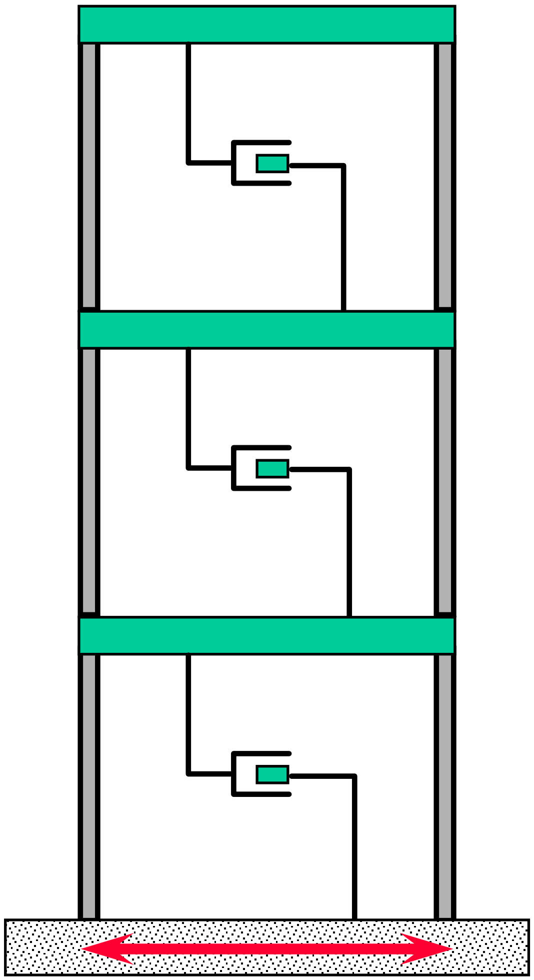
TOPIC 6 Structural Dynamics III

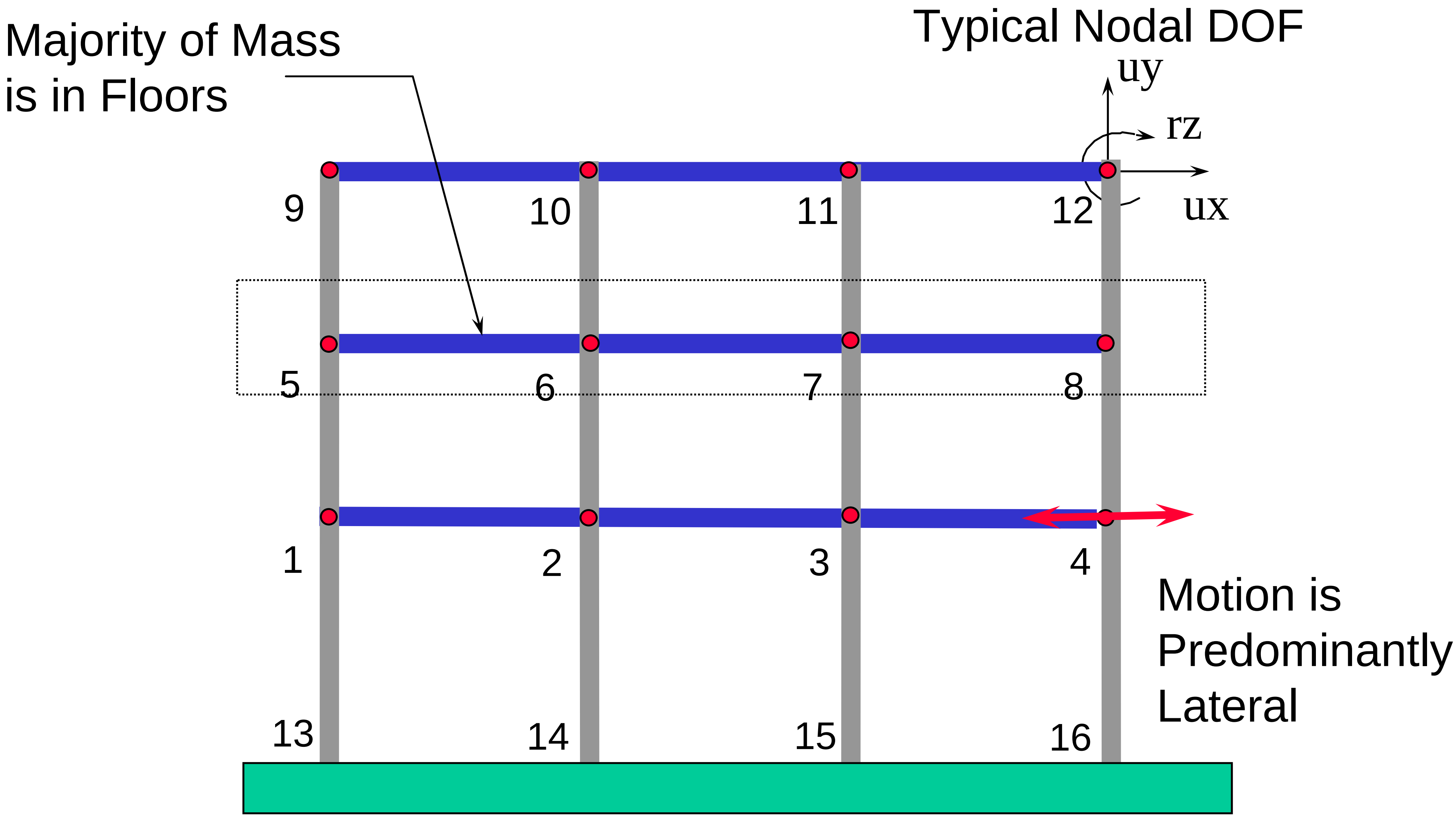
Analysis of Elastic MDOF Systems



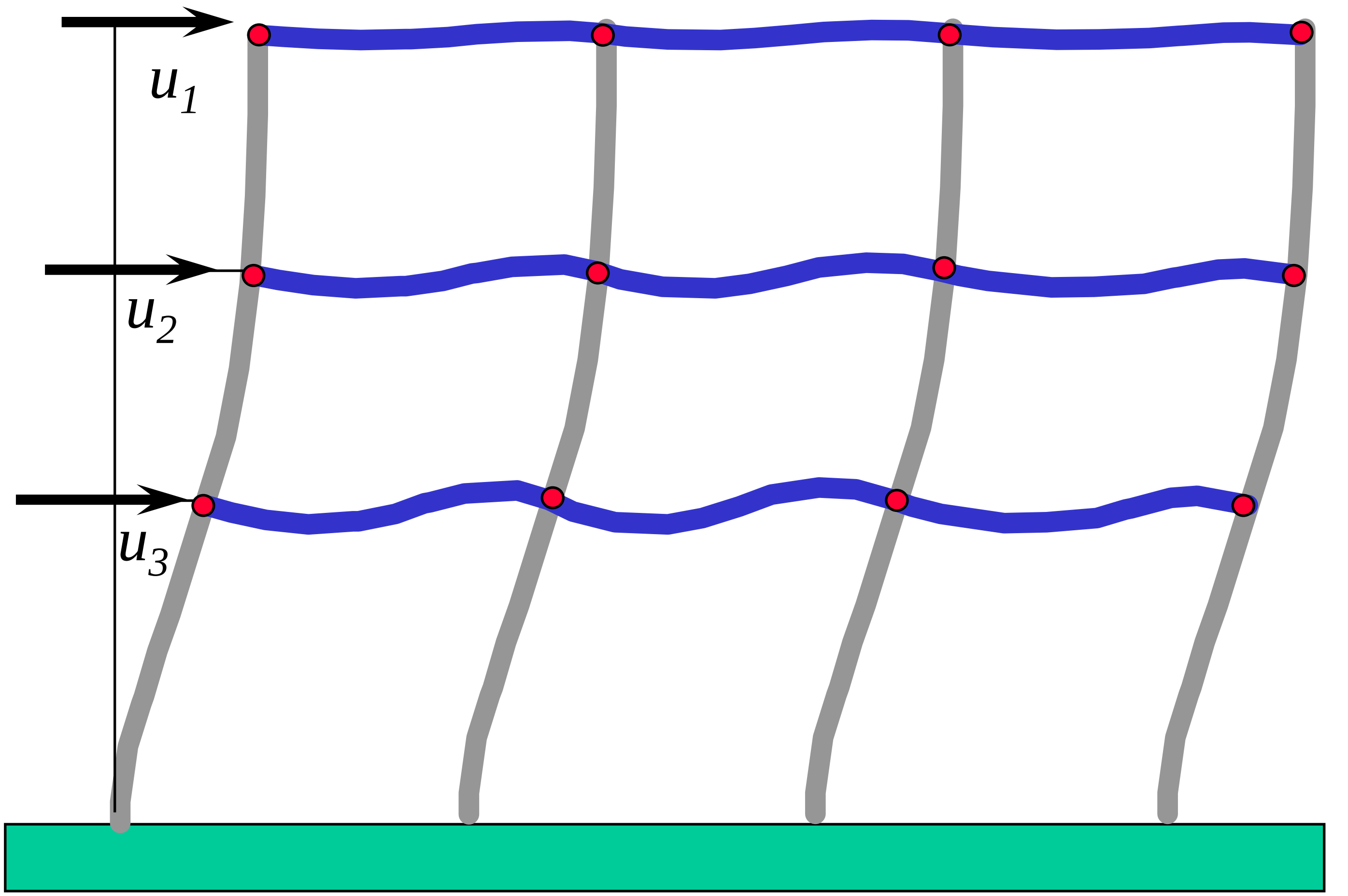
TOPIC 6   Structural Dynamics III
Analysis of Elastic MDOF Systems

- Equations of Motion for MDOF Systems
- Uncoupling of Equations through use of Natural Mode Shapes
- Solution of Uncoupled Equations
- Recombination of Computed Response
- Modal Response Spectrum Analysis (By Example)
- Use of Reduced Number of Modes

