A First Course in Finite Elements

Introduction

- The **finite element method** has become a powerful tool for the numerical solution of a wide range of engineering problems.
- Applications range from deformation and stress analysis of automotive, aircraft, building, and bridge structures to field analysis of heat flux, fluid flow, magnetic flux, seepage, and other flow problems.

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Introduction

- With the advances in computer technology and CAD systems, complex problems can be modeled with relative ease.
- Several alternative configurations can be tried out on a computer before the first prototype is built.
- All of this suggests that we need to keep pace with these developments by understanding the basic theory, modeling techniques, and computational aspects of the finite element method.

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Introduction

- In this method of analysis, a complex region defining a continuum is discretized into simple geometric shapes called **finite elements**.
- The material properties and the governing relationships are considered over these elements and expressed in terms of unknown values at element corners.
- An assembly process, considering the loading and constraints, results in a set of equations.
- Solution of these equations gives an approximate behavior of the continuum.

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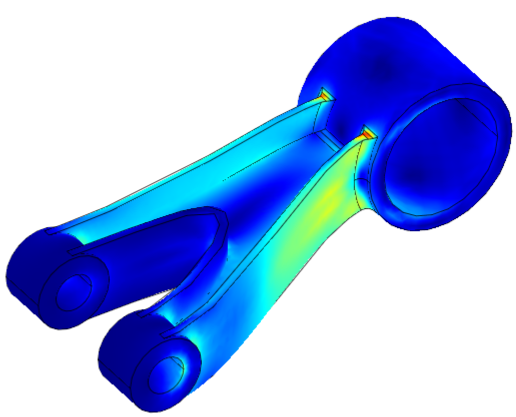

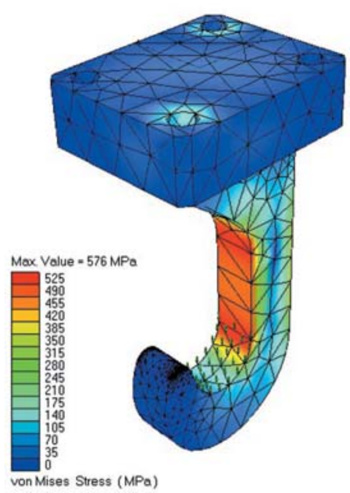

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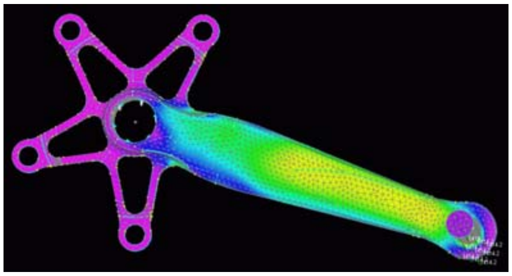

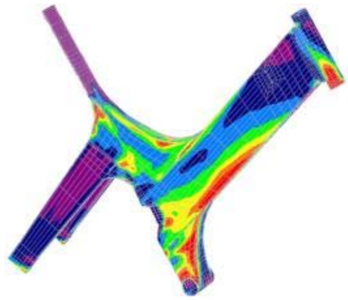

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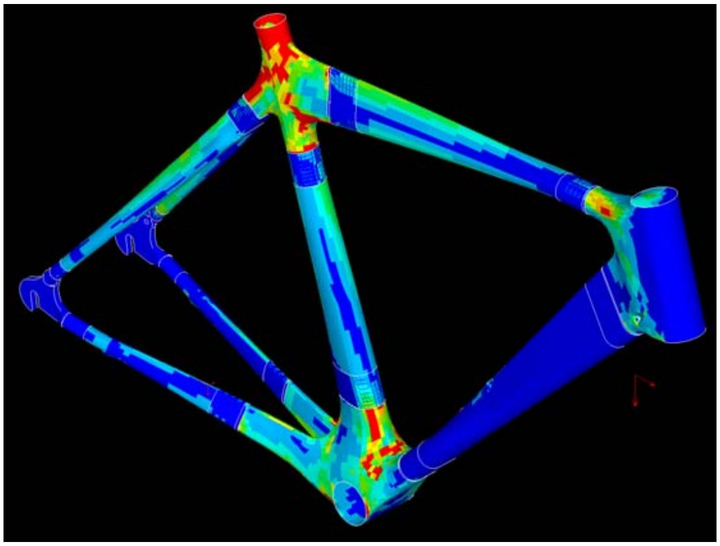

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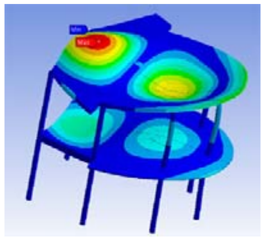

