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TORSION OF COMPOUND BARS—A RELAXATION SOLUTION

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Summary—This paper deals with a finite-difference solution of the torsion problem of nonhomogeneous and compound prismatic bars. General, governing equations for both problems are developed and the boundary conditions for an interface between parts composed of homogeneous but different materials are stated. The case of multiply connected regions is discussed and integral conditions, analogous to the conditions in multiply connected homogeneous bars, are developed.

Examples illustrating various types of problems are worked out and the accuracy of the method demonstrated by comparison with some known solutions.

NOTATION

xyz	co-ordinates
uvw	displacement components in xyz direction
τ_{xz}, τ_{yz}	shear stresses
θ	twist per unit length
G	modulus of rigidity
ϕ	Prandtl's stress function
T	torque
s, n	tangent and normal directions to curve
α	ratio of G_2 to G_1

1. INTRODUCTION

The increasing use of geometrically complicated composite sections in practice has prompted this investigation of the torsion of composite bars. Composite sections are used in reinforced concrete and aircraft, as well as other more specialized applications, and the methods presented here should prove useful in solving the torsion problem in these fields.

Several problems on the torsion of compound prismatic bars have been solved analytically by Muskhelishvilli,[1] Gorgidze,[2] Mitra,[3] Takeyama,[4] Suhareviki,[5] Sherman,[6] Cowan,[7] Craven[8] and others. The problems solved generally dealt with cross-sectional shapes that could be mapped conformally, with relative ease.

While these analytical solutions of the torsion problem for both homogeneous as well as composite bars are exceedingly important, it is necessary to use relaxation methods to solve the problem of even a homogeneous bar when the geometry of the cross-section becomes complicated as instanced by numerous publications by Southwell,[9] Shaw,[10] Allen,[11] Dobie[12] and others[13–16]. Since the homogeneous bar can be easily treated by relaxation methods, it seemed logical to attempt to extend these methods to cover the case of simply and multiply connected composite bars when the geometry of the cross-section is complicated. Indeed, the treatment of this more general case also turns out to be simple.

It should be pointed out that the purpose of the examples which were chosen for this paper is to illustrate how the method can be applied to the various general classes of problems, and not to illustrate the handling of especially complicated boundaries. More complex boundary shapes can be dealt with as usual by using a finer mesh size which will require more time to obtain the required solutions. It is anticipated that any digital computer programmes available for solution of the Poisson equation could be easily adapted to the type of problems discussed.

Equations will be developed for simply connected cross-sections, where the material has a continuously varying G, and then for those that are composed of two distinct materials joined together, without slipping, at an interface. The additional conditions required in the case of a multiply connected cross-section will also be derived, and examples will be shown for each class of problem. Finally, the membrane analogy will be discussed in detail and other analogies pointed out.

2. GOVERNING EQUATIONS FOR CONTINUOUSLY VARYING G

Consider Fig. 1 and a material which has a continuously varying G in cross-section, but whose G is independent of z, i.e.

$$G = G(x, y). \tag{1}$$

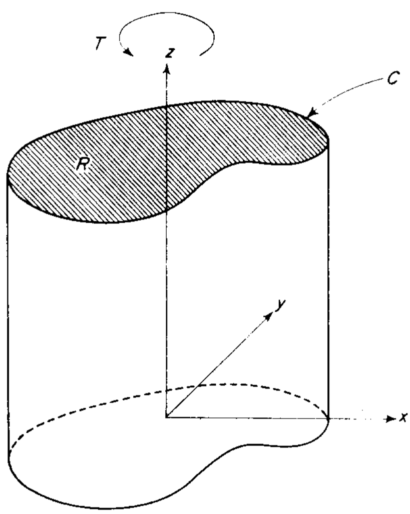

Fig. 1. Torsion of a bar. Co-ordinates used.

