TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

Swiss Mathematician and physicist Leonhard Euler discovered the wave equation in three space dimensions.

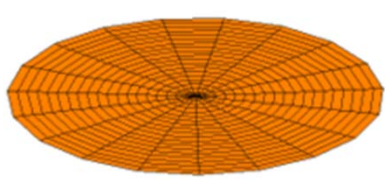
Leonhard Euler (1707 – 1783) was a pioneering Swiss mathematician and physicist. He made important discoveries in fields as diverse as infinitesimal calculus and graph theory. He is also renowned for his work in mechanics, fluid dynamics, optics, astronomy, and music theory

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The wave equation is an important second-order linear partial differential equation for the description of waves – as they occur in physics – such as sound waves, light waves and water waves.

It arises in fields like acoustics, electromagnetics, and fluid dynamics.

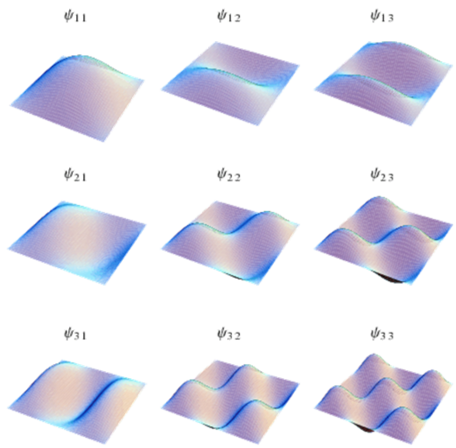


A solution of the wave equation in two dimensions with a zero-displacement boundary condition along the entire outer edge.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

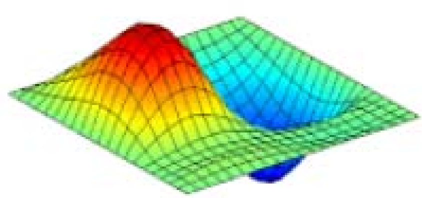
Consider the motion of a rectangular membrane (in the absence of gravity) using the two-dimensional wave equation. Plots of the spatial part for modes are illustrated below.



TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

Consider the motion of a rectangular membrane (in the absence of gravity) using the two-dimensional wave equation.



TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The governing equations of motion that describe the propagation of waves in situations involving two independent variables appear typically as

$$\nabla \cdot \left(k \nabla u(x,y,t)\right) - \rho \frac{\partial^2 u}{\partial t^2} + f(x,y,t) = 0 \qquad \text{in } \Omega$$

$$u = g(s,t) \qquad \text{on } \Gamma_1$$

$$k\frac{\partial u}{\partial n} + \alpha(s,t)u = q(s,t) \qquad \text{on } \Gamma_2$$

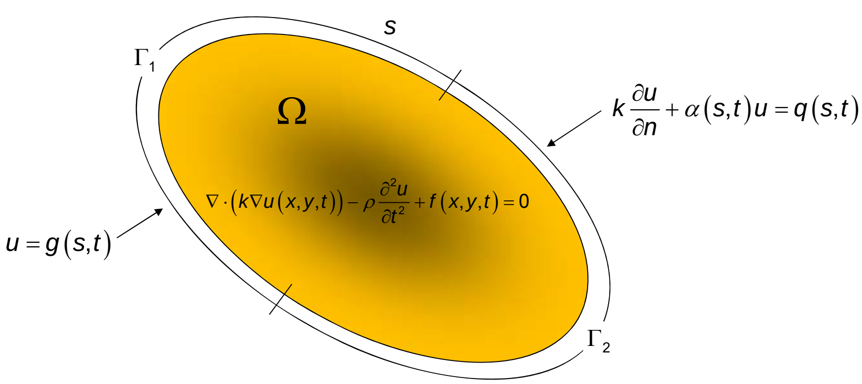
$$u(x,y,0) = c(x,y) \qquad \text{in } \Omega$$

$$\frac{\partial u(x,y,0)}{\partial t} = d(x,y) \qquad \text{in } \Omega$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

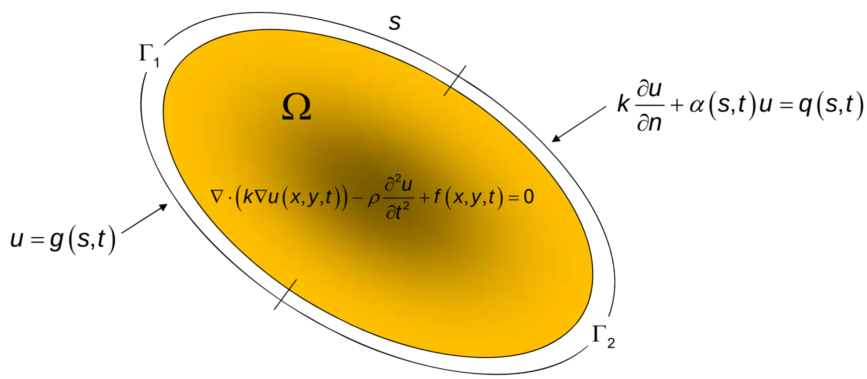
Where Ω is the interior domain, and Γ_1 and Γ_2 form the boundary of the domain.



TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

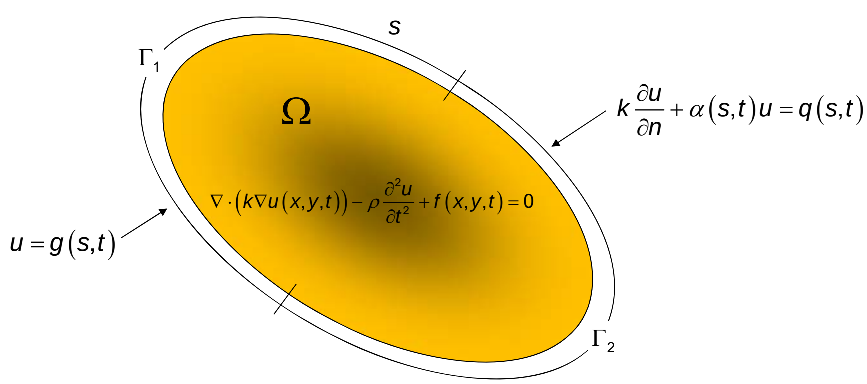
The physical constants are *k* and ρ, and *s* is a linear measure of the position on the boundary.



TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The type of boundary condition specified on Γ_2 results from a local balance between internal and external forces.



TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

This equation is similar in form to the parabolic initial-boundary value problem presented in the previous section with the very important change in the time derivative term from a first to a second derivative, and with the addition of a second initial condition on the velocity.

With these changes, the problem changes from a parabolic initial-boundary value problem to a hyperbolic initial boundary value problem.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The basic steps of discretization, interpolation, elemental formulation, assembly, constraints, solution, and computation of derived variables are presented in this section as they relate to the two-dimensional hyperbolic initial-boundary value problem.

The Galerkin method, in connection with the corresponding weak formulation to be developed, will be used to generate the finite element model.

Discretization - Referred to the material in Chapter 3.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The Galerkin Finite Element Method

Interpolation - The solution is assumed to be expressible in terms of the nodally based interpolation functions $n_i(x, y)$ introduced and discussed in Section 3.2.

In the present setting, these interpolation functions are used with the semidiscretization.

$$u(x,y,t) = \sum_{i=1}^{N+1} u_i(t) n_i(x,y)$$

The $n_i(x, y)$ are nodally based interpolation functions and can be linear, quadratic, or as otherwise desired.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The Galerkin Finite Element Method

Elemental Formulation - The starting point for the elemental formulation is the weak formulation of the initial-boundary value problem.

The first step in developing the weak formulation is to multiply the differential equation by an arbitrary test function $v(x, y)$ vanishing on Γ_1.

The result is then integrated over the domain Ω to obtain

$$\iint_{\Omega} v\left(\nabla \cdot \left(k \nabla u \right) - \rho \frac{\partial^2 u}{\partial t^2} + f \right) d\Omega = 0$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The Galerkin Finite Element Method

Elemental Formulation - Using the two-dimensional form of the divergence theorem to integrate the first term by parts, the results after rearranging are:

$$\iint_{\Omega} \nabla v \cdot k \nabla u \, d\Omega + \iint_{\Omega} v \rho \frac{\partial^2 u}{\partial t^2} d\Omega = \int_{\Gamma} v\mathbf{n} \cdot k \nabla u \, d\Gamma + \iint_{\Omega} vf \, d\Omega$$

where $\Gamma = \Gamma_1 + \Gamma_2$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The Galerkin Finite Element Method

Elemental Formulation - Recalling that v vanishes on Γ_1 and that $k\partial u/\partial x = k\mathbf{n} \cdot \nabla u = q - hu$ on Γ_2, it follows that

$$\iint_{\Omega} \nabla v \cdot k \nabla u \, d\Omega + \iint_{\Omega} v \rho \frac{\partial^2 u}{\partial t^2} d\Omega = \int_{\Gamma} v\left(q - hu\right) d\Gamma + \iint_{\Omega} vf \, d\Omega$$

This equation is the required weak formulation for the two-dimensional diffusion problem.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The Galerkin Finite Element Method

Elemental Formulation - Substituting the approximation of $u(x, y, t)$ into the weak formulation and taking $v = n_k$, $k = 1, 2, \ldots$ yields:

$$\iint_{\Omega^-}\left(\frac{\partial n_k}{\partial x}k\sum u_i\frac{\partial n_i}{\partial x}+\frac{\partial n_k}{\partial y}k\sum u_i\frac{\partial n_i}{\partial y}\right)d\Omega$$

$$+\iint_{\Omega^-} n_k\rho\sum \ddot{u}_i n_i\, d\Omega+\int_{\Gamma^-} n_k h\sum u_i n_i\, d\Gamma$$

$$=\iint_{\Omega^-} n_k f\, d\Omega+\int_{\Gamma_2^-} n_k q\, d\Gamma \qquad k=1, 2, \ldots$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The Galerkin Finite Element Method