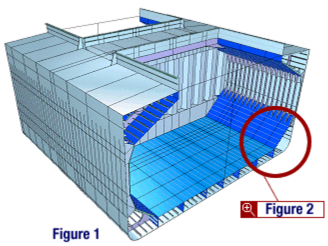

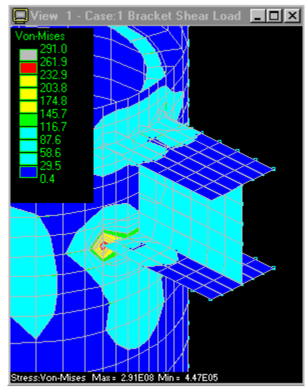

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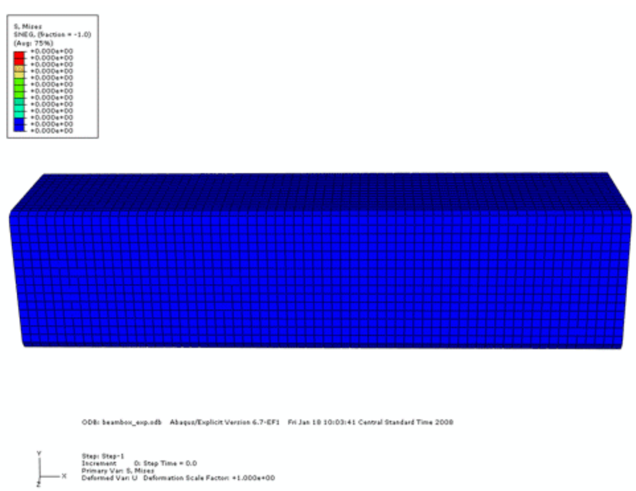

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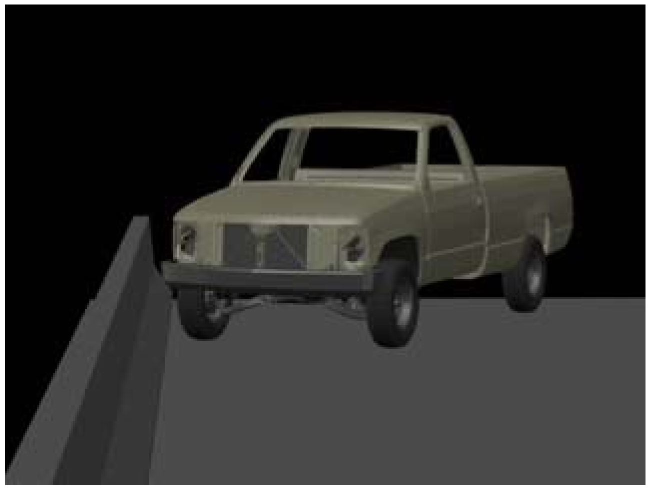

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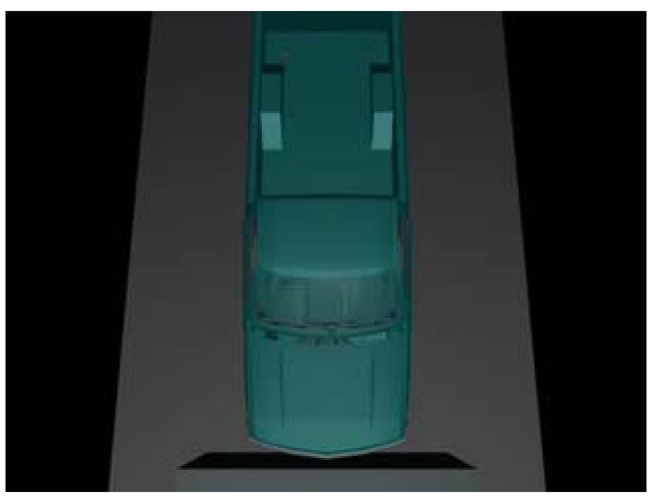

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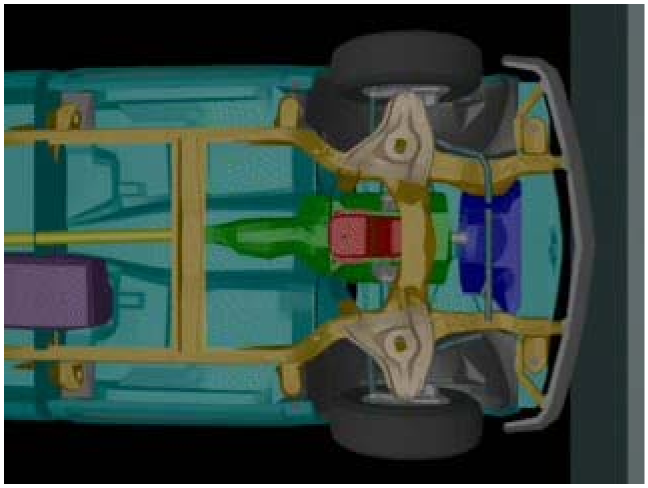