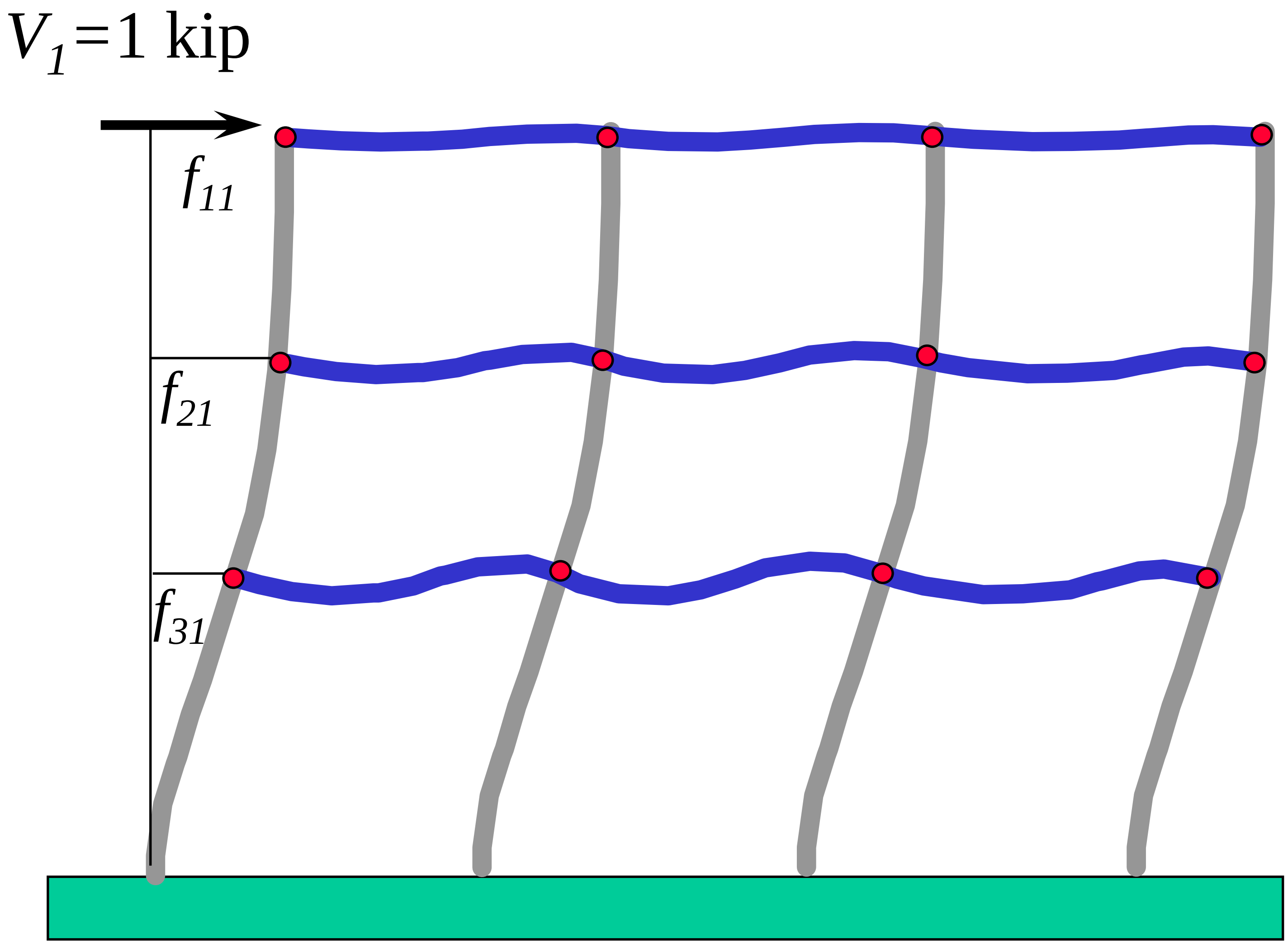
Planar Frame with 36 Degrees of Freedom



Planar Frame with 36 Static Degrees of Freedom but with only THREE Dynamic DOF

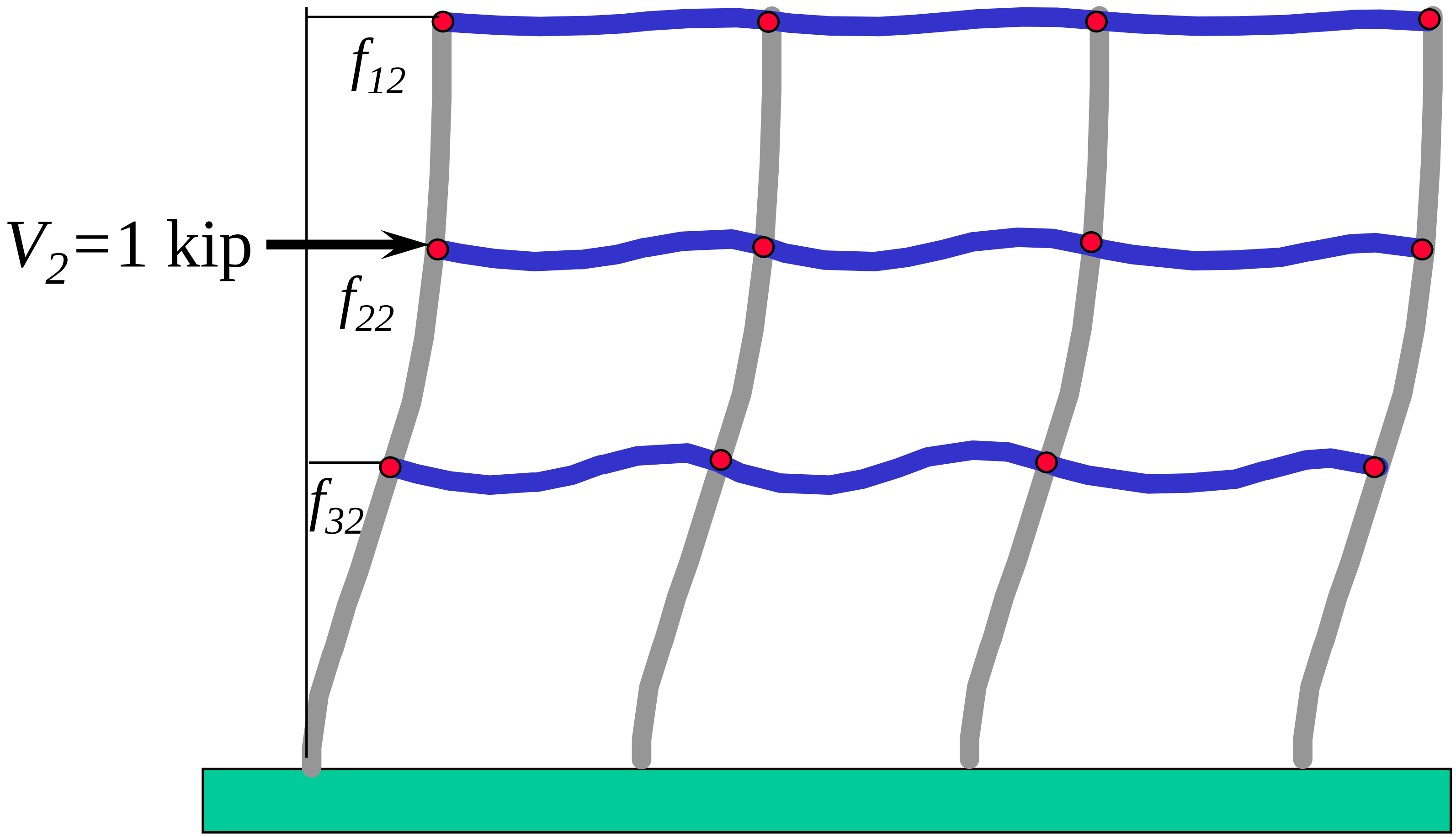


Development of Flexibility Matrix



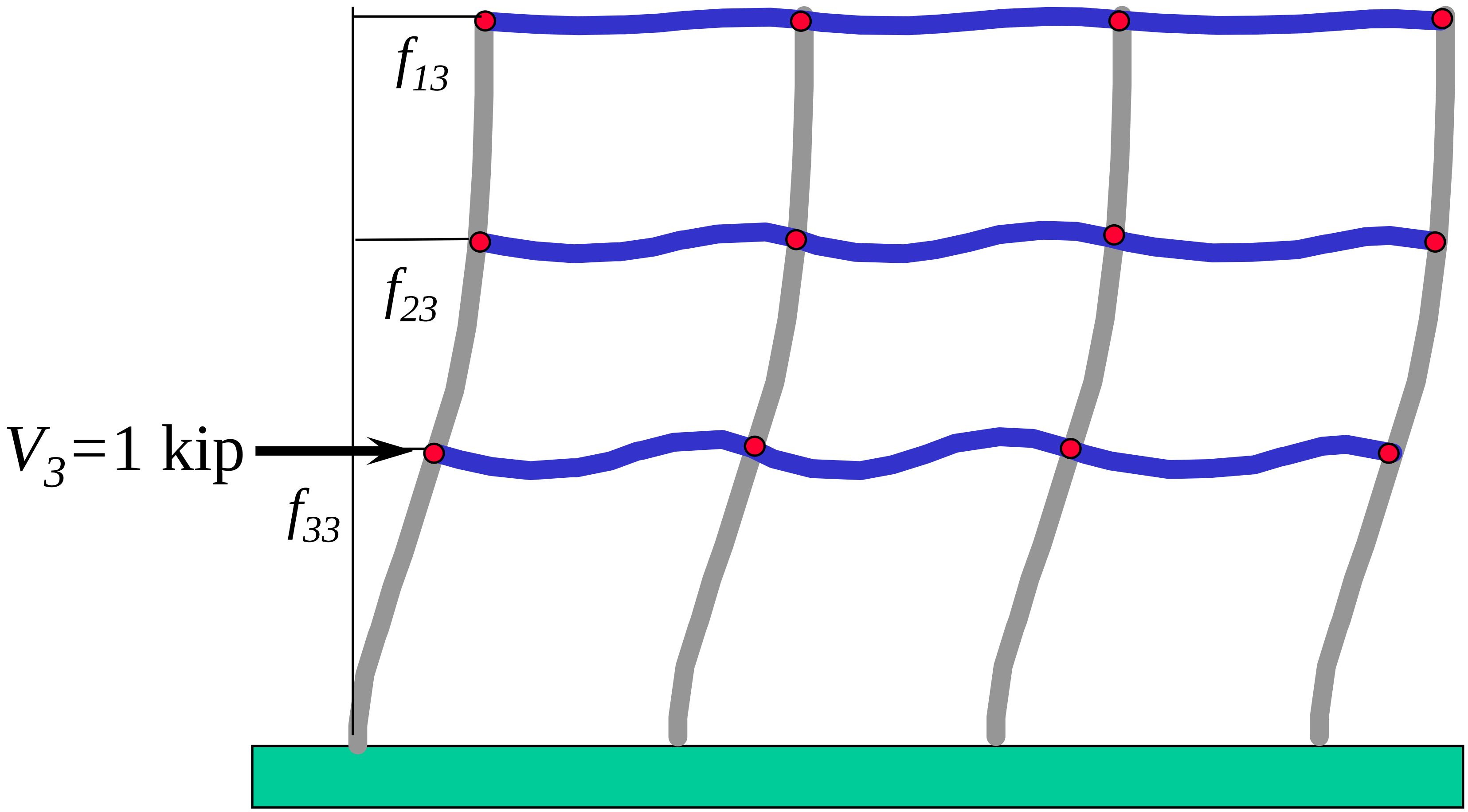
$$\begin{bmatrix} f_{11} & \bullet & \bullet \\ f_{21} & \bullet & \bullet \\ f_{31} & \bullet & \bullet \end{bmatrix}$$

Development of Flexibility Matrix



$$\begin{bmatrix} \bullet & f_{12} & \bullet \\ \bullet & f_{22} & \bullet \\ \bullet & f_{32} & \bullet \end{bmatrix}$$

Development of Flexibility Matrix



$$\begin{bmatrix} \bullet & \bullet & f_{13} \\ \bullet & \bullet & f_{23} \\ \bullet & \bullet & f_{33} \end{bmatrix}$$

Concept of Linear Combination of Shapes (Flexibility)

$$U = \begin{Bmatrix} f_{11} \\ f_{21} \\ f_{31} \end{Bmatrix} V_1 + \begin{Bmatrix} f_{12} \\ f_{22} \\ f_{33} \end{Bmatrix} V_2 + \begin{Bmatrix} f_{13} \\ f_{23} \\ f_{33} \end{Bmatrix} V_3$$

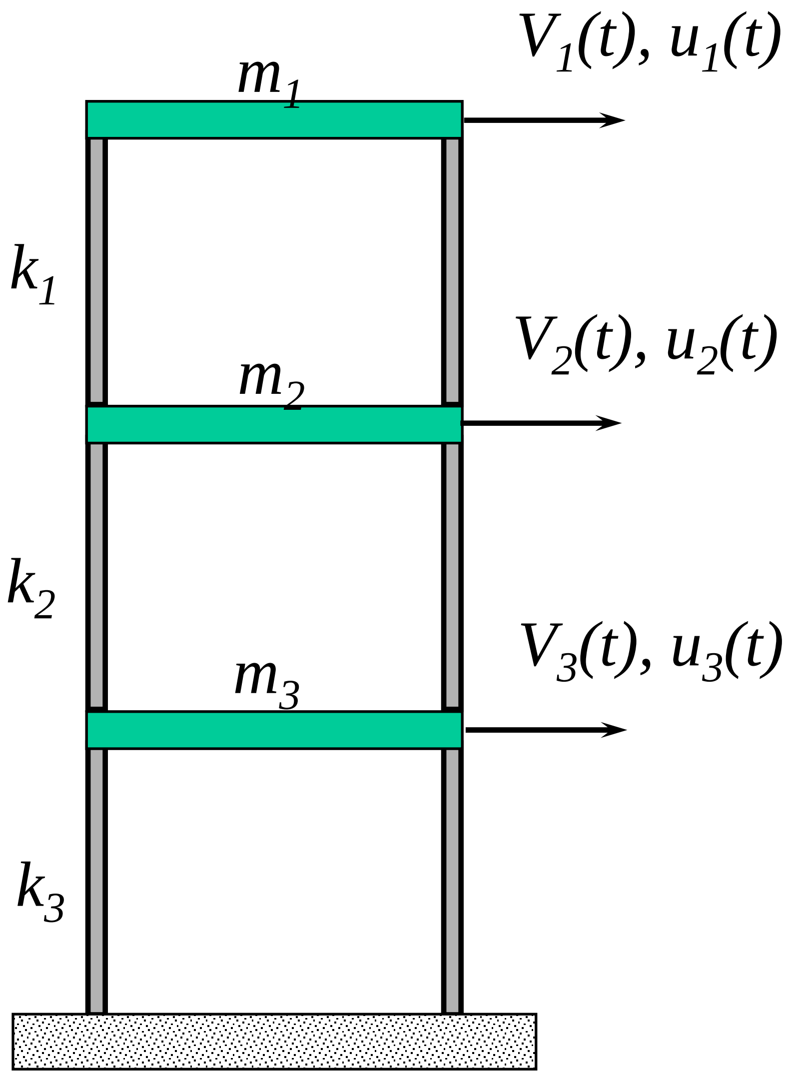
$$U = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix}$$

$$FV = U$$

$$KU = V$$

$$K = F^{-1}$$

Idealized Structural Property Matrices



$$K = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 + k_3 \end{bmatrix}$$

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

$$V(t) = \begin{Bmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \end{Bmatrix} \qquad U(t) = \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix}$$

Note: Damping to be shown later

Static Condensation

$$\begin{bmatrix} K_{n,n} & K_{n,m} \\ K_{m,n} & K_{m,m} \end{bmatrix} \begin{bmatrix} r_n \\ r_m \end{bmatrix} = \begin{bmatrix} F_n \\ 0 \end{bmatrix}$$

DOF with mass

Massless DOF

1 $$K_{n,n} r_n + K_{n,m} r_m = F_n$$

2 $$K_{m,n} r_n + K_{m,m} r_m = 0$$

Static Condensation (continued)

Rearrange 2 $$r_m = -K_{m,m}^{-1} K_{m,n} r_n$$

Plug into 1 $$K_{n,n} r_n - K_{n,m} K_{m,m}^{-1} K_{m,n} r_n = F_n$$

Simplify $$\left\{ K_{n,n} - K_{n,m} K_{m,m}^{-1} K_{m,n} \right\} r_n = F_n$$

$$\hat{K} r_n = F_n$$

$$\boxed{\hat{K} = K_{n,n} - K_{n,m} K_{m,m}^{-1} K_{m,n}}$$

Condensed Stiffness Matrix

Coupled Equations of Motion for Undamped Forced Vibration

$$M\ddot{U}(t) + KU(t) = V(t)$$

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \ddot{u}_3(t) \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = \begin{Bmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \end{Bmatrix}$$

DOF 1 $\quad m_1\ddot{u}_1(t) + k_1u_1(t) - k_1u_2(t) = V_1(t)$

DOF 2 $\quad m_2\ddot{u}_2(t) - k_1u_1(t) + k_1u_2(t) + k_2u_2(t) - k_2u_3(t) = V_2(t)$

DOF 3 $\quad m_3\ddot{u}_3(t) - k_2u_2(t) + k_2u_3(t) + k_3u_3(t) = V_3(t)$

Solutions for System in Undamped Free Vibration

$$M\ddot{u}(t) + Ku(t) = 0$$

Assume $u(t) = \phi \sin \omega t \qquad \ddot{u}(t) = -\omega^2 \phi \sin \omega t$

Then $K\phi - \omega^2 M\phi = 0$ has three (n) solutions:

$$\phi_1 = \begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \\ \varphi_{31} \end{Bmatrix}, \quad \omega_1 \qquad \phi_2 = \begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \\ \varphi_{32} \end{Bmatrix}, \quad \omega_2 \qquad \phi_3 = \begin{Bmatrix} \varphi_{13} \\ \varphi_{23} \\ \varphi_{33} \end{Bmatrix}, \quad \omega_3$$

Natural Frequency

Natural Mode Shape

Solutions for System in Undamped Free Vibration

$$K\phi = \omega^2 M\phi$$ For a SINGLE Mode

$$K\Phi = M\Phi\Omega^2$$ For ALL Modes

Where: $\Phi = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \end{bmatrix}$

$$\Omega^2 = \begin{bmatrix} \omega_1^2 & & \\ & \omega_2^2 & \\ & & \omega_3^2 \end{bmatrix}$$

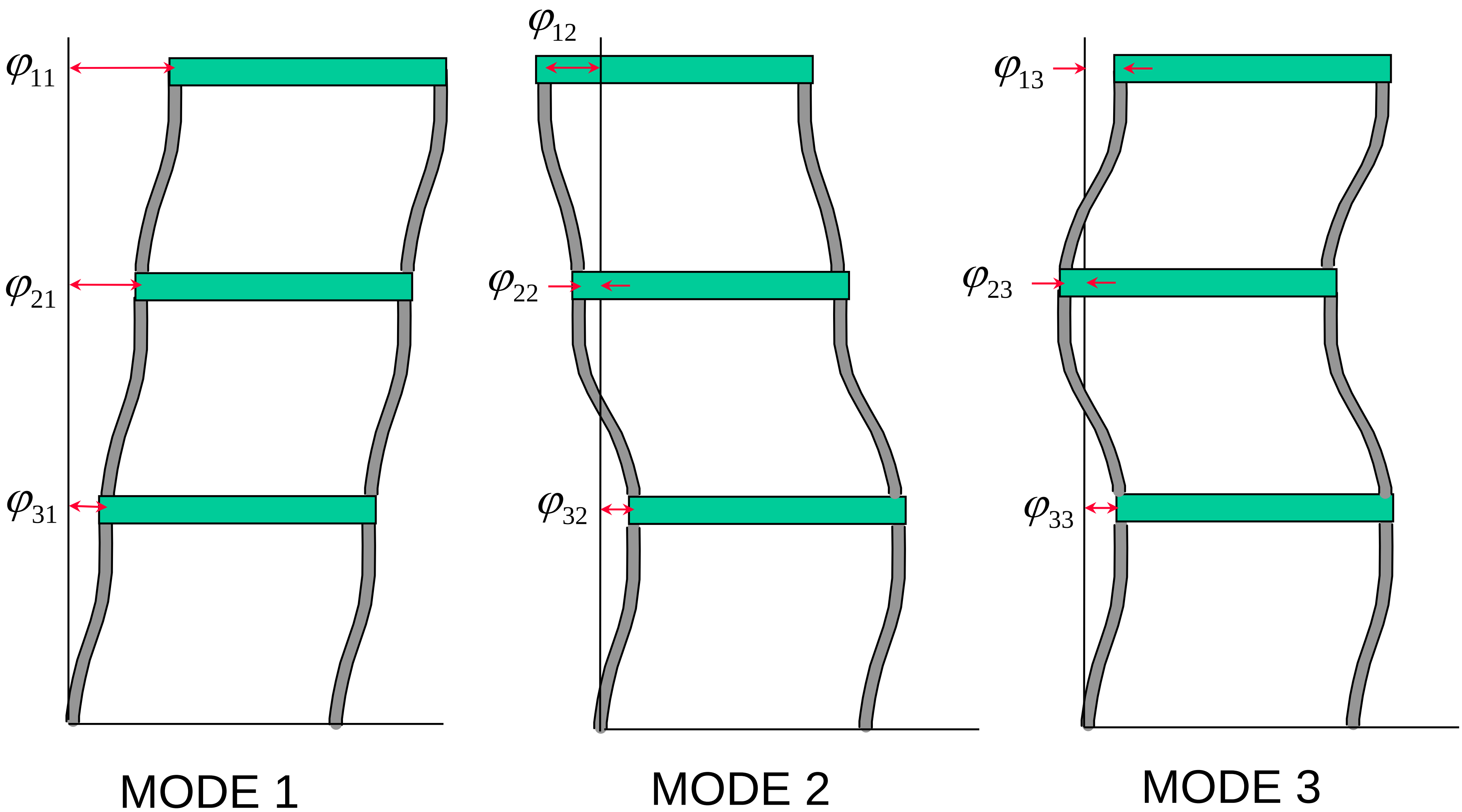
Note: Mode shape has arbitrary scale. Usually

$$\varphi_{1i} = 1.0$$

or

$$\Phi^T M\Phi = I$$

Mode Shapes for Idealized 3-Story Frame



Concept of Linear Combination of Mode Shapes
(Change of Coordinates)

$$U = \Phi Y$$

Mode Shape

$$U = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{Bmatrix} \qquad U = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} Y_1 + \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{33} \end{Bmatrix} Y_2 + \begin{Bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{Bmatrix} Y_3$$

Modal Coordinate

$$KU = V$$

$$K\Phi Y = V$$

$$\boxed{\Phi^T K \Phi Y = \Phi^T V}$$

Orthogonality Conditions

$$\Phi = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \end{bmatrix}$$

Generalized Mass

$$\Phi^T M \Phi = \begin{bmatrix} m_1^* & & \\ & m_2^* & \\ & & m_3^* \end{bmatrix}$$

Generalized Stiffness

$$\Phi^T K \Phi = \begin{bmatrix} k_1^* & & \\ & k_2^* & \\ & & k_3^* \end{bmatrix}$$

Generalized Damping

$$\Phi^T C \Phi = \begin{bmatrix} c_1^* & & \\ & c_2^* & \\ & & c_3^* \end{bmatrix}$$

Generalized Force

$$\Phi^T V(t) = \begin{Bmatrix} V_1^*(t) \\ V_2^*(t) \\ V_3^*(t) \end{Bmatrix}$$

Development of Uncoupled Equations of Motion

MDOF Equation of Motion: $M\ddot{u} + C\dot{u} + Ku = V(t)$

Transformation of Coordinates: $u = \Phi y$

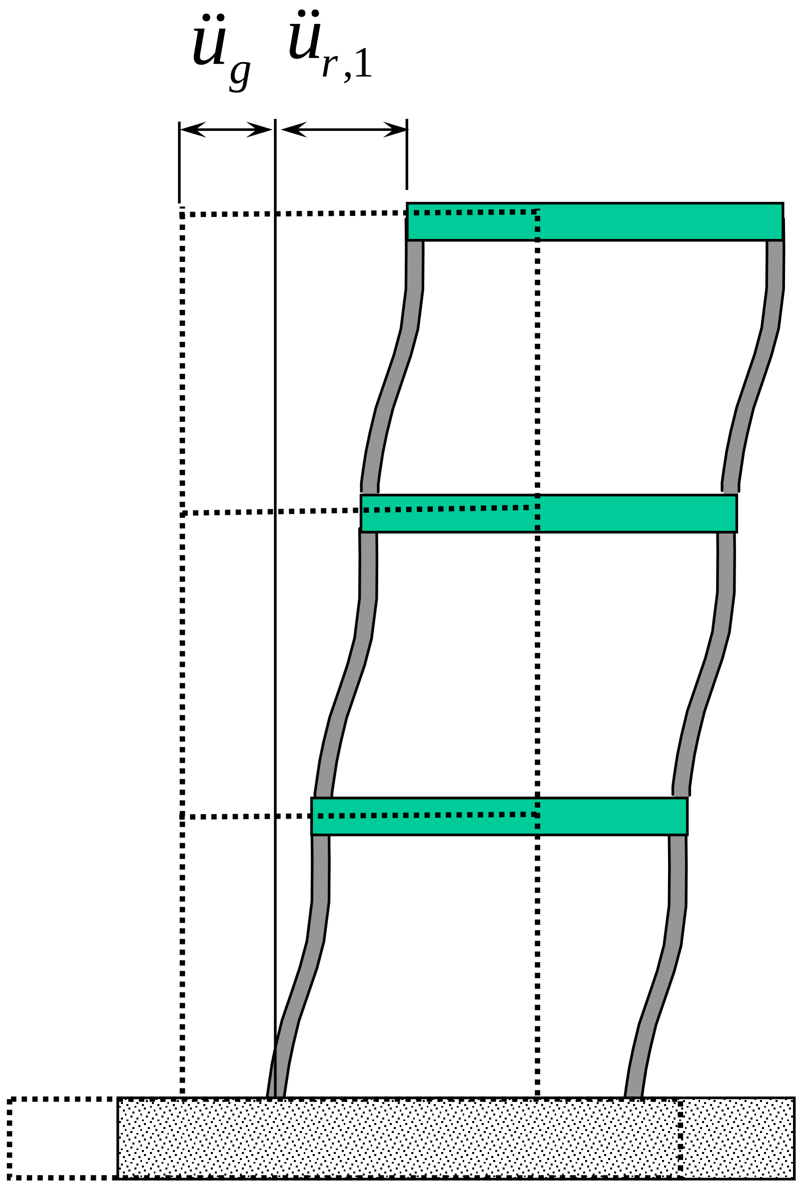
Substitution: $M\Phi\ddot{y} + C\Phi\dot{y} + K\Phi y = V(t)$