Elemental Formulation – Where Ω^- and Γ_2^- represent the elemental areas approximating Ω, and the collection of the elemental edges approximating Γ_2,respectively.

$$\iint_{\Omega^-}\left(\frac{\partial n_k}{\partial x}k\sum u_i\frac{\partial n_i}{\partial x}+\frac{\partial n_k}{\partial y}k\sum u_i\frac{\partial n_i}{\partial y}\right)d\Omega$$

$$+\iint_{\Omega^-} n_k\rho\sum \ddot{u}_i n_i\, d\Omega+\int_{\Gamma^-} n_k h\sum u_i n_i\, d\Gamma$$

$$=\iint_{\Omega^-} n_k f\, d\Omega+\int_{\Gamma_2^-} n_k q\, d\Gamma \qquad k=1, 2, \ldots$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The Galerkin Finite Element Method

Elemental Formulation – This N x N set of linear algebraic equations can be written as

$$\sum_{i=1}^{N}\left(A_{ki}u_i + B_{ki}\ddot{u}_i\right) = F_k(t) \qquad k = 1,2,\ldots,N$$

$$A_{ki} = \iint_{\Omega^-}\left(\frac{\partial n_k}{\partial x}k\frac{\partial n_i}{\partial x} + \frac{\partial n_k}{\partial y}k\frac{\partial n_i}{\partial y}\right)d\Omega + \int_{\Gamma^-} n_k h n_i \, d\Gamma$$

$$B_{ki} = \iint_{\Omega^-} n_k \rho n_i \, d\Omega$$

$$F_k = \iint_{\Omega^-} n_k f \, d\Omega + \int_{\Gamma_2^-} n_k q \, d\Gamma$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The Galerkin Finite Element Method

Elemental Formulation – Note that assembly is contained implicitly within the formulation.

$$\sum_{i=1}^{N}\left(A_{ki}u_i + B_{ki}\ddot{u}_i\right) = F_k(t) \qquad k = 1,2,\ldots,N$$

$$A_{ki} = \iint_{\Omega^-}\left(\frac{\partial n_k}{\partial x}k\frac{\partial n_i}{\partial x} + \frac{\partial n_k}{\partial y}k\frac{\partial n_i}{\partial y}\right)d\Omega + \int_{\Gamma^-} n_k h n_i \, d\Gamma$$

$$B_{ki} = \iint_{\Omega^-} n_k \rho n_i \, d\Omega$$

$$F_k = \iint_{\Omega^-} n_k f \, d\Omega + \int_{\Gamma_2^-} n_k q \, d\Gamma$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The Galerkin Finite Element Method

Elemental Formulation – In terms of the corresponding elementally based interpolations

$$u_e(x,y) = \mathbf{N}^T \mathbf{u}_e = \mathbf{u}_e{}^T \mathbf{N}$$

The finite element model can be expressed as

$$\mathbf{Au} + \mathbf{B\ddot{u}} = \mathbf{F}$$

$$\mathbf{A} = \sum_e \mathbf{k_G} + \sum_e{}' \mathbf{a_G} \qquad \mathbf{B} = \sum_e \mathbf{r_G}$$

$$\mathbf{F} = \sum_e \mathbf{f_G} + \sum_e{}' \mathbf{q_G}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The Galerkin Finite Element Method

Elemental Formulation – In terms of the corresponding elementally based interpolations

$$u_e(x,y) = \mathbf{N}^T \mathbf{u}_e = \mathbf{u}_e{}^T \mathbf{N}$$

The finite element model can be expressed as

$$\mathbf{k}_e = \iint_{A_e} \left(\frac{\partial \mathbf{N}}{\partial x} k \frac{\partial \mathbf{N}^T}{\partial x} + \frac{\partial \mathbf{N}}{\partial y} k \frac{\partial \mathbf{N}^T}{\partial y} \right) dA$$

$$\mathbf{r}_e = \iint_{A_e} \mathbf{N} \rho \mathbf{N}^T \, dA \qquad \mathbf{a}_e = \int_{\gamma_{2e}} \mathbf{N} \alpha \mathbf{N}^T \, ds$$

$$\mathbf{f}_e = \iint_{A_e} \mathbf{N} f \, dA \qquad \mathbf{q}_e = \int_{\gamma_{2e}} \mathbf{N} q \, ds$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The Galerkin Finite Element Method

Elemental Formulation – The initial conditions for the system of first-order differential equations are obtained from the initial conditions prescribed for the original initial-boundary value problem.

Generally $\mathbf{u}(0)$ is determined by evaluating the function $c(x, y)$ at the nodes to obtain:

$$\mathbf{u}(0) = \mathbf{u}_0 = \langle c_1 \quad c_2 \quad c_3 \quad \ldots \quad c_{N-1} \quad c_N \rangle^{\mathrm{T}}$$

where $c_i = c(x_i, y_i)$ with (x_i, y_i) the coordinates of the i^{th} node.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The Galerkin Finite Element Method

Elemental Formulation – The initial conditions for the system of first-order differential equations are obtained from the initial conditions prescribed for the original initial-boundary value problem.

Generally $\dot{\mathbf{u}}(0)$ is determined by evaluating the function $d(x, y)$ at the nodes to obtain

$$\dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0 = \langle d_1 \quad d_2 \quad d_3 \quad \ldots \quad d_{N-1} \quad d_N \rangle^{\mathrm{T}}$$

where $d_i = d(x_i, y_i)$ with (x_i, y_i) the coordinates of the i^{th} node.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The Galerkin Finite Element Method

Constraints - The constraints arise from the boundary conditions specified on Γ_1.

Generally the values of the constraints are determined from the *q* function with the constrained value of *u* at a node on Γ_1 being taken as the value of *q* at that point.

These constraints are then enforced on the assembled equations, resulting in the final global constrained set of linear first-order differential equations.

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f} \qquad \mathbf{u}(0) = \mathbf{u}_0 \qquad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The Galerkin Finite Element Method

Solution - The system of equations is precisely the same in form and character as the corresponding equations developed in Section 4.2.2 for the one-dimensional wave problem.

The analytical method as well as the numerical methods using the central difference and Newmark algorithms can be used for integrating the above set of equations.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The Galerkin Finite Element Method

Solution - The central difference algorithm will be conditionally stable with the critical time step depending on the maximum eigenvalue of the associated problem $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{v} = 0$.

The Newmark algorithm will be unconditionally stable for $\delta = 0.5$ and $\alpha = 0.25(\delta + 0.5)^2$.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

The Galerkin Finite Element Method

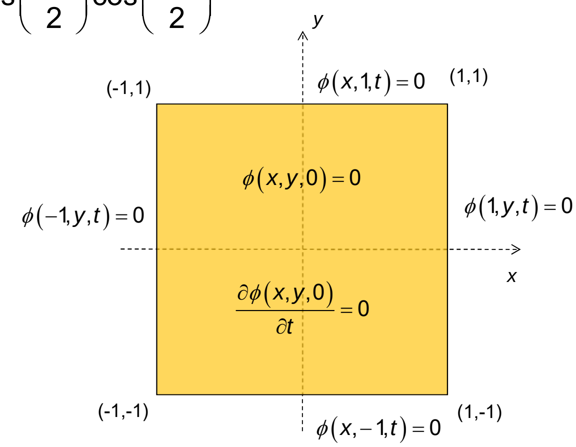
Derived variables - In a physical situation governed by a wave equation, the derived variables are usually the internal forces computed according to $\mathbf{F_e} = \mathbf{k_e u_e}$ per element for each time step.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

Consider the two-dimensional wave problem shown below.

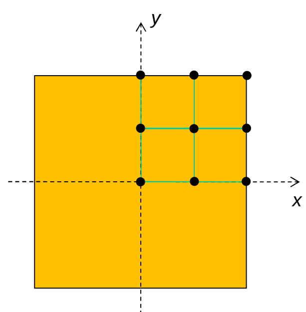
$$T\nabla^2\phi - \rho\frac{\partial^2\phi}{\partial t^2} = p_0\cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right)$$



TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

Discretization. Due to symmetry, we will model the top right-most quadrant of the membrane using four equally-sized 4-noded quadrilaterals.



$$\nabla^2\phi - \psi\frac{\partial^2\phi}{\partial t^2} = \frac{p_0}{T}\cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right)$$

$$\phi(\pm 1,y,t)=0 \qquad \psi = \frac{\rho}{T}$$

$$\phi(x,\pm 1,t)=0$$

$$\phi(x,y,0)=0 \qquad \frac{\partial\phi(x,y,0)}{\partial t}=0$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

Elemental Formulation – This *N* x *N* set of linear algebraic equations can be written as:

$$\sum_{i=1}^{N}\left(A_{ki}\phi_i + \psi B_{ki}\ddot{\phi}_i\right) = F_k(t) \qquad k = 1,2,\ldots,N$$

$$A_{ki} = \iint_{\Omega^-}\left(\frac{\partial n_k}{\partial x}\frac{\partial n_i}{\partial x} + \frac{\partial n_k}{\partial y}\frac{\partial n_i}{\partial y}\right)d\Omega$$

$$B_{ki} = \iint_{\Omega^-} n_k \psi n_i \, d\Omega$$

$$F_k = \iint_{\Omega^-} n_k f \, d\Omega$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