Following the notation used by Timoshenko and Goodier,[17] and using Prandtl's stress function ϕ, which defines the two stress components by

$$\tau_{xz} = \frac{\partial \phi}{\partial y}, \quad \tau_{yz} = -\frac{\partial \phi}{\partial x}, \tag{2}$$

and, with the other stresses equal to zero, we find that the equilibrium equations are automatically satisfied. If, in addition, we take the displacement components given by

$$u = -\theta yz, \quad v = \theta xz,$$

$$\frac{\partial w}{\partial x} = \frac{1}{G}\frac{\partial \phi}{\partial y} + \theta y, \quad \frac{\partial w}{\partial y} = -\frac{1}{G}\frac{\partial \phi}{\partial x} - \theta x, \tag{3}$$

all the relations between stresses and strains are determined correctly. To ensure continuity of the axial displacement w, we have, by differentiating the final two expressions of (3), that

$$\left\{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right\} - \frac{1}{G}\left\{\frac{\partial \phi}{\partial x}\frac{\partial G}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial G}{\partial y}\right\} = -2G\theta,$$

or

$$\frac{\partial}{\partial x}\left(\frac{1}{G}\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{G}\frac{\partial \phi}{\partial y}\right) = -2\theta. \tag{4}$$

In the case of constant G this reduces to the familiar Poisson equation

$$\nabla^2 \phi \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta. \tag{4a}$$

As the shear stress normal to the external boundary must be zero, it follows from the definition (2) that the stress function on this boundary must have a constant value. This constant can be arbitrarily fixed as zero. The sufficient and necessary boundary condition on the external boundary C becomes

$$\phi = 0. \tag{5}$$

To determine the torque T acting on the section, we integrate

$$T = \iint_R \{x\tau_{yz} - y\tau_{xz}\}\,dx\,dy,$$

which, by substitution of (2) and (5), yields

$$T = 2\iint_R \phi\,dx\,dy. \tag{6}$$

For purposes of computation it is convenient to rewrite the above relationships in a non-dimensional form. Substituting

$$x = x'L, \quad y = y'L, \quad G = G'G_0, \quad \phi = \phi'(G_0\theta L^2),$$

in which L represents some typical dimension of the section, and G_0 some typical value of G, we have, in place of (4) and (5),

$$\left\{\frac{\partial^2 \phi'}{\partial x'^2} + \frac{\partial^2 \phi'}{\partial y'^2}\right\} - \frac{1}{G'}\left\{\frac{\partial \phi'}{\partial x'}\frac{\partial G'}{\partial x'} + \frac{\partial \phi'}{\partial y'}\frac{\partial G'}{\partial y'}\right\} = -2G', \tag{7}$$

$$\phi' = 0 \quad \text{or} \quad G. \tag{8}$$

The stresses and the torque are given by

$$\tau_{s'z} = -(G_0\theta L)\frac{\partial \phi'}{\partial n'}, \tag{9}$$

$$T = 2G_0\theta L^4\iint_R \phi'\,dx'\,dy'. \tag{10}$$

3. GOVERNING EQUATIONS FOR A DISCONTINUOUS G

We shall now consider the section shown in Fig. 2, where a discontinuity occurs in the variation of G at the interface C_2, which separates regions R_1 and R_2. Clearly, if G_1 and G_2 refer to the values of G in the respective regions, two functions, ϕ_1 and ϕ_2, satisfying equation (4) in their domains, are required.

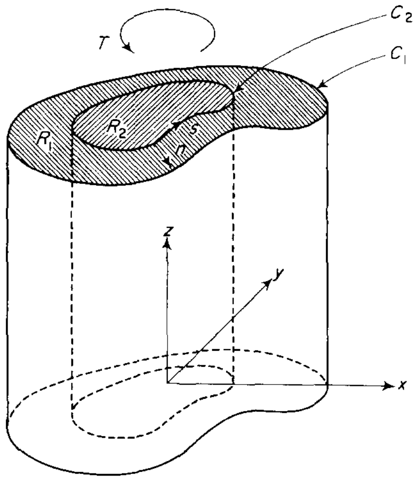

FIG. 2. Torsion of a composite bar.

On the external boundary, whether C_2 cuts it or not, the boundary condition (5) still has to be imposed. However, to specify the problem completely, additional requirements have to be imposed on the interface. These conditions must ensure that: (a) the shear stresses normal to the interface are the same in each region; and (b) the axial displacements are compatible on the interface. The first of these can be expressed as

$$\frac{\partial \phi_1}{\partial s} = \frac{\partial \phi_2}{\partial s} \quad \text{on} \quad C_2, \tag{11}$$

which can be satisfied by making

$$\phi_1 = \phi_2 \quad \text{on} \quad C_2, \tag{11a}$$

as the addition of arbitrary constants does not affect the results.

