conditions to the system equations gives:

\[
\begin{bmatrix}
2 & 0 & -2 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & -2 \\
k & -2 & 0 & 3 & -1 & 1 \\
0 & 0 & -1 & 1 & 1 & 1 \\
0 & 0 & -1 & 1 & 1 & 1 \\
0 & -2 & 1 & -1 & -1 & 3
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_6
\end{bmatrix}
= 
\begin{bmatrix}
F_{u_1} \\
F_{u_2} \\
F_{u_3} \\
F_{u_4} \\
P_1 \\
P_2
\end{bmatrix}
\]

### Stiffness Matrix for a Bar Element

**Example 6 - Plane Truss Problem**

Applying the boundary conditions to the system equations gives:

\[
\begin{bmatrix}
3 & -1 \\
-2 & 0 & 3 & -1 & 1 \\
0 & 0 & -1 & 1 & 1 \\
0 & 0 & -1 & 1 & 1 \\
0 & -2 & 1 & -1 & -1 & 3
\end{bmatrix}
\begin{bmatrix}
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_6
\end{bmatrix}
= 
\begin{bmatrix}
P_1 \\
P_2 \\
P_3 \\
P_4
\end{bmatrix}
\]

Solving this set of equations is fairly easy. The solution is:

\[
u_2 = \frac{P_1 - P_2}{k}, \quad v_2 = \frac{P_1 - 3P_2}{k}
\]
**Stiffness Matrix for a Bar Element**

**Example 6 - Plane Truss Problem**

Using the force-displacement relationship the force in each member may be computed.

Member (element) 1:

\[
\begin{bmatrix}
    f_{x1} \\
    f_{y1} \\
    f_{x2} \\
    f_{y2}
\end{bmatrix} = k
\begin{bmatrix}
    Cu_1 + Sv_1 - Cu_2 - Sv_2 \\
    0 \\
    -Cu_1 - Sv_1 + Cu_2 + Sv_2 \\
    0
\end{bmatrix}
\]

- \( f_{x1} = k(-Cu_2) = k\left(-\frac{P_1 - P_2}{k}\right) = -(P_1 - P_2) \quad f_{y1} = 0 \)

- \( f_{x2} = k(Cu_2) = k\left(\frac{P_1 - P_2}{k}\right) = P_1 - P_2 \quad f_{y2} = 0 \)

---

**Stiffness Matrix for a Bar Element**

**Example 6 - Plane Truss Problem**

Using the force-displacement relationship the force in each member may be computed.

Member (element) 2:

\[
\begin{bmatrix}
    f_{x2} \\
    f_{y2} \\
    f_{x3} \\
    f_{y3}
\end{bmatrix} = k
\begin{bmatrix}
    Cu_2 + Sv_2 - Cu_3 - Sv_3 \\
    0 \\
    -Cu_2 - Sv_2 + Cu_3 + Sv_3 \\
    0
\end{bmatrix}
\]

- \( f_{x2} = k(Cu_2 + Sv_2) \)

\[
= k\left(\frac{P_1 - P_2}{k}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{P_1 - 3P_2}{k}\right)\left(\frac{1}{\sqrt{2}}\right) = -\sqrt{2}P_2
\]

- \( f_{x3} = k(-Cu_2 - Sv_2) \)

\[
= k\left(\frac{P_1 - P_2}{k}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{P_1 - 3P_2}{k}\right)\left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2}P_2
\]
**Stiffness Matrix for a Bar Element**

**Example 6 - Plane Truss Problem**

Using the force-displacement relationship the force in each member may be computed.

Member (element) 3: \[ f_{1x} = 0 \quad f_{1y} = 0 \]
\[ f_{3x} = 0 \quad f_{3y} = 0 \]

The solution to this simple problem can be readily checked by using simple static equilibrium equations.

---

**Stiffness Matrix for a Bar Element**

**Example 7 - Plane Truss Problem**

Consider the two bar truss shown below.

Determine the displacement in the y direction of node 1 and the axial force in each element.

Assume \( E = 210 \text{ GPa} \) and \( A = 6 \times 10^{-4} \text{ m}^2 \)
### Stiffness Matrix for a Bar Element

**Example 7 - Plane Truss Problem**

The global elemental stiffness matrix for **element 1** is:

\[
\cos \theta^{(1)} = \frac{3}{5} = 0.6 \quad \sin \theta^{(1)} = \frac{4}{5} = 0.8
\]

\[
k^{(1)} = \frac{210 \times 10^6 (6 \times 10^{-4})}{5} = \begin{bmatrix}
0.36 & 0.48 & -0.36 & -0.48 \\
0.48 & 0.64 & -0.48 & -0.64 \\
-0.36 & -0.48 & 0.36 & 0.48 \\
-0.48 & -0.64 & 0.48 & 0.64
\end{bmatrix}
\]

Simplifying the above expression gives:

\[
k^{(1)} = 25,200 \begin{bmatrix}
u_1 & v_1 & u_2 & v_2 \\
0.36 & 0.48 & -0.36 & -0.48 \\
0.48 & 0.64 & -0.48 & -0.64 \\
-0.36 & -0.48 & 0.36 & 0.48 \\
-0.48 & -0.64 & 0.48 & 0.64
\end{bmatrix}
\]

---

### Stiffness Matrix for a Bar Element

**Example 7 - Plane Truss Problem**

The global elemental stiffness matrix for **element 2** is:

\[
\cos \theta^{(2)} = 0 \quad \sin \theta^{(2)} = 1
\]

\[
k^{(2)} = \frac{(210 \times 10^6)(6 \times 10^{-4})}{4} = \begin{bmatrix}
0 & 0 & 0 & -1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1
\end{bmatrix}
\]

Simplifying the above expression gives:

\[
k^{(2)} = 25,200 \begin{bmatrix}
u_1 & v_1 & u_2 & v_3 \\
0 & 0 & 0 & 0 \\
0 & 1.25 & 0 & -1.25 \\
0 & 0 & 0 & 0 \\
0 & -1.25 & 0 & 1.25
\end{bmatrix}
\]
**Stiffness Matrix for a Bar Element**

**Example 7 - Plane Truss Problem**

The total global equations are:

\[
\begin{pmatrix}
F_{1x} \\
F_{1y} \\
F_{2x} \\
F_{2y} \\
F_{3x} \\
F_{3y}
\end{pmatrix} = 25,200
\begin{pmatrix}
0.36 & 0.48 & -0.36 & -0.48 & 0 & 0 \\
0.48 & 1.89 & -0.48 & -0.64 & 0 & -1.25 \\
-0.36 & -0.48 & 0.36 & 0.48 & 0 & 0 \\
-0.48 & -0.64 & 0.48 & 0.64 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1.25 & 0 & 0 & 1.25
\end{pmatrix}
\begin{pmatrix}
u_1 \\
v_1 \\
u_2 \\
u_2 \\
u_3 \\
u_3
\end{pmatrix}
\]

