Chapter 2 – In	troduction to the Stiffness
(D	isplacement) Method
A First Course in the Finite Element Method UNITY DIFFIL LOAN	<ul> <li>Learning Objectives</li> <li>To define the stiffness matrix</li> <li>To derive the stiffness matrix for a spring element</li> <li>To demonstrate how to assemble stiffness matrices into a global stiffness matrix</li> <li>To illustrate the concept of direct stiffness method to obtain the global stiffness matrix and solve a spring assemblage problem</li> <li>To describe and apply the different kinds of boundary conditions relevant for spring assemblages</li> <li>To show how the potential energy approach can be used to both derive the stiffness matrix for a spring and solve a spring assemblage problem</li> </ul>

This section introduces some of the basic concepts on which the **direct stiffness method** is based.

The linear spring is simple and an instructive tool to illustrate the basic concepts.

The steps to develop a finite element model for a linear spring follow our general 8 step procedure.

- 1. Discretize and Select Element Types Linear spring elements
- 2. Select a Displacement Function Assume a variation of the displacements over each element.
- 3. Define the Strain/Displacement and Stress/Strain Relationships - use elementary concepts of equilibrium and compatibility.

- **4.** Derive the Element Stiffness Matrix and Equations -Define the stiffness matrix for an element and then consider the derivation of the stiffness matrix for a linearelastic spring element.
- 5. Assemble the Element Equations to Obtain the Global or Total Equations and Introduce Boundary Conditions - We then show how the total stiffness matrix for the problem can be obtained by superimposing the stiffness matrices of the individual elements in a direct manner.

The term *direct stiffness method* evolved in reference to this method.

# The Stiffness (Displacement) Method

- 6. Solve for the Unknown Degrees of Freedom (or Generalized Displacements) Solve for the nodal displacements.
- 7. Solve for the Element Strains and Stresses The reactions and internal forces association with the bar element.
- 8. Interpret the Results



2. Select a Displacement Function - A displacement function *u*(*x*) is assumed.

$$u = a_1 + a_2 x$$

In general, the number of coefficients in the displacement function is equal to the total number of degrees of freedom associated with the element. We can write the displacement function in matrix forms as:

$$\boldsymbol{U} = \begin{bmatrix} 1 & \boldsymbol{X} \end{bmatrix}_{1 \times 2} \begin{cases} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_2 \end{cases}_{2 \times 1}$$

We can express *u* as a function of the nodal displacements  $u_i$  by evaluating *u* at each node and solving for  $a_1$  and  $a_2$ .

$$u(x = 0) = u_1 = a_1$$
  

$$u(x = L) = u_2 = a_2L + a_1$$
Boundary Conditions

Solving for a2:

$$a_2 = \frac{u_2 - u_1}{L}$$

Substituting  $a_1$  and  $a_2$  into *u* gives:

$$u = \left(\frac{u_2 - u_1}{L}\right)x + u_1 = \left(1 - \frac{x}{L}\right)u_1 + \left(\frac{x}{L}\right)u_2$$

# The Stiffness (Displacement) MethodIn matrix form: $u = \left[ \left( 1 - \frac{x}{L} \right) \quad \frac{x}{L} \right] \left\{ \begin{matrix} u_1 \\ u_2 \end{matrix} \right\}$ Or in another form: $u = \left[ N_1 \quad N_2 \right] \left\{ \begin{matrix} u_1 \\ u_2 \end{matrix} \right\}$ Where $N_1$ and $N_2$ are defined as: $N_1 = 1 - \frac{x}{L}$ $N_1 = 1 - \frac{x}{L}$ $N_2 = \frac{x}{L}$ The functions $N_i$ are called *interpolation functions*<br/>because they describe how the assumed displacement<br/>function varies over the domain of the element. In this case<br/>the interpolation functions are linear.







$$T = f_{2x} = k(u_2 - u_1) \implies f_{2x} = k(-u_1 + u_2)$$

We can write the last two force-displacement relationships in matrix form as:

 $\begin{cases} f_{1x} \\ f_{2x} \end{cases} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{cases} u_1 \\ u_2 \end{cases}$ 