The second condition is satisfied if

$$\frac{\partial w_1}{\partial s} = \frac{\partial w_2}{\partial s} \quad \text{on} \quad C_2, \tag{12}$$

which by the use of (3) can be shown to be equivalent to

$$\frac{1}{G_1}\frac{\partial \phi_1}{\partial n} = \frac{1}{G_2}\frac{\partial \phi_2}{\partial n} \quad \text{on} \quad C_2, \tag{12a}$$

n being the direction of the normal to C_2.

It can be seen that the stress function will be continuous across the interface, but its derivatives will in general be discontinuous there. The problem thus becomes very similar to those encountered in porous media flow, or in the determination of magnetic fields, when the permeabilities differ abruptly. It is well known that phenomena of refraction of the flow lines occur in such instances.

Considering the case of a bar made of two materials, each having constant elastic properties, and using the non-dimensional notation of the previous section, we have

$$\nabla^2 \phi_1' = -2 \quad \text{in} \quad R_1, \tag{13}$$

$$\nabla^2 \phi_2' = -2\alpha \quad \text{in} \quad R_2, \tag{14}$$

with $\phi' = 0$ on external boundary and

$$\phi_1' = \phi_2', \tag{15}$$

$$\alpha \frac{\partial \phi_1'}{\partial n'} = \frac{\partial \phi_2'}{\partial n'}$$

on interface, in which arbitrarily

$$G_1 = G_0$$

and

$$\alpha = \frac{G_2}{G_1}.$$

These relations will be used in the illustrative examples shown later.

It should be noted that the methods of the previous section, using a continuously varying G, could be used to solve this problem of a discontinuous G at an interface, by approximating to the discontinuity in G by a continuous but steep, variation. This is particularly easy to do in finite-difference treatment, and will be illustrated later. The results obtained by using this procedure, however, are not as accurate (for a given mesh size) as those obtained by assuming the G discontinuous, as does the method of this section.

4. MULTIPLY CONNECTED CROSS-SECTIONS

In the problems examined so far the condition of single-valued displacements is automatically satisfied, as could be shown by integration of the expressions (3). If, however, referring to Fig. 2, the region enclosed by C_2 were empty, an additional condition would be required. Such a condition, usually expressed in a form of a line integral, is well known for the problem of torsion of multiply connected, homogeneous sections (see Timoshenko and Goodier[17]). For the problem on hand, where G_1 may be variable, an equivalent condition has to be established. We shall derive this by taking the case of a hole as a limiting case of the problem discussed in the previous section. Now we shall take G_2 as being equal to a constant k in R_2, and let this constant tend to zero.

Clearly, one of the conditions on the "interface" now is that ϕ_1 is equal to a constant. Condition (12a) becomes insufficient in the limit, merely giving $\partial\phi_2/\partial n = 0$, which is already implicit, no condition being imposed on $\partial\phi_1/\partial n$. However, integrating (12a) around the interface C_2, we have

$$\oint_{C_1} \frac{1}{G_1} \frac{\partial \phi_1}{\partial n} = \oint_{C_2} \frac{1}{G_2} \frac{\partial \phi_2}{\partial n},$$

where G_2 is a constant, while G_1 is a function of x and y. Applying Green's Theorem in two dimensions to the right-hand side, we obtain

$$\oint_{C_2} \frac{1}{G_1}\frac{\partial \phi_1}{\partial n} = \frac{1}{G_2}\iint_{A_{C_2}} \left\{\frac{\partial^2 \phi_2}{\partial x^2}+\frac{\partial^2 \phi_2}{\partial y^2}\right\} dx\,dy,$$

where A_{C_2} is the area inside the curve C_2. Since G_2 is constant, equation (4a) applies inside and on C_2. Therefore we obtain after substitution

$$\oint_{C_2} \frac{1}{G_1}\frac{\partial \phi_1}{\partial n} = \frac{1}{G_2}\iint_{A_{C_2}} (-2G_2\theta)\,dx\,dy$$

$$\oint_{C_2} \frac{1}{G_1}\frac{\partial \phi_1}{\partial n} = -2\theta A_{C_2}. \tag{16}$$

It can be seen that this equation reduces to the familiar form when G_1 is a constant; namely,

$$\oint_{C_2} \frac{\partial \phi_1}{\partial n} = -2G\theta A_{C_2}.$$

A sufficient number of boundary conditions is now available for a unique solution of multiply connected sections.

