TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

The governing balance equations that describe diffusion processes in situations involving two independent variables appear typically as

$$\nabla \cdot \left(k \nabla u(x,y,t)\right) + \rho \frac{\partial u}{\partial t} + f(x,y,t) = 0 \qquad \text{in } \Omega$$

$$u = g(s,t) \qquad \text{on } \Gamma_1$$

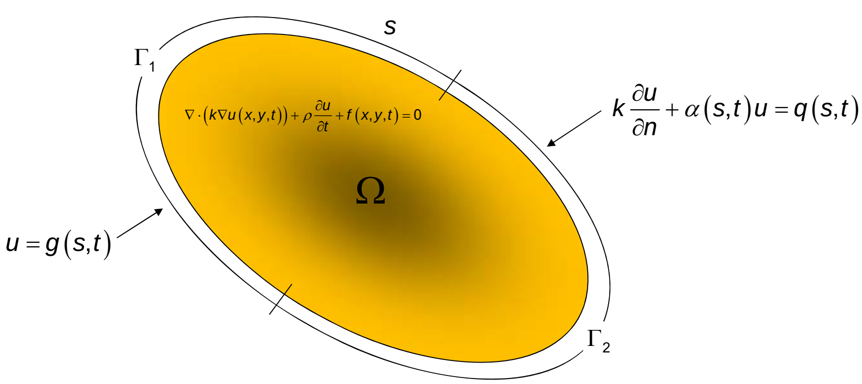
$$k \frac{\partial u}{\partial n} + \alpha(s,t) u = q(s,t) \qquad \text{on } \Gamma_2$$

$$u(x,y,0) = c(x,y) \qquad \text{in } \Omega$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

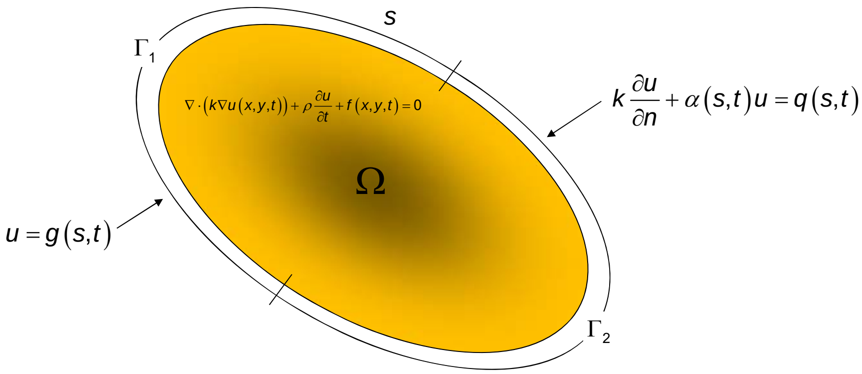
Where Ω is the interior domain, and Γ_1 and Γ_2 form the boundary of the domain.



TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

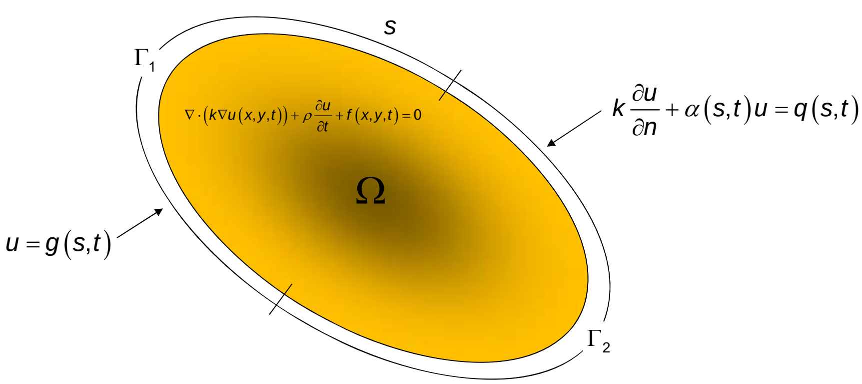
The physical constants are k and ρ, and s is a linear measure of the position on the boundary.



TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

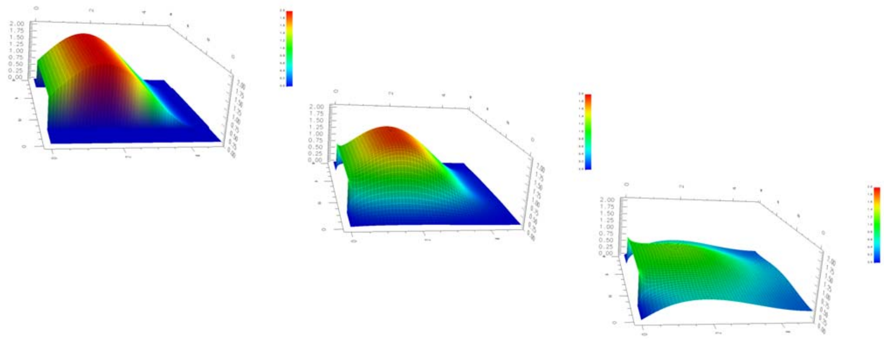
The type of boundary condition specified on Γ_2 results from a local balance between conduction in the interior and convection into the exterior.



TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

In two dimensions, the influence of the boundary conditions is shown. The left side of the simulation domain (x = 0) is used as a constant source of diffusion particles, thereby creating a steady flow into the area. All other boundaries are modeled by a Neumann boundary condition.



TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

This equation is similar in form to the elliptic boundary value problem studied in Chapter 3, with the very important addition of the time derivative term in the differential equation and the corresponding initial condition.

With these additions, the problem changes from an elliptic boundary value problem to a parabolic initial boundary value problem.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

The basic steps of discretization, interpolation, elemental formulation, assembly, constraints, solution, and computation of derived variables are presented in this section as they relate to the two-dimensional parabolic initial-boundary value problem.

The Galerkin method, in connection with the corresponding weak formulation to be developed, will be used to generate the finite element model.

Discretization - For discretization, please refer to the material in Chapter 3.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

The Galerkin Finite Element Method

Interpolation - The solution is assumed to be expressible in terms of the nodally based interpolation functions $n_i(x, y)$ introduced and discussed in Section 3.2.

In the present setting, these interpolation functions are used with the semidiscretization

$$u(x,y,t) = \sum_{1}^{N+1} u_i(t) n_i(x,y)$$

The $n_i(x, y)$ are nodally based interpolation functions and can be linear, quadratic, or as otherwise desired.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

The Galerkin Finite Element Method

Elemental Formulation - The starting point for the elemental formulation is the weak formulation of the initial-boundary value problem.

The first step in developing the weak formulation is to multiply the differential equation by an arbitrary test function $v(x, y)$ vanishing on Γ_1.

The result is then integrated over the domain Ω to obtain

$$\iint_{\Omega} v\left(\nabla\cdot\left(k\nabla u\right)+\rho\frac{\partial u}{\partial t}+f\right)d\Omega=0$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

The Galerkin Finite Element Method

Elemental Formulation - Using the two-dimensional form of the divergence theorem to integrate the first term by parts, there results after rearranging

$$\boxed{\iint_{\Omega}\nabla v\cdot k\nabla u\,d\Omega}+\iint_{\Omega} v\rho\frac{\partial u}{\partial t}d\Omega$$

$$=\boxed{\int_{\Gamma} v\mathbf{n}\cdot k\nabla u\,d\Gamma}+\iint_{\Omega} vf\,d\Omega$$

where $\Gamma = \Gamma_1 + \Gamma_2$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

The Galerkin Finite Element Method

Elemental Formulation - Recalling that v vanishes on Γ_1 and that $k\partial u/\partial x = k\mathbf{n}\cdot\nabla u = q - hu$ on Γ_2, it follows that

$$\iint_{\Omega} \nabla v \cdot k\nabla u \, d\Omega + \iint_{\Omega} v\rho \frac{\partial u}{\partial t} d\Omega$$

$$= \int_{\Gamma} v\left(q - hu\right) d\Gamma + \iint_{\Omega} vf \, d\Omega$$

This equation is the required weak formulation for the two-dimensional diffusion problem.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

The Galerkin Finite Element Method

Elemental Formulation - Substituting the approximation of $u(x, y, t)$ into the weak formulation and taking $v = n_k$, k = 1, 2, ... yields

$$\iint_{\Omega^-} \left(\frac{\partial n_k}{\partial x} k \sum u_i \frac{\partial n_i}{\partial x} + \frac{\partial n_k}{\partial y} k \sum u_i \frac{\partial n_i}{\partial y} \right) d\Omega$$

$$+ \iint_{\Omega^-} n_k \rho \sum \dot{u}_i n_i \, d\Omega + \int_{\Gamma^-} n_k h \sum u_i n_i \, d\Gamma$$

$$= \iint_{\Omega^-} n_k f \, d\Omega + \int_{\Gamma_2^-} n_k q \, d\Gamma \qquad k = 1, 2, \dots$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

The Galerkin Finite Element Method

Elemental Formulation – Where Ω^- and Γ_2^- represent the elemental areas approximating Ω, and the collection of the elemental edges approximating Γ_2,respectively.

$$\iint_{\Omega^-}\left(\frac{\partial n_k}{\partial x}k\sum u_i\frac{\partial n_i}{\partial x}+\frac{\partial n_k}{\partial y}k\sum u_i\frac{\partial n_i}{\partial y}\right)d\Omega$$

$$+\iint_{\Omega^-} n_k\rho\sum \dot{u}_i n_i\, d\Omega+\int_{\Gamma^-} n_k h\sum u_i n_i\, d\Gamma$$

$$=\iint_{\Omega^-} n_k f\, d\Omega+\int_{\Gamma_2^-} n_k q\, d\Gamma \qquad k=1,\,2,\,\ldots$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