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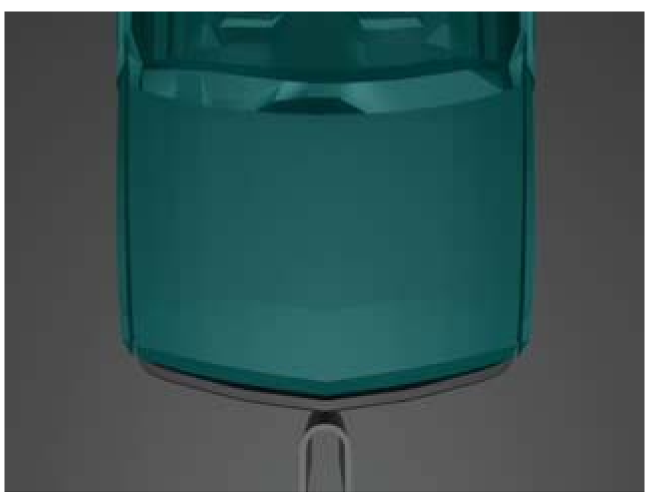

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Historical Background

- Basic ideas of the finite element method originated from advances in aircraft structural analysis.
- In 1941, Hrenikoff presented a solution of elasticity problems using the “frame work method.”
- Courant’s paper, which used piecewise polynomial interpolation over triangular subregions to model torsion problems, appeared in 1943.

- Turner et al. derived stiffness matrices for truss, beam, and other elements and presented their findings in 1956.
- The term “finite element” was first coined and used by Clough in 1960.

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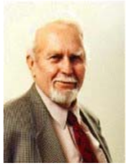
Historical Background

- In the early 1960s, engineers used the method for approximate solution of problems in stress analysis, fluid flow, heat transfer, and other areas.
- A book by Argyris in 1955 on energy theorems and matrix methods laid a foundation for further developments in finite element studies.

- The first book on finite elements by Zienkiewicz and Chung was published in 1967.



- In the late 1960s and early 1970s, finite element analysis was applied to nonlinear problems and large deformations.

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Historical Background

- Until the early 1950s, matrix methods and the associated finite element method were not readily adaptable for solving complicated problems because of the large number of algebraic equations that resulted.

- Hence, even though the finite element method was being used to describe complicated structures, the resulting large number of equations associated with the finite element method of structural analysis made the method extremely difficult and impractical to use.

- With the advent of the computer, the solution of thousands of equations in a matter of minutes became possible.

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Historical Background

- Mathematical foundations were laid in the 1970s.

- New element development, convergence studies, and other related areas fall in this category.

- Today, developments in distributed or multi-node computers and availability of powerful microcomputers have brought this method within reach of students and engineers working in small industries.

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General Steps of the Finite Element Method

- Typically, for the structural stress-analysis problem, the engineer seeks to determine ***displacements*** and ***stresses*** throughout the structure, which is in equilibrium and is subjected to applied loads.

- For many structures, it is difficult to determine the distribution of deformation using conventional methods, and thus the finite element method is necessarily used.

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General Steps of the Finite Element Method

- There are two general approaches associated with the finite element method.

- One approach, called the ***force***, or ***flexibility method***, uses internal forces as the unknowns of the problem.

- To obtain the governing equations, first the equilibrium equations are used. Then necessary additional equations are found by introducing compatibility equations.

- The result is a set of algebraic equations for determining the redundant or unknown forces.

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General Steps of the Finite Element Method

- The second approach, called the ***displacement***, or ***stiffness method***, assumes the displacements of the nodes as the unknowns of the problem.

- The governing equations are expressed in terms of nodal displacements using the equations of equilibrium and an applicable law relating forces to displacements.

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General Steps of the Finite Element Method

- These two approaches result in different unknowns (forces or displacements) in the analysis and different matrices associated with their formulations (flexibilities or stiffnesses).

- It has been shown that, for computational purposes, the displacement (or stiffness) method is more desirable because its formulation is simpler for most structural analysis problems.

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General Steps of the Finite Element Method

- The ***finite element method*** involves modeling the structure using small interconnected elements called ***finite elements***.
- A displacement function is associated with each finite element.

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General Steps of the Finite Element Method

- Every interconnected element is linked, directly or indirectly, to every other element through common (or shared) interfaces, including nodes and/or boundary lines and/or surfaces.
- The total set of equations describing the behavior of each node results in a series of algebraic equations best expressed in matrix notation.