It can be observed that as G_2 tends to zero, ϕ_2 must become constant in region R_2 to satisfy the governing equation (4), and the analogy of the "floating disc" is again applicable, as it is in the case of homogeneous sections.

5. MEMBRANE AND OTHER ANALOGIES

It has been pointed out that the torsion problem of a homogeneous bar is analogous, mathematically, with several other physical problems. These include the membrane under constant pressure, the problem of viscous flow of fluid in a tube, flow of a current in a conductor of variable thickness, and in general any of the physical problems which satisfy Poisson's or Laplace's equation. These analogies can be extended to the case of a composite bar in torsion with rewarding results. We shall discuss only one of these, namely, the membrane analogy. It is well known that the deflections of a membrane subjected to a constant pressure and tension satisfy the equation

$$\nabla^2 z = -\frac{q}{T}$$

(in which q is a pressure, T the membrane tension and Z the deflection). It can easily be shown that in the case of a membrane with variable tension the equivalent expression is

$$\frac{\partial}{\partial x}\left(T\frac{\partial z}{\partial x}\right)+\frac{\partial}{\partial y}\left(T\frac{\partial z}{\partial y}\right) = -q.$$

The analogy with equation (4) is now obvious if

$$T \equiv \frac{1}{G}, \quad q \equiv 2\theta, \quad z \equiv \phi.$$

Discontinuous variation of G can be interpreted as a discontinuous variation of the membrane tension and because

$$\oint T \frac{\partial z}{\partial n} = -Aq$$

by statics, all the other relationships for interface conditions, as well as for multiply connected regions, can be deduced. It should be observed that the variation of T implies existence of distributed forces parallel to the xy planes—which in the case of an interface becomes a line force. As a possible experimental solution the membrane analogy does not appear practicable.

6. EXAMPLES

The governing equations and their respective boundary conditions can be easily transformed into suitable finite-difference relationships and a solution can then be obtained by application of relaxation methods. As the techniques of both are well known and described in many texts (e.g. Allen[11]) it is not

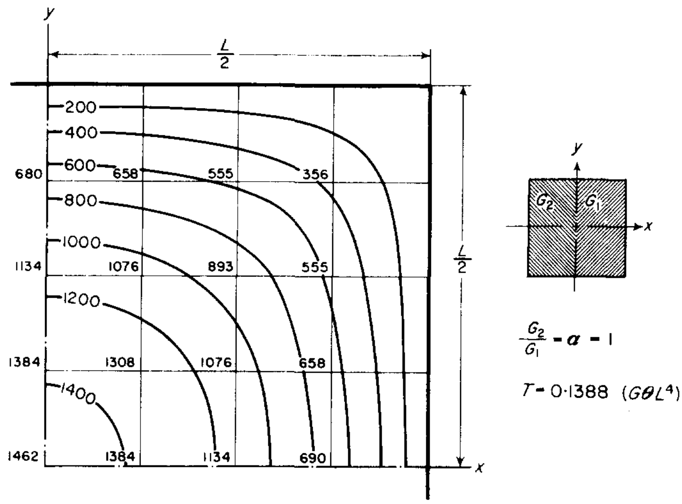

FIG. 3(a). Example A. Torsion of a square composite bar; $\alpha = 1$; values of $\phi' \times 10^4$.

proposed to elaborate on these matters here. The general treatment of Poisson's equation is now virtually standard for the usual Neumann and Dirichlet boundary conditions and the main problems encountered here are those pertaining to the interface between the two materials. Again, as the conditions at the interface are essentially similar to those encountered in the treatment of seepage or magnetic field problems in media of different permeabilities, techniques or relaxation treatment are well known. These, used for the first time by Christopherson,[18] are excellently described in Allen's text, and no special difficulty was encountered in their application in the examples solved.

Examples A and B (Figs. 3(a–c) and 4) show problems in which straight and curved interfaces between two materials occur respectively. The accuracy of the relaxation solution can be seen from the table below, in which the stiffness of the bars of example A obtained from a relatively coarse mesh solution are compared with values computed by a series solution developed by Muskhelishvilli[1].