Elemental Formulation – In terms of the corresponding elementally based interpolations

$$\phi_e(x,y) = \mathbf{N}^T \phi_e = \phi_e{}^T \mathbf{N}$$

The finite element model can be expressed as

$$\mathbf{k}_e = \iint_{A_e}\left(\frac{\partial \mathbf{N}}{\partial x}\frac{\partial \mathbf{N}^T}{\partial x} + \frac{\partial \mathbf{N}}{\partial y}\frac{\partial \mathbf{N}^T}{\partial y}\right)dA$$

$$\mathbf{r}_e = \iint_{A_e} \mathbf{N}\mathbf{N}^T \, dA$$

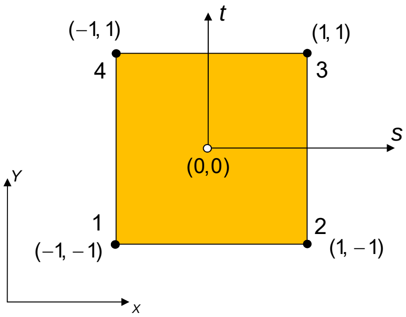
$$\mathbf{f}_e = \iint_{A_e} \mathbf{N} f \, dA$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

Interpolation - In matrix notation, the distribution of the function over the element in local (*s*, *t*) coordinates is:

$$\phi_e(x,y) = \boldsymbol{\phi}_\mathbf{e}^{\mathbf{T}}\mathbf{N} = \mathbf{N}^{\mathbf{T}}\boldsymbol{\phi}_\mathbf{e}$$



$$N_1(s,t) = \frac{(1-s)(1-t)}{4}$$

$$N_2(s,t) = \frac{(1+s)(1-t)}{4}$$

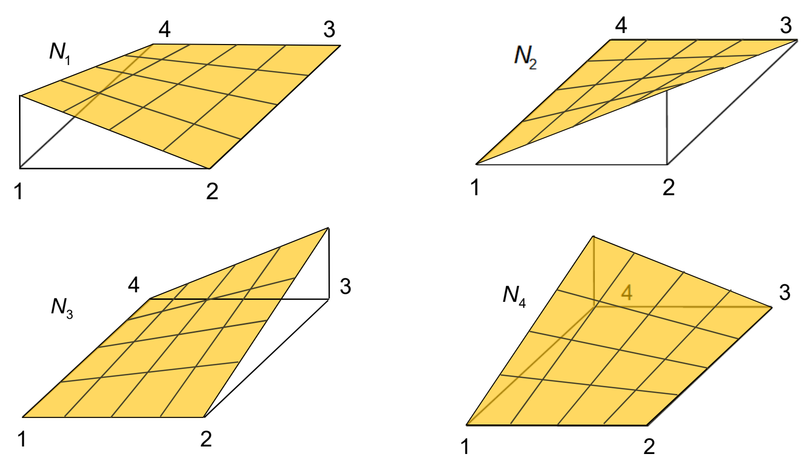
$$N_3(s,t) = \frac{(1+s)(1+t)}{4}$$

$$N_4(s,t) = \frac{(1-s)(1+t)}{4}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

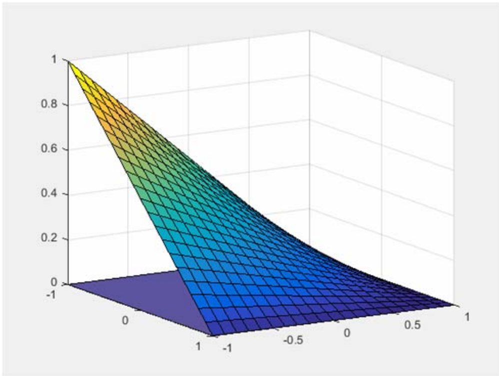
Rectangular Elements - The elemental interpolations for a rectangular element are:



TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

Rectangular Elements - The shape functions are visually deceiving. There is no curvature in directions parallel to any side; however, there is a twist due to the *xy* term in the element representation.



TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

Interpolation - This integrals may be transformed into the local coordinate space using $as = x - x_0$ and $bt = y - y_0$, for a rectangular element $2a \times 2b$ in size:

$$\iint_{A_e} F(x,y)\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} F(x_0 + as, y_0 + bt)\frac{\partial(\xi,\eta)}{\partial(s,t)}\,ds\,dt$$

where the Jacobian $\partial(\xi,\eta)/\partial(s,t)$ has a value of $A_e/4$.

The resulting global system stiffness matrix $\mathbf{k_e}$ is:

$$\mathbf{k_e} = \frac{1}{6}\left[\frac{b}{a}\begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix} + \frac{a}{b}\begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix}\right]$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

Rectangular Elements - Evaluation of $\mathbf{k_e}$

If the element is square, then $a = b$, then $\mathbf{k_e}$ becomes:

$$\mathbf{k_e} = \frac{1}{6}\left[\frac{b}{a}\begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix} + \frac{a}{b}\begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix}\right]$$

$$\mathbf{k_e} = \frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

The elemental $\mathbf{m_e}$ matrices are:

$$\mathbf{m_e} = \iint_{A_e} \mathbf{N}\mathbf{N}^{\mathbf{T}}\, dA = \int_{-1}^{1}\int_{-1}^{1} F\left(x_0 + as, y_0 + bt\right)\frac{\partial(\xi,\eta)}{\partial(s,t)}\, ds\, dt$$

where the Jacobian $\partial(\xi,\eta)/\partial(s, t)$ has a value of $A_e/4$.

The resulting global system stiffness matrix $\mathbf{m_e}$ is:

$$\mathbf{m_e} = \frac{A_e}{36}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

Transforming the integral into the non-dimensional coordinates (s, t) yields:

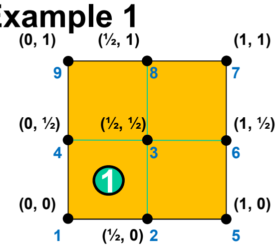
$$\mathbf{f_e} \approx \left(\int_{-1}^{1} \int_{-1}^{1} \mathbf{N}\mathbf{N}^T \frac{A_e}{4} \, ds \, dt \right) \mathbf{f}$$

$$\mathbf{f_e} \approx \frac{A_e}{36} \begin{Bmatrix} 4f_1 + 2f_2 + f_3 + 2f_4 \\ 2f_1 + 4f_2 + 2f_3 + f_4 \\ f_1 + 2f_2 + 4f_3 + 2f_4 \\ 2f_1 + f_2 + 2f_3 + 4f_4 \end{Bmatrix}$$

The resulting 4 x 1 elemental load vector contributes to the global system equations at those locations corresponding to the four nodes defining the element.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1



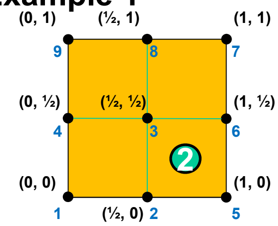
For element 1:

$$\mathbf{k}_1 = \frac{1}{6} \begin{array}{c} \begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix} \end{array} \begin{array}{c} \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{array} \qquad A_1 = \frac{1}{4}$$

$$\mathbf{m}_1 = \frac{1}{144} \begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix} \begin{array}{c} \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{array} \qquad \mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix}_1 = \frac{1}{2} \begin{Bmatrix} 2 \\ \sqrt{2} \\ 1 \\ \sqrt{2} \end{Bmatrix} \qquad \mathbf{f}_1 = \begin{Bmatrix} 0.05089 \\ 0.02659 \\ 0.02659 \\ 0.01389 \end{Bmatrix} \begin{array}{c} \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{array}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1



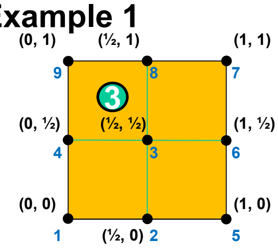
For element 2:

$$\mathbf{k}_2 = \frac{1}{6}\begin{array}{c}\left[\begin{array}{cccc} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{array}\right]\end{array}\begin{array}{c}\boxed{2}\\ \boxed{5}\\ \boxed{6}\\ \boxed{3}\end{array} \qquad A_2 = \frac{1}{4}$$

$$\mathbf{m}_2 = \frac{1}{144}\left[\begin{array}{cccc} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{array}\right]\begin{array}{c}\boxed{2}\\ \boxed{5}\\ \boxed{6}\\ \boxed{3}\end{array} \qquad \mathbf{f} = \begin{Bmatrix} f_2 \\ f_5 \\ f_6 \\ f_3 \end{Bmatrix}_2 = \frac{1}{2}\begin{Bmatrix} \sqrt{2} \\ 0 \\ 0 \\ 1 \end{Bmatrix} \qquad \mathbf{f}_2 = \begin{Bmatrix} 0.04539 \\ 0.01329 \\ 0.02371 \\ 0.00694 \end{Bmatrix}\begin{array}{c}\boxed{2}\\ \boxed{5}\\ \boxed{6}\\ \boxed{3}\end{array}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1



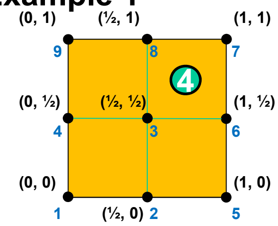
For element 3:

$$\mathbf{k}_3 = \frac{1}{6}\left[\begin{array}{cccc} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{array}\right]\begin{array}{c}\boxed{4}\\ \boxed{3}\\ \boxed{8}\\ \boxed{9}\end{array} \qquad A_3 = \frac{1}{4}$$

$$\mathbf{m}_3 = \frac{1}{144}\left[\begin{array}{cccc} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{array}\right]\begin{array}{c}\boxed{4}\\ \boxed{3}\\ \boxed{8}\\ \boxed{9}\end{array} \qquad \mathbf{f} = \begin{Bmatrix} f_4 \\ f_3 \\ f_8 \\ f_9 \end{Bmatrix}_3 = \frac{1}{2}\begin{Bmatrix} \sqrt{2} \\ 1 \\ 0 \\ 0 \end{Bmatrix} \qquad \mathbf{f}_3 = \begin{Bmatrix} 0.04048 \\ 0.01185 \\ 0.01185 \\ 0.00347 \end{Bmatrix}\begin{array}{c}\boxed{3}\\ \boxed{6}\\ \boxed{7}\\ \boxed{8}\end{array}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1



