The governing balance equations that describe diffusion processes in situations involving two independent variables appear typically as:

\[
\nabla \cdot \left( k \nabla u(x,y,t) \right) + \rho \frac{\partial u}{\partial t} + f(x,y,t) = 0 \quad \text{in } \Omega \\

u = g(s,t) \quad \text{on } \Gamma_1 \\
k \frac{\partial u}{\partial n} + \alpha(s,t) u = q(s,t) \quad \text{on } \Gamma_2 \\
u(x,y,0) = c(x,y) \quad \text{in } \Omega 
\]

Where \( \Omega \) is the interior domain, and \( \Gamma_1 \) and \( \Gamma_2 \) form the boundary of the domain.

The physical constants are \( k \) and \( \rho \), and \( s \) is a linear measure of the position on the boundary.

In two dimensions, the influence of the boundary conditions is shown. The left side of the simulation domain (\( x = 0 \)) is used as a constant source of diffusion particles, thereby creating a steady flow into the area. All other boundaries are modeled by a Neumann boundary condition.

This equation is similar in form to the elliptic boundary value problem studied in Chapter 3, with the very important addition of the time derivative term in the differential equation and the corresponding initial condition.

With these additions, the problem changes from an elliptic boundary value problem to a parabolic initial boundary value problem.
The basic steps of discretization, interpolation, elemental formulation, assembly, constraints, solution, and computation of derived variables are presented in this section as they relate to the two-dimensional parabolic initial-boundary value problem.

The Galerkin method, in connection with the corresponding weak formulation to be developed, will be used to generate the finite element model.

Discretization - For discretization, please refer to the material in Chapter 3.

Elemental Formulation - The starting point for the elemental formulation is the weak formulation of the initial-boundary value problem.

The first step in developing the weak formulation is to multiply the differential equation by an arbitrary test function \( v(x, y) \) vanishing on \( \Gamma_1 \).

The result is then integrated over the domain \( \Omega \) to obtain

\[
\int_{\Omega} \left( \nabla \cdot (k \nabla u) + \rho \frac{\partial u}{\partial t} + f \right) d\Omega = 0
\]

This equation is the required weak formulation for the two-dimensional diffusion problem.

Elemental Formulation - Recalling that \( v \) vanishes on \( \Gamma_1 \) and that \( k \frac{\partial u}{\partial x} = -\nabla \cdot u = q - hu \) on \( \Gamma_2 \), it follows that

\[
\int_{\Omega} \nabla \cdot (k \nabla u) d\Omega + \int_{\Omega} \rho \frac{\partial u}{\partial t} d\Omega = \int_{\Gamma} v(q - hu) d\Gamma + \int_{\Omega} f d\Omega
\]

This equation is the required weak formulation for the two-dimensional diffusion problem.
TIME-DEPENDENT PROBLEMS
Two-Dimensional Diffusion
The Galerkin Finite Element Method
Elemental Formulation – Where \( \Omega \) and \( \Gamma_2 \) represent the elemental areas approximating \( \Omega \), and the collection of the elemental edges approximating \( \Gamma_2 \), respectively.

\[
\int_\Omega \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \, d\Omega + \int_{\Gamma_2} \eta_n \frac{\partial u}{\partial n} \, dl = \int_\Omega \rho \frac{\partial u}{\partial t} \, d\Omega + \int_{\Gamma_2} h \frac{\partial u}{\partial n} \, dl \quad k = 1, 2, \ldots
\]

TIME-DEPENDENT PROBLEMS
Two-Dimensional Diffusion
The Galerkin Finite Element Method
Elemental Formulation – Note that assembly is contained implicitly within the formulation.

\[
\sum_{i=1}^N (A_{ik} u_i + B_{ik} \dot{u}_i) = F_i(t) \quad k = 1, 2, \ldots, N
\]

TIME-DEPENDENT PROBLEMS
Two-Dimensional Diffusion
The Galerkin Finite Element Method
Elemental Formulation – In terms of the corresponding elementally based interpolations

\[
u_i(x,y) = \mathbf{N}' \mathbf{u}_i = \mathbf{u}_i \, \mathbf{N}
\]

The finite element model can be expressed as

\[
A \mathbf{u} + \mathbf{B} \dot{\mathbf{u}} = \mathbf{F}
\]

where

\[
A = \sum_{k} k_0 + \sum_{x} a_0 \quad \mathbf{B} = \sum_{x} t_0 \quad \mathbf{F} = \sum_{x} f_0 + \sum_{x} q_0
\]

TIME-DEPENDENT PROBLEMS
Two-Dimensional Diffusion
The Galerkin Finite Element Method
Elemental Formulation – The initial conditions for the system of first-order differential equations are obtained from the initial conditions prescribed for the original initial-boundary value problem.