## The Stiffness Method – Spring Example 1

Therefore the force-displacement equations for this spring system are:

$$F_{1x} = k_1 u_1 - k_1 u_3 \qquad F_{2x} = -k_2 u_3 + k_2 u_2$$
  
Element 1  
$$F_{3x} = \left(-k_1 u_1 + k_1 u_3\right) + \left(k_2 u_3 - k_2 u_2\right)$$

In matrix form the above equations are:

$$\begin{cases} F_{1x} \\ F_{2x} \\ F_{3x} \end{cases} = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{cases} u_1 \\ u_2 \\ u_3 \end{cases} \qquad \mathbf{F} = \mathbf{K}\mathbf{d}$$

where **F** is the *global nodal force vector*, **d** is called the *global nodal displacement vector*, and **K** is called the *global stiffness matrix*.



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The Stiffness Method – Spring Example 1 Apply the force equilibrium equations at each node.  $\begin{cases}
f_{1x}^{(1)} \\ 0 \\ f_{2x}^{(1)} \\ f_{3x}^{(2)} \\ f_{2x}^{(2)} \\ f_{3x}^{(2)} \\ f_{2x}^{(2)} \\ f_{3x}^{(2)} \\ f_{2x}^{(2)} \\ f_{3x}^{(2)} \\ f_{2x}^{(2)} \\ f_{3x}^{(2)} \\ f_{3x}^{$ 





**The Stiffness Method – Spring Example 1** Consider the equations we developed for the two-spring system. We will consider node 1 to be fixed  $u_1 = 0$ . The equations describing the elongation of the spring system become:  $\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{bmatrix}$ Expanding the matrix equations gives:  $F_{1x} = -k_1 u_3$   $F_{2x} = -k_2 u_3 + k_2 u_2$   $F_{3x} = -k_2 u_2 + (k_1 + k_2) u_3$ Solve for  $u_2$  and  $u_3$ 

# The Stiffness Method – Spring Example 1

The second and third equation may be written in matrix form as:

 $\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_{2x} \\ F_{3x} \end{bmatrix}$ 

Once we have solved the above equations for the unknown nodal displacements, we can use the first equation in the original matrix to find the support reaction.

$$F_{1x} = -k_1 u_3$$

For homogeneous boundary conditions, we can delete the row and column corresponding to the zero-displacement degrees-of-freedom.

## The Stiffness Method – Spring Example 1

Let's again look at the equations we developed for the twospring system.

However, this time we will consider a nonhomogeneous boundary condition at node 1:  $u_1 = \delta$ .

The equations describing the elongation of the spring system become:

$$\begin{bmatrix} k_{1} & 0 & -k_{1} \\ 0 & k_{2} & -k_{2} \\ -k_{1} & -k_{2} & k_{1} + k_{2} \end{bmatrix} \begin{bmatrix} \delta \\ u_{2} \\ u_{3} \end{bmatrix} = \begin{bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{bmatrix}$$

Expanding the matrix equations gives:

$$F_{1x} = k_1 \delta - k_1 u_3 \qquad F_{2x} = -k_2 u_3 + k_2 u_2$$
$$F_{3x} = -k_1 \delta + k_1 u_3 + k_2 u_3 - k_2 u_2$$

# The Stiffness Method – Spring Example 1

By considering the second and third equations because they have known nodal forces we get:

$$F_{2x} = -k_2 u_3 + k_2 u_2 \qquad \qquad F_{3x} = -k_1 \delta + k_1 u_3 + k_2 u_3 - k_2 u_2$$

In matrix form the above equations are:

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{cases} F_{2x} \\ F_{3x} + k_1 \delta \end{bmatrix}$$

For nonhomogeneous boundary conditions, we must transfer the terms from the stiffness matrix to the right-hand-side force vector before solving for the unknown displacements.