Premultiply by Φ^T: $\Phi^T M\Phi\ddot{y} + \Phi^T C\Phi\dot{y} + \Phi^T K\Phi y = \Phi^T V(t)$

Using Orthogonality Conditions: Uncoupled Equations of Motion are:

$$\begin{bmatrix} m_1^* & & \\ & m_2^* & \\ & & m_3^* \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{Bmatrix} + \begin{bmatrix} c_1^* & & \\ & c_2^* & \\ & & c_3^* \end{bmatrix} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{Bmatrix} + \begin{bmatrix} k_1^* & & \\ & k_2^* & \\ & & k_3^* \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} V_1^*(t) \\ V_2^*(t) \\ V_3^*(t) \end{Bmatrix}$$

Earthquake "Loading" for MDOF System

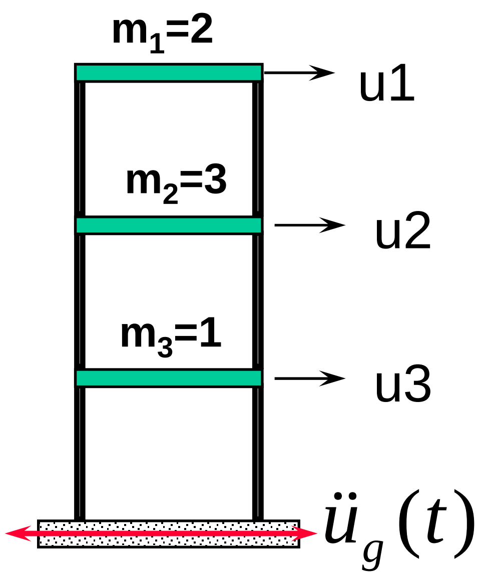


$$F_I(t) = M\begin{Bmatrix} \ddot{u}_g(t)+\ddot{u}_{R,1}(t) \\ \ddot{u}_g(t)+\ddot{u}_{R,2}(t) \\ \ddot{u}_g(t)+\ddot{u}_{R,3}(t) \end{Bmatrix} =$$

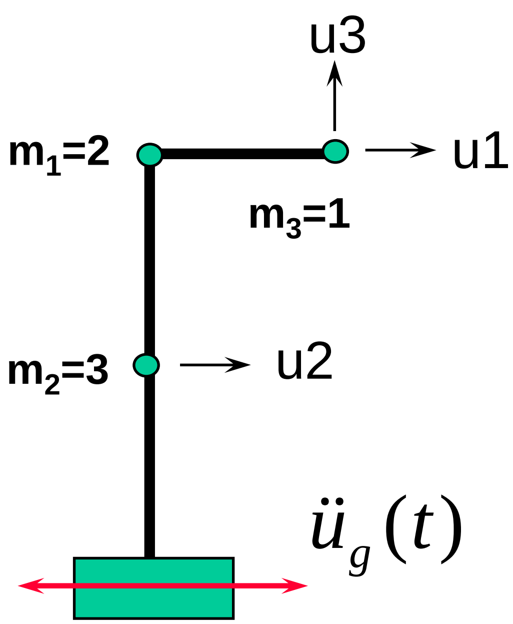
$$M\begin{Bmatrix} 1.0 \\ 1.0 \\ 1.0 \end{Bmatrix}\ddot{u}_g(t) + M\begin{Bmatrix} \ddot{u}_{R,1}(t) \\ \ddot{u}_{R,2}(t) \\ \ddot{u}_{R,3}(t) \end{Bmatrix}$$

Move to RHS as $V_{EFF}(t) = -MR\,\ddot{u}_g(t)$

Modal Earthquake Loading $V^*(t) = -\Phi^T M R \ddot{u}_g(t)$



$$M = \begin{bmatrix} 2 & & \\ & 3 & \\ & & 1 \end{bmatrix} \qquad R = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$



$$M = \begin{bmatrix} 2+1 & & \\ & 3 & \\ & & 1 \end{bmatrix} \qquad R = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

Development of Uncoupled Equations of Motion (Explicit Form)

MODE 1 $$m_1^* \ddot{y}_1 + c_1^* \dot{y}_1 + k_1^* y_1 = V_1^*(t)$$

MODE 2 $$m_2^* \ddot{y}_2 + c_2^* \dot{y}_2 + k_2^* y_2 = V_2^*(t)$$

MODE 3 $$m_3^* \ddot{y}_3 + c_3^* \dot{y}_3 + k_3^* y_3 = V_3^*(t)$$

Simplify by Dividing Through by m^* and defining $\xi_i = \dfrac{c_i^*}{2 m_i^* \omega_i}$

MODE 1 $$\ddot{y}_1 + 2\xi_1 \omega_1 \dot{y}_1 + \omega_1^2 y_1 = V_1^*(t) / m_1^*$$

MODE 2 $$\ddot{y}_2 + 2\xi_2 \omega_2 \dot{y}_2 + \omega_2^2 y_2 = V_2^*(t) / m_2^*$$

MODE 3 $$\ddot{y}_3 + 2\xi_3 \omega_3 \dot{y}_3 + \omega_3^2 y_3 = V_3^*(t) / m_3^*$$

Definition of Modal Participation Factor

Typical Modal Equation:

$$\ddot{y}_i + 2\xi_i w_i \dot{y}_i + w_i^2 y_i = \frac{V_i^*(t)}{m_i^*}$$

recall $\quad V_i^*(t) = -\phi_i^T M r \ddot{v}_g(t)$

Right hand side = $\boxed{-\dfrac{\phi_i^T M r}{m_i^*}} \ddot{v}_g(t)$

Modal Participation Factor P_i

Concept of Effective Modal Mass

For each Mode *i* $$\overline{M}_i = P_i^2 m_i^*$$

Development of a Modal Damping Matrix

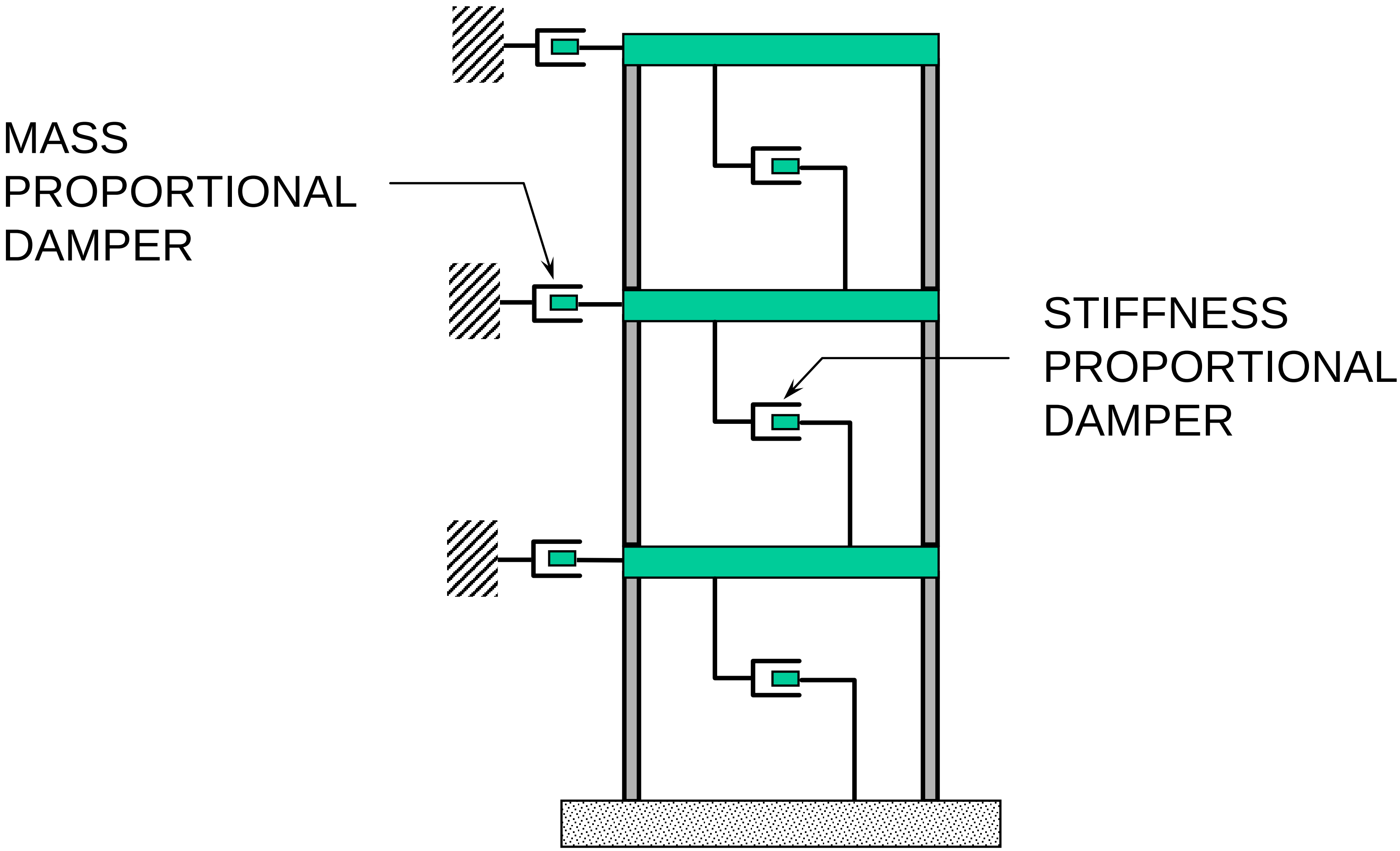
In Previous Development, We have Assumed:

$$\Phi^T C \Phi = \begin{bmatrix} c_1^* & & \\ & c_2^* & \\ & & c_3^* \end{bmatrix}$$

Two Methods Described Herein:

- Rayleigh "Proportional Damping"
- Wilson "Discrete Modal Damping"

Rayleigh Proportional Damping



$$C = \alpha\, M + \beta\, K$$

Rayleigh Proportional Damping

$$C = \alpha\, M + \beta\, K$$

For Modal Equations to Be Uncoupled:

$$2\omega_n \xi_n = \phi_n^T C \phi_n$$

Assumes

$$\Phi^T M \Phi = I$$

Using Orthogonality Conditions:

$$2\omega_n \xi_n = \alpha + \beta \omega_n^2$$

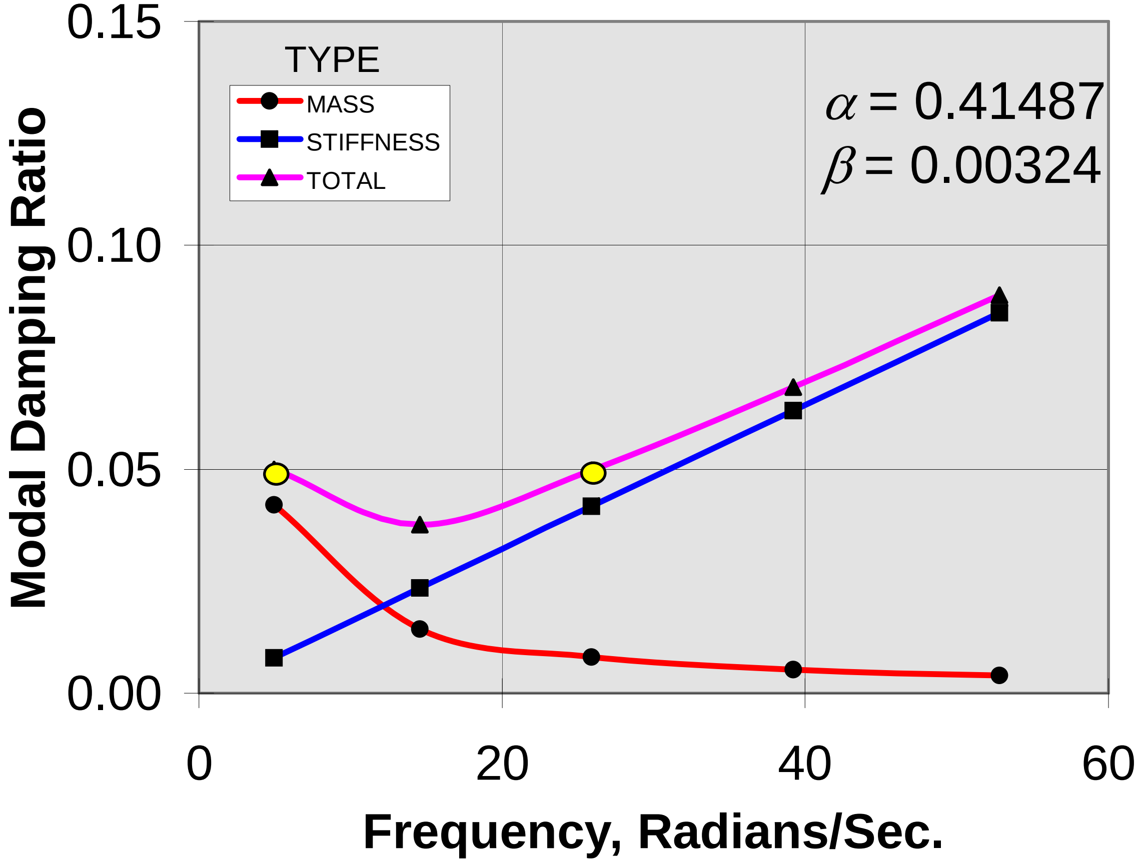
$$\xi_n = \frac{1}{2\omega_n}\alpha + \frac{\omega_n}{2}\beta$$

Rayleigh Proportional Damping (Example)