Generally \( \mathbf{u}(0) \) is determined by evaluating the function \( c(x, y) \) at the nodes to obtain

\[
\mathbf{u}(0) = \mathbf{u}_i = [c_1, c_2, \ldots, c_n, \ldots, c_m]'
\]

where \( c_i = c(x_i, y_i) \) with \( (x_i, y_i) \) the coordinates of the \( i \)th node.
Two-Dimensional Diffusion

The Galerkin Finite Element Method

Constraints - The constraints arise from the boundary conditions specified on $\Gamma_1$.

Generally the values of the constraints are determined from the $q$ function with the constrained value of $u$ at a node on $\Gamma_1$, being taken as the value of $q$ at that point.

These constraints are then enforced on the assembled equations, resulting in the final global constrained set of linear first-order differential equations.

\[ Mu + Ku = f \]
\[ u(0) = u_i \]

Solution - The system of equations is precisely the same in form and character as the corresponding equations developed for one-dimensional diffusion.

An analytical method as well as the numerical methods of Euler and improved Euler or Crank-Nicolson can be used for integrating the above set of equations.

The Euler method will be conditionally stable with the critical time step depending on the maximum eigenvalue of the associated problem $\lambda(M - K)v = 0$.

The improved Euler or Crank-Nicolson algorithm will be unconditionally stable.

Derived variables - The derived variables will be time dependent, and depending on the particular problem being considered, may need to be computed at each time step.

The computations would be per element and would be carried out using the techniques described in Chapter 3.

Two-Dimensional Wave Equations

Swiss Mathematician and physicist Leonhard Euler discovered the wave equation in three space dimensions.

Leonhard Euler (1707 – 1783) was a pioneering Swiss mathematician and physicist. He made important discoveries in fields as diverse as infinitesimal calculus and graph theory. He is also renowned for his work in mechanics, fluid dynamics, optics, astronomy, and music theory.

The wave equation is an important second-order linear partial differential equation for the description of waves – as they occur in physics – such as sound waves, light waves and water waves.

It arises in fields like acoustics, electromagnetics, and fluid dynamics.

A solution of the wave equation in two dimensions with a zero-displacement boundary condition along the entire outer edge.
Consider the motion of a rectangular membrane (in the absence of gravity) using the two-dimensional wave equation. Plots of the spatial part for modes are illustrated below.

The governing equations of motion that describe the propagation of waves in situations involving two independent variables appear typically as

\[ \nabla \cdot (k \nabla u(x,y,t)) - \rho \frac{\partial^2 u}{\partial t^2} + f(x,y,t) = 0 \quad \text{in } \Omega \]

\[ u = g(s,t) \quad \text{on } \Gamma_1 \]

\[ k \frac{\partial u}{\partial n} + \alpha(s,t)u = q(s,t) \quad \text{on } \Gamma_2 \]

\[ u(x,y,0) = c(x,y) \quad \text{in } \Omega \]

\[ \frac{\partial u(x,y,0)}{\partial t} = d(x,y) \quad \text{in } \Omega \]

The physical constants are \( k \) and \( \rho \), and \( s \) is a linear measure of the position on the boundary.

The type of boundary condition specified on \( \Gamma_2 \) results from a local balance between internal and external forces.
TIME-DEPENDENT PROBLEMS
Two-Dimensional Wave Equations
This equation is similar in form to the parabolic initial-boundary value problem presented in the previous section with the very important change in the time derivative term from a first to a second derivative, and with the addition of a second initial condition on the velocity.

With these changes, the problem changes from a parabolic initial-boundary value problem to a hyperbolic initial-boundary value problem.

TIME-DEPENDENT PROBLEMS
Two-Dimensional Wave Equations
The basic steps of discretization, interpolation, elemental formulation, assembly, constraints, solution, and computation of derived variables are presented in this section as they relate to the two-dimensional hyperbolic initial-boundary value problem.

The Galerkin method, in connection with the corresponding weak formulation to be developed, will be used to generate the finite element model.

Discretization - Referred to the material in Chapter 3.

TIME-DEPENDENT PROBLEMS
Two-Dimensional Wave Equations
The Galerkin Finite Element Method
Interpolation - The solution is assumed to be expressible in terms of the nodally based interpolation functions \( n_i(x, y) \) introduced and discussed in Section 3.2.

In the present setting, these interpolation functions are used with the semidiscretization

\[ u(x, y, t) = \sum_{i=1}^{N} n_i(t) u_i(x, y) \]

The \( n_i(x, y) \) are nodally based interpolation functions and can be linear, quadratic, or as otherwise desired.

TIME-DEPENDENT PROBLEMS
Two-Dimensional Wave Equations
The Galerkin Finite Element Method
Elemental Formulation - The starting point for the elemental formulation is the weak formulation of the initial-boundary value problem.

The first step in developing the weak formulation is to multiply the differential equation by an arbitrary test function \( v(x, y) \) vanishing on \( \Gamma_1 \).