The displacement boundary conditions are:

\[u_1 = \delta \quad u_2 = v_2 = u_3 = v_3 = 0\]

**Stiffness Matrix for a Bar Element**

**Example 7 - Plane Truss Problem**

The total global equations are:

\[
\begin{pmatrix}
F_{1x} \\
P \\
F_{2x} \\
F_{2y} \\
F_{3x} \\
F_{3y}
\end{pmatrix} = 25,200
\begin{pmatrix}
0.36 & 0.48 & -0.36 & -0.48 & 0 & 0 \\
0.48 & 1.89 & -0.48 & -0.64 & 0 & -1.25 \\
-0.36 & -0.48 & 0.36 & 0.48 & 0 & 0 \\
-0.48 & -0.64 & 0.48 & 0.64 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1.25 & 0 & 0 & 1.25
\end{pmatrix}
\begin{pmatrix}
\delta \\
v_1 \\
v_1 \\
v_2 \\
v_2 \\
v_3
\end{pmatrix}
\]

By applying the boundary conditions the force-displacement equations reduce to:

\[P = 25,200(0.48\delta + 1.89v_1)\]
**Stiffness Matrix for a Bar Element**

**Example 7 - Plane Truss Problem**

Solving the equation gives: \( v_1 = (2.1 \times 10^{-5})P - 0.25\delta \)

By substituting \( P = 1,000 \text{ kN} \) and \( \delta = -0.05 \text{ m} \) in the above equation gives:
\[
v_1 = 0.0337m
\]

The local element forces for element 1 are:
\[
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} = 25,200 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ 0 & 0.6 & 0.8 \end{bmatrix} \begin{bmatrix} u_1 = -0.05 \\ v_1 = 0.0337 \\ u_2 \\ v_2 \end{bmatrix}
\]

The element forces are: \( f_{1x} = -76.6 \text{ kN} \) \( f_{2x} = 76.7 \text{ kN} \)

**Tension**

---

**Stiffness Matrix for a Bar Element**

**Example 7 - Plane Truss Problem**

The local element forces for element 2 are:
\[
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} = 31,500 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 = -0.05 \\ v_1 = 0.0337 \\ u_3 \\ v_3 \end{bmatrix}
\]

The element forces are: \( f_{1x} = 1,061 \text{ kN} \) \( f_{2x} = -1,061 \text{ kN} \)

**Compression**
Stiffness Matrix for a Bar Element
Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space

Let's derive the transformation matrix for the stiffness matrix for a bar element in three-dimensional space as shown below:

The coordinates at node 1 are $x_1$, $y_1$, and $z_1$, and the coordinates of node 2 are $x_2$, $y_2$, and $z_2$. Also, let $\theta_x$, $\theta_y$, and $\theta_z$ be the angles measured from the global $x$, $y$, and $z$ axes, respectively, to the local axis.
Stiffness Matrix for a Bar Element

Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space

The three-dimensional vector representing the bar element is given as:

\[ \mathbf{d} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = u'\mathbf{i}' + v'\mathbf{j}' + w'\mathbf{k}' \]

Taking the dot product of the above equation with \( \mathbf{i}' \) gives:

\[ u(\mathbf{i} \cdot \mathbf{i}') + v(\mathbf{j} \cdot \mathbf{i}') + w(\mathbf{k} \cdot \mathbf{i}') = u' \]

By the definition of the dot product we get:

\[ C_x = \frac{x_2 - x_1}{L}, \quad C_y = \frac{y_2 - y_1}{L}, \quad C_z = \frac{z_2 - z_1}{L} \]

where \( L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \)

\[ C_x = \cos \theta_x, \quad C_y = \cos \theta_y, \quad C_z = \cos \theta_z \]

where \( C_x, C_y, \) and \( C_z \) are projections of \( \mathbf{i}' \) on to \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k}, \) respectively.
**Stiffness Matrix for a Bar Element**

**Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space**

Therefore:  
\[ u' = C_x u + C_y v + C_z w \]

The transformation between local and global displacements is:

\[
\begin{bmatrix}
  u'_1 \\
  v'_1 \\
  w'_1 \\
  u'_2 \\
  v'_2 \\
  w'_2
\end{bmatrix} = \begin{bmatrix}
  C_x & C_y & C_z & 0 & 0 & 0 \\
  0 & 0 & C_x & C_y & C_z & 0 \\
  0 & 0 & 0 & C_x & C_y & C_z
\end{bmatrix}\begin{bmatrix}
  u_1 \\
  v_1 \\
  w_1 \\
  u_2 \\
  v_2 \\
  w_2
\end{bmatrix}
\]

\[ d' = T^* d \]

\[
T^* = \begin{bmatrix}
  C_x & C_y & C_z & 0 & 0 & 0 \\
  0 & 0 & C_x & C_y & C_z & 0 \\
  0 & 0 & 0 & C_x & C_y & C_z
\end{bmatrix}
\]

**Stiffness Matrix for a Bar Element**

**Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space**

The transformation from the local to the global stiffness matrix is:

\[
k = T^* k T
\]

\[
k = \frac{AE}{L} \begin{bmatrix}
  C_x^2 & C_x C_y & C_x C_z & -C_x^2 & -C_x C_y & -C_x C_z \\
  C_x C_y & C_y^2 & C_y C_z & -C_x C_y & -C_y^2 & -C_y C_z \\
  C_x C_z & C_y C_z & C_z^2 & -C_x C_z & -C_y C_z & -C_z^2 \\
  -C_x^2 & -C_x C_y & -C_x C_z & C_x^2 & C_x C_y & C_x C_z \\
  -C_x C_y & -C_y^2 & -C_y C_z & C_x C_y & C_y^2 & C_y C_z \\
  -C_x C_z & -C_y C_z & -C_z^2 & C_x C_z & C_y C_z & C_z^2
\end{bmatrix}
\]
Stiffness Matrix for a Bar Element

Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space

The global stiffness matrix can be written in a more convenient form as:

\[ k = \frac{AE}{L} \begin{bmatrix} \lambda & -\lambda \\ -\lambda & \lambda \end{bmatrix} \]

Stiffness Matrix for a Bar Element

Example 8 – Space Truss Problem

Consider the space truss shown below. The modulus of elasticity, \( E = 1.2 \times 10^6 \) psi for all elements. Node 1 is constrained from movement in the \( y \) direction.