Structural Frequencies

Mode	ω
1	**4.94**
2	14.6
3	**25.9**
4	39.2
5	52.8

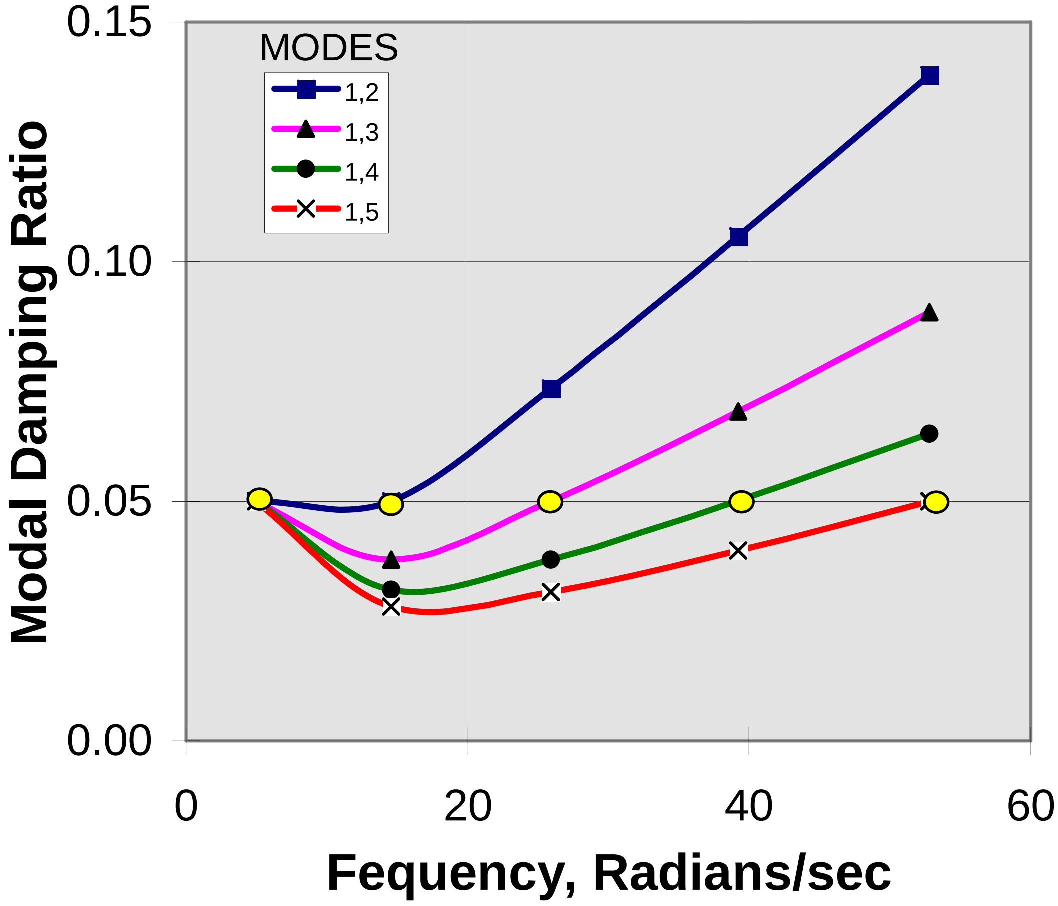
5% Critical in Modes 1 and 3



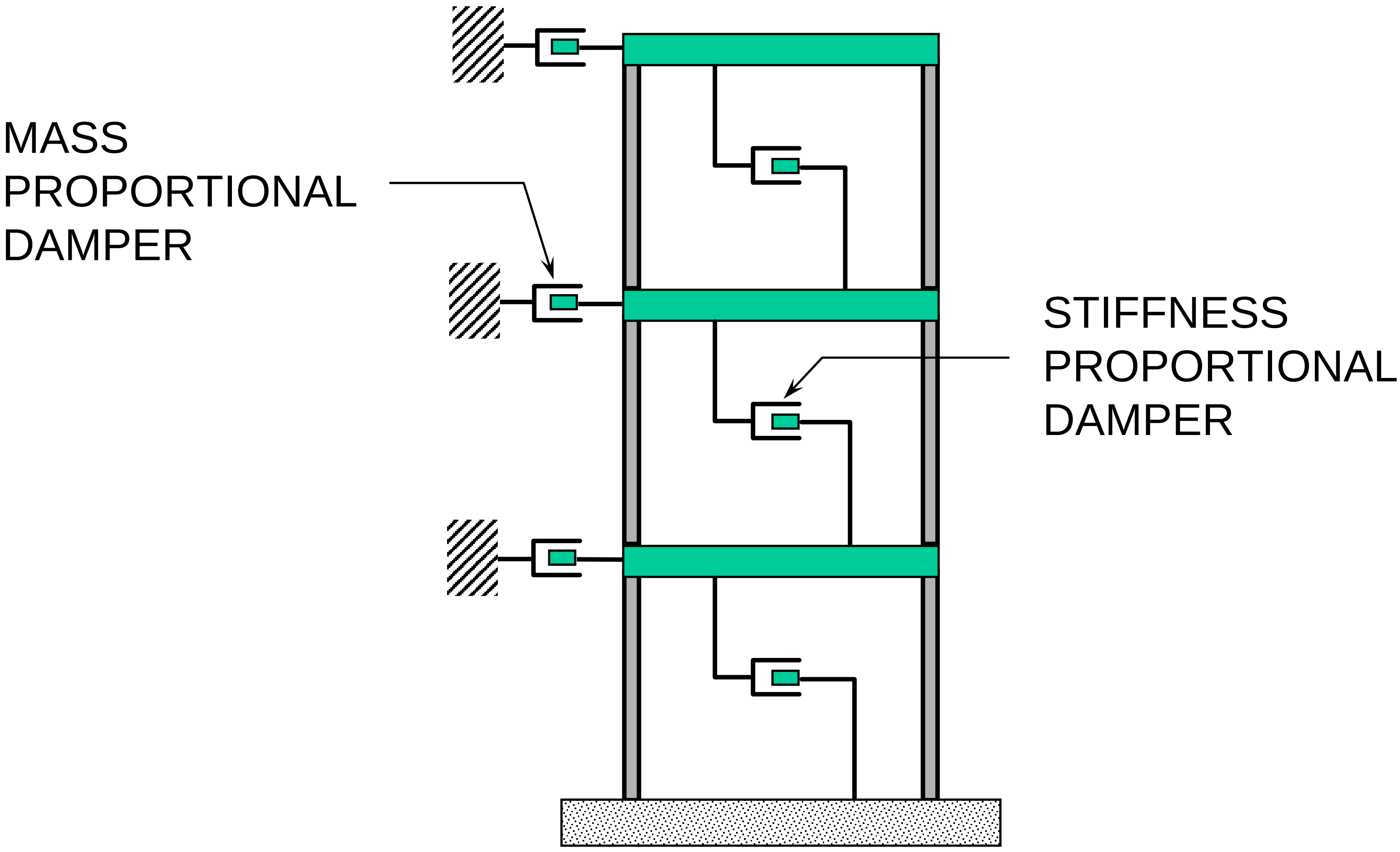
Rayleigh Proportional Damping (Example)
5% Damping in Modes 1 & 2, 1 & 3, 1 & 4, or 1 & 5

Proportionality Factors
(5% each indicated mode)

Modes	α	β
1 & 2	.36892	0.00513
1 & 3	.41487	0.00324
1 & 4	.43871	0.00227
1 & 5	.45174	0.00173



Rayleigh Proportional Damping



$$C = \alpha M + \beta K$$

Wilson Damping

Directly Specify Modal Damping Values ξ_i^*

$$\Phi^T C \Phi = \begin{bmatrix} c_1^* & & \\ & c_2^* & \\ & & c_3^* \end{bmatrix} = \begin{bmatrix} 2m_1^*\omega_1\xi_1^* & & \\ & 2m_2^*\omega_2\boxed{\xi_2^*} & \\ & & 2m_3^*\omega_3\xi_3^* \end{bmatrix}$$

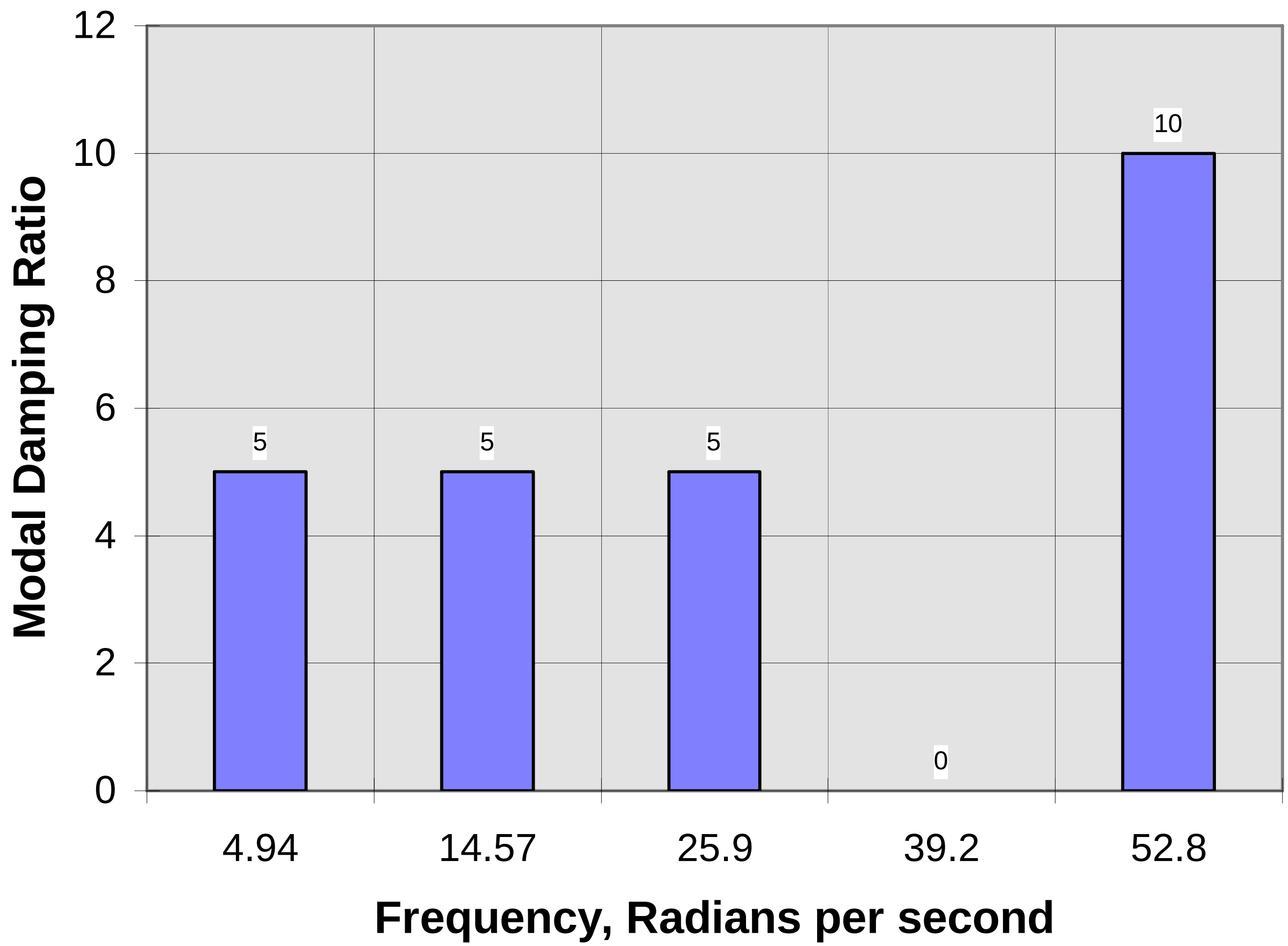
FORMATION OF EXPLICIT DAMPING MATRIX FROM “WILSON” MODAL DAMPING

$$\Phi^T C \Phi = \begin{bmatrix} 2\xi_1\omega_1 & & & & \\ & 2\xi_2\omega_2 & & & \\ & & \bullet\bullet\bullet & & \\ & & & 2\xi_{n-1}\omega_{n-1} & \\ & & & & 2\xi_n\omega_n \end{bmatrix} = c$$

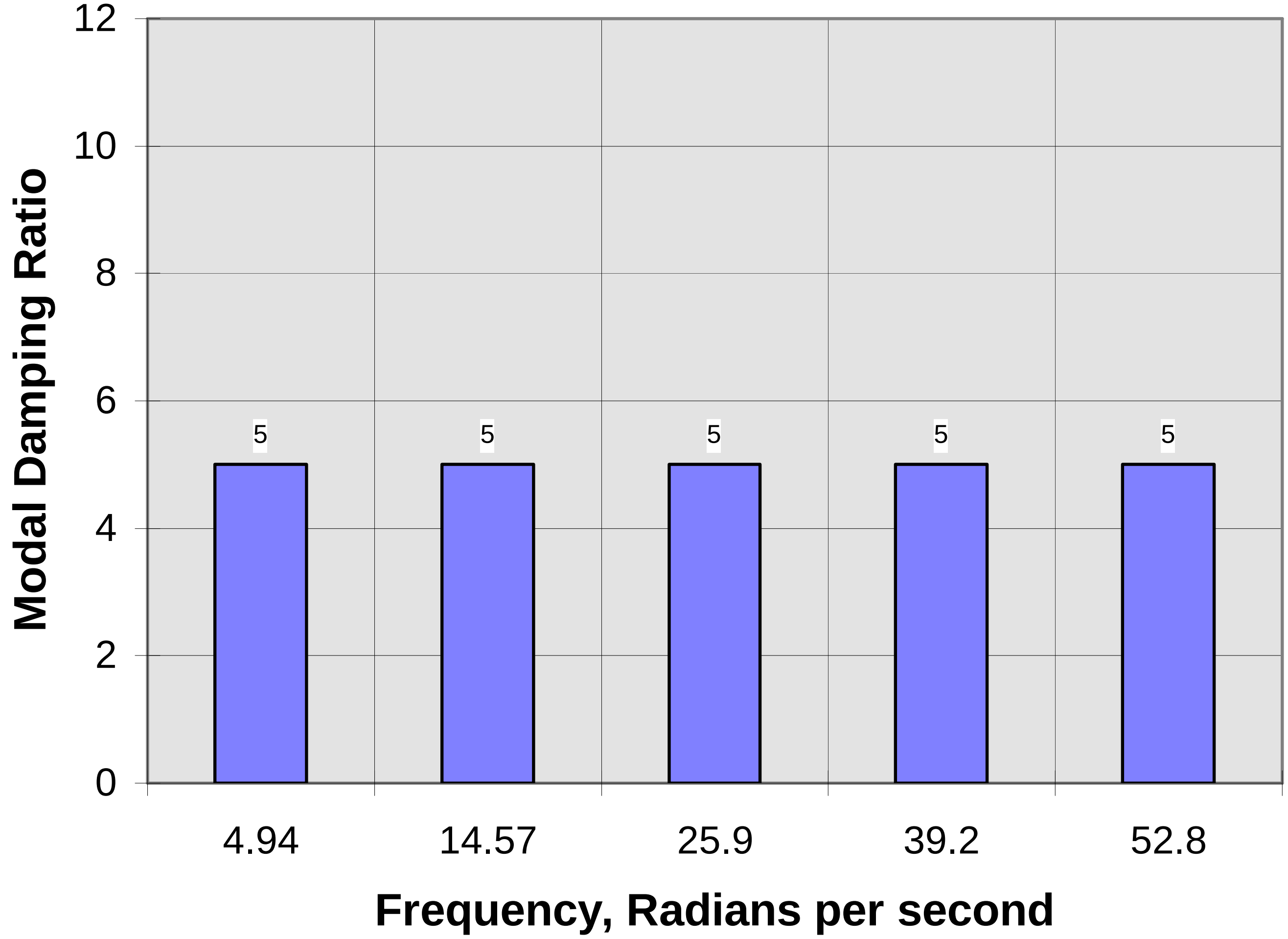
$$\left(\Phi^T\right)^{-1} c \ \Phi^{-1} = C$$

$$C = M\left[\sum_{i=1}^{n} 2\xi_i\omega_i\phi_i^T\phi_i\right]M$$

Wilson Damping (Example)
5% Damping in Modes 1 and 2, 3
10% in Mode 5, Zero in Mode 4



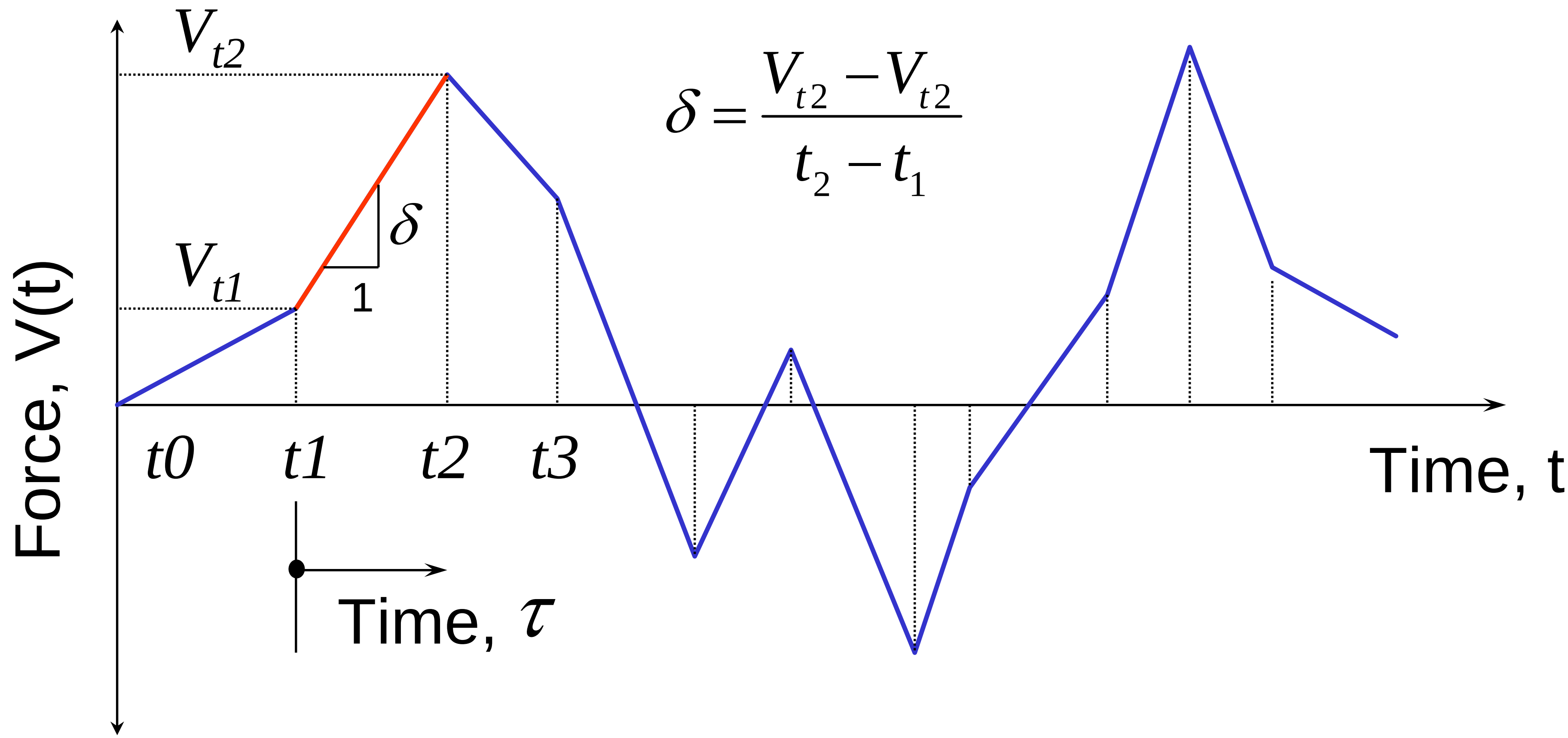
Wilson Damping (Example)
5% Damping in all Modes