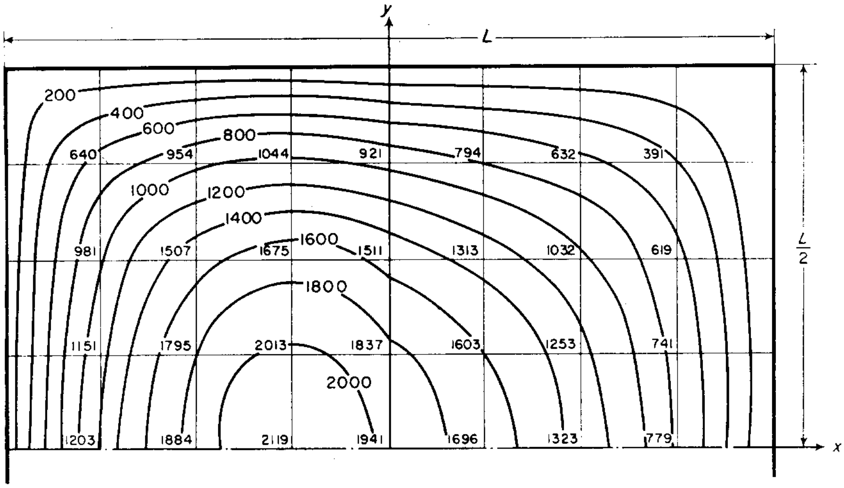

FIG. 3(b). Example A. Torsion of a square composite bar; $\alpha = 2$; $T = 0{\cdot}1941\ (G\theta L^4)$; values of $\phi' \times 10^4$.

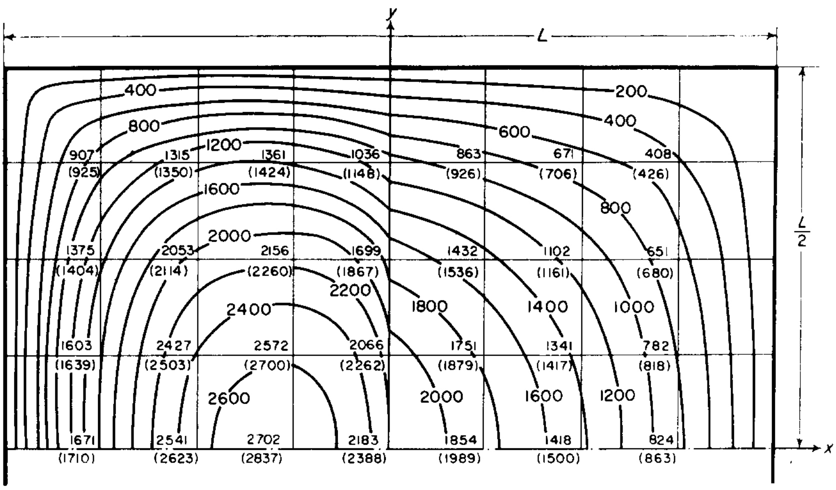

FIG. 3(c). Example A. Torsion of a square composite bar; $\alpha = 3$; $T = 0{\cdot}2358\ (G_0\theta L^4)$; values of $\phi' \times 10^4$.

$$T = D(G_0\theta L^4)$$

Values of	Torsional rigidity, D			
	By relaxation	Timoshenko	Muskhelishvilli*	(% discrepancy)
$\alpha = 1$	0·1388	0·1406	0·1407	1·28
$\alpha = 2$	0·1941		0·1972	1·57
$\alpha = 3$	0·2358		0·2399	1·71

* Only two terms of the series solution were used, and these values are always greater than the true value. The error increases with increasing α.

To test the alternative method of satisfying the interface conditions, in one of the examples [see Fig. 3(c)], G was assumed to vary in a continuous way from G_1 to G_2 within a distance of one mesh length. The finite difference expression of equation (4) can now be used directly at all mesh points. Values in parentheses, [Fig. 3(c)], refer to an assumed continuous variation of G, while the others are for a discontinuous variation of G.