To simplify the stiffness matrices for the three elements, we will express each element in the following form:

\[ k = \frac{AE}{L} \begin{bmatrix} \lambda & -\lambda \\ -\lambda & \lambda \end{bmatrix} \]
Stiffness Matrix for a Bar Element

Example 8 – Space Truss Problem

Consider element 1:

\[ L^{(1)} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \]

\[ L^{(1)} = \sqrt{(-72)^2 + (36)^2} = 80.5 \text{ in} \]

\[ C_x = \frac{-72}{80.5} = -0.89 \]

\[ C_y = \frac{36}{80.5} = 0.45 \]

\[ C_z = 0 \]

\[ \lambda = \begin{bmatrix} 0.79 & -0.40 & 0 \\ -0.40 & 0.20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ k = \frac{(0.302 \text{ in}^2)(1.2 \times 10^6 \text{ psi})}{80.5 \text{ in}} \begin{bmatrix} u_{x,1} & u_{y,1} & u_{z,1} \\ -u_{x,1} & -u_{y,1} & -u_{z,1} \end{bmatrix} \begin{bmatrix} \lambda \\ -\lambda \\ \lambda \end{bmatrix} \]
Stiffness Matrix for a Bar Element

Example 8 – Space Truss Problem

Consider element 2:

\[ L^{(2)} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2} \]

\[ L^{(2)} = \sqrt{(-72)^2 + (36)^2 + (72)^2} = 108 \text{ in} \]

\[ C_x = \frac{-72}{108} = -0.667 \]
\[ C_y = \frac{36}{108} = 0.33 \]
\[ C_z = \frac{72}{108} = 0.667 \]

\[ \lambda = \begin{bmatrix} 0.45 & -0.22 & -0.45 \\ -0.22 & 0.11 & 0.45 \\ -0.45 & 0.45 & 0.45 \end{bmatrix} \]

\[ k = \frac{(0.729 \text{ in}^2)(1.2 \times 10^6 \text{ psi})}{108 \text{ in}} \begin{bmatrix} \lambda & -\lambda & -\lambda \\ -\lambda & \lambda & -\lambda \\ -\lambda & -\lambda & \lambda \end{bmatrix} \]
Stiffness Matrix for a Bar Element

Example 8 – Space Truss Problem

Consider element 3:

\[ L^{(3)} = \sqrt{(x_4 - x_1)^2 + (y_4 - y_1)^2 + (z_4 - z_1)^2} \]

\[ L^{(3)} = \sqrt{(-72)^2 + (-48)^2} = 86.5 \text{ in} \]

\[ C_x = \frac{-72}{86.5} = -0.833 \]

\[ C_y = 0 \]

\[ C_z = \frac{-48}{86.5} = -0.550 \]

\[ \lambda = \begin{bmatrix} 0.69 & 0 & 0.46 \\ 0 & 0 & 0 \\ 0.46 & 0 & 0.30 \end{bmatrix} \]

Stiffness Matrix for a Bar Element

Example 8 – Space Truss Problem

Consider element 3:

\[ k = \frac{(0.187 \text{ in}^2)(1.2 \times 10^6 \text{ psi})}{86.5 \text{ in}} \begin{bmatrix} v_{1x} & v_{1y} & v_{1z} \\ v_{2x} & v_{2y} & v_{2z} \\ v_{3x} & v_{3y} & v_{3z} \end{bmatrix} \begin{bmatrix} \lambda \\ -\lambda \\ \lambda \end{bmatrix} \]

The boundary conditions are:

\[ u_2 = v_2 = w_2 = 0 \]

\[ u_3 = v_3 = w_3 = 0 \]

\[ u_4 = v_4 = w_4 = 0 \]

\[ v_1 = 0 \]
Stiffness Matrix for a Bar Element

Example 8 – Space Truss Problem

Canceling the rows and the columns associated with the boundary conditions reduces the global stiffness matrix to:

\[ K = \begin{bmatrix} 9,000 & -2,450 \\ -2,450 & 4,450 \end{bmatrix} \]

The global force-displacement equations are:

\[
\begin{bmatrix} 9,000 & -2,450 \\ -2,450 & 4,450 \end{bmatrix} \begin{bmatrix} u_i \\ w_i \end{bmatrix} = \begin{bmatrix} 0 \\ -1,000 \end{bmatrix}
\]

Solving the equation gives:

\[ u_i = -0.072 \text{ in} \quad w_i = -0.264 \text{ in} \]

Stiffness Matrix for a Bar Element

Example 8 – Space Truss Problem

It can be shown, that the local forces in an element are:

\[
\begin{bmatrix} f'_x \\ f'_y \\ f'_z \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} -C_x & -C_y & -C_z \\ C_x & C_y & C_z \\ -C_x & -C_y & -C_z \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix}
\]

The stress in an element is:

\[
\sigma = \frac{E}{L} \begin{bmatrix} -C_x & -C_y & -C_z \\ C_x & C_y & C_z \\ -C_x & -C_y & -C_z \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix}
\]
Stiffness Matrix for a Bar Element

Example 8 – Space Truss Problem

The stress in element 1 is:

\[
\sigma^{(1)} = \frac{1.2 \times 10^6}{80.5} \begin{bmatrix}
0.89 & 0.45 & 0 & -0.89 & 0.45 & 0
\end{bmatrix}
\begin{bmatrix}
-0.072 \\
0 \\
-0.264 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\sigma^{(1)} = -955 \text{ psi}
\]

The stress in element 2 is:

\[
\sigma^{(2)} = \frac{1.2 \times 10^6}{108} \begin{bmatrix}
0.667 & -0.33 & -0.667 & -0.667 & 0.33 & 0.667
\end{bmatrix}
\begin{bmatrix}
-0.072 \\
0 \\
-0.264 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\sigma^{(2)} = 1,423 \text{ psi}
\]

The stress in element 3 is:

\[
\sigma^{(3)} = \frac{1.2 \times 10^6}{86.5} \begin{bmatrix}
0.83 & 0.55 & 0 & -0.83 & 0 & -0.55
\end{bmatrix}
\begin{bmatrix}
-0.072 \\
0 \\
-0.264 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\sigma^{(3)} = 2,843 \text{ psi}
\]
Stiffness Matrix for a Bar Element

Inclined, or Skewed Supports

If a support is inclined, or skewed, at some angle $\alpha$ for the global $x$ axis, as shown below, the boundary conditions on the displacements are not in the global $x$-$y$ directions but in the $x'$-$y'$ directions.

Stiffness Matrix for a Bar Element

Inclined, or Skewed, Supports

We must transform the local boundary condition of $v'_3 = 0$ (in local coordinates) into the global $x$-$y$ system.
Stiffness Matrix for a Bar Element

Inclined, or Skewed, Supports

Therefore, the relationship between the components of the displacement in the local and the global coordinate systems at node 3 is:

\[
\begin{bmatrix}
    u' \\
    v'
\end{bmatrix} = \begin{bmatrix}
    \cos \alpha & \sin \alpha \\
    -\sin \alpha & \cos \alpha
\end{bmatrix}
\begin{bmatrix}
    u_3 \\
    v_3
\end{bmatrix}
\]

We can rewrite the above expression as:

\[
\begin{bmatrix}
    d'_3
\end{bmatrix} = [t_3] \begin{bmatrix}
    d_3
\end{bmatrix}
\]

where

\[
[t_3] = \begin{bmatrix}
    \cos \alpha & \sin \alpha \\
    -\sin \alpha & \cos \alpha
\end{bmatrix}
\]

We can apply this sort of transformation to the entire displacement vector as:

\[
\{d'\} = [T_1] \{d\} \quad \text{or} \quad \{d'\} = [T_1]^T \{d''\}
\]

Stiffness Matrix for a Bar Element

Inclined, or Skewed, Supports

Where the matrix \([T_1]^T\) is:

\[
[T_1]^T = \begin{bmatrix}
    [I] & [0] & [0] \\
    [0] & [I] & [0] \\
    [0] & [0] & [t_3]
\end{bmatrix}
\]

Both the identity matrix \([I]\) and the matrix \([t_3]\) are 2 x 2 matrices.