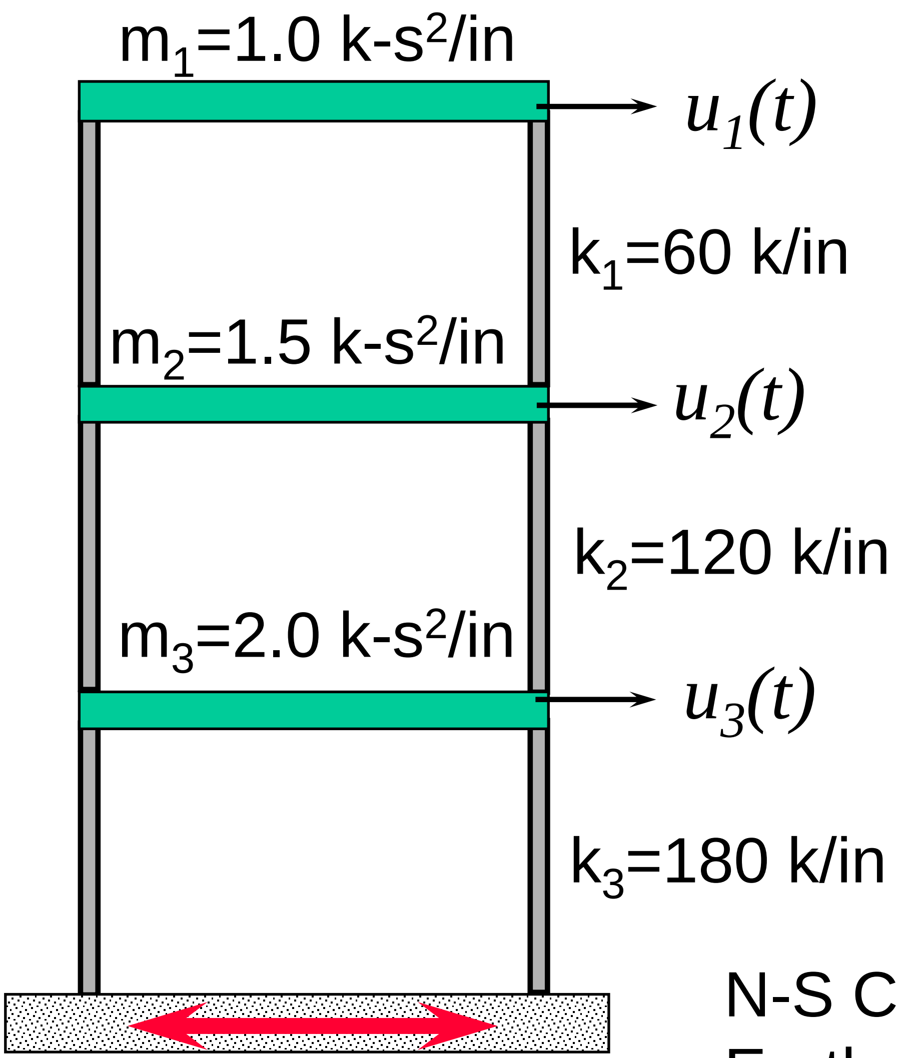
Solution of MDOF Equations of Motion

- Explicit (Step by Step) Integration of Coupled Equations
- Explicit Integration of FULL SET of Uncoupled Equations
- Explicit Integration of PARTIAL SET of Uncoupled Equations
- Modal Response Spectrum Analysis

Computed Response for Piecewise Linear Loading



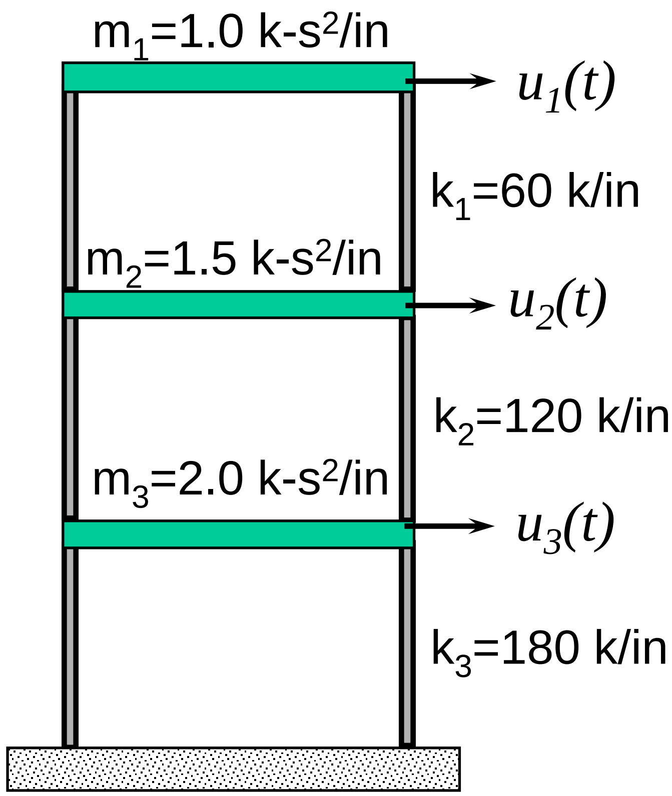
EXAMPLE of MDOF Response of Structure Responding to 1940 El Centro Earthquake



Assume Wilson Damping with 5% critical in each mode.

N-S Component of 1940 El Centro Earthquake. Maximum acceleration = 0.35 g

Example (Continued)

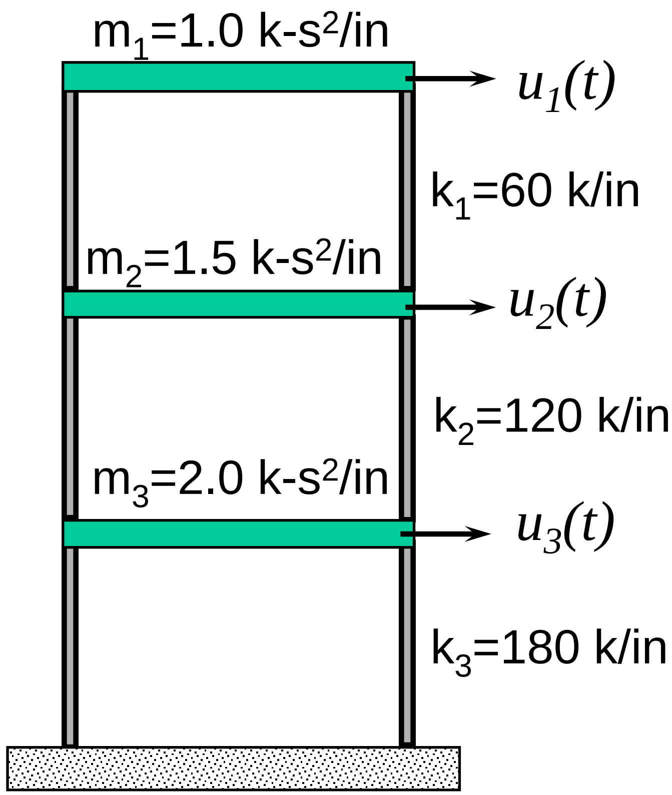


Form Property Matrices:

$$M = \begin{bmatrix} 1.0 & & \\ & 1.5 & \\ & & 2.0 \end{bmatrix} \text{kip} - \text{s}^2 / \text{in}$$

$$K = \begin{bmatrix} 60 & -60 & 0 \\ -60 & 180 & -120 \\ 0 & -120 & 300 \end{bmatrix} \text{kip} / \text{in}$$

Example (Continued)



Solve Eigenvalue Problem:

$$K\Phi = M\Phi\Omega^2$$

$$\Omega^2 = \begin{bmatrix} 21.0 & & \\ & 96.6 & \\ & & 212.4 \end{bmatrix} \sec^{-2}$$

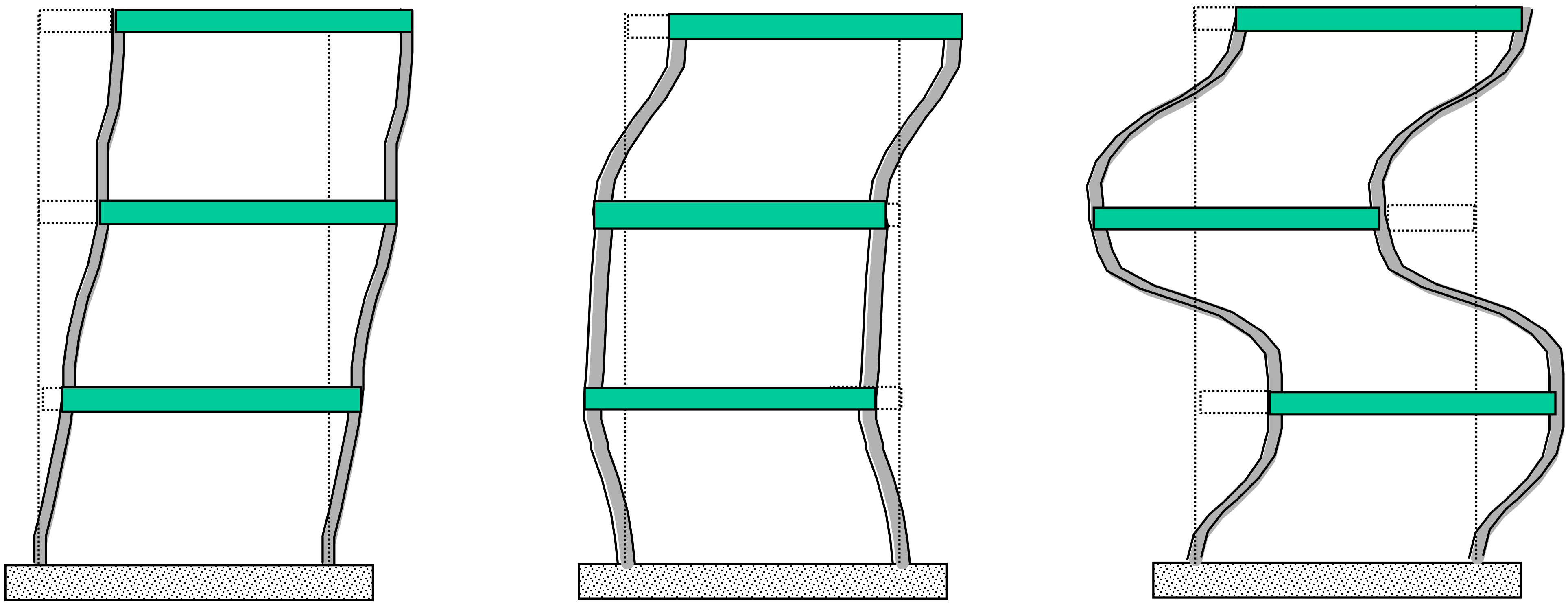
$$\Phi = \begin{bmatrix} 1.000 & 1.000 & 1.000 \\ 0.644 & -0.601 & -2.57 \\ 0.300 & -0.676 & 2.47 \end{bmatrix}$$

Normalization of Modes using $\Phi^T M \Phi = I$

$$\Phi = \begin{bmatrix} 0.749 & 0.638 & 0.208 \\ 0.478 & -0.384 & -0.534 \\ 0.223 & -0.431 & 0.514 \end{bmatrix} \quad \text{vs} \quad \begin{bmatrix} 1.000 & 1.000 & 1.000 \\ 0.644 & -0.601 & -2.57 \\ 0.300 & -0.676 & 2.47 \end{bmatrix}$$

Example (Continued)

Mode Shapes and Periods of Vibration



MODE 1

ω=4.58 rad/sec

T=1.37 sec

MODE 2

ω=9.83 rad/sec

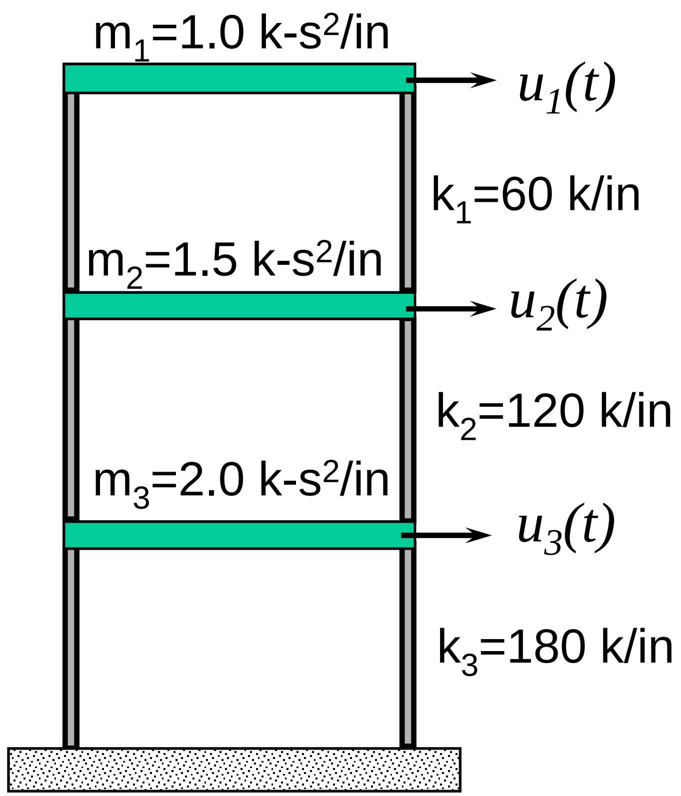
T=0.639 sec

MODE 3

ω=14.57 rad/sec

T=0.431 sec

Example (Continued)



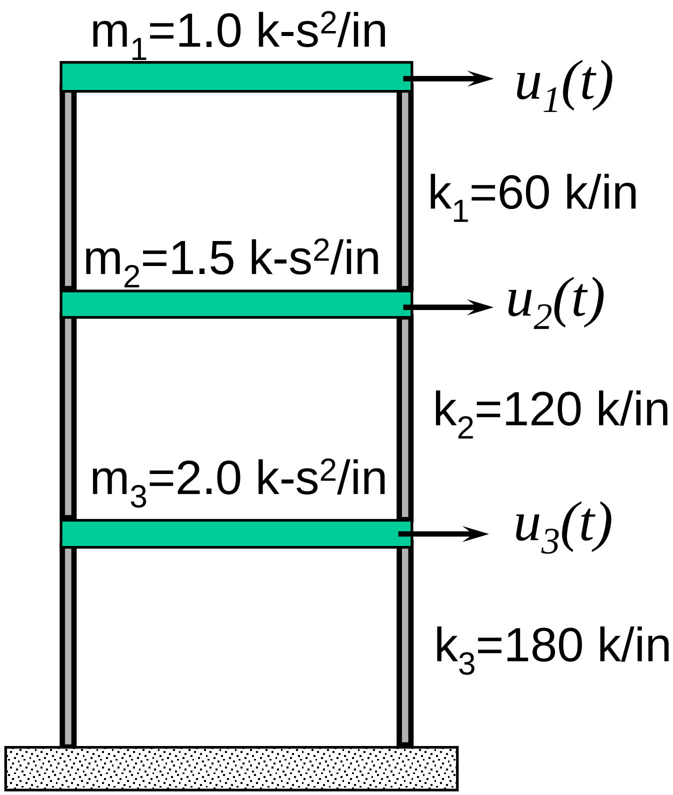
$$\omega_n = \begin{Bmatrix} 4.58 \\ 9.83 \\ 14.57 \end{Bmatrix} \text{rad / sec}$$

$$T_n = \begin{Bmatrix} 1.37 \\ 0.639 \\ 0.431 \end{Bmatrix} \text{sec}$$

Compute Generalized Mass:

$$M^* = \Phi^T M \Phi = \begin{bmatrix} 1.801 & & \\ & 2.455 & \\ & & 23.10 \end{bmatrix} \text{kip} - \text{sec}^2 / \text{in}$$

Example (Continued)



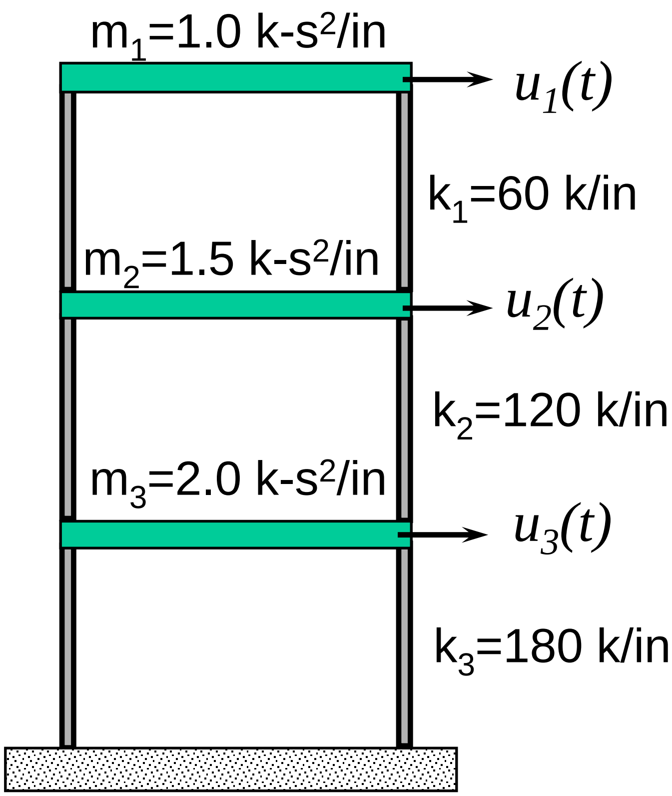
Compute Generalized Loading:

$$V^*(t) = -\Phi^T M R \ddot{v}_g(t)$$

$$V_n^* = \begin{Bmatrix} 2.566 \\ -1.254 \\ 2.080 \end{Bmatrix} \ddot{v}_g(t)$$

Example (Continued)