For element 4:

$$\mathbf{k}_4 = \frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix}\begin{array}{c}\boxed{3}\\ \boxed{6}\\ \boxed{7}\\ \boxed{8}\end{array} \qquad A_4 = \frac{1}{4}$$

$$\mathbf{m}_4 = \frac{1}{144}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix}\begin{array}{c}\boxed{3}\\ \boxed{6}\\ \boxed{7}\\ \boxed{8}\end{array} \qquad \mathbf{f} = \begin{Bmatrix} f_3 \\ f_6 \\ f_7 \\ f_8 \end{Bmatrix}_4 = \frac{1}{2}\begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \mathbf{f}_4 = \begin{Bmatrix} 0.04539 \\ 0.02371 \\ 0.01329 \\ 0.00694 \end{Bmatrix}\begin{array}{c}\boxed{3}\\ \boxed{8}\\ \boxed{9}\\ \boxed{4}\end{array}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

The assembled $\mathbf{K_G}$, $\mathbf{M_G}$, and $\mathbf{F}$ matrices are:

$$\mathbf{K_G} = \frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 8 & -2 & -2 & -1 & -2 & 0 & 0 & 0 \\ -2 & -2 & 16 & -2 & -2 & -2 & -2 & -2 & -2 \\ -1 & -2 & -2 & 8 & 0 & 0 & 0 & -2 & -1 \\ 0 & -1 & -2 & 0 & 4 & -1 & 0 & 0 & 0 \\ 0 & -2 & -2 & 0 & -1 & 8 & -1 & -2 & 0 \\ 0 & 0 & -2 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -2 & -2 & 0 & -2 & -1 & 8 & -1 \\ 0 & 0 & -2 & -1 & 0 & 0 & 0 & -1 & 4 \end{bmatrix}\begin{array}{c}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\\ \phi_6\\ \phi_7\\ \phi_8\\ \phi_9\end{array}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

The assembled $\mathbf{K_G}$, $\mathbf{M_G}$, and $\mathbf{F}$ matrices are:

$$\mathbf{M_G} = \frac{1}{144}\begin{array}{c}\begin{bmatrix} 4 & 2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 8 & 4 & 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 16 & 4 & 1 & 4 & 1 & 4 & 1 \\ 2 & 1 & 4 & 8 & 0 & 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 2 & 8 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 4 & 2 & 0 \\ 0 & 0 & 4 & 1 & 0 & 1 & 2 & 8 & 2 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 2 & 4 \end{bmatrix}\end{array}\begin{array}{c} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{array} \qquad \mathbf{F} = \begin{Bmatrix} 0.05089 \\ 0.07197 \\ 0.10178 \\ 0.07197 \\ 0.01329 \\ 0.01880 \\ 0.00347 \\ 0.01880 \\ 0.01329 \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

Applying nonhomogeneous boundary conditions that $\{\phi_5, \ldots, \phi_9\} = \{0, 0, 0, 0, 0\}$; the constrained equations are:

$$\mathbf{K_G}\phi + \psi \mathbf{M_G}\ddot{\phi} = \mathbf{F}$$

$$\frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 8 & -2 & -2 & -1 & -2 & 0 & 0 & 0 \\ -2 & -2 & 16 & -2 & -2 & -2 & -2 & -2 & -2 \\ -1 & -2 & -2 & 8 & 0 & 0 & 0 & -2 & -1 \\ 0 & -1 & -2 & 0 & 4 & -1 & 0 & 0 & 0 \\ 0 & -2 & -2 & 0 & -1 & 8 & -1 & -2 & 0 \\ 0 & 0 & -2 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -2 & -2 & 0 & -2 & -1 & 8 & -1 \\ 0 & 0 & -2 & -1 & 0 & 0 & 0 & -1 & 4 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \frac{\psi}{144}\begin{bmatrix} 4 & 2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 8 & 4 & 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 16 & 4 & 1 & 4 & 1 & 4 & 1 \\ 2 & 1 & 4 & 8 & 0 & 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 2 & 8 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 4 & 2 & 0 \\ 0 & 0 & 4 & 1 & 0 & 1 & 2 & 8 & 2 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 2 & 4 \end{bmatrix}\begin{Bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \\ \ddot{\phi}_3 \\ \ddot{\phi}_4 \\ \ddot{\phi}_5 \\ \ddot{\phi}_6 \\ \ddot{\phi}_7 \\ \ddot{\phi}_8 \\ \ddot{\phi}_9 \end{Bmatrix} = \begin{Bmatrix} 0.05089 \\ 0.07197 \\ 0.10178 \\ 0.07197 \\ 0.01329 \\ 0.01880 \\ 0.00347 \\ 0.01880 \\ 0.01329 \end{Bmatrix}$$

$$\left\{\ddot{\phi}_5, \cdots, \ddot{\phi}_9\right\} = 0$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

Assembly. With both the boundary conditions essential, **BT**=0 and **bt**=0. It follows that the assembled equations are:

$$\mathbf{K_G}\phi + \psi\mathbf{M_G}\ddot{\phi} = \mathbf{F}$$

$$\frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 8 & -2 & -2 \\ -2 & -2 & 16 & -2 \\ -1 & -2 & -2 & 8 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} + \frac{\psi}{144}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 8 & 4 & 1 \\ 1 & 4 & 16 & 4 \\ 2 & 1 & 4 & 8 \end{bmatrix}\begin{Bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \\ \ddot{\phi}_3 \\ \ddot{\phi}_4 \end{Bmatrix} = \begin{Bmatrix} 0.05089 \\ 0.07197 \\ 0.10178 \\ 0.07197 \end{Bmatrix}$$

Let assume the ψ = 1 for this example.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

For the Newmark algorithm, the critical time step is associated with the largest eigenvalue of the $(\mathbf{K} - \omega^2\mathbf{M})\phi = 0$ system.

$$h_{cr} < \frac{2}{\omega_{\max}}$$

The eigenvalue problem is:

$$\left(\mathbf{K_G} + \omega^2\mathbf{M_G}\right)\phi = 0$$

$$\left(\frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 8 & -2 & -2 \\ -2 & -2 & 16 & -2 \\ -1 & -2 & -2 & 8 \end{bmatrix} - \frac{\omega^2}{144}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 8 & 4 & 1 \\ 1 & 4 & 16 & 4 \\ 2 & 1 & 4 & 8 \end{bmatrix}\right)\phi = 0$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

The eigenvalues can be found using a variety of available solution techniques. In Matlab, use **`[V,D] = eig(A).`**

For example, **`[V,D] = eig(M_G\K_G)`** gives eigenvalues **D:**

$$\omega_1^2 = 5.19332 \quad \omega_2^2 = 34.28571 \quad \omega_3^2 = 34.28571 \quad \omega_4^2 = 63.37811$$

The eigenvectors **V** are:

$$V_1 = \begin{Bmatrix} 0.66667 \\ 0.47140 \\ 0.33333 \\ 0.47140 \end{Bmatrix} \quad V_2 = \begin{Bmatrix} 0.89443 \\ 0.00000 \\ -0.44721 \\ 0.00000 \end{Bmatrix} \quad V_3 = \begin{Bmatrix} 0.09022 \\ -0.70350 \\ -0.04511 \\ 0.70350 \end{Bmatrix} \quad V_4 = \begin{Bmatrix} -0.66667 \\ 0.47140 \\ -0.33333 \\ 0.47140 \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

For the example the largest eigenvalue is 63.37811, the critical time step is:

$$h_{cr} < \frac{2}{\omega_{\max}} < \frac{2}{\sqrt{63.37811}} < 0.2512\,\text{sec}.$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

Newmark's Method– The equations to be solved at the first step can be written as:

$$\left(\mathbf{M}+\alpha h^2\mathbf{K}\right)\phi_1=\mathbf{M}\left(\phi_0+h\dot{\phi}_0+c_1h^2\ddot{\phi}_0\right)+\alpha h^2\mathbf{F}_1$$

$$\dot{\phi}_1=\dot{\phi}_0+\frac{\delta\left(\phi_1-\phi_0-h\dot{\phi}_0\right)}{\alpha h}+c_2h\ddot{\phi}_0$$

$$\ddot{\phi}_1=\frac{\phi_1-\phi_0-h\dot{\phi}_0}{\alpha h^2}-\frac{c_1\ddot{\phi}_0}{\alpha}$$

$$c_1=\frac{1}{2}-\alpha$$

$$c_2=1-\frac{\delta}{2\alpha}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

Newmark's Method– Evaluating the differential equation at t = 0 yields

$\phi_0(0)=0$

$$\mathbf{M}\ddot{\phi}_0=\mathbf{F}-\mathbf{K}\phi_0 \qquad \Rightarrow \qquad \ddot{\phi}_0=\mathbf{M}^{-1}\left(\mathbf{F}-\mathbf{K}\phi_0\right)$$

$$\ddot{\phi}_0=\begin{bmatrix} 47.02041 & -11.75510 & 2.93878 & -11.75510 \\ -11.75510 & 23.51020 & -5.87755 & 2.93878 \\ 2.93878 & -5.87755 & 11.75510 & -5.87755 \\ -11.75510 & 2.93878 & -5.87755 & 23.51020 \end{bmatrix}\begin{Bmatrix} 0.05089 \\ 0.07197 \\ 0.10178 \\ 0.07197 \end{Bmatrix}$$

$$=\begin{Bmatrix} 1.00000 \\ 0.70711 \\ 0.50000 \\ 0.70711 \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