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Basic Ingredients - Discrete Problems

The basic steps or building blocks of any application of FEM to a mathematical or physical problem are:

1. Discretization
2. Interpolation
3. Elemental Description or Formulation
4. Assembly
5. Constraints
6. Solution
7. Computation of Derived Variables

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Basic Ingredients - Discrete Problems

Whether the FEM is used for

(1) setting up and solving a problem by hand,
(2) using an existing FEM program to solve a problem, or
(3) generating or writing a program to solve a class of problems;

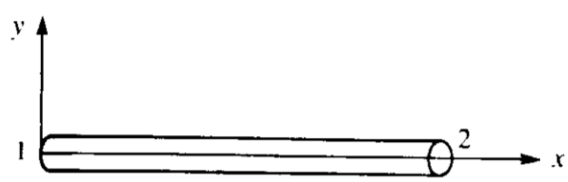
These basic steps are important in setting-up and solving the FEM model.

Step 1 - Discretize and Select Element Types

- Step 1 involves dividing the body into an equivalent system of finite elements with associated nodes and choosing the most appropriate element type.
- The total number of elements used and their variation in size and type within a given body are primarily matters of engineering judgment.
- The elements must be made small enough to give usable results and yet large enough to reduce computational effort.
- Small elements (and possibly higher-order elements) are generally desirable where the results are changing rapidly, such as where changes in geometry occur, whereas large elements can be used where results are relatively constant.

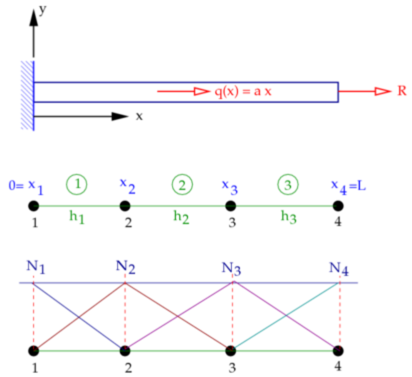
Step 1 - Discretize and Select Element Types

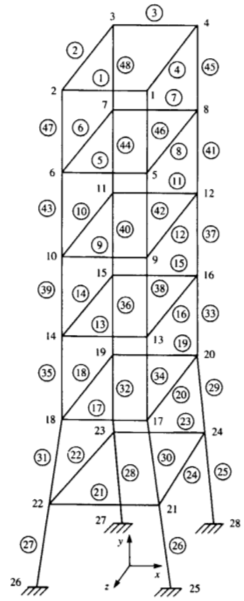
- Primary ***line elements*** consist of bar (or truss) and ***beam elements***.
- They have a cross-sectional area but are usually represented by line segments.



- The simplest line element (called a ***linear element***) has two nodes, one at each end, although higher-order elements having three nodes or more (called ***quadratic***, ***cubic***, etc. ***elements***) also exist.

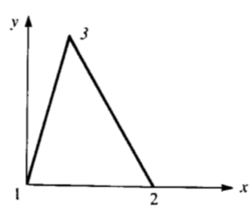
Step 1 - Discretize and Select Element Types

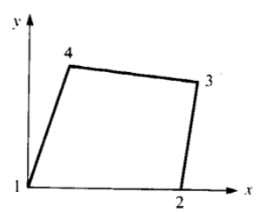




Step 1 - Discretize and Select Element Types

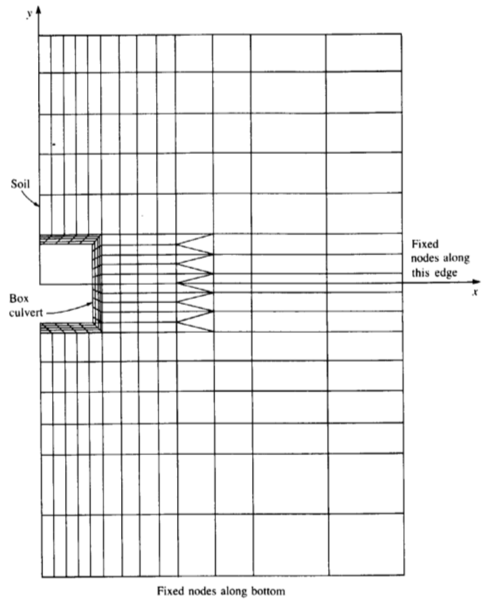
- The basic two-dimensional (or plane) elements are loaded by forces in their own plane (plane stress or plane strain conditions). They are triangular or quadrilateral elements.





- The simplest two-dimensional elements have corner nodes only (linear elements) with straight sides or boundaries although there are also higher-order elements, typically with mid-side nodes (called ***quadratic elements***) and curved sides.