The force vector can be transformed by using the same transformation.

\[
\{f'\} = [T_1] \{f\}
\]

In global coordinates, the force-displacement equations are:

\[
\{f\} = [K] \{d\} \]
**Stiffness Matrix for a Bar Element**

Inclined, or Skewed, Supports

Applying the skewed support transformation to both sides of the equation gives:

$$[T_i]\{f\} = [T_i][K][d]$$

By using the relationship between the local and the global displacements, the force-displacement equations become:

$$\{f\}' = [T_i][K][T_i]^T \{d\}'$$

Therefore the global equations become:

$$\begin{bmatrix} F_u \\ F_v \\ F_x \\ F_y \\ F_{x'} \\ F_{y'} \end{bmatrix} = [T_i][K][T_i]^T \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

---

**Stiffness Matrix for a Bar Element**

**Example 9 – Space Truss Problem**

Consider the plane truss shown below. Assume $E = 210$ GPa, $A = 6 \times 10^{-4}$ m$^2$ for element 1 and 2, and $A = \sqrt{2}(6 \times 10^{-4})$ m$^2$ for element 3.

Determine the stiffness matrix for each element.

$$k = \frac{AE}{L} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$
**Stiffness Matrix for a Bar Element**

**Example 9 – Space Truss Problem**

The global elemental stiffness matrix for **element 1** is:

\[
\begin{bmatrix}
C^2 & CS & -C^2 & -CS \\
CS & S^2 & -CS & -S^2 \\
-C^2 & -CS & C^2 & CS \\
-CS & -S^2 & CS & S^2 \\
\end{bmatrix}
\]

\[
k = \frac{AE}{L}
\]

\[
\theta(1) = 0 \quad \text{sin} \theta(1) = 1
\]

\[
k^{(1)} = \frac{(210 \times 10^6 \text{ kN} / \text{m}^2)(6 \times 10^{-4} \text{m}^2)}{1 \text{ m}}
\]

\[
\begin{bmatrix}
u_1 & v_1 & u_2 & v_2 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 \\
\end{bmatrix}
\]

---

**Stiffness Matrix for a Bar Element**

**Example 9 – Space Truss Problem**

The global elemental stiffness matrix for **element 2** is:

\[
\begin{bmatrix}
C^2 & CS & -C^2 & -CS \\
CS & S^2 & -CS & -S^2 \\
-C^2 & -CS & C^2 & CS \\
-CS & -S^2 & CS & S^2 \\
\end{bmatrix}
\]

\[
k = \frac{AE}{L}
\]

\[
\theta(2) = 1 \quad \text{sin} \theta(2) = 0
\]

\[
k^{(2)} = \frac{(210 \times 10^6 \text{ kN} / \text{m}^2)(6 \times 10^{-4} \text{m}^2)}{1 \text{ m}}
\]

\[
\begin{bmatrix}
u_2 & v_2 & u_3 & v_3 \\
1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Stiffness Matrix for a Bar Element

Example 9 – Space Truss Problem

The global elemental stiffness matrix for element 3 is:

\[
k = \frac{AE}{L} \begin{bmatrix}
C^2 & CS & -C^2 & -CS \\
CS & S^2 & -CS & -S^2 \\
-C^2 & -CS & C^2 & CS \\
-CS & -S^2 & CS & S^2 \\
\end{bmatrix}
\]

\[
\cos \theta^{(3)} = \frac{\sqrt{2}}{2} \quad \sin \theta^{(3)} = \frac{\sqrt{2}}{2}
\]

\[
k^{(3)} = \frac{(210 \times 10^6 \text{kN} / \text{m}^2)(6\sqrt{2} \times 10^{-4} \text{m}^3)}{2\sqrt{2} \text{m}}
\]

\[
\begin{bmatrix}
u_x & v_y & u_3 & v_3 \\
1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 \\
-1 & -1 & 1 & 1 \\
\end{bmatrix}
\]


Stiffness Matrix for a Bar Element

Example 9 – Space Truss Problem

Using the direct stiffness method, the global stiffness matrix is:

\[
K = 1260 \times 10^6 \text{ N/m}
\]

\[
\begin{bmatrix}
0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 \\
0.5 & 1.5 & 0 & -1 & -0.5 & -0.5 \\
0 & 0 & 1 & 0 & -1 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 \\
-0.5 & -0.5 & -1 & 0 & 1.5 & 0.5 \\
-0.5 & -0.5 & 0 & 0 & 0.5 & 0.5 \\
\end{bmatrix}
\]

We must transform the global displacements into local coordinates. Therefore the transformation \([T,\text{]}\) is:

\[
[T,\text{]} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
**Stiffness Matrix for a Bar Element**

Example 9 – Space Truss Problem

The first step in the matrix transformation to find the product of \([T_1][K]\).

\[
[T_1][K] = 1,260 \times 10^5 \frac{N}{m}
\]

\[
[T_1][K] = \begin{bmatrix}
0.5 & 0.5 & 0 & -0.5 & -0.5 \\
0.5 & 1.5 & 0 & -1 & -0.5 \\
0 & 0 & 1 & 1 & -1 \\
0 & -1 & 1 & 0 & 0 \\
-0.707 & -0.707 & 0 & 1.414 & 0.707 \\
0 & 0 & 0.707 & 0 & -0.707 \\
\end{bmatrix}
\]

**Stiffness Matrix for a Bar Element**

Example 9 – Space Truss Problem

The next step in the matrix transformation to find the product of \([T_1][K][T_1]^T\).

\[
[T_1][K][T_1]^T = 1,260 \times 10^5 \frac{N}{m}
\]

\[
[T_1][K][T_1]^T = \begin{bmatrix}
0.5 & 0.5 & 0 & -0.707 & 0 \\
0.5 & 1.5 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
-0.707 & -0.707 & 0 & 0.707 & 0 \\
0 & 0 & 0.707 & 0 & 0 \\
\end{bmatrix}
\]
**Stiffness Matrix for a Bar Element**