Newmark's Method– Taking α = 0.25, δ = 0.5, and h = 0.01 seconds yields:

$$(\mathbf{M}+\alpha h^2\mathbf{K})=\begin{bmatrix}0.02779 & 0.01388 & 0.00694 & 0.01388\\ 0.01388 & 0.05559 & 0.02777 & 0.00694\\ 0.00694 & 0.02777 & 0.11118 & 0.02777\\ 0.01388 & 0.00694 & 0.02777 & 0.05559\end{bmatrix} \qquad \alpha h^2\mathbf{F}=\begin{Bmatrix}0.12723\\0.17993\\0.25446\\0.17993\end{Bmatrix}10^{-5}$$

At step 1:

$\phi_0(0)=0$ $\qquad$ $\dot{\phi}_0(0)=0$

$$(\mathbf{M}+\alpha h^2\mathbf{K})\phi_1=\mathbf{M}\left(\cancel{\phi_0}+h\cancel{\dot{\phi}_0}+c_1h^2\ddot{\phi}_0\right)+\alpha h^2\mathbf{F}_1$$

$$\begin{bmatrix}0.02779 & 0.01388 & 0.00694 & 0.01388\\ 0.01388 & 0.05559 & 0.02777 & 0.00694\\ 0.00694 & 0.02777 & 0.11118 & 0.02777\\ 0.01388 & 0.00694 & 0.02777 & 0.05559\end{bmatrix}\phi_1=\mathbf{M}\left(0.25h^2\ddot{\phi}_0\right)\begin{Bmatrix}0.12723\\0.17993\\0.25446\\0.17993\end{Bmatrix}10^{-5}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

Newmark's Method– The solution for $\boldsymbol{\phi}_1$ is

$$\phi_1=10^{-5}\langle 4.9994 \quad 3.5351 \quad 2.4997 \quad 3.5351\rangle^T$$

Now solve for the velocity at t = h

$\dot{\phi}_0(0)=0$ $\qquad$ $\dot{\phi}_0(0)=0$

$$\dot{\phi}_1=\cancel{\dot{\phi}_0}+\frac{\delta\left(\phi_1-\cancel{\phi_0}-h\cancel{\dot{\phi}_0}\right)}{\alpha h}+c_2h\ddot{\phi}_0$$

$\phi_0(0)=0$

$$\dot{\phi}_1=\frac{\delta\phi_1}{\alpha h}+c_2h\ddot{\phi}_0=\frac{\delta}{\alpha h}\begin{Bmatrix}4.99935\\3.53507\\2.49968\\3.53507\end{Bmatrix}10^{-5}+c_2h\begin{Bmatrix}1.00000\\0.70711\\0.50000\\0.70711\end{Bmatrix}$$

$$=\langle 0.01000 \quad 0.00707 \quad 0.005 \quad 0.00707\rangle^T$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

Newmark's Method– The equation for acceleration at $t = 0$ is

$\phi_0(0) = 0$

$$\ddot{\phi}_1 = \frac{\phi_1 - \phi_0 - h\dot{\phi}_0}{\alpha h^2} - \frac{c_1\ddot{\phi}_0}{\alpha} = \frac{\phi_1}{\alpha h^2} - \frac{c_1\ddot{\phi}_0}{\alpha}$$

$\dot{\phi}_0(0) = 0$

Now solve for the acceleration at $t = h$

$$\ddot{\phi}_1 = \frac{\delta}{\alpha h}\begin{Bmatrix} 4.99935 \\ 3.53507 \\ 2.49968 \\ 3.53507 \end{Bmatrix} 10^{-5} - c_2 h \begin{Bmatrix} 1.00000 \\ 0.70711 \\ 0.50000 \\ 0.70711 \end{Bmatrix}$$

$$\ddot{\phi}_1 = \langle 0.99974 \quad 0.70692 \quad 0.49987 \quad 0.70692 \rangle^T$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

Newmark's Method – For n = 2:

$$\phi_2 = 10^{-4}\langle 1.99948 \quad 1.41385 \quad 0.99974 \quad 1.41385 \rangle^T$$

$$\dot{\phi}_2 = \langle 0.01999 \quad 0.01414 \quad 0.01000 \quad 0.01414 \rangle^T$$

$$\ddot{\phi}_2 = \langle 0.99896 \quad 0.70637 \quad 0.49948 \quad 0.70637 \rangle^T$$

For n = 3:

$$\phi_3 = 10^{-4}\langle 4.49786 \quad 3.18047 \quad 2.24893 \quad 3.18047 \rangle^T$$

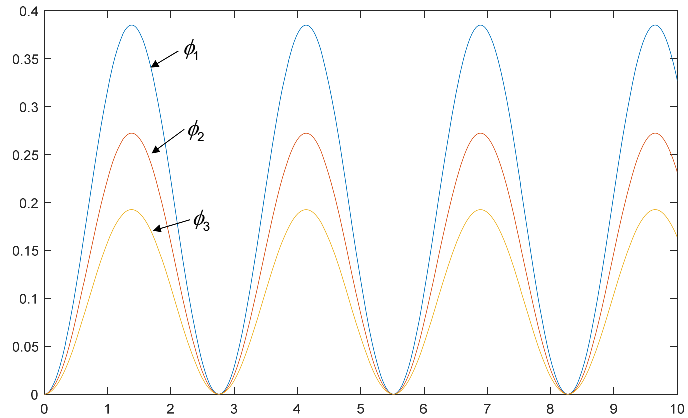
$$\dot{\phi}_3 = \langle 0.02998 \quad 0.02120 \quad 0.01499 \quad 0.02120 \rangle^T$$

$$\ddot{\phi}_3 = \langle 0.99766 \quad 0.70546 \quad 0.49883 \quad 0.70546 \rangle^T$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

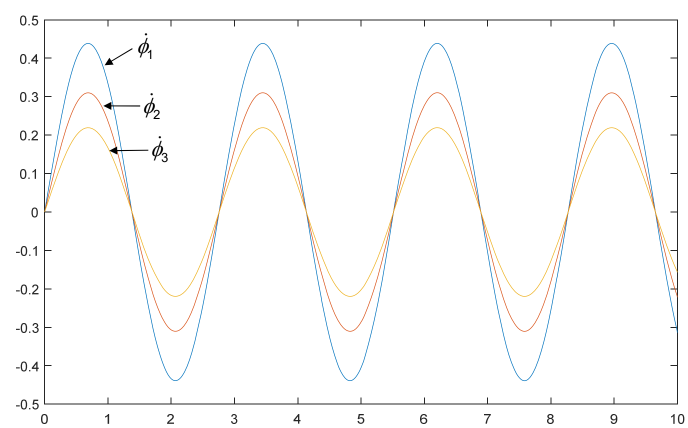
The values of ϕ_1, ϕ_2, and ϕ_3 for $0 < t < 10$ sec. are show below:



TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

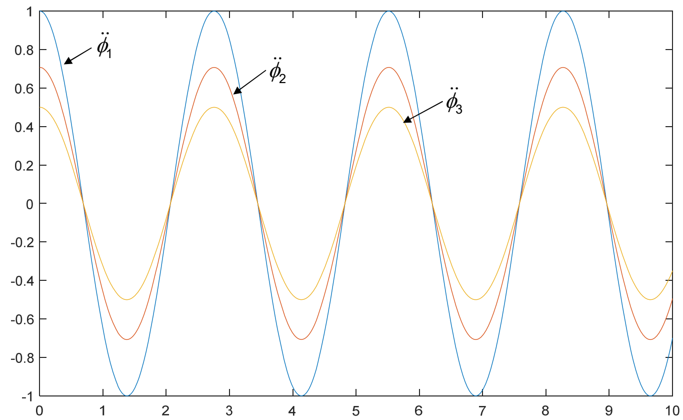
The velocity values of ϕ_1, ϕ_2, and ϕ_3 for $0 < t < 10$ sec. are:



TIME-DEPENDENT PROBLEMS

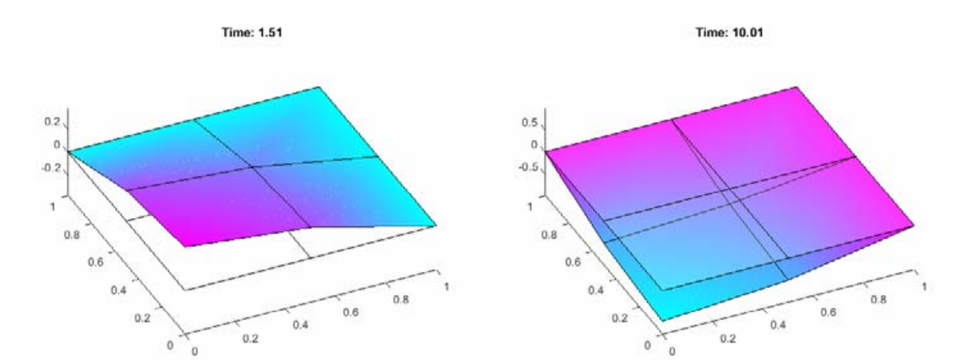
Two-Dimensional Wave Equations - Example 1

The acceleration values of ϕ_1, ϕ_2, and ϕ_3 for $0 < t < 10$ sec. are:



TIME-DEPENDENT PROBLEMS

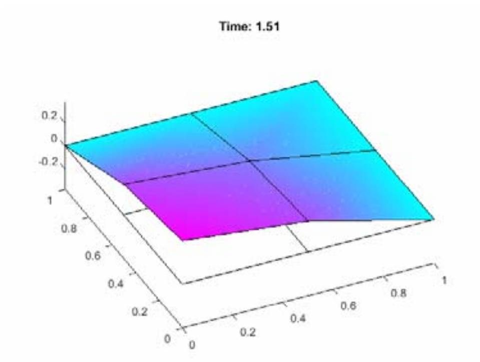
Two-Dimensional Wave Equations - Example 1