Write Uncoupled (Modal) Equations of Motion:



$$\ddot{y}_1 + 2\xi_1\omega_1\dot{y}_1 + \omega_1^2 y_1 = V_1^*(t) / m_1^*$$

$$\ddot{y}_2 + 2\xi_2\omega_2\dot{y}_2 + \omega_2^2 y_2 = V_2^*(t) / m_2^*$$

$$\ddot{y}_3 + 2\xi_3\omega_3\dot{y}_3 + \omega_3^2 y_3 = V_3^*(t) / m_3^*$$

$$\ddot{y}_1 + 0.458\dot{y}_1 + 21.0 y_1 = 1.425\ddot{v}_g(t)$$

$$\ddot{y}_2 + 0.983\dot{y}_2 + 96.6 y_2 = -0.511\ddot{v}_g(t)$$

$$\ddot{y}_3 + 1.457\dot{y}_3 + 212.4 y_3 = 0.090\ddot{v}_g(t)$$

Modal Participation Factors

$$\begin{array}{ll} \mathit{Mode} & 1 \\ \mathit{Mode} & 2 \\ \mathit{Mode} & 3 \end{array} \qquad \boxed{\begin{array}{c} 1.425 \\ -0.511 \\ 0.090 \end{array}} \qquad \boxed{\begin{array}{c} 1.911 \\ -0.799 \\ 0.435 \end{array}}$$

Modal Scaling $\qquad \phi_{i,1} = 1.0 \qquad \phi_i^T M \phi_i = 1.0$

Modal Participation Factors

$$1.425\begin{Bmatrix} 1.000 \\ 0.644 \\ 0.300 \end{Bmatrix} = 1.911\begin{Bmatrix} 0.744 \\ 0.480 \\ 0.223 \end{Bmatrix}$$

using $\phi_{1,1} = 1$ using $\phi_1{}^T M \phi_1 = 1$

Effective Modal Mass

$$\overline{M_n} = P_n^{\,2} m_n$$

	$\overline{m}_n$	%	Accum%
Mode 1	3.66	81	81
Mode 2	0.64	14	95
Mode 3	0.20	5	100%
	4.50	100%	

Example (Continued)

Solving Modal Equation via NONLIN:

For MODE 1:

$$\ddot{y}_1 + 2\xi_1\omega_1\dot{y}_1 + \omega_1^2 y_1 = V_1^*(t)/m_1^*$$

$$1.00\ddot{y}_1 + 0.458\dot{y}_1 + 21.0y_1 = 1.425\ddot{v}_g(t)$$

M = 1.00 kip-sec^2/in

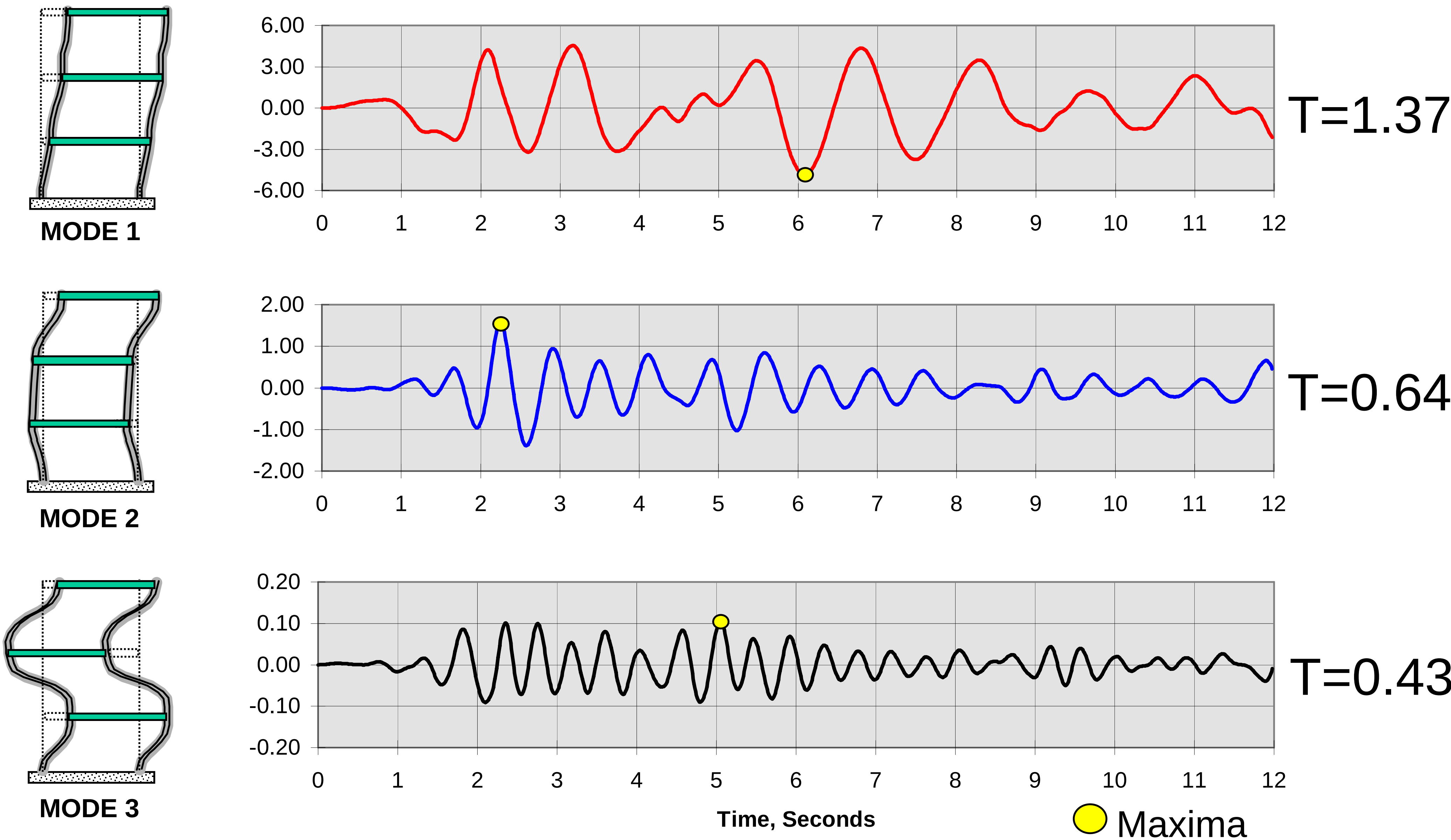
C = 0.458 kip-sec/in

K_1 = 21.0 kips/inch

Scale Ground Acceleration by Factor 1.425

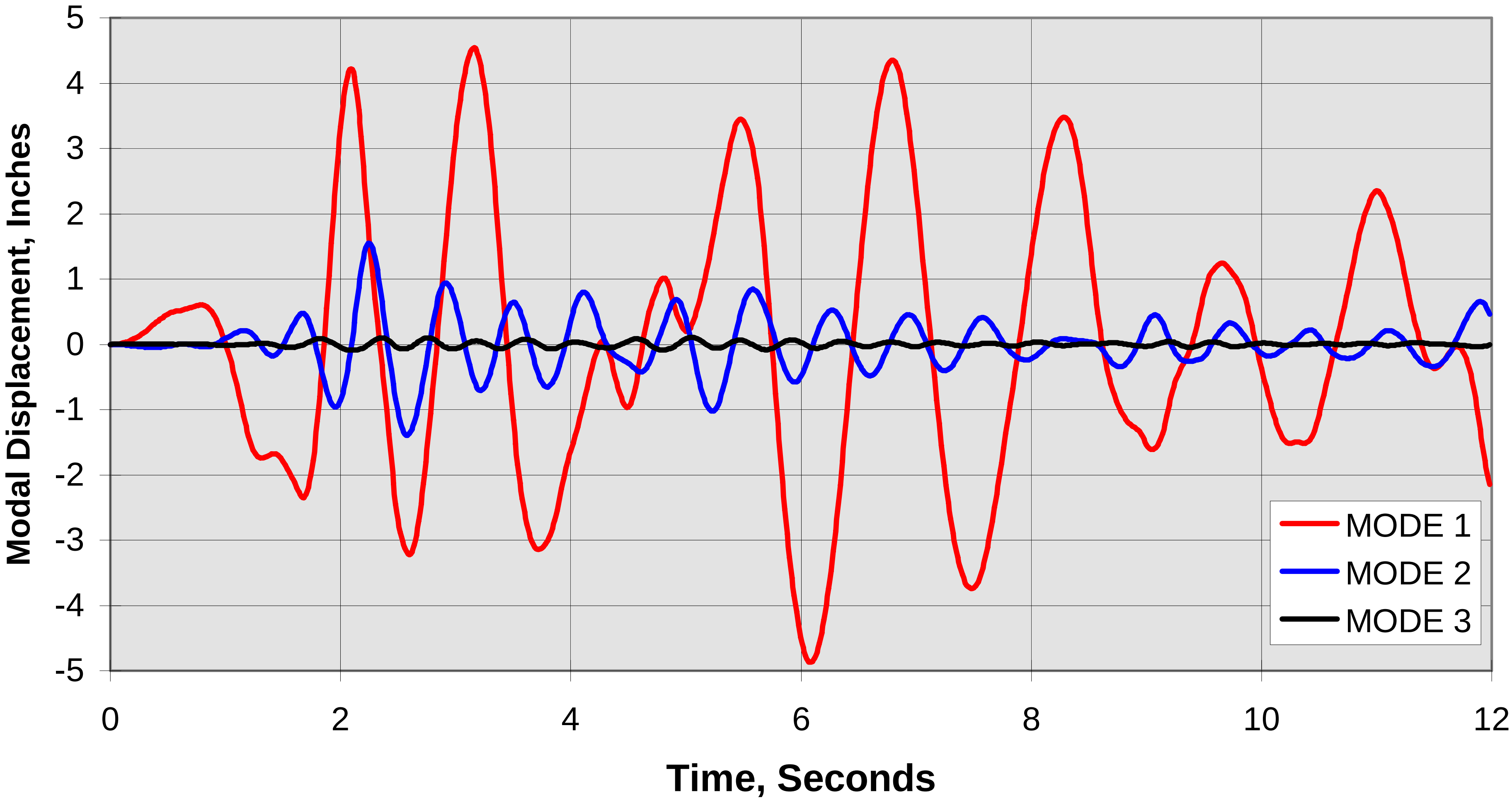
Example (Continued)

MODAL Displacement Time Histories (From NONLIN)



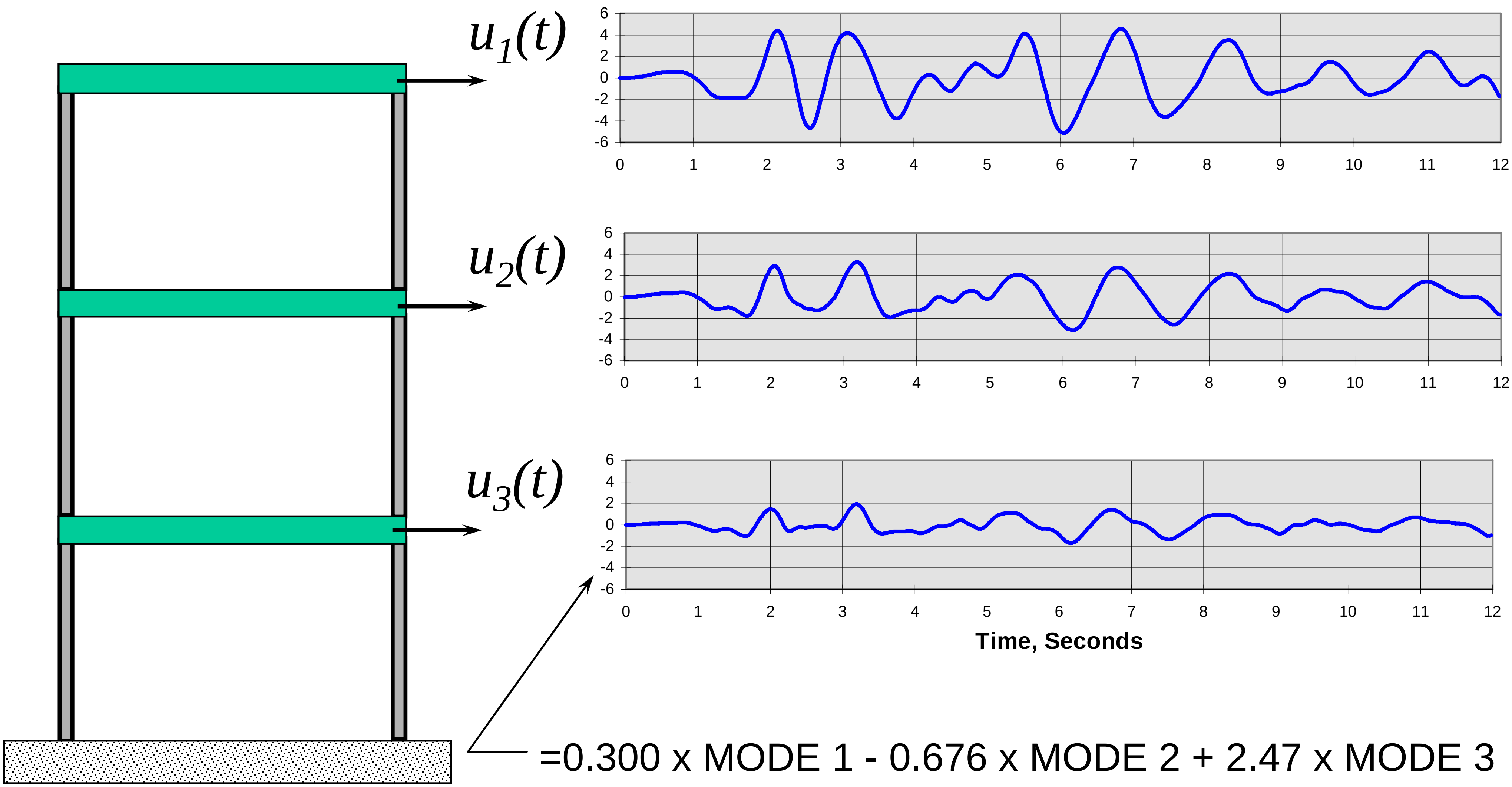
Example (Continued)

MODAL Response Time Histories:



Example (Continued)

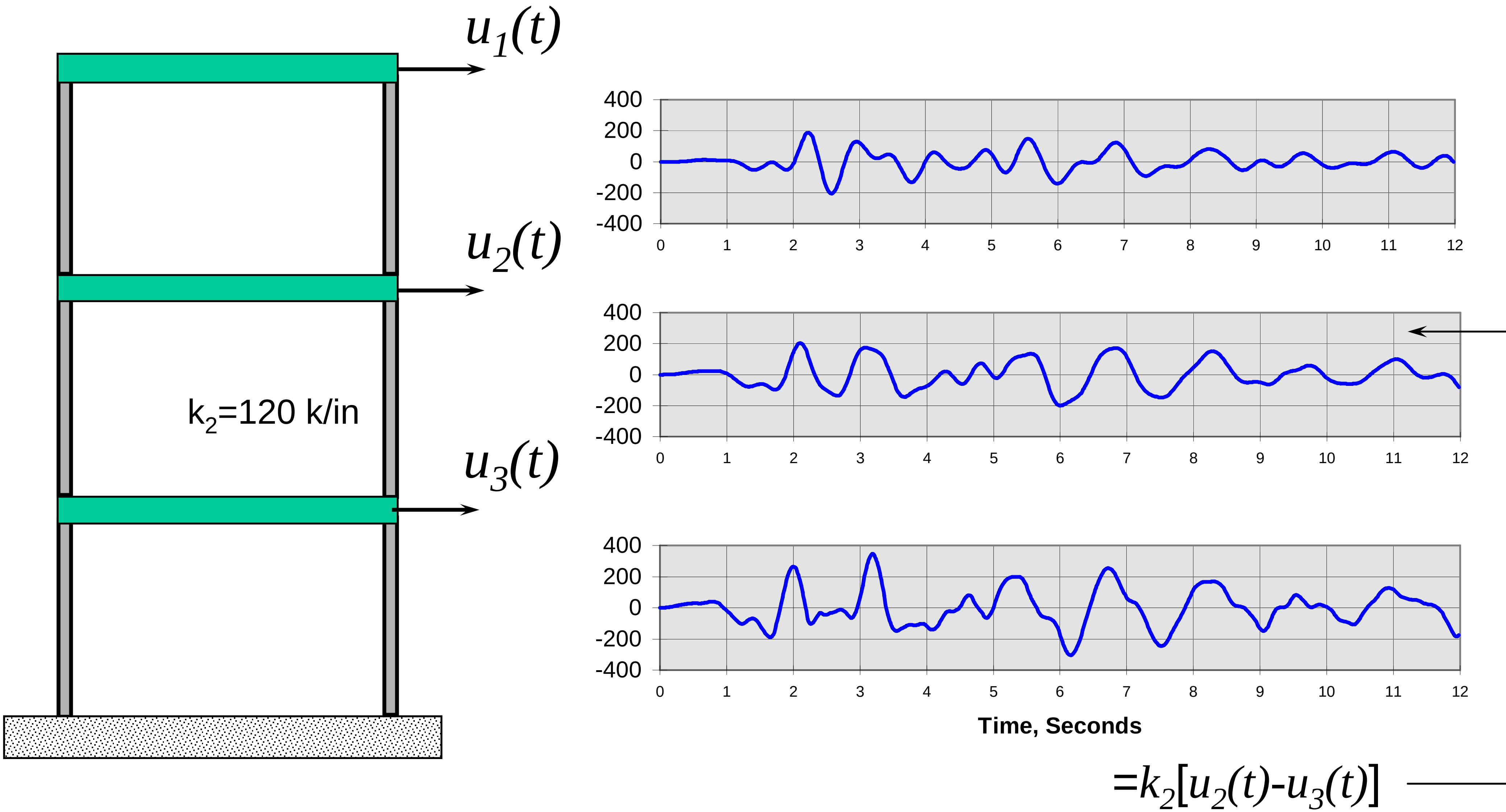
Compute Story Displacement Time-Histories: $u(t) = \Phi y(t)$