The Galerkin Finite Element Method

Elemental Formulation – This *N* x *N* set of linear algebraic equations can be written as:

$$\sum_{i=1}^{N}\left(A_{ki}u_i+B_{ki}\dot{u}_i\right)=F_k(t) \qquad k=1,2,\ldots,N$$

$$A_{ki}=\iint_{\Omega^-}\left(\frac{\partial n_k}{\partial x}k\frac{\partial n_i}{\partial x}+\frac{\partial n_k}{\partial y}k\frac{\partial n_i}{\partial y}\right)d\Omega+\int_{\Gamma^-} n_k h n_i\, d\Gamma$$

$$B_{ki}=\iint_{\Omega^-} n_k\rho n_i\, d\Omega$$

$$F_k=\iint_{\Omega^-} n_k f\, d\Omega+\int_{\Gamma_2^-} n_k q\, d\Gamma$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

The Galerkin Finite Element Method

Elemental Formulation – Note that assembly is contained implicitly within the formulation.

$$\sum_{i=1}^{N}\left(A_{ki}u_i + B_{ki}\dot{u}_i\right) = F_k(t) \qquad k = 1,2,\ldots,N$$

$$A_{ki} = \iint_{\Omega^-}\left(\frac{\partial n_k}{\partial x}k\frac{\partial n_i}{\partial x} + \frac{\partial n_k}{\partial y}k\frac{\partial n_i}{\partial y}\right)d\Omega + \int_{\Gamma^-} n_k h n_i \, d\Gamma$$

$$B_{ki} = \iint_{\Omega^-} n_k \rho n_i \, d\Omega$$

$$F_k = \iint_{\Omega^-} n_k f \, d\Omega + \int_{\Gamma_2^-} n_k q \, d\Gamma$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

The Galerkin Finite Element Method

Elemental Formulation – In terms of the corresponding elementally based interpolations

$$u_e(x,y) = \mathbf{N}^T\mathbf{u}_e = \mathbf{u}_e{}^T\mathbf{N}$$

The finite element model can be expressed as

$$\mathbf{Au} + \mathbf{B\dot{u}} = \mathbf{F}$$

$$\mathbf{A} = \sum_e \mathbf{k_G} + \sum_e{}' \mathbf{a_G} \qquad \mathbf{B} = \sum_e \mathbf{r_G}$$

$$\mathbf{F} = \sum_e \mathbf{f_G} + \sum_e{}' \mathbf{q_G}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

The Galerkin Finite Element Method

Elemental Formulation – In terms of the corresponding elementally based interpolations

$$u_e(x,y) = \mathbf{N}^T \mathbf{u}_e = \mathbf{u}_e{}^T \mathbf{N}$$

The finite element model can be expressed as

$$\mathbf{k}_e = \iint_{A_e} \left(\frac{\partial \mathbf{N}}{\partial x} k \frac{\partial \mathbf{N}^T}{\partial x} + \frac{\partial \mathbf{N}}{\partial y} k \frac{\partial \mathbf{N}^T}{\partial y} \right) dA$$

$$\mathbf{r}_e = \iint_{A_e} \mathbf{N} \rho \mathbf{N}^T \, dA \qquad \mathbf{a}_e = \int_{\gamma_{2e}} \mathbf{N} \alpha \mathbf{N}^T \, ds$$

$$\mathbf{f}_e = \iint_{A_e} \mathbf{N} f \, dA \qquad \mathbf{q}_e = \int_{\gamma_{2e}} \mathbf{N} q \, ds$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

The Galerkin Finite Element Method

Elemental Formulation – The initial conditions for the system of first-order differential equations are obtained from the initial conditions prescribed for the original initial-boundary value problem.

Generally $\mathbf{u}(0)$ is determined by evaluating the function $c(x, y)$ at the nodes to obtain

$$\mathbf{u}(0) = \mathbf{u}_0 = \langle c_1 \quad c_2 \quad c_3 \quad \ldots \quad c_{N-1} \quad c_N \rangle^{\mathrm{T}}$$

where $c_i = c(x_i, y_i)$ with (x_i, y_i) the coordinates of the i^{th} node.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

The Galerkin Finite Element Method

Constraints - The constraints arise from the boundary conditions specified on Γ_1.

Generally the values of the constraints are determined from the *q* function with the constrained value of *u* at a node on Γ_1 being taken as the value of *q* at that point.

These constraints are then enforced on the assembled equations, resulting in the final global constrained set of linear first-order differential equations.

$$\mathbf{M}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f} \qquad\qquad \mathbf{u}(0) = \mathbf{u}_0$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

The Galerkin Finite Element Method

Solution - The system of equations is precisely the same in form and character as the corresponding equations developed for one-dimensional diffusion.

An analytical method as well as the numerical methods of Euler and improved Euler or Crank-Nicolson can be used for integrating the above set of equations.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

The Galerkin Finite Element Method

Solution - The Euler method will be conditionally stable with the critical time step depending on the maximum eigenvalue of the associated problem ($\mathbf{K}$ - $\lambda\mathbf{M}$)$\mathbf{v}$ = 0.

The improved Euler or Crank-Nicolson algorithm will be unconditionally stable.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

The Galerkin Finite Element Method