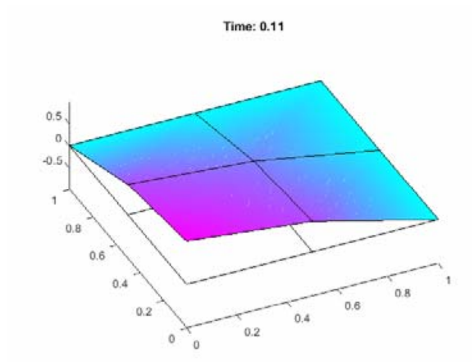
$\phi(x,y,t)$ $\qquad$ $\dfrac{\partial\phi(x,y,t)}{\partial t}$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1



$\phi(x,y,t)$

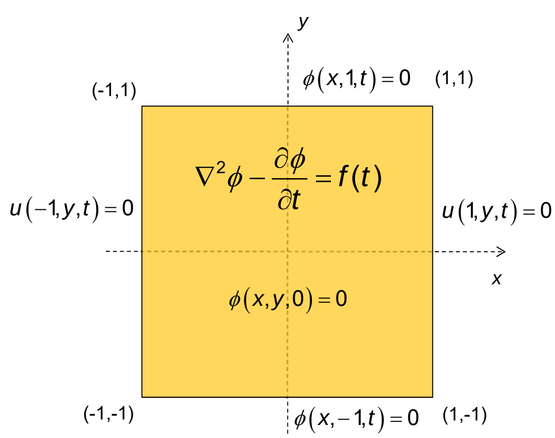


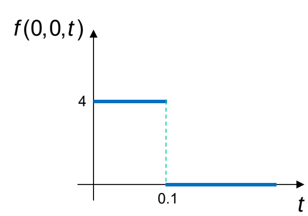
$$\frac{\partial^2\phi(x,y,t)}{\partial t^2}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

Consider the two-dimensional wave problem shown below.

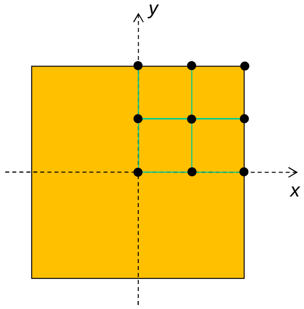




TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

Discretization. Due to symmetry, we will model the top right-most quadrant of the membrane using four equally-sized 4-noded quadrilaterals.



$$\nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = f(t)$$

$$\phi(\pm 1, y, t) = 0 \qquad \phi(x, \pm 1, t) = 0$$

$$\phi(x, y, 0) = 0 \qquad \frac{\partial \phi(x, y, 0)}{\partial t} = 0$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

Rectangular Elements - Evaluation of $\mathbf{k_e}$

As in Example 1, all the $\mathbf{k_e}$ **are:**

$$\mathbf{k}_1 = \frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix}\begin{array}{c} \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{array} \qquad \mathbf{k}_2 = \frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix}\begin{array}{c} \boxed{2} \\ \boxed{5} \\ \boxed{6} \\ \boxed{3} \end{array}$$

$$\mathbf{k}_3 = \frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix}\begin{array}{c} \boxed{3} \\ \boxed{6} \\ \boxed{7} \\ \boxed{8} \end{array} \qquad \mathbf{k}_4 = \frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix}\begin{array}{c} \boxed{3} \\ \boxed{8} \\ \boxed{9} \\ \boxed{4} \end{array}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

The elemental $\mathbf{m_e}$ matrices are:

$$\mathbf{m_e} = \iint_{A_e} \mathbf{NN^T}\, dA \;= \int_{-1}^{1}\int_{-1}^{1} F\left(x_0 + as, y_0 + bt\right)\frac{\partial(\xi,\eta)}{\partial(s,t)}\, ds\, dt$$

where the Jacobian $\partial(\xi,\eta)/\partial(s, t)$ has a value of $A_e/4$.

The resulting global system stiffness matrix $\mathbf{m_e}$ is:

$$\mathbf{m_e} = \frac{A_e}{36}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

Rectangular Elements - Evaluation of $\mathbf{m_e}$

As in Example 1, all the $\mathbf{m_e}$ **are:**

$$\mathbf{m}_1 = \frac{1}{144}\begin{array}{cc}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix} & \begin{array}{c} \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{array}\end{array} \qquad \mathbf{m}_2 = \frac{1}{144}\begin{array}{cc}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix} & \begin{array}{c} \boxed{2} \\ \boxed{5} \\ \boxed{6} \\ \boxed{3} \end{array}\end{array}$$

$$\mathbf{m}_3 = \frac{1}{144}\begin{array}{cc}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix} & \begin{array}{c} \boxed{3} \\ \boxed{6} \\ \boxed{7} \\ \boxed{8} \end{array}\end{array} \qquad \mathbf{m}_4 = \frac{1}{144}\begin{array}{cc}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix} & \begin{array}{c} \boxed{3} \\ \boxed{8} \\ \boxed{9} \\ \boxed{4} \end{array}\end{array}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

Transforming the integral into the non-dimensional coordinates (s, t) yields:

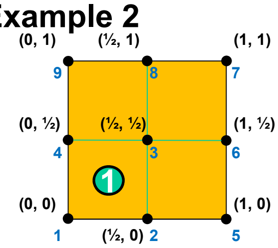
$$\mathbf{f_e} \approx \left(\int_{-1}^{1} \int_{-1}^{1} \mathbf{N}\mathbf{N}^T \frac{A_e}{4} ds\, dt \right) \mathbf{f}$$

$$\mathbf{f_e} \approx \frac{A_e}{36} \begin{Bmatrix} 4f_1 + 2f_2 + f_3 + 2f_4 \\ 2f_1 + 4f_2 + 2f_3 + f_4 \\ f_1 + 2f_2 + 4f_3 + 2f_4 \\ 2f_1 + f_2 + 2f_3 + 4f_4 \end{Bmatrix}$$

The resulting 4 x 1 elemental load vector contributes to the global system equations at those locations corresponding to the four nodes defining the element.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2



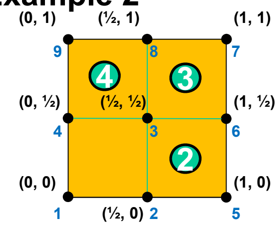
For element 1:

$$\mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}_{t \le 0.1} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}_{t > 0.1}$$

$$\mathbf{f}_1 = \begin{Bmatrix} 0.02778 \\ 0.01389 \\ 0.00694 \\ 0.01389 \end{Bmatrix}_{t \le 0.1} \begin{matrix} \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{matrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}_{t > 0.1} \begin{matrix} \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{matrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2



For elements 2 to 4:

$$\mathbf{f} = \begin{Bmatrix} f_2 \\ f_5 \\ f_6 \\ f_3 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \mathbf{f}_2 = \begin{array}{c} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \end{array} \begin{array}{c} \boxed{2} \\ \boxed{5} \\ \boxed{6} \\ \boxed{3} \end{array}$$

$$\mathbf{f} = \begin{Bmatrix} f_3 \\ f_6 \\ f_7 \\ f_8 \end{Bmatrix}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \mathbf{f}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \begin{array}{c} \boxed{3} \\ \boxed{6} \\ \boxed{7} \\ \boxed{8} \end{array} \qquad \mathbf{f} = \begin{Bmatrix} f_3 \\ f_8 \\ f_9 \\ f_4 \end{Bmatrix}_4 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \mathbf{f}_4 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \begin{array}{c} \boxed{3} \\ \boxed{8} \\ \boxed{9} \\ \boxed{4} \end{array}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

The assembled $\mathbf{K_G}$, $\mathbf{M_G}$, and $\mathbf{F}$ matrices are:

$$\mathbf{K_G} = \frac{1}{6} \begin{bmatrix} 4 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 8 & -2 & -2 & -1 & -2 & 0 & 0 & 0 \\ -2 & -2 & 16 & -2 & -2 & -2 & -2 & -2 & -2 \\ -1 & -2 & -2 & 8 & 0 & 0 & 0 & -2 & -1 \\ 0 & -1 & -2 & 0 & 4 & -1 & 0 & 0 & 0 \\ 0 & -2 & -2 & 0 & -1 & 8 & -1 & -2 & 0 \\ 0 & 0 & -2 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -2 & -2 & 0 & -2 & -1 & 8 & -1 \\ 0 & 0 & -2 & -1 & 0 & 0 & 0 & -1 & 4 \end{bmatrix} \begin{array}{c} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{array}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

The assembled $\mathbf{K_G}$, $\mathbf{M_G}$, and $\mathbf{F}$ matrices are:

$$\mathbf{M_G} = \frac{1}{144}\begin{bmatrix} 4 & 2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 8 & 4 & 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 16 & 4 & 1 & 4 & 1 & 4 & 1 \\ 2 & 1 & 4 & 8 & 0 & 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 2 & 8 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 4 & 2 & 0 \\ 0 & 0 & 4 & 1 & 0 & 1 & 2 & 8 & 2 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 2 & 4 \end{bmatrix} \begin{matrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{matrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

The assembled $\mathbf{K_G}$, $\mathbf{M_G}$, and $\mathbf{F}$ matrices are:

$$\mathbf{F} = \begin{Bmatrix} 0.02778 \\ 0.01389 \\ 0.00694 \\ 0.01389 \\ 0.00000 \\ 0.00000 \\ 0.00000 \\ 0.00000 \\ 0.00000 \end{Bmatrix}_{t \le 0.1} \begin{matrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{matrix} \qquad \mathbf{F} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}_{t > 0.1} \begin{matrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{matrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