=0.300 x MODE 1 - 0.676 x MODE 2 + 2.47 x MODE 3

Example (Continued)

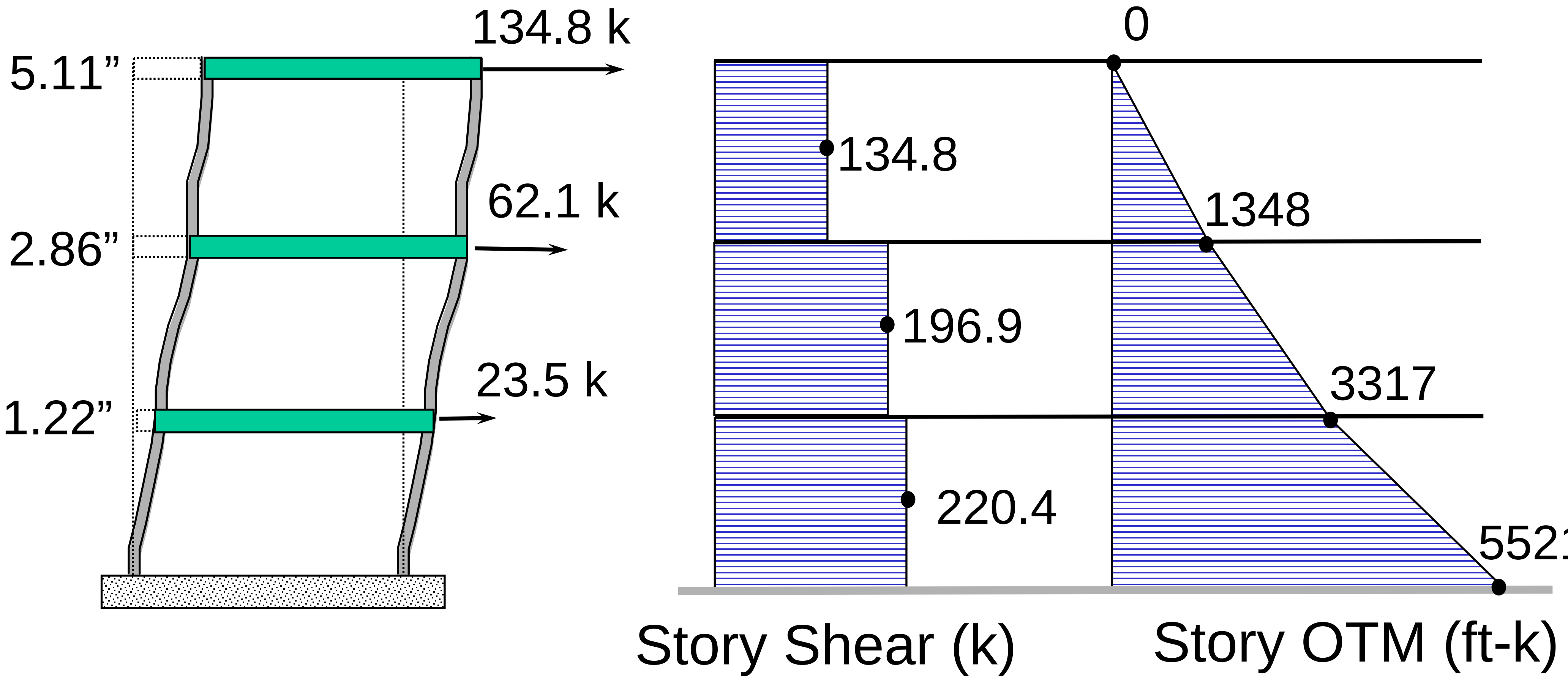
Compute Story Shear Time-Histories:



Example (Continued)

Displacements and Forces at time of Maximum Displacement

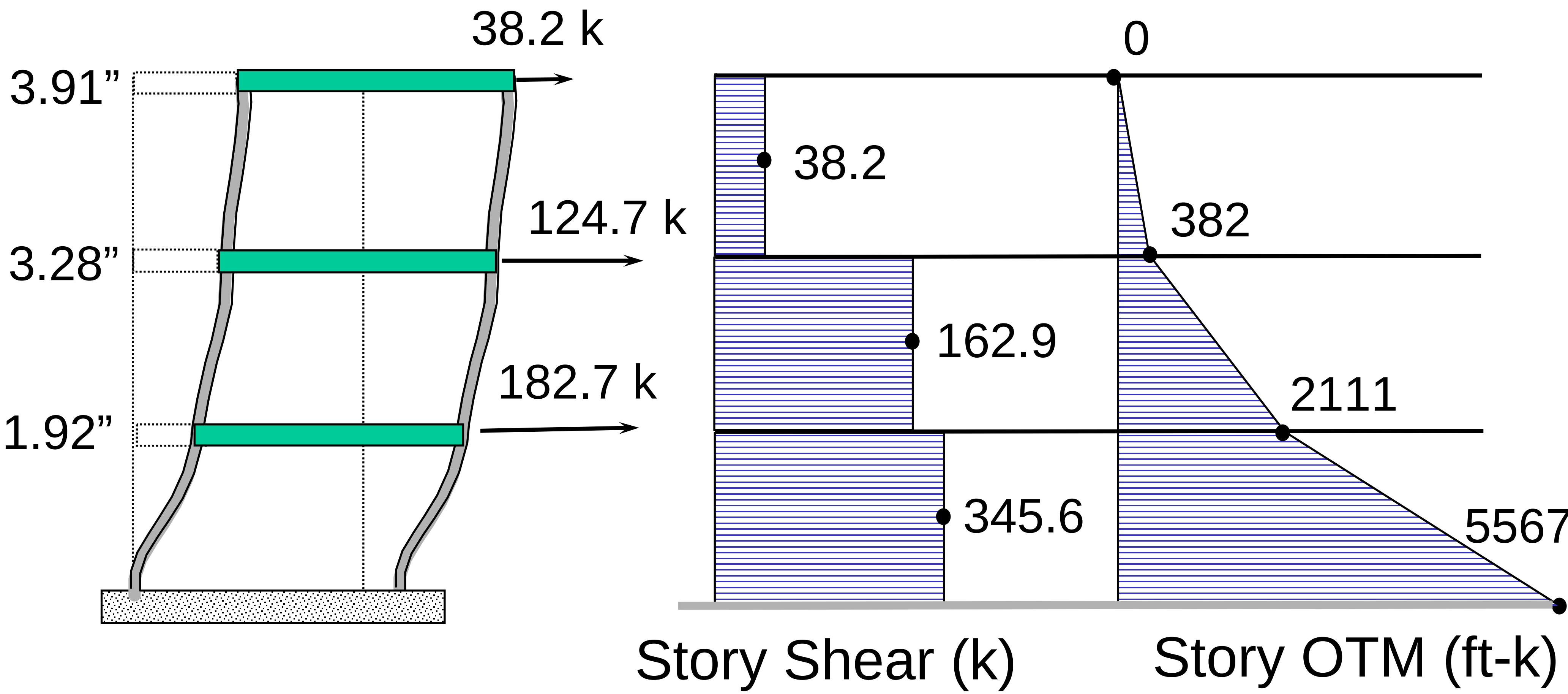
t = 6.04 seconds



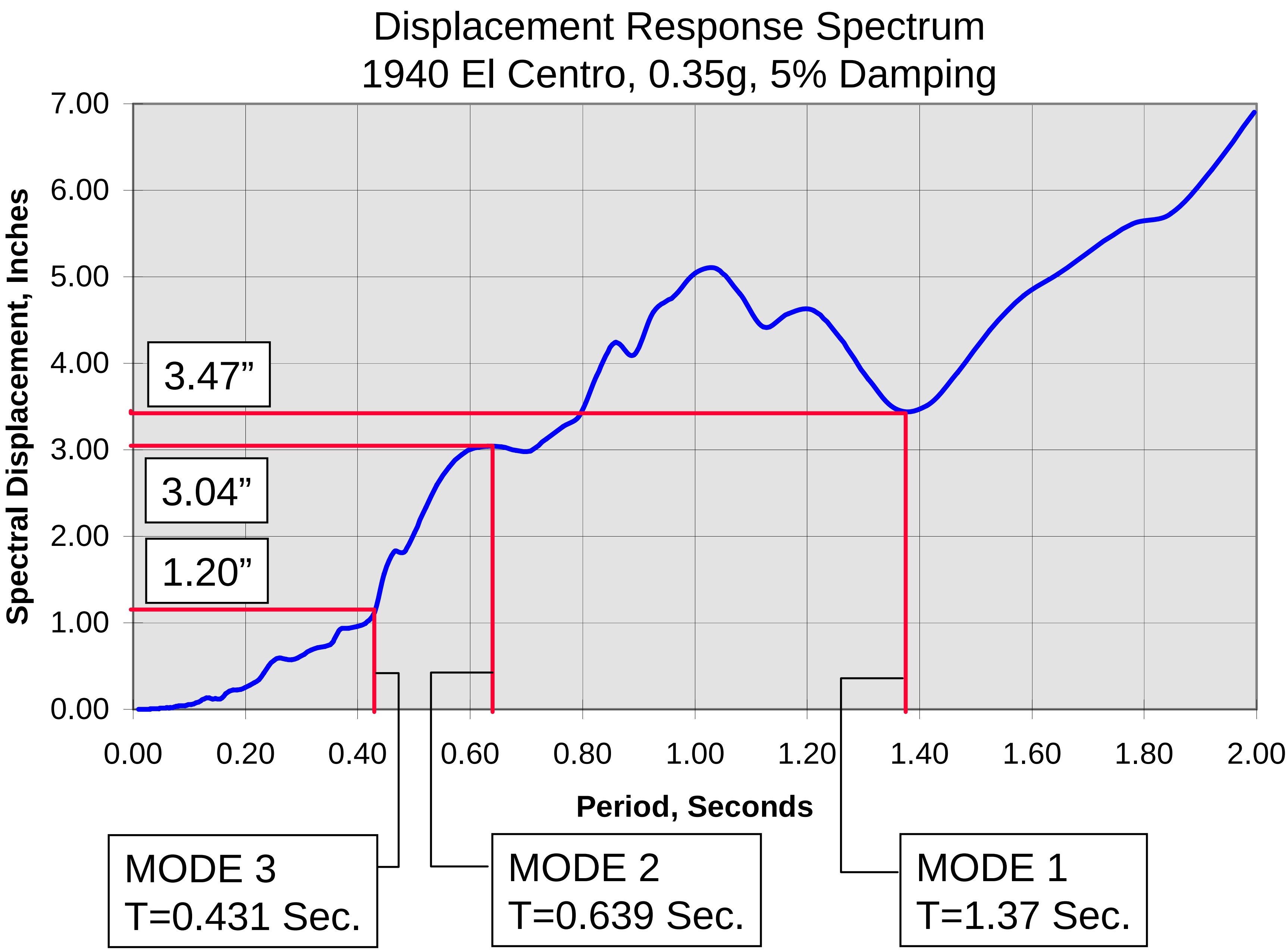
Example (Continued)

Displacements and Forces at time of Maximum Shear

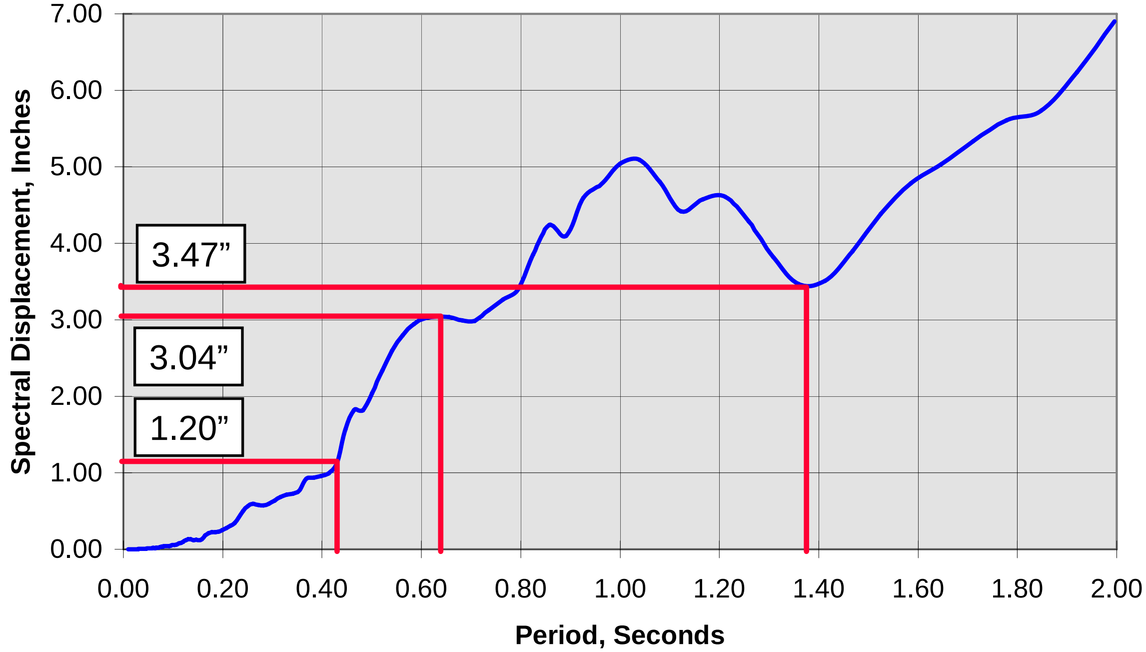
t = 3.18 seconds



Example (Response Spectrum Method)



Example (Continued)