The resulting expression is then integrated over the domain \( \Omega \) to obtain

\[ \int_{\Omega} \left( \nabla \cdot (k \nabla u) - \rho \frac{\partial^2 u}{\partial t^2} + f \right) d\Omega = 0 \]

TIME-DEPENDENT PROBLEMS
Two-Dimensional Wave Equations
The Galerkin Finite Element Method
Elemental Formulation - Using the two-dimensional form of the divergence theorem to integrate the first term by parts, there results after rearranging

\[ \int_{\Omega} \nabla \cdot \left( k \nabla u \right) d\Omega + \int_{\Omega} \rho \frac{\partial^2 u}{\partial t^2} d\Omega = \int_{\Gamma} n \cdot k \nabla u d\Gamma' + \int_{\Omega} f d\Omega \]

where \( \Gamma = \Gamma_1 + \Gamma_2 \)

This equation is the required weak formulation for the two-dimensional diffusion problem.
TIME-DEPENDENT PROBLEMS
Two-Dimensional Wave Equations
The Galerkin Finite Element Method
Elemental Formulation – Substituting the approximation of
\[ u(x, y, t) = N u_\alpha(x) \] into the weak formulation and taking \( v = n_k, k = 1, 2, \ldots \) yields
\[
\int_{\Omega_e} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k \sum_{\alpha} a_k \frac{\partial u_\alpha}{\partial x} \right) d\Omega_e + \int_{\Gamma_e} n_k u \, ds_e = \int_{\Omega_e} f e d\Omega_e + \int_{\Gamma_e} q e d\Gamma_e \quad k = 1, 2, \ldots
\]
Two-Dimensional Wave Equations
The Galerkin Finite Element Method

Elemental Formulation – The initial conditions for the system of first-order differential equations are obtained from the initial conditions prescribed for the original initial-boundary value problem.

Generally \( u(0) \) is determined by evaluating the function \( c(x, y) \) at the nodes to obtain

\[
\begin{bmatrix}
\mathbf{u}(0)
\end{bmatrix} = \begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
\vdots \\
c_N
\end{bmatrix}
\]

where \( c_i = c(x_i, y_i) \) with \( (x_i, y_i) \) the coordinates of the \( i^{\text{th}} \) node.

Constraints - The constraints arise from the boundary conditions specified on \( \Gamma_c \).

Generally the values of the constraints are determined from the \( q \) function with the constrained value of \( u \) at a node on \( \Gamma_c \), being taken as the value of \( q \) at that point.

These constraints are then enforced on the assembled equations, resulting in the final global constrained set of linear first-order differential equations.

\[
M \ddot{u} + K u = f \quad u(0) = u_0 \quad \dot{u}(0) = \dot{u}_0
\]

Solution - The central difference algorithm will be conditionally stable with the critical time step depending on the maximum eigenvalue of the associated problem \((K - \omega^2M)v = 0\).

The Newmark algorithm will be unconditionally stable for \( \delta = 0.5 \) and \( \alpha = 0.25(\delta + 0.5)^2 \).

Derived variables - In a physical situation governed by a wave equation, the derived variables are usually the internal forces computed according to \( F_e = k \mathbf{u}_e \) per element for each time step.
TIME-DEPENDENT PROBLEMS
Two-Dimensional Wave Equations
The Galerkin Finite Element Method
Closure - Time-dependent problems are inherently more difficult and expensive to solve than their corresponding steady-state counterparts.

The expense of generating the global matrices is higher for the time-dependent problems because of the necessity of computing the mass matrices.

The main extra expense, however, is in solving the resulting time-dependent global equations.

For an analytical approach to the solution, additional expense is incurred in terms of having to determine eigenvalues and eigenvectors.

The actual amount of expense depends on the specific form of the stiffness and mass matrices and the algorithm used, but in any case it is significantly in excess of the expense of solving the single set of linear algebraic equations associated with the steady-state problem.

For a time domain integration technique, the additional expense is clearly related to the number of time steps necessary to trace out the desired time history.

In addition to several matrix multiplications and additions, each step can involve the solution of a set of linear algebraic equations.

In some instances this expense can be minimized by using a decomposition that can be reused for the computation of the solution at each new time.

In this regard recall that the Euler and central difference algorithms require that the size of the time step not exceed a value proportional to the inverse of the largest eigenvalue.

For large systems this critical step size can be very small resulting in many applications of the algorithm to trace out the time history.

The unconditionally stable Crank-Nicolson and Newmark algorithms, on the other hand, can be used with arbitrary step size that has been chosen so as to accurately integrate the lower modes, with significant improvement in the expense relative to the conditionally stable Euler and central difference algorithms.

There are of course other algorithms available that are specifically tailored to address other numerical issues.

End of Chapter 4c