Step 1 - Discretize and Select Element Types



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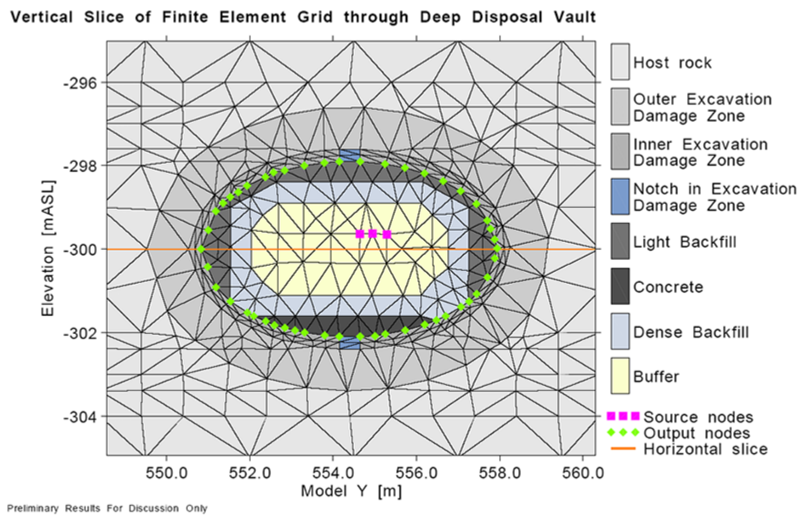

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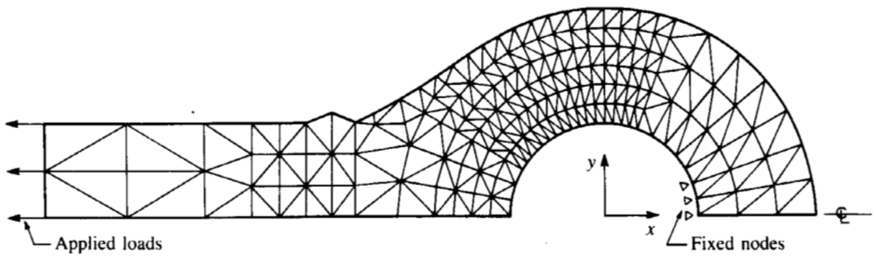

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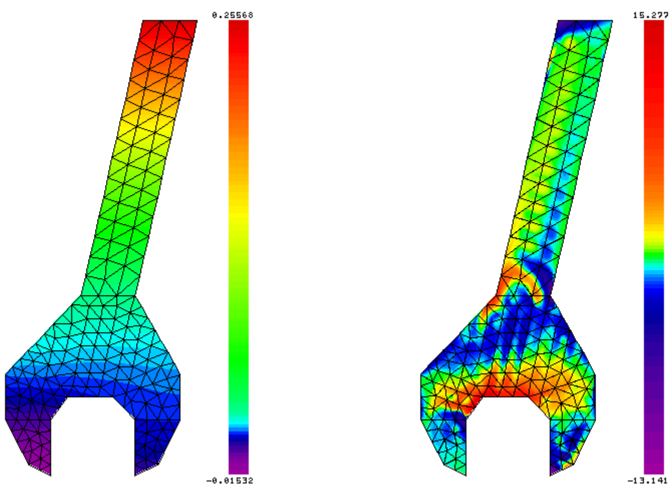

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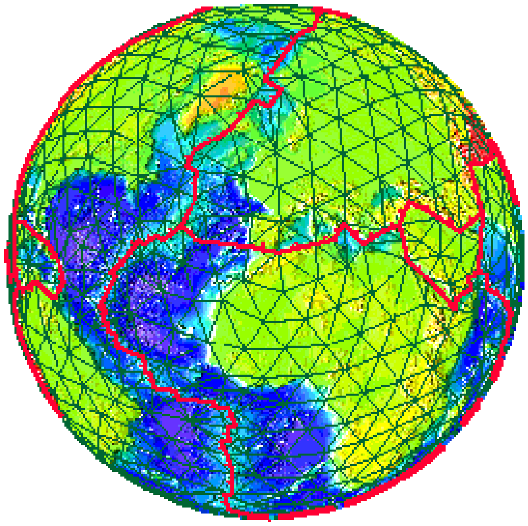
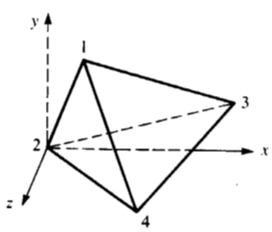
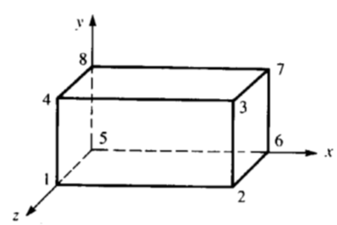