Modal Equations of Motion

$$\ddot{y}_1 + 0.458\dot{y}_1 + 21.0y_1 = 1.425\ddot{v}_g(t)$$

$$\ddot{y}_2 + 0.983\dot{y}_2 + 96.6y_2 = -0.511\ddot{v}_g(t)$$

$$\ddot{y}_3 + 1.457\dot{y}_3 + 212.4y_3 = 0.090\ddot{v}_g(t)$$

Modal Maxima

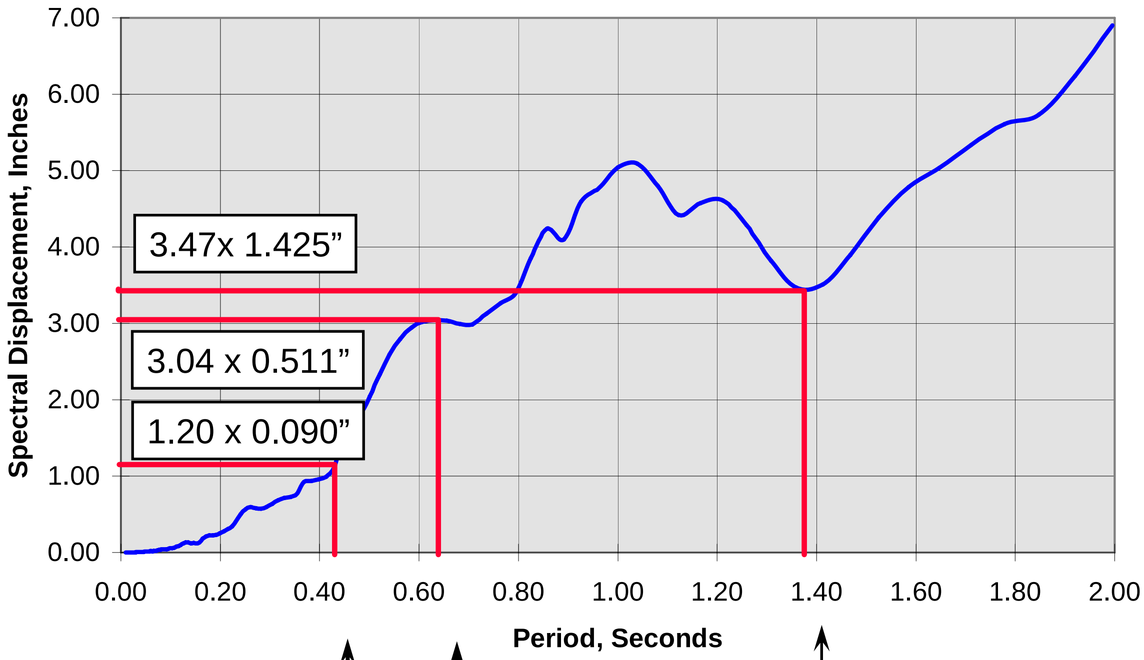
$$\bar{y}_1 = 1.425 * 3.47 = 4.94''$$

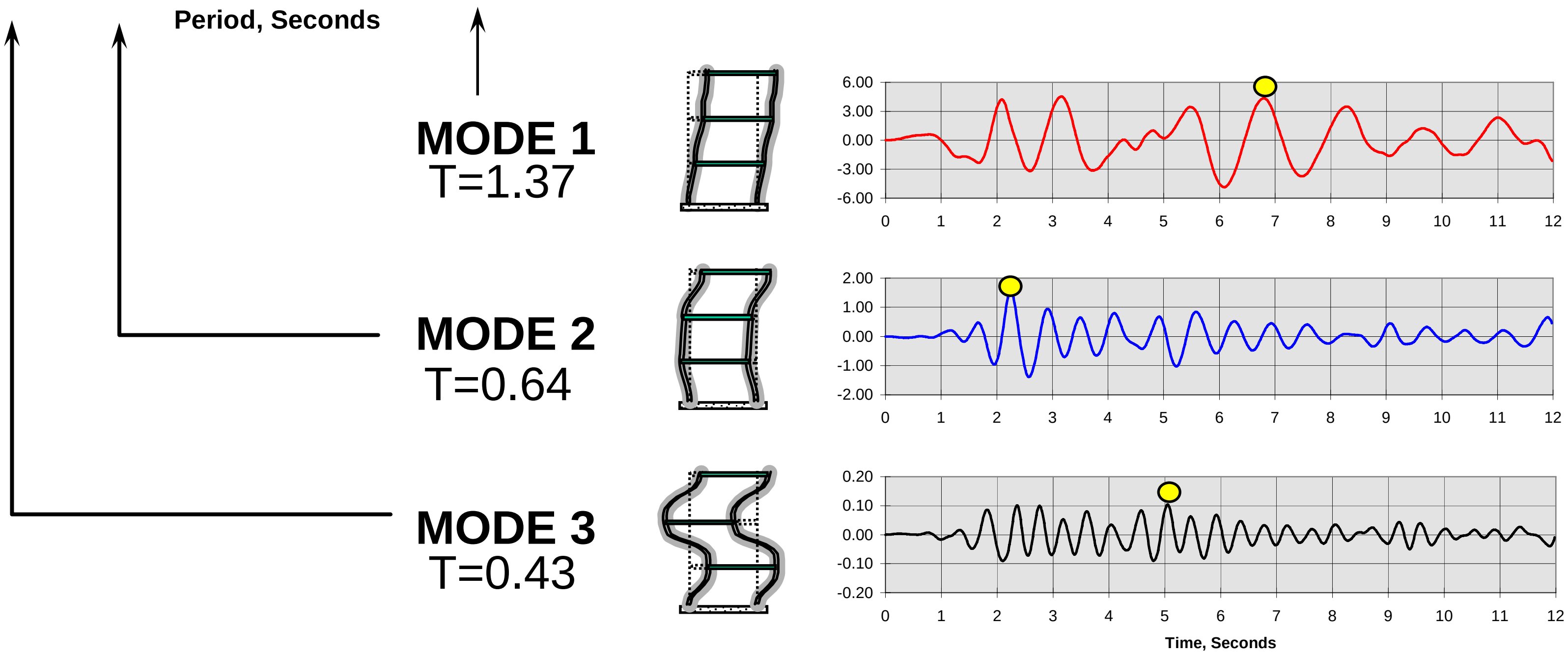
$$\bar{y}_2 = 0.511 * 3.04 = 1.55''$$

$$\bar{y}_3 = 0.090 * 1.20 = 0.108''$$

Example (Continued)

The **Scaled** Response Spectrum gives the Same Results as the Previous Time-Histories

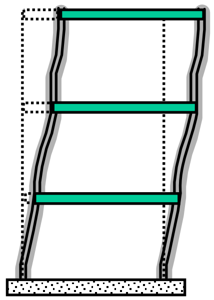




Example (Continued)

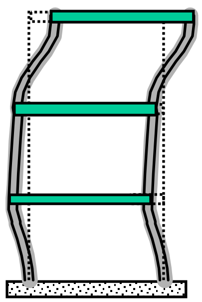
Computing Story Displacements

MODE 1



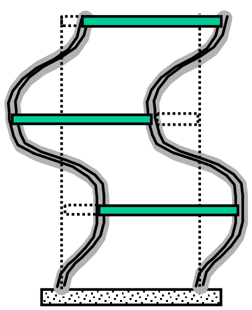
$$\begin{Bmatrix} 1.000 \\ 0.644 \\ 0.300 \end{Bmatrix} 4.940 = \begin{Bmatrix} 4.940 \\ 3.181 \\ 1.482 \end{Bmatrix}$$

MODE 2



$$\begin{Bmatrix} 1.000 \\ -0.601 \\ -0.676 \end{Bmatrix} 1.550 = \begin{Bmatrix} 1.550 \\ -0.931 \\ -1.048 \end{Bmatrix}$$

MODE 3



$$\begin{Bmatrix} 1.000 \\ -2.570 \\ 2.470 \end{Bmatrix} 0.108 = \begin{Bmatrix} 0.108 \\ -0.278 \\ 0.267 \end{Bmatrix}$$

Example (Continued)

Modal Combination Techniques (For Displacement)

Sum of Absolute Values

$$\begin{Bmatrix} 4.940 + 1.550 + 0.108 \\ 3.181 + 0.931 + 1.048 \\ 1.482 + 1.048 + 0.267 \end{Bmatrix} = \begin{Bmatrix} 6.60 \\ 5.16 \\ 2.80 \end{Bmatrix}$$

Square Root of the Sum of the Squares

$$\begin{Bmatrix} \sqrt{4.940^2 + 1.550^2 + 0.108^2} \\ \sqrt{3.181^2 + 0.931^2 + 1.048^2} \\ \sqrt{1.482^2 + 1.048^2 + 0.267^2} \end{Bmatrix} = \begin{Bmatrix} 5.18 \\ 3.48 \\ 1.84 \end{Bmatrix}$$

“Exact”

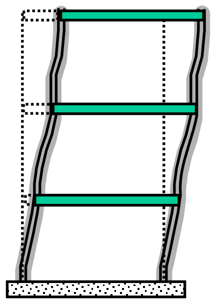
$$\begin{Bmatrix} 5.11 \\ 2.86 \\ 1.22 \end{Bmatrix}$$

At time of Max. Displacement

Example (Continued)

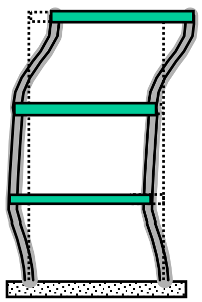
Computing Interstory Drifts

MODE 1



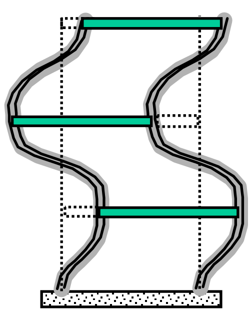
$$\begin{Bmatrix} 4.940-3.181 \\ 3.181-1.482 \\ 1.482-0 \end{Bmatrix} = \begin{Bmatrix} 1.759 \\ 1.699 \\ 1.482 \end{Bmatrix}$$

MODE 2



$$\begin{Bmatrix} 1.550-(-0.931) \\ -0.931-(-1.048) \\ -1.048-0 \end{Bmatrix} = \begin{Bmatrix} 2.481 \\ 0.117 \\ -1.048 \end{Bmatrix}$$

MODE 3

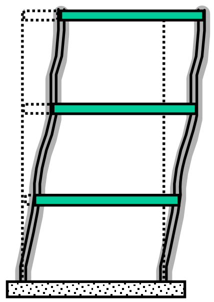


$$\begin{Bmatrix} 0.108-(-0.278) \\ -0.278-0.267 \\ 0.267-0 \end{Bmatrix} = \begin{Bmatrix} 0.386 \\ -0.545 \\ 0.267 \end{Bmatrix}$$

Example (Continued)

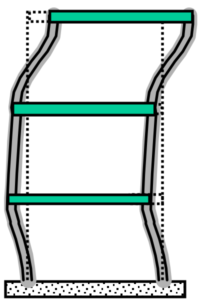
Computing Interstory Shears

MODE 1



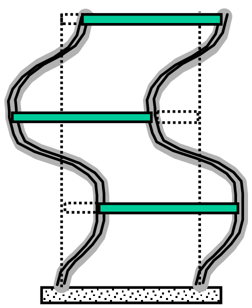
$$\begin{Bmatrix} 1.759(60) \\ 1.699(120) \\ 1.482(180) \end{Bmatrix} = \begin{Bmatrix} 105.5 \\ 203.9 \\ 266.8 \end{Bmatrix}$$

MODE 2



$$\begin{Bmatrix} 2.481(60) \\ 0.117(120) \\ -1.048(180) \end{Bmatrix} = \begin{Bmatrix} 148.9 \\ 14.0 \\ -188.6 \end{Bmatrix}$$

MODE 3



$$\begin{Bmatrix} 0.386(60) \\ -0.545(120) \\ 0.267(180) \end{Bmatrix} = \begin{Bmatrix} 23.2 \\ -65.4 \\ 48.1 \end{Bmatrix}$$

Example (Continued)

Computing Interstory Shears: SRSS Combination

$$\begin{Bmatrix} \sqrt{106^2 + 149^2 + 23.2^2} \\ \sqrt{204^2 + 14^2 + 65.4^2} \\ \sqrt{267^2 + 189^2 + 48.1^2} \end{Bmatrix} = \begin{Bmatrix} 220 \\ 215 \\ 331 \end{Bmatrix}$$

“Exact”

$$\begin{Bmatrix} 38.2 \\ 163 \\ 346 \end{Bmatrix}$$

At time of Max. Shear

“Exact”

$$\begin{Bmatrix} 135 \\ 197 \\ 220 \end{Bmatrix}$$

At time of Max. Displacement

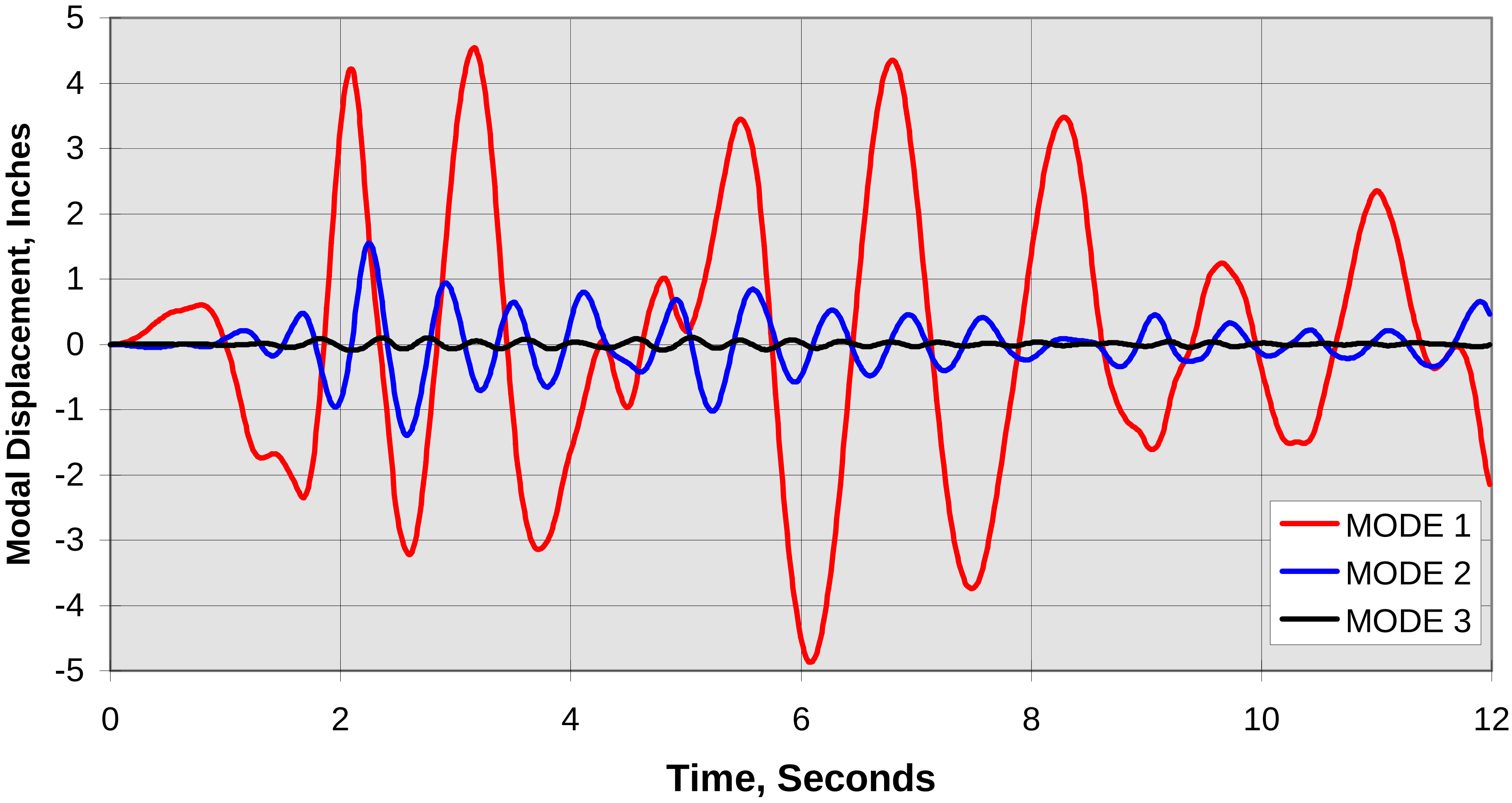
“Exact”

$$\begin{Bmatrix} 207 \\ 203 \\ 346 \end{Bmatrix}$$

Envelope = Maximum per Story

Using Less than Full (Possible) Number of Natural Modes

MODAL Response Time Histories:



Using Less than Full Number of Natural Modes

Time-History for **MODE** 1

$$y(t)=\begin{bmatrix} y_1(t1) & y_1(t2) & y_1(t3) & y_1(t4) & y_1(t5) & y_1(t6) & y_1(t7) & y_1(t8) & \dots & y_1(tn) \\ y_2(t1) & y_2(t2) & y_2(t3) & y_2(t4) & y_2(t5) & y_2(t6) & y_2(t7) & y_2(t8) & \dots & y_2(tn) \\ y_3(t1) & y_3(t2) & y_3(t3) & y_3(t4) & y_3(t5) & y_3(t6) & y_3(t7) & y_3(t8) & \dots & y_3(tn) \end{bmatrix}$$

Transformation:

$$\underset{3 \times nt}{u(t)} = \underbrace{\underset{3 \times 3}{\begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \end{bmatrix}} \underset{3 \times nt}{y(t)}}_{3 \times nt}$$

Time-History for **DOF** 1

$$u(t)=\begin{bmatrix} u_1(t1) & u_1(t2) & u_1(t3) & u_1(t4) & u_1(t5) & u_1(t6) & u_1(t7) & u_1(t8) & \dots & u_1(tn) \\ u_2(t1) & u_2(t2) & u_2(t3) & u_2(t4) & u_2(t5) & u_2(t6) & u_2(t7) & u_2(t8) & \dots & u_2(tn) \\ u_3(t1) & u_3(t2) & u_3(t3) & u_3(t4) & u_3(t5) & u_3(t6) & u_3(t7) & u_3(t8) & \dots & u_3(tn) \end{bmatrix}$$

Using Less than Full (Possible) Number of Natural Modes

Time-History for **MODE** 1

$$y(t) = \begin{bmatrix} y_1(t1) & y_1(t2) & y_1(t3) & y_1(t4) & y_1(t5) & y_1(t6) & y_1(t7) & y_1(t8) & \dots & y_1(tn) \\ y_2(t1) & y_2(t2) & y_2(t3) & y_2(t4) & y_2(t5) & y_2(t6) & y_2(t7) & y_2(t8) & \dots & y_2(tn) \end{bmatrix}$$

Mode 3 **NOT** Analyzed

Transformation:

$$\boxed{u(t) = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} y(t)}$$

3 x nt   3 x 2   2 x nt

3 x nt

Time-History for **DOF** 1

$$u(t) = \begin{bmatrix} u_1(t1) & u_1(t2) & u_1(t3) & u_1(t4) & u_1(t5) & u_1(t6) & u_1(t7) & u_1(t8) & \dots & u_1(tn) \\ u_2(t1) & u_2(t2) & u_2(t3) & u_2(t4) & u_2(t5) & u_2(t6) & u_2(t7) & u_2(t8) & \dots & u_2(tn) \\ u_3(t1) & u_3(t2) & u_3(t3) & u_3(t4) & u_3(t5) & u_3(t6) & u_3(t7) & u_3(t8) & \dots & u_3(tn) \end{bmatrix}$$

Using Less than Full (Possible) Number of Natural Modes

(Modal Response Spectrum Technique)

Sum of Absolute Values

$$\begin{Bmatrix} 4.940+1.550+0.108 \\ 3.181+0.931+(-0.278) \\ 1.482+1.048+0.267 \end{Bmatrix} = \begin{Bmatrix} 6.60 \\ 3.83 \\ 2.80 \end{Bmatrix} \qquad \begin{Bmatrix} 6.49 \\ 4.112 \\ 2.53 \end{Bmatrix}$$

"Exact"

$$\begin{Bmatrix} 5.11 \\ 2.86 \\ 1.22 \end{Bmatrix}$$

Square Root of the Sum of the Squares

$$\begin{Bmatrix} \sqrt{4.940^2+1.550^2+0.108^2} \\ \sqrt{3.181^2+0.931^2+(-0.278)^2} \\ \sqrt{1.482^2+1.048^2+0.267^2} \end{Bmatrix} = \begin{Bmatrix} 5.18 \\ 3.33 \\ 1.84 \end{Bmatrix} \qquad \begin{Bmatrix} 5.18 \\ 3.31 \\ 1.82 \end{Bmatrix}$$

At time of Max. Displacement

3 modes 2 modes

NEHRP Provisions allow an Approximate Modal Analysis Technique Called the "EQUIVALENT LATERAL FORCE PROCEDURE"

- Empirical Period of Vibration
- Smoothed Response Spectrum
- Compute Total Base Shear **V** as if SDOF
- Distribute **V** Along Height assuming "Regular" Geometry
- Compute Displacements and Member Forces using Standard Procedures