**Example 9 – Space Truss Problem**

The displacement boundary conditions are: \( u_i = v_i = v_2 = v'_3 = 0 \)

\[
\begin{bmatrix}
F_{1x} \\
F_{1y} \\
F_{2x} \\
F_{2y} \\
F_{3x} \\
F_{3y}
\end{bmatrix} = 1.260 \times 10^6 \text{N/m}
\begin{bmatrix}
0.5 & 0.5 & 0 & 0 & -0.707 & 0 \\
0.5 & 1.5 & 0 & -1 & -0.707 & 0 \\
0 & 0 & 1 & 0 & -0.707 & 0.707 \\
0 & -1 & 0 & 1 & 0 & 0 \\
-0.707 & -0.707 & -0.707 & 0 & 1.5 & -0.5 \\
0 & 0 & 0.707 & 0 & -0.5 & 0.5
\end{bmatrix}
\begin{bmatrix}
u_1 \\
v_1 \\
u_2 \\
v_2 \\
u_3 \\
v'_3
\end{bmatrix}
\]

By applying the boundary conditions the global force-displacement equations are:

\[
1.260 \times 10^6 \text{N/m} \begin{bmatrix}
1 & 0 & 0 & -0.707 & 0 \\
-0.707 & 1.5 & 0 & 0 & 0 \\
0 & -0.707 & 1.5 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1.5 & -0.5 \\
0 & 0 & 0 & -0.5 & 0.5
\end{bmatrix}
\begin{bmatrix}
u_2 \\
u'_3
\end{bmatrix} = 
\begin{bmatrix}
F_{2x} = 1,000 \text{ kN} \\
F_{3x} = 0
\end{bmatrix}
\]

Solving the equation gives: \( u_2 = 11.91 \text{ mm} \) \( u'_3 = 5.61 \text{ mm} \)
**Stiffness Matrix for a Bar Element**

### Example 9 – Space Truss Problem

The global nodal forces are calculated as:

\[
\begin{bmatrix}
F_{x1} \\
F_{y1} \\
F_{x2} \\
F_{y2} \\
F'_{3x} \\
F'_{3y}
\end{bmatrix} = 1,260 \times 10^2 N/mm \begin{bmatrix}
0.5 & 0.5 & 0 & 0 & -0.707 & 0 & 0 \\
0.5 & 1.5 & 0 & -1 & -0.707 & 0 & 0 \\
0 & 0 & 1 & 0 & -0.707 & 0.707 & 11.91 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 \\
-0.707 & -0.707 & -0.707 & 0 & 1.5 & -0.5 & 5.61 \\
0 & 0 & 0.707 & 0 & -0.5 & 0.5 & 0
\end{bmatrix} \begin{bmatrix}
\Delta u_1 \\
\Delta v_1 \\
\Delta u_2 \\
\Delta v_2 \\
\Delta u_3 \\
\Delta v_3
\end{bmatrix} = 0
\]

Therefore:

\[
F_{x1} = -500 \text{kN} \quad F_{y1} = -500 \text{kN} \\
F_{xy} = 0 \quad F'_{3y} = 707 \text{kN}
\]

---

### Stiffness Matrix for a Bar Element

#### Potential Energy Approach to Derive Bar Element Equations

Let’s derive the equations for a bar element using the principle of minimum potential energy.

The total potential energy, \( \pi_p \), is defined as the sum of the internal strain energy \( U \) and the potential energy of the external forces \( \Omega \):

\[
\pi_p = U + \Omega
\]

The differential internal work (strain energy) \( dU \) in a one-dimensional bar element is:

\[
dU = \sigma(\Delta y)(\Delta z)(\Delta x)d\varepsilon_x
\]
Stiffness Matrix for a Bar Element

Potential Energy Approach to Derive Bar Element Equations

If we let the volume of the element approach zero, then:

\[ dU = \sigma_x d\varepsilon_x dV \]

Summing the differential energy over the whole bar gives:

\[ U = \int_{V} \left[ \int_{V} \sigma_x d\varepsilon_x \right] dV = \int_{V} \left[ \int_{V} E\varepsilon_x d\varepsilon_x \right] dV = \int_{V} \frac{1}{2} E\varepsilon_x^2 dV \]

For a linear-elastic material (Hooke’s law) as shown below:

\[ \sigma_x = E\varepsilon_x \quad U = \int_{V} \frac{1}{2} \sigma_x \varepsilon_x dV \]

Stiffness Matrix for a Bar Element

Potential Energy Approach to Derive Bar Element Equations

The internal strain energy statement becomes

\[ U = \frac{1}{2} \int_{V} \sigma_x \varepsilon_x dV \]

The potential energy of the external forces is:

\[ \Omega = -\int_{V} X_b u dV - \int_{S} T_x u_s dS - \sum_{i=1}^{M} f_{ix} u_i \]

where \( X_b \) is the body force (force per unit volume), \( T_x \) is the traction (force per unit area), and \( f_{ix} \) is the nodal concentrated force. All of these forces are considered to act in the local \( x \) direction.
Stiffness Matrix for a Bar Element

Potential Energy Approach to Derive Bar Element Equations

Apply the following steps when using the principle of minimum potential energy to derive the finite element equations.

1. Formulate an expression for the total potential energy.

2. Assume a displacement pattern.

3. Obtain a set of simultaneous equations minimizing the total potential energy with respect to the displacement parameters.

Consider the following bar element, as shown below:

\[ \pi_p = \frac{A}{2} \int_0^L \sigma_x e_x \, dx - \int_0^L f_x u_1 - f_{2x} u_2 \, dx \]

- \[ \int_X B_u \, dV - \int_S T_x u_s \, dS \]

We can approximate the axial displacement as:

\[ u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad N_1 = 1 - \frac{x}{L}, \quad N_2 = \frac{x}{L} \]
**Stiffness Matrix for a Bar Element**

Potential Energy Approach to Derive Bar Element Equations

Using the stress-strain relationships, the axial strain is:

\[
\varepsilon_x = \frac{du}{dx} = \begin{bmatrix}
  dN_1 & dN_2
\end{bmatrix} \begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}
\]

where \( N_1 \) and \( N_2 \) are the interpolation functions gives as:

\[
\varepsilon_x = \begin{bmatrix}
  -\frac{1}{L} & \frac{1}{L}
\end{bmatrix} \begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}
\]

and

\[
\{\varepsilon_x\} = [B]\{d\}
\]

\[
B = \begin{bmatrix}
  -\frac{1}{L} & \frac{1}{L}
\end{bmatrix}
\]

The axial stress-strain relationship is:

\[
\{\sigma_x\} = [D]\{\varepsilon_x\}
\]

**Stiffness Matrix for a Bar Element**

Potential Energy Approach to Derive Bar Element Equations

Where \([D] = [E]\) for the one-dimensional stress-strain relationship and \(E\) is the modulus of elasticity.

Therefore, stress can be related to nodal displacements as:

\[
\{\sigma_x\} = [D][B]\{d\}
\]

The total potential energy expressed in matrix form is:

\[
\pi_p = \frac{A}{2} \int_0^L \{\sigma_x\}^T \{\varepsilon_x\} \, dx - \{d\}^T \{P\} - \int_V \{u\}^T \{X_b\} \, dV - \int_S \{u\}^T \{T_x\} \, dS
\]

where \(\{P\}\) represented the concentrated nodal loads.
Stiffness Matrix for a Bar Element

Potential Energy Approach to Derive Bar Element Equations

If we substitute the relationship between \( \hat{u} \) and \( \hat{d} \) into the energy equations we get:

\[
\pi_p = \frac{A L}{2} \left[ \{d\}^T [B]^T [D]^T [B] \{d\} - \{d\}^T \{P\} \right]
\]

\[
- \int \{d\}^T [N]^T \{X_b\} dV - \int \{d\}^T [N_s]^T \{T_x\} dS
\]

In the above expression for potential energy \( \pi_p \) is a function of the \( d \), that is: \( \pi_p = \pi_p(u_1, u_2) \).

However, \([B]\) and \([D]\) and the nodal displacements \( u \) are not a function of \( x \).

Stiffness Matrix for a Bar Element

Potential Energy Approach to Derive Bar Element Equations

Integration the energy expression with respect to \( x \) gives:

\[
\pi_p = \frac{AL}{2} \{d\}^T [B]^T [D]^T [B] \{d\} - \{d\}^T \{f\}
\]

where

\[
\{f\} = \{P\} + \int [N]^T \{X_b\} dV + \int [N_s]^T \{T_x\} dS
\]

We can define the surface tractions and body-force matrices as:

\[
\{f_s\} = \int [N]^T \{T_x\} dS \quad \{f_b\} = \int [N]^T \{X_b\} dV
\]
**Stiffness Matrix for a Bar Element**

**Potential Energy Approach to Derive Bar Element Equations**

Minimization of $\pi_p$ with respect to each nodal displacement requires that:

$$\frac{\partial \pi_p}{\partial u_i} = 0 \quad \frac{\partial \pi_p}{\partial u_2} = 0$$

For convenience, let’s define the following

$$\{U^*\} = \{d\}^T [B]^T [D]^T [B]{\{d\}}$$

$$\{U^*\} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} -1 \frac{1}{L} \frac{1}{L} \end{bmatrix} \begin{bmatrix} E \frac{-1}{L} \frac{1}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

**Simplifying the above expression gives:**

$$U^* = \frac{E}{L^2} (u_1^2 - 2u_1 u_2 + u_2^2)$$

The loading on a bar element is given as:

$$\{d\}^T \{f\} = u_1 f_{1x} + u_2 f_{2x}$$

Therefore, the minimum potential energy is:

$$\frac{\partial \pi_p}{\partial u_1} = \frac{AE}{2L} (2u_1 - 2u_2) - f_{1x} = 0$$

$$\frac{\partial \pi_p}{\partial u_2} = \frac{AE}{2L} (-2u_1 + 2u_2) - f_{2x} = 0$$
**Stiffness Matrix for a Bar Element**

**Potential Energy Approach to Derive Bar Element Equations**

The above equations can be written in matrix form as:

\[
\frac{\partial \pi_p}{\partial (d)} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} f_{1x} \\ f_{2x} \end{bmatrix} = 0
\]

The stiffness matrix for a bar element is:

\[
[k] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\]

This form of the stiffness matrix obtained from the principle of minimum potential energy is identical to the stiffness matrix derived from the equilibrium equations.

---

**Example 10 - Bar Problem**

Consider the bar shown below:

The energy equivalent nodal forces due to the distributed load are:

\[
\{f_0\} = \int_S \{N\}^T \{T_x\} dS \\
\{f_0\} = \begin{\{f_{1x}\} \\ f_{2x} \end{\{f_0\}} = \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix} \{Cx\} dx
\]
**Stiffness Matrix for a Bar Element**

**Example 10 - Bar Problem**

\[
\begin{align*}
\{f_{1x}\} &= \int_0^L \left\{1 - \frac{x}{L}\right\} \{Cx\} \, dx = \left\{\frac{Cx^2}{2} - \frac{Cx^3}{3L}\right\}_0^L = \left\{\frac{CL^2}{6} \right\} \\
\{f_{2x}\} &= \int_0^L \left\{\frac{x}{L}\right\} \{Cx\} \, dx = \left\{\frac{Cx^3}{3L}\right\}_0^L = \left\{\frac{CL^2}{3} \right\}
\end{align*}
\]

The total load is the area under the distributed load curve, or:

\[F = \frac{1}{2}(L)(CL) = \frac{CL^2}{2}\]

The equivalent nodal forces for a linearly varying load are:

\[f_{1x} = \frac{F}{3} = \frac{CL^2}{3}\] of the total load

\[f_{2x} = \frac{2F}{3} = \frac{2CL^2}{3}\] of the total load

**Stiffness Matrix for a Bar Element**

**Example 11 - Bar Problem**

Consider the axially loaded bar shown below. Determine the axial displacement and axial stress. Let \(E = 30 \times 10^6 \text{ psi}\), \(A = 2 \text{ in}^2\), and \(L = 60 \text{ in}\). Use (a) one and (b) two elements in the finite element solutions.
**Stiffness Matrix for a Bar Element**

Example 11 - Bar Problem

The one-element solution:

The distributed load can be converted into equivalent nodal forces using:

\[
\{F_0\} = \int [N]^T \{T_x\} \, dS
\]

\[
\begin{align*}
\{F_0\} &= \begin{bmatrix} F_{1x} \\ F_{2x} \end{bmatrix} = \int_0^L \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix} \{-10x\} \, dx
\end{align*}
\]

**Stiffness Matrix for a Bar Element**

Example 11 - Bar Problem

The one-element solution:

\[
\begin{align*}
\{F_{1x}\} &= \int_0^L \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix} \{-10x\} \, dx = \begin{bmatrix} \frac{10L^2}{2} \\ \frac{10L^2}{3} \end{bmatrix} = \begin{bmatrix} \frac{10L^2}{6} \\ \frac{10L^2}{3} \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\{F_{2x}\} &= \begin{bmatrix} -6,000 \text{ lb} \\ -12,000 \text{ lb} \end{bmatrix}
\end{align*}
\]
**Stiffness Matrix for a Bar Element**

**Example 11 - Bar Problem**

*The one-element solution:*

\[
\mathbf{k}^{(1)} = 10^6 \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\]

The element equations are:

\[
10^6 \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} = \begin{bmatrix}
-6,000 \\
R_{2x} - 12,000
\end{bmatrix}
\quad u_2 = 0.006 \text{ in}
\]

The second equation gives:

\[
-10^6(u_1) = R_{2x} - 12,000 \quad \Rightarrow \quad R_{2x} = 18,000 \text{ lb}
\]

---

**Stiffness Matrix for a Bar Element**

**Example 11 - Bar Problem**

*The one-element solution:*

The axial stress-strain relationship is: \( \{\sigma_x\} = [D]\{\varepsilon_x\} \)

\[
\{\sigma_x\} = E[B]\{d\}
\]

\[
= E\left[-\frac{1}{L} \quad \frac{1}{L}\right]\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} = E\left(\frac{u_2 - u_1}{L}\right)
\]

\[
= 30 \times 10^6 \left(\frac{0 + 0.006}{60}\right) = 3,000 \text{ psi (T)}
\]
Stiffness Matrix for a Bar Element

Example 11 - Bar Problem

The two-element solution:

The distributed load can be converted into equivalent nodal forces.

For element 1, the total force of the triangular-shaped distributed load is:

\[ \frac{1}{2} (30 \text{ in.})(300 \text{ lb/in.}) = -4,500 \text{ lb} \]

Based on equations developed for the equivalent nodal force of a triangular distributed load, develop in the one-element problem, the nodal forces are:

\[
\begin{align*}
\begin{bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{bmatrix} &= \begin{bmatrix} -\frac{1}{3} (4,500) \\ -\frac{2}{3} (4,500) \end{bmatrix} = \begin{bmatrix} -1,500 \text{ lb} \\ -3,000 \text{ lb} \end{bmatrix}
\end{align*}
\]
**Stiffness Matrix for a Bar Element**

**Example 11 - Bar Problem**

*The two-element solution:*

For element 2, the applied force is in two parts: a triangular-shaped distributed load and a uniform load. The uniform load is:

\[(30 \text{ in})(300 \text{ lb/in}) = -9,000 \text{ lb}\]

The nodal forces for element 2 are:

\[
\begin{bmatrix}
  f_{2x}^{(2)} \\
  f_{3x}^{(2)}
\end{bmatrix} = \begin{bmatrix}
  -\frac{1}{2} (9,000) + \frac{1}{3} (4,500) \\
  -\frac{1}{2} (9,000) + \frac{2}{3} (4,500)
\end{bmatrix} = \begin{bmatrix}
  -6,000 \text{ lb} \\
  -7,500 \text{ lb}
\end{bmatrix}
\]

**Stiffness Matrix for a Bar Element**

**Example 11 - Bar Problem**

*The two-element solution:*

The final nodal force vector is:

\[
\begin{bmatrix}
  F_{1x} \\
  F_{2x} \\
  F_{3x}
\end{bmatrix} = \begin{bmatrix}
  f_{1x}^{(1)} \\
  f_{2x}^{(1)} + f_{2x}^{(2)} \\
  f_{3x}^{(2)}
\end{bmatrix} = \begin{bmatrix}
  -1,500 \\
  -9,000 \\
  R_{3x} - 7,500
\end{bmatrix}
\]

The element stiffness matrices are:

\[
k^{(1)} = k^{(2)} = \frac{2AE}{L} \begin{bmatrix}
  1 & 2 & 3 \\
  2 & 1 & -1 \\
  3 & -1 & 1
\end{bmatrix}
\]
**Stiffness Matrix for a Bar Element**

**Example 11 - Bar Problem**

*The two-element solution:*

The assembled global stiffness matrix is:

\[
K = 2 \times 10^6 \begin{bmatrix}
1 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{bmatrix}
\]

The assembled global force-displacement equations are:

\[
2 \times 10^6 \begin{bmatrix}
1 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1,500 \\ -9,000 \\ R_{3x} - 7,500 \end{bmatrix}
\]

After the eliminating the row and column associated with \( u_{3x} \), we get:

\[
2 \times 10^6 \begin{bmatrix}
1 & -1 \\
-1 & 2
\end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -1,500 \\ -9,000 \end{bmatrix}
\]

Solving the equation gives:

\[
u_1 = -0.006 \text{ in}
\]

\[
u_2 = -0.00525 \text{ in}
\]

Solving the last equation gives:

\[-2 \times 10^6 u_2 = R_{3x} - 7,500 \quad \Rightarrow \quad R_{3x} = 18,000\]
**Stiffness Matrix for a Bar Element**

Example 11 - Bar Problem

*The two-element solution:*

The axial stress-strain relationship is:

\[
\sigma_x^{(1)} = E \left[ -\frac{1}{L} \begin{array}{c} 1 \\ 1 \end{array} \right] \begin{bmatrix} d_{1x} \\ d_{2x} \end{bmatrix}
\]

\[
= E \left[ -\frac{1}{30} \begin{array}{c} 1 \\ 1 \end{array} \right] \begin{bmatrix} -0.006 \\ -0.00525 \end{bmatrix} = 750 \text{ psi (T)}
\]

---

**Stiffness Matrix for a Bar Element**

Example 11 - Bar Problem

*The two-element solution:*

The axial stress-strain relationship is:

\[
\sigma_x^{(2)} = E \left[ -\frac{1}{L} \begin{array}{c} 1 \\ 1 \end{array} \right] \begin{bmatrix} d_{2x} \\ d_{5x} \end{bmatrix}
\]

\[
= E \left[ -\frac{1}{30} \begin{array}{c} 1 \\ 1 \end{array} \right] \begin{bmatrix} -0.00525 \\ 0 \end{bmatrix} = 5,250 \text{ psi (T)}
\]
**Stiffness Matrix for a Bar Element**

**Comparison of Finite Element Solution to Exact Solution**

In order to be able to judge the accuracy of our finite element models, we will develop an exact solution for the bar element problem.

The exact solution for the displacement may be obtained by:

\[ u = \frac{1}{AE} \int_0^L P(x) dx \]

where the force \( P \) is shown on the following free-body diagram.

\[ P(x) = \frac{1}{2} [x(10x)] = 5x^2 \]

Therefore:

\[ u = \frac{1}{AE} \int_0^L P(x) dx \quad u = \frac{1}{AE} \int_0^x 5x^2 dx = \frac{5x^3}{3AE} + C_1 \]

Applying the boundary conditions:

\[ u(L) = 0 = \frac{5x^3}{3AE} + C_1 \quad \Rightarrow \quad C_1 = -\frac{5L^3}{3AE} \]

The exact solution for axial displacement is:

\[ u(L) = \frac{5}{3AE} \left( x^3 - L^3 \right) \quad \sigma(x) = \frac{P(x)}{A} = \frac{5x^2}{A} \]
**Stiffness Matrix for a Bar Element**

Comparison of Finite Element Solution to Exact Solution

A plot of the exact solution for displacement as compared to several different finite element solutions is shown below.

![Displacement Plot](image)

**Stiffness Matrix for a Bar Element**

Comparison of Finite Element Solution to Exact Solution

A plot of the exact solution for axial stress as compared to several different finite element solutions is shown below.

![Stress Plot](image)
Comparison of Finite Element Solution to Exact Solution

A plot of the exact solution for axial stress at the fixed end \(x = L\) as compared to several different finite element solutions is shown below.

![Plot of exact solution and finite element solutions for axial stress](image)

Galerkin's Residual Method and Its Application to a One-Dimensional Bar

There are a number of weighted residual methods. However, the Galerkin's method is more well-known and will be the only weighted residual method discussed in this course.

In weighted residual methods, a trial or approximate function is chosen to approximate the independent variable (in our case, displacement) in a problem defined by a differential equation.

The trial function will not, in general, satisfy the governing differential equation.

Therefore, the substitution of the trial function in the differential equation will create a residual over the entire domain of the problem.
Stiffness Matrix for a Bar Element
Galerkin’s Residual Method and Its Application to a One-Dimensional Bar

Therefore, the substitution of the trial function in the differential equation will create a residual over the entire domain of the problem.

\[ \int V R dV = \text{minimum} \]

In the residual methods, we require that a weighted value of the residual be a minimum over the entire domain of the problem.

The weighting function \( W \) allows the weighted integral of the residuals to go to zero.

\[ \int V R W dV = 0 \]

---

Stiffness Matrix for a Bar Element
Galerkin’s Residual Method and Its Application to a One-Dimensional Bar

Using Galerkin’s weighted residual method, we require the weighting functions to be the interpolation functions \( N_i \).

Therefore:

\[ \int V R N_i dV = 0 \quad i = 1, 2, \ldots, n \]
Stiffness Matrix for a Bar Element
Example 12 - Bar Element Formulation

Let's derive the bar element formulation using Galerkin’s method. The governing differential equation is:

$$\frac{d}{dx} \left( AE \frac{du}{dx} \right) = 0$$

Applying Galerkin’s method we get:

$$\int_0^L \frac{d}{dx} \left( AE \frac{du}{dx} \right) N_i \, dx = 0 \quad i = 1, 2, \ldots, n$$

We now apply integration by parts using the following general formula:

$$\int u \, dv = uv - \int v \, du$$

Stiffness Matrix for a Bar Element
Example 12 - Bar Element Formulation

If we assume the following:

$$u = N_i, \quad du = \frac{dN_i}{dx} \, dx$$

$$dv = \frac{d}{dx} \left( AE \frac{du}{dx} \right) \, dx, \quad v = AE \frac{du}{dx}$$

then integration by parts gives:

$$\int u \, dv = uv - \int v \, du$$

$$\int_0^L \frac{d}{dx} \left( AE \frac{du}{dx} \right) N_i \, dx = \left[ N_i AE \frac{du}{dx} \right]_0^L - \int_0^L AE \frac{du}{dx} \frac{dN_i}{dx} \, dx = 0$$
Stiffness Matrix for a Bar Element

Example 12 - Bar Element Formulation

Recall that:
\[
\frac{du}{dx} = \frac{dN_1}{dx} u_1 + \frac{dN_2}{dx} u_2 \quad \frac{du}{dx} = \begin{bmatrix} -\frac{1}{L} & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\]

Our original weighted residual expression, with the approximation for \( u \) becomes:
\[
AE \int_0^L dN_i \left[ -\frac{1}{L} \begin{bmatrix} 1 \\ L \end{bmatrix} \right] dx \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \int_0^L \begin{bmatrix} 1 \\ L \end{bmatrix} \frac{dN_i}{dx} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = N_i AE \frac{d\mathbf{u}}{dx} \bigg|_0^L
\]

Stiffness Matrix for a Bar Element

Example 12 - Bar Element Formulation

Substituting \( N_i \) for the weighting function \( N_i \) gives:
\[
AE \int_0^L dN_i \left[ -\frac{1}{L} \begin{bmatrix} 1 \\ L \end{bmatrix} \right] dx \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \int_0^L \begin{bmatrix} 1 \\ L \end{bmatrix} \frac{dN_i}{dx} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{AE L}{L^2} (u_1 - u_2)
\]

\[
\int_0^L N_i AE \frac{d\mathbf{u}}{dx} \bigg|_0^L = AE \frac{d\mathbf{u}}{dx} \bigg|_{x=0} = AE \varepsilon, \quad \sigma|_{x=0} = A \sigma|_{x=0} = f_{1x}
\]

\[
\Rightarrow \frac{AE L}{L^2} (u_1 - u_2) = f_{1x}
\]
### Stiffness Matrix for a Bar Element

**Example 12 - Bar Element Formulation**

Substituting $N_2$ for the weighting function $N_i$ gives:

$$
AE \int_0^L \left[ \frac{1}{L} \right] \left[ \begin{array}{c} -1 \\ 1 \\ \frac{1}{L} \\ \frac{1}{L} \end{array} \right] \left[ \begin{array}{c} u_1 \\ u_2 \\ \frac{1}{L} \\ \frac{1}{L} \end{array} \right] dx \left[ \begin{array}{c} u_1 \\ u_2 \end{array} \right] = \left[ N_2 AE \frac{du}{dx} \right]_0^L
$$

$$
AE \int_0^L \left[ \frac{1}{L} \right] \left[ \begin{array}{c} -1 \\ 1 \\ \frac{1}{L} \\ \frac{1}{L} \end{array} \right] dx \left[ \begin{array}{c} u_1 \\ u_2 \end{array} \right] = \frac{AE L}{L^2} (-u_1 + u_2)
$$

$$
\left| N_2 AE \frac{du}{dx} \right|_0^L = AE \frac{du}{dx} \bigg|_{x=L} = AE \varepsilon_x \bigg|_{x=L} = A \sigma_x \bigg|_{x=L} = f_{2x}
$$

$$
\Rightarrow \frac{AE}{L} (-u_1 + u_2) = f_{2x}
$$

### Stiffness Matrix for a Bar Element

**Example 12 - Bar Element Formulation**

Writing the last two equations in matrix form gives:

$$
\begin{bmatrix}
AE \\
L
\end{bmatrix}
\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
= \begin{bmatrix}
f_{1x} \\
f_{2x}
\end{bmatrix}
$$

This element formulation is identical to that developed from equilibrium and the minimum potential energy approach.
Stiffness Matrix for a Bar Element

Problems:

2. Verify the global stiffness matrix for a three-dimensional bar. **Hint:** First, expand $T^*$ to a 6 x 6 square matrix, then expand $k$ to 6 x 6 square matrix by adding the appropriate rows and columns of zeros, and finally, perform the matrix triple product $k = T^*k'T$.

3. Do problems 3.4, 3.10, 3.12, 3.15a,b, 3.18, 3.23, 3.37, 3.43, 3.48, 3.50, and 3.55 on pages 146 - 165 in your textbook “A First Course in the Finite Element Method” by D. Logan.

4. Use SAP2000 and solve problems 3.63 and 3.64.

End of Chapter 3a