Step 1 - Discretize and Select Element Types

- The most common three-dimensional elements are tetrahedral and hexahedral (or ***brick***) ***elements***; they are used when it becomes necessary to perform a three-dimensional stress analysis.
- The basic three dimensional elements have corner nodes only and straight sides, whereas higher-order elements with mid-edge nodes (and possible mid-face nodes) have curved surfaces for their sides





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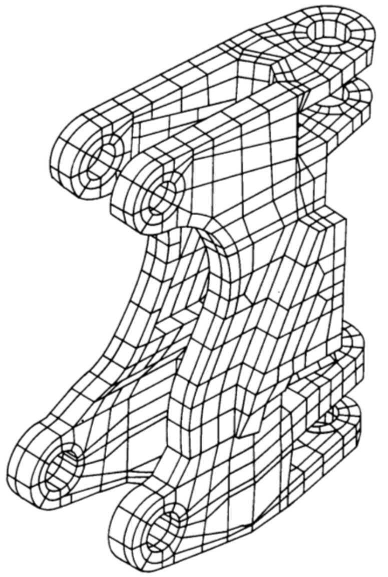

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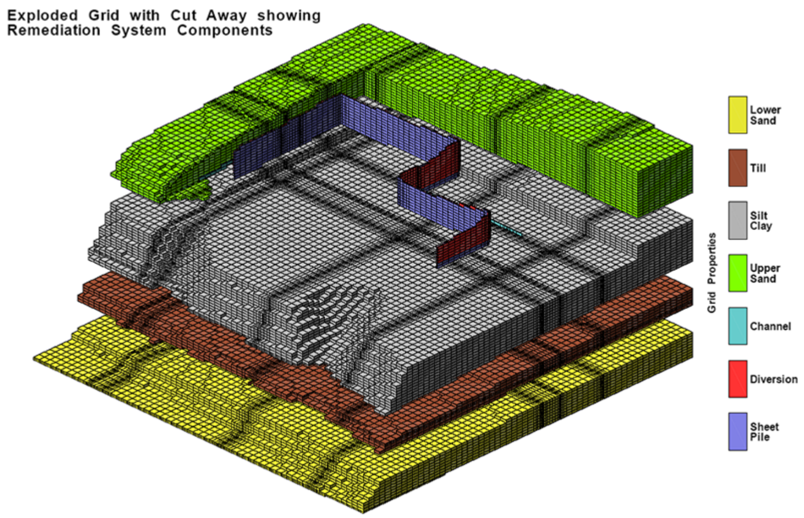

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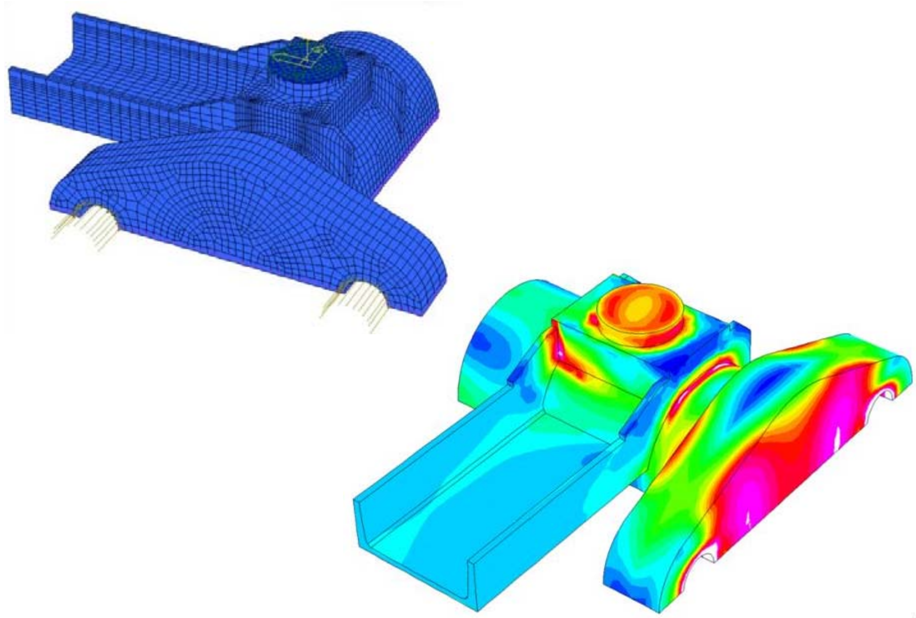
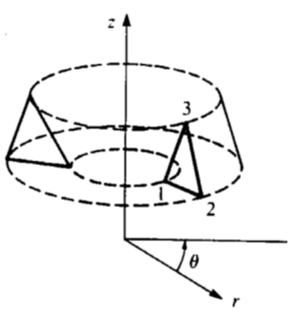


Step 1 - Discretize and Select Element Types

- The ***axisymmetric element*** is developed by rotating a triangle or quadrilateral about a fixed axis located in the plane of the element through 360°.
- This element can be used when the geometry and loading of the problem are axisymmetric.