Applying nonhomogeneous boundary conditions that $\{\phi_5, \ldots, \phi_9\} = \{0, 0, 0, 0, 0\}$; the constrained equations are:

$$\mathbf{K_G}\phi + \mathbf{M_G}\ddot{\phi} = \mathbf{F}$$

$$\frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 8 & -2 & -2 & -1 & -2 & 0 & 0 & 0 \\ -2 & -2 & 16 & -2 & -2 & -2 & -2 & -2 & -2 \\ -1 & -2 & -2 & 8 & 0 & 0 & 0 & -2 & -1 \\ 0 & -1 & -2 & 0 & 4 & -1 & 0 & 0 & 0 \\ 0 & -2 & -2 & 0 & -1 & 8 & -1 & -2 & 0 \\ 0 & 0 & -2 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -2 & -2 & 0 & -2 & -1 & 8 & -1 \\ 0 & 0 & -2 & -1 & 0 & 0 & 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \boxed{\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}} \end{Bmatrix} + \frac{1}{144}\begin{bmatrix} 4 & 2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 8 & 4 & 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 16 & 4 & 1 & 4 & 1 & 4 & 1 \\ 2 & 1 & 4 & 8 & 0 & 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 2 & 8 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 4 & 2 & 0 \\ 0 & 0 & 4 & 1 & 0 & 1 & 2 & 8 & 2 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 2 & 4 \end{bmatrix} \begin{Bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \\ \ddot{\phi}_3 \\ \ddot{\phi}_4 \\ \ddot{\phi}_5 \\ \ddot{\phi}_6 \\ \ddot{\phi}_7 \\ \ddot{\phi}_8 \\ \ddot{\phi}_9 \end{Bmatrix} = \begin{Bmatrix} 0.02778 \\ 0.01389 \\ 0.00694 \\ 0.01389 \\ 0.00000 \\ 0.00000 \\ 0.00000 \\ 0.00000 \\ 0.00000 \end{Bmatrix}_{t \le 0.1}$$

$$\{\ddot{\phi}_5, \cdots, \ddot{\phi}_9\} = 0$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

Assembly. With both the boundary conditions essential, **BT**=0 and **bt**=0. It follows that the assembled equations are:

$$\mathbf{K_G}\phi + \mathbf{M_G}\ddot{\phi} = \mathbf{F}$$

$$\frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 8 & -2 & -2 \\ -2 & -2 & 16 & -2 \\ -1 & -2 & -2 & 8 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} + \frac{1}{144}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 8 & 4 & 1 \\ 1 & 4 & 16 & 4 \\ 2 & 1 & 4 & 8 \end{bmatrix} \begin{Bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \\ \ddot{\phi}_3 \\ \ddot{\phi}_4 \end{Bmatrix} = \begin{Bmatrix} 0.027778 \\ 0.013889 \\ 0.006944 \\ 0.013889 \end{Bmatrix}_{t \le 0.1}$$

$$\frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 8 & -2 & -2 \\ -2 & -2 & 16 & -2 \\ -1 & -2 & -2 & 8 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} + \frac{1}{144}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 8 & 4 & 1 \\ 1 & 4 & 16 & 4 \\ 2 & 1 & 4 & 8 \end{bmatrix} \begin{Bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \\ \ddot{\phi}_3 \\ \ddot{\phi}_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}_{t > 0.1}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

For the Newmark algorithm, the critical time step is associated with the largest eigenvalue of the ($\mathbf{K} - \omega^2\mathbf{M})\phi = 0$ system.

$$h_{cr} < \frac{2}{\omega_{\max}}$$

The eigenvalue problem is the same as Example 1:

$$\left(\mathbf{K_G} + \omega^2\mathbf{M_G}\right)\phi = 0$$

$$\left(\frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 8 & -2 & -2 \\ -2 & -2 & 16 & -2 \\ -1 & -2 & -2 & 8 \end{bmatrix} - \frac{1}{144}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 8 & 4 & 1 \\ 1 & 4 & 16 & 4 \\ 2 & 1 & 4 & 8 \end{bmatrix}\right)\phi = 0$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

The eigenvalues can be found using a variety of available solution techniques. In Matlab, use **`[V,D] = eig(A).`**

For example, **`[V,D] = eig(`$\mathbf{M_G}$`\`$\mathbf{K_G}$`)`** gives eigenvalues **`D`:**

$$\omega_1^2 = 5.19332 \quad \omega_2^2 = 34.28571 \quad \omega_3^2 = 34.28571 \quad \omega_4^2 = 63.37811$$

The eigenvectors **`V`** are:

$$V_1 = \begin{Bmatrix} 0.66667 \\ 0.47140 \\ 0.33333 \\ 0.47140 \end{Bmatrix} \quad V_2 = \begin{Bmatrix} 0.89443 \\ 0.00000 \\ -0.44721 \\ 0.00000 \end{Bmatrix} \quad V_3 = \begin{Bmatrix} 0.09022 \\ -0.70350 \\ -0.04511 \\ 0.70350 \end{Bmatrix} \quad V_4 = \begin{Bmatrix} -0.66667 \\ 0.47140 \\ -0.33333 \\ 0.47140 \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

For the example the largest eigenvalue is 63.37811, the critical time step is:

$$h_{cr} < \frac{2}{\omega_{\max}} < \frac{2}{\sqrt{63.37811}} < 0.2512 \text{ sec.}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

Newmark's Method – The equations to be solved at the first step can be written as:

$$\left(\mathbf{M} + \alpha h^2 \mathbf{K}\right)\phi_1 = \mathbf{M}\left(\phi_0 + h\dot{\phi}_0 + c_1 h^2 \ddot{\phi}_0\right) + \alpha h^2 \mathbf{F}_1$$

$$\dot{\phi}_1 = \dot{\phi}_0 + \frac{\delta\left(\phi_1 - \phi_0 - h\dot{\phi}_0\right)}{\alpha h} + c_2 h \ddot{\phi}_0$$

$$\ddot{\phi}_1 = \frac{\phi_1 - \phi_0 - h\dot{\phi}_0}{\alpha h^2} - \frac{c_1 \ddot{\phi}_0}{\alpha}$$

$$c_1 = \frac{1}{2} - \alpha$$

$$c_2 = 1 - \frac{\delta}{2\alpha}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

Newmark's Method – Evaluating the differential equation at t = 0 yields

$$\mathbf{M}\ddot{\phi}_0 = \mathbf{F} - \mathbf{K}\phi_0 \qquad \Rightarrow \qquad \ddot{\phi}_0 = \mathbf{M}^{-1}\left(\mathbf{F} - \mathbf{K}\overset{\phi_0(0)=0}{\cancel{\phi_0}}\right)$$

$$\ddot{\phi}_0 = \begin{bmatrix} 47.02041 & -11.75510 & 2.93878 & -11.75510 \\ -11.75510 & 23.51020 & -5.87755 & 2.93878 \\ 2.93878 & -5.87755 & 11.75510 & -5.87755 \\ -11.75510 & 2.93878 & -5.87755 & 23.51020 \end{bmatrix} \begin{Bmatrix} 0.027778 \\ 0.013889 \\ 0.006944 \\ 0.013889 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1.00000 \\ 0.00000 \\ 0.00000 \\ 0.00000 \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

Newmark's Method – Taking α = 0.25, δ = 0.5, and h = 0.01 seconds yields:

$$\left(\mathbf{M} + \alpha h^2 \mathbf{K}\right) = \begin{bmatrix} 0.02779 & 0.01388 & 0.00694 & 0.01388 \\ 0.01388 & 0.05559 & 0.02777 & 0.00694 \\ 0.00694 & 0.02777 & 0.11118 & 0.02777 \\ 0.01388 & 0.00694 & 0.02777 & 0.05559 \end{bmatrix} \qquad \alpha h^2 \mathbf{F} = \begin{Bmatrix} 6.94444 \\ 3.47222 \\ 1.73611 \\ 3.47222 \end{Bmatrix} 10^{-7}$$

At step 1:

$$\left(\mathbf{M} + \alpha h^2 \mathbf{K}\right)\phi_1 = \mathbf{M}\left(\overset{\phi_0(0)=0}{\cancel{\phi_0}} + h\overset{\dot{\phi}_0(0)=0}{\cancel{\dot{\phi}_0}} + c_1 h^2 \ddot{\phi}_0\right) + \alpha h^2 \mathbf{F}_1$$

$$\begin{bmatrix} 0.02779 & 0.01388 & 0.00694 & 0.01388 \\ 0.01388 & 0.05559 & 0.02777 & 0.00694 \\ 0.00694 & 0.02777 & 0.11118 & 0.02777 \\ 0.01388 & 0.00694 & 0.02777 & 0.05559 \end{bmatrix} \phi_1 = \mathbf{M}\left(0.25h^2\ddot{\phi}_0\right) \begin{Bmatrix} 6.94444 \\ 3.47222 \\ 1.73611 \\ 3.47222 \end{Bmatrix} 10^{-7}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

Newmark's Method – The solution for $\boldsymbol{\phi}_1$ is

$$\phi_1 = \langle 4.996\text{E-05} \quad 1.284\text{E-08} \quad 6.595\text{E-12} \quad 1.284\text{E-08} \rangle^T$$

Now solve for the velocity at t = h

$\dot{\phi}_0(0) = 0$ $\dot{\phi}_0(0) = 0$

$$\dot{\phi}_1 = \dot{\phi}_0 + \frac{\delta\left(\phi_1 - \phi_0 - h\dot{\phi}_0\right)}{\alpha h} + c_2 h \ddot{\phi}_0$$

$\phi_0(0) = 0$

$$= \langle 9.9914\text{E-03} \quad 2.5670\text{E-06} \quad 1.3191\text{E-09} \quad 2.5670\text{E-06} \rangle^T$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

Newmark's Method – The equation for acceleration at t = 0 is

$\phi_0(0) = 0$

$$\ddot{\phi}_1 = \frac{\phi_1 - \phi_0 - h\dot{\phi}_0}{\alpha h^2} - \frac{c_1 \ddot{\phi}_0}{\alpha} = \frac{\phi_1}{\alpha h^2} - \frac{c_1 \ddot{\phi}_0}{\alpha}$$