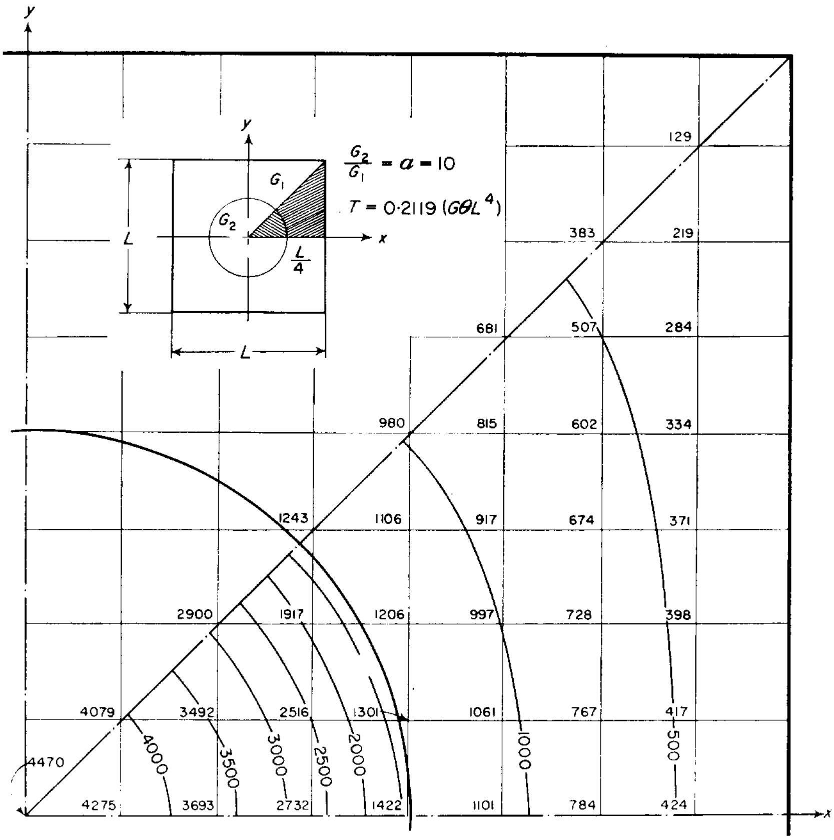

FIG. 4. Example B. Torsion of a square bar with a circular insert; $\alpha = 10$; values of $\phi' \times 10^4$.

The coarseness of the mesh resulted in an error of about 6 per cent in the final stiffness as compared with the exact solution, i.e. it proved to be less accurate than the method employing the correct interface conditions. The procedure may, however, be advantageous when numerous curved interfaces occur.

In example C (Fig. 5) the case of a multiply connected region is considered in which the interface between two materials cuts the hole boundary. The procedure followed here was essentially the same as described by Shaw[10] for the treatment of multiply connected, homogeneous shafts. Solutions with a constant and arbitrary value of the stress function on the hole boundary are first obtained using the governing equation (4) and excluding the non-homogeneous term. Then a solution of the full equation with a zero value of the stress function of the hole boundary is computed, and the final solution obtained by a linear combination of these. This combined solution satisfies the integral condition (16). Again no special difficulties were encountered.

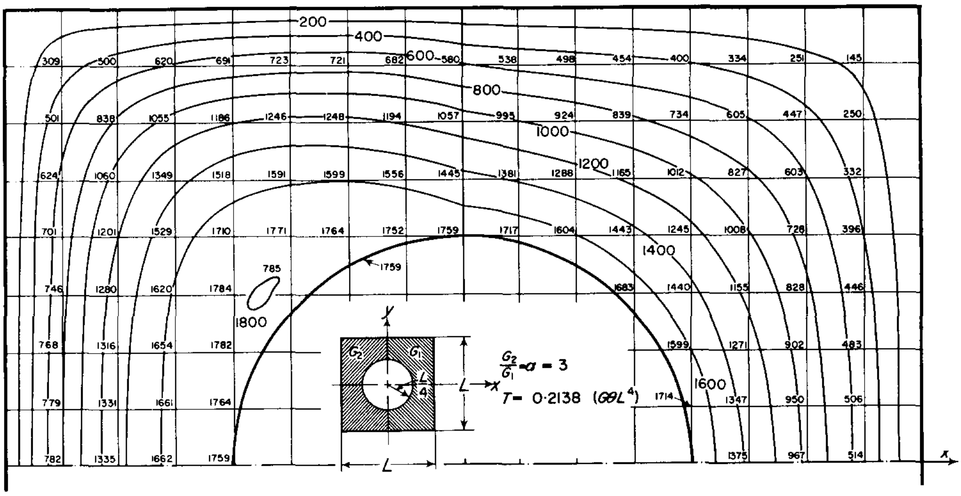

FIG. 5. Example C. Torsion of a composite bar with a circular hole.

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