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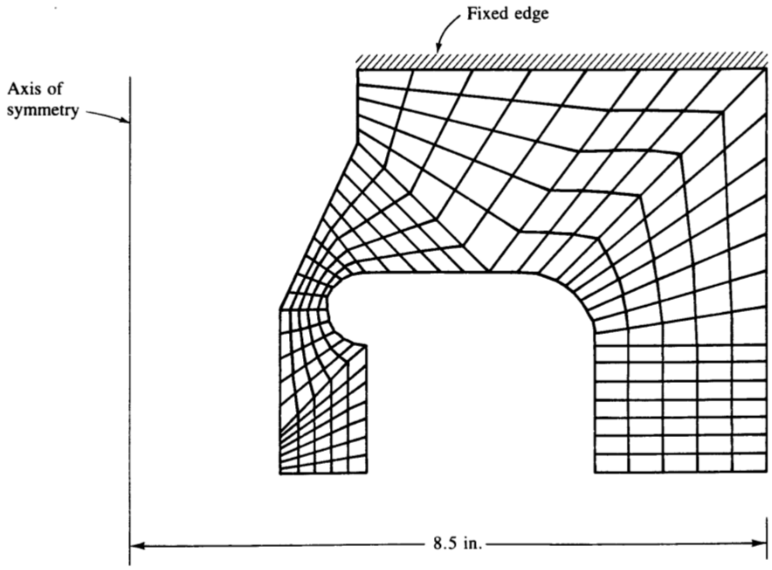

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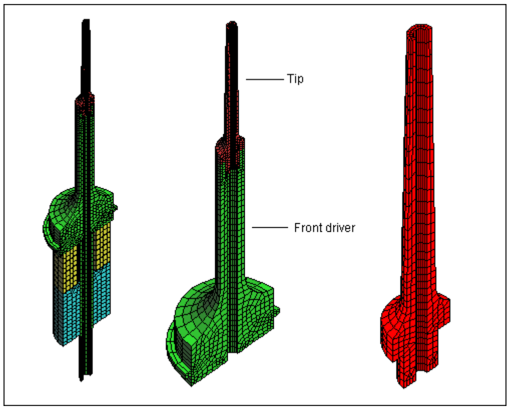
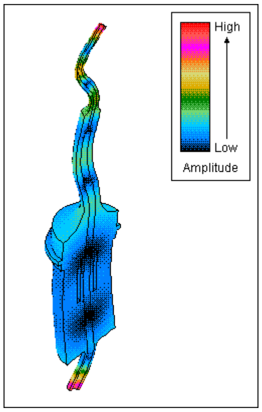


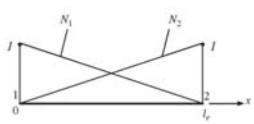


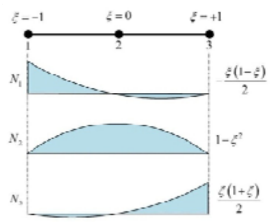
Step 2 - Select a Displacement Function

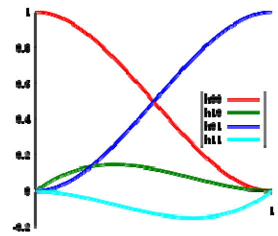
Step 2 involves choosing a displacement function within each element.

The function is defined within the element using the nodal values of the element.

Linear, quadratic, and cubic polynomials are frequently used functions because they are simple to work with in finite element formulation.



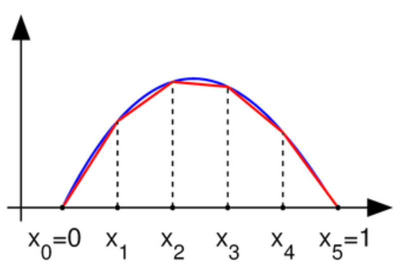


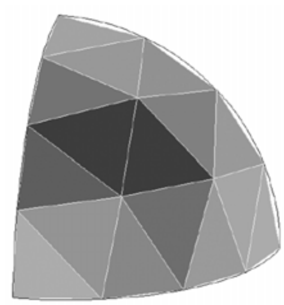


Step 2 - Select a Displacement Function

The functions are expressed in terms of the nodal unknowns (in the two-dimensional problem, in terms of an *x* and a *y* component).