$\dot{\phi}_0(0) = 0$

Now solve for the acceleration at t = h

$$\ddot{\phi}_1 = \langle 0.99829 \quad 5.1341\text{E-04} \quad 2.6381\text{E-07} \quad 5.1341\text{E-04} \rangle^T$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

Newmark's Method – For n = 2:

$$\phi_2 = \langle 1.9966\text{E-04} \quad 1.0259\text{E-07} \quad 7.9076\text{E-11} \quad 1.0259\text{E-07} \rangle^T$$

$$\dot{\phi}_2 = \langle 0.01995 \quad 1.5385\text{E-05} \quad 1.3177\text{E-08} \quad 1.5385\text{E-05} \rangle^T$$

$$\ddot{\phi}_2 = \langle 0.99316 \quad 2.0501\text{E-03} \quad 2.1078\text{E-06} \quad 2.0501\text{E-03} \rangle^T$$

For n = 3:

$$\phi_3 = \langle 4.4859\text{E-04} \quad 4.2268\text{E-07} \quad 4.8051\text{E-10} \quad 4.2268\text{E-07} \rangle^T$$

$$\dot{\phi}_3 = \langle 0.02984 \quad 4.8633\text{E-05} \quad 6.7109\text{E-08} \quad 4.8633\text{E-05} \rangle^T$$

$$\ddot{\phi}_3 = \langle 0.98464 \quad 4.5996\text{E-03} \quad 8.6787\text{E-06} \quad 4.5996\text{E-03} \rangle^T$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 2

Newmark's Method – For n = 11, *t* > 0.1 sec.

$$\mathbf{F} = \left\{ \begin{matrix} 0.027778 \\ 0.013889 \\ 0.006944 \\ 0.013889 \end{matrix} \right\}_{t \le 0.1} \begin{matrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{matrix} \quad \Longrightarrow \quad \mathbf{F} = \left\{ \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \right\}_{t > 0.1} \begin{matrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{matrix}$$

$$\phi_{11} = 10^{-3} \langle 5.81661 \quad 0.06193 \quad 0.00055 \quad 0.06193 \rangle^T$$

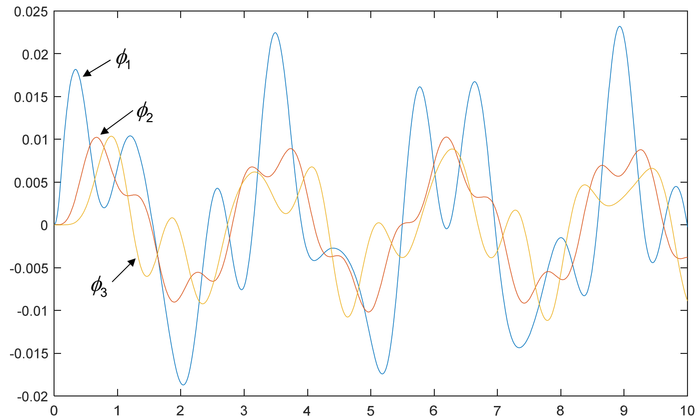
$$\dot{\phi}_{11} = 10^{-3} \langle 97.58691 \quad 2.19325 \quad 0.02862 \quad 2.19325 \rangle^T$$

$$\ddot{\phi}_{11} = \langle -0.19688 \quad 0.05772 \quad 0.00126 \quad 0.05772 \rangle^T$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

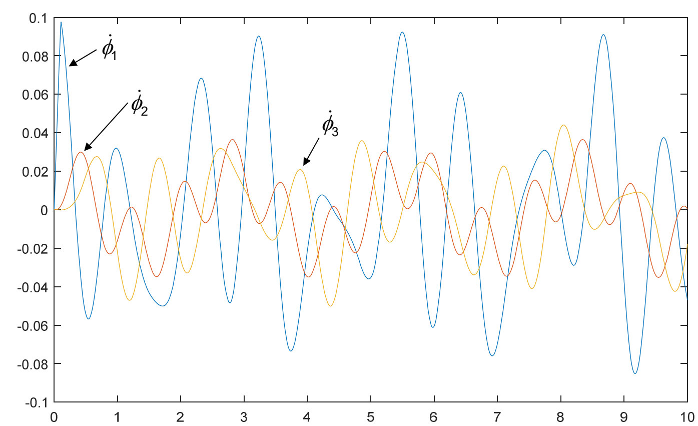
The values of ϕ_1, ϕ_2, and ϕ_3 for $0 < t < 10$ sec. are show below:



TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1

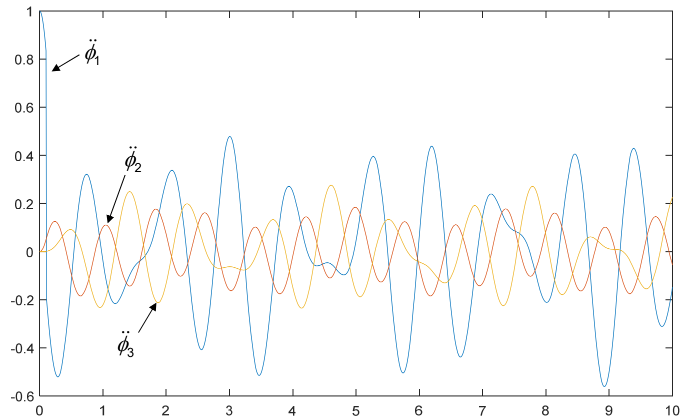
The velocity values of ϕ_1, ϕ_2, and ϕ_3 for $0 < t < 10$ sec. are:



TIME-DEPENDENT PROBLEMS

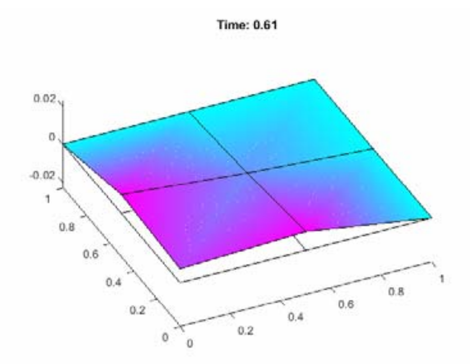
Two-Dimensional Wave Equations - Example 1

The acceleration values of ϕ_1, ϕ_2, and ϕ_3 for 0 < t < 10 sec. are:

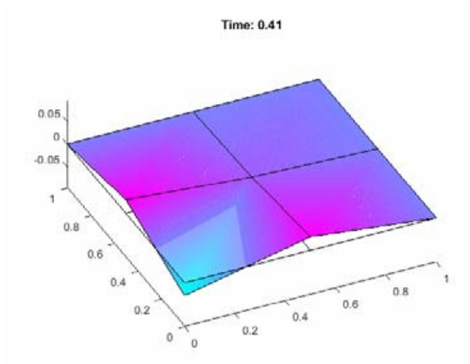


TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1



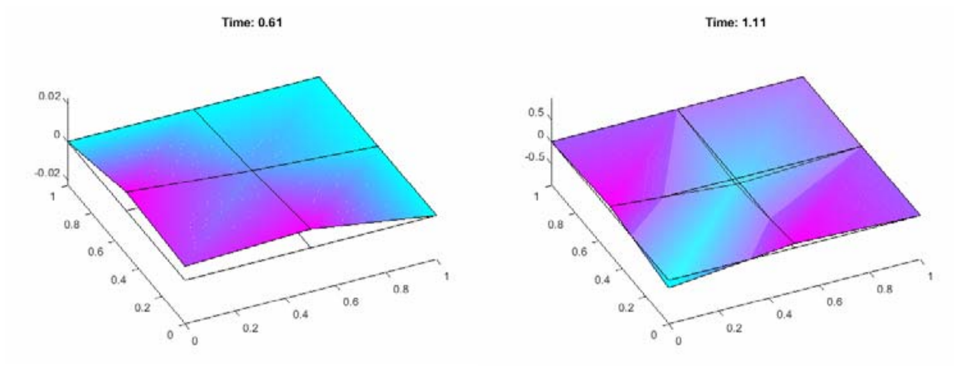
$\phi(x, y, t)$



$$\frac{\partial \phi(x, y, t)}{\partial t}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations - Example 1



$\phi(x,y,t)$ $\qquad$ $\dfrac{\partial^2\phi(x,y,t)}{\partial t^2}$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

Closure - Time-dependent problems are inherently more difficult and expensive to solve than their corresponding steady-state counterparts.

The expense of generating the global matrices is higher for the time-dependent problems because of the necessity of computing the mass matrices.

The main extra expense, however, is in solving the resulting time-dependent global equations.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

Closure - For an analytical approach to the solution, additional expense is incurred in terms of having to determine eigenvalues and eigenvectors.

The actual amount of expense depends on the specific form of the stiffness and mass matrices and the algorithm used, but in any case it is significantly in excess of the expense of solving the single set of linear algebraic equations associated with the steady-state problem.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

Closure - For a time domain integration technique, the additional expense is clearly related to the number of time steps necessary to trace out the desired time history.

In addition to several matrix multiplications and additions, *each step* can involve the solution of a set of linear algebraic equations.

In some instances this expense can be minimized by using a decomposition that can be reused for the computation of the solution at each new time.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

Closure - In this regard recall that the Euler and central difference algorithms require that the size of the time step not exceed a value proportional to the inverse of the largest eigenvalue.

For large systems this critical step size can be very small resulting in many applications of the algorithm to trace out the time history.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Wave Equations

Closure - The unconditionally stable Crank-Nicolson and Newmark algorithms, on the other hand, can be used with arbitrary step size that has been chosen so as to accurately integrate the lower modes, with significant improvement in the expense relative to the conditionally stable Euler and central difference algorithms.

There are of course other algorithms available that are specifically tailored to address other numerical issues.

End of
Chapter 4d