# The Stiffness Method – Spring Example 1

Once we have solved the above equations for the unknown nodal displacements, we can use the first equation in the original matrix to find the support reaction.

$$F_{1x} = k_1 \delta - k_1 u_3$$







The Stiffness Method – Spring Example 2						
Substituting for the known force at node 4 ( $F_{4x} = 5,000 \ lb$ ) gives: $\begin{bmatrix} 3000 & -2000 \\ -2000 & 5000 \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5,000 \end{bmatrix}$						
Solving for $u_3$ and $u_4$ gives:						
$u_3 = \frac{10}{11}$ in $u_4 = \frac{15}{11}$ in						



# The Stiffness Method – Spring Example 2Next, use the local element equations to obtain the force in<br/>each spring.For element 1: $\begin{cases} f_{1x} \\ f_{3x} \end{cases} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{cases} 0 \\ 10/11 \end{cases}$ The local forces are: $f_{1x} = -\frac{10,000}{11} lb$ $f_{3x} = \frac{10,000}{11} lb$ A free-body diagram of the spring element 1 is shown below. $\frac{10,000}{11} - \frac{1}{10} - \frac{10,000}{11} - \frac{10$











# The Stiffness Method – Spring Example 3Next, use the local element equations to obtain the force in<br/>each spring.For element 1: $\begin{cases} f_{1x} \\ f_{2x} \end{cases} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{cases} 0 \\ 0.005 \end{cases}$ $f_{1x} = -1.0kN$ $f_{2x} = 1.0kN$ For element 2: $\begin{cases} f_{2x} \\ f_{3x} \end{cases} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{cases} 0.005 \\ 0.01 \end{cases}$ $f_{2x} = -1.0kN$ $f_{3x} = 1.0kN$

The Stiffness Method – Spring Example 3						
Next, use the loca each spring.	al element equations to obtain the force in					
For element 3:	$ \begin{cases} f_{3x} \\ f_{4x} \end{cases} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{cases} 0.01 \\ 0.015 \end{cases} $					
	$f_{3x} = -1.0kN$ $f_{4x} = 1.0kN$					
For element 4:	$ \begin{cases} f_{4x} \\ f_{5x} \end{cases} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{cases} 0.015 \\ 0.02 \end{cases} $					
	$f_{4x} = -1.0kN$ $f_{5x} = 1.0kN$					



# **The Stiffness Method – Spring Example 4** Using the direct stiffness method: the elemental stiffness matrices for each element are: $\mathbf{k}^{(1)} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}_2^1 \quad \mathbf{k}^{(2)} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}_2^2 \quad \mathbf{k}^{(3)} = \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix}_4^2$ Using the concept of superposition (the direct stiffness method), the global stiffness matrix is: $\mathbf{k} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix}_4^1$

# The Stiffness Method – Spring Example 4

Applying the boundary conditions  $(u_1 = u_3 = u_4 = 0)$  the stiffness matrix becomes:

	1	2	3	4	
<b>K</b> =	$k_1$	$-k_1$	0	0	1
	$-k_{1}$	$k_1 + k_2 + k_3$	-k <sub>2</sub>	$-k_3$	2
	0	-k <sub>2</sub>	<i>k</i> <sub>2</sub>	0	3
	0	$-k_3$	0	<i>k</i> <sub>3</sub>	4
I					

# The Stiffness Method – Spring Example 4

Applying the known forces ( $F_{2x}$  = P) gives:

$$\boldsymbol{P} = \left(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3\right)\boldsymbol{u}_2$$

Solving the equation gives: *u* 

$$I_2 = \frac{P}{k_1 + k_2 + k_3}$$

Solving for the forces gives:

$$F_{1x} = -k_1 u_2$$
  $F_{3x} = -k_2 u_2$   $F_{4x} = -k_3 u_2$ 

# Potential Energy Approach to Derive Spring Element Equations

One of the alternative methods often used to derive the element equations and the stiffness matrix for an element is based on the *principle of minimum potential energy*.

This method has the advantage of being more general than the methods involving nodal and element equilibrium equations, along with the stress/strain law for the element.

The principle of minimum potential energy is more adaptable for the determination of element equations for complicated elements (those with large numbers of degrees of freedom) such as the plane stress/strain element, the axisymmetric stress element, the plate bending element, and the threedimensional solid stress element.

# **Total Potential Energy**

The total potential energy  $\pi_p$  is defined as the sum of the internal strain energy **U** and the potential energy of the external forces  $\Omega$ :

$$\pi_{p} = U + \Omega$$

Strain energy is the capacity of the internal forces (or stresses) to do work through deformations (strains) in the structure.

The potential energy of the external forces  $\Omega$  is the capacity of forces such as body forces, surface traction forces, and applied nodal forces to do work through the deformation of the structure.

# Total Potential Energy

Recall the force-displacement relationship for a linear spring:

$$F = kx$$

The differential internal work (or strain energy) dU in the spring is the internal force multiplied by the change in displacement which the force moves through:

$$dU = Fdx = (kx)dx$$







# **Total Potential Energy**

We can replace *G* with the total potential energy  $\pi_p$  and the coordinate *x* with a discrete value  $d_i$ . To minimize  $\pi_p$  we first take the **variation** of  $\pi_p$  (we will not cover the details of variational calculus):

$$\delta \pi_{p} = \frac{\partial \pi_{p}}{\partial d_{1}} \delta d_{1} + \frac{\partial \pi_{p}}{\partial d_{2}} \delta d_{2} + \dots + \frac{\partial \pi_{p}}{\partial d_{n}} \delta d_{n}$$

The principle states that equilibrium exist when the  $d_i$  define a structure state such that  $\delta \pi_p = 0$  for arbitrary admissible variations  $\delta d_i$  from the equilibrium state.







# Total Potential Energy – Example 5

The total potential energy is defined as the sum of the internal strain energy U and the potential energy of the external forces  $\Omega$ :

$$\pi_{\rho} = U + \Omega$$
  $U = \frac{1}{2}kx^2$   $\Omega = -Fx$ 

The variation of  $\pi_p$  with respect to x is:

$$\delta \pi_{p} = \frac{\partial \pi_{p}}{\partial \mathbf{x}} \delta \mathbf{x} = \mathbf{0}$$

Since  $\delta x$  is arbitrary and might not be zero, then:  $\frac{\partial \pi_p}{\partial x} = 0$ 



# **Total Potential Energy**

Let's derive the spring element equations and stiffness matrix using the principal of minimum potential energy. Consider the linear spring subjected to nodal forces shown below:



The total potential energy  $\pi_p$ 

$$\pi_{p} = \frac{1}{2} k (u_{2} - u_{1})^{2} - f_{1x} u_{1} - f_{2x} u_{2}$$

Expanding the above express gives:

$$\pi_{p} = \frac{1}{2} k \left( u_{2}^{2} - 2u_{1}u_{2} + u_{1}^{2} \right) - f_{1x}u_{1} - f_{2x}u_{2}$$

# Total Potential Energy

Let's derive the spring element equations and stiffness matrix using the principal of minimum potential energy. Consider the linear spring subjected to nodal forces shown below:



Therefore:

$$\frac{\partial \pi_p}{\partial u_1} = \frac{k}{2} \left( -2u_2 + 2u_1 \right) - f_{1x} = 0$$
$$\frac{\partial \pi_p}{\partial u_2} = \frac{k}{2} \left( 2u_2 - 2u_1 \right) - f_{2x} = 0$$

# **Total Potential Energy**

Let's derive the spring element equations and stiffness matrix using the principal of minimum potential energy. Consider the linear spring subjected to nodal forces shown below:



$$k\left(-U_1+U_2\right)=f_{2x}$$

In matrix form the equations are:

$$\begin{cases} f_{1x} \\ f_{2x} \end{cases} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{cases} u_1 \\ u_2 \end{cases}$$



## Total Potential Energy – Example 6

Obtain the total potential energy of the spring system shown below and find its minimum value.

The potential energy  $\pi_p$  for element 3 is:

$$\pi_{\rho}^{(3)} = \frac{1}{2} k_3 (u_2 - u_4)^2 - f_{2x} u_2 - f_{4x} u_4$$

The total potential energy  $\pi_p$  for the spring system is:

$$\pi_p = \sum_{e=1}^3 \pi_p^{(e)}$$



 $\begin{aligned} \text{Total Potential Energy} - \text{Example 6} \\ \text{In matrix form:} & \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{cases} f_{1x} \\ f_{3x}^{(1)} + f_{3x}^{(2)} \\ f_{4x}^{(2)} + f_{4x}^{(3)} \end{bmatrix} \\ \end{aligned}$   $\begin{aligned} \text{Using the following force equilibrium equations:} \\ f_{1x}^{(1)} = F_{1x} \\ f_{2x}^{(3)} = F_{2x} \\ f_{3x}^{(1)} + f_{3x}^{(2)} = F_{3x} \\ f_{4x}^{(2)} + f_{4x}^{(3)} = F_{4x} \end{aligned}$ 