Hence, the finite element method is one in which a continuous quantity, such as the displacement throughout the body, is approximated by a discrete model composed of a set of piecewise-continuous functions defined within each finite domain or finite element.





Step 3 - Define the Strain/Displacement and Stress/Strain Relationships

Strain/displacement and stress/strain relationships are necessary for deriving the equations for each finite element.

For one-dimensional small strain deformation, say, in the *x* direction, we have strain ε_x, related to displacement ***u*** by

$$\varepsilon_x = \frac{du}{dx}$$

Stresses must be related to the strains through the stress/strain law (generally called the ***constitutive law***). The simplest of stress/strain laws, Hooke's law, often used in stress analysis, is given by:

$$\sigma_x = E\varepsilon_x$$

Step 4 - Derive the Element Stiffness Matrix and Equations

Direct Equilibrium Method - According to this method, the stiffness matrix and element equations relating nodal forces to nodal displacements are obtained using force equilibrium conditions for a basic element, along with force/deformation relationships.

This method is most easily adaptable to line or one-dimensional elements (spring, bar, and beam elements)

Step 4 - Derive the Element Stiffness Matrix and Equations

Work or Energy Methods - To develop the stiffness matrix and equations for two- and three-dimensional elements, it is much easier to apply a work or energy method.

The ***principle of virtual work*** (using virtual displacements), the principle of minimum potential energy, and Castigliano's theorem are methods frequently used for the purpose of derivation of element equations.

We will present the principle of minimum potential energy (probably the most well known of the three energy methods mentioned here)

Step 4 - Derive the Element Stiffness Matrix and Equations

Methods of Weighted Residuals - The methods of weighted residuals are useful for developing the element equations (particularly popular is Galerkin's method).

These methods yield the same results as the energy methods, wherever the energy methods are applicable.

They are particularly useful when a ***functional*** such as potential energy is not readily available.

The weighted residual methods allow the finite element method to be applied directly to any differential equation

Step 5 - Assemble the Element Equations and Introduce Boundary Conditions

The individual element equations generated in Step 4 can now be added together using a method of superposition (called the ***direct stiffness method***) whose basis is nodal force equilibrium (to obtain the global equations for the whole structure).

Implicit in the direct stiffness method is the concept of continuity, or compatibility, which requires that the structure remain together and that no tears occur anywhere in the structure.

The final assembled or global equation written in matrix form is:

$$\{F\} = [K]\{d\}$$

Step 6 - Solve for the Unknown Degrees of Freedom (or Generalized Displacements)

Once the element equations are assembled and modified to account for the boundary conditions, a set of simultaneous algebraic equations that can be written in expanded matrix form as:

$$\begin{Bmatrix} F_1 \\ F_2 \\ \cdot \\ F_n \end{Bmatrix}_{n \times 1} = \begin{bmatrix} K_{11} & K_{12} & \cdot & K_{1n} \\ K_{21} & K_{22} & \cdot & K_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ K_{n1} & K_{n2} & \cdot & K_{nn} \end{bmatrix}_{n \times n} \begin{Bmatrix} d_1 \\ d_2 \\ \cdot \\ d_n \end{Bmatrix}_{n \times 1}$$

where ***n*** is the structure total number of unknown nodal degrees of freedom.

These equations can be solved for the ***d***'s by using an elimination method (such as Gauss's method) or an iterative method (such as Gauss Seidel's method)

Step 7 - Solve for the Element Strains and Stresses

For the structural stress-analysis problem, important secondary quantities of strain and stress (or moment and shear force) can be obtained in terms of the displacements determined in Step 6.

Step 8 - Interpret the Results

The final goal is to interpret and analyze the results for use in the design/analysis process.

Determination of locations in the structure where large deformations and large stresses occur is generally important in making design/analysis decisions.

Advantages of the Finite Element Method

The finite element method has been applied to numerous problems, both structural and non-structural. This method has a number of advantages that have made it very popular.

1. Model irregularly shaped bodies quite easily
2. Handle general load conditions without difficulty
3. Model bodies composed of several different materials because the element equations are evaluated individually
4. Handle unlimited numbers and kinds of boundary conditions

Advantages of the Finite Element Method

5. Vary the size of the elements to make it possible to use small elements where necessary

6. Alter the finite element model relatively easily and cheaply

7. Include dynamic effects

8. Handle nonlinear behavior existing with large deformations and nonlinear materials

The finite element method of structural analysis enables the designer to detect stress, vibration, and thermal problems during the design process and to evaluate design changes ***before*** the construction of a possible prototype.

End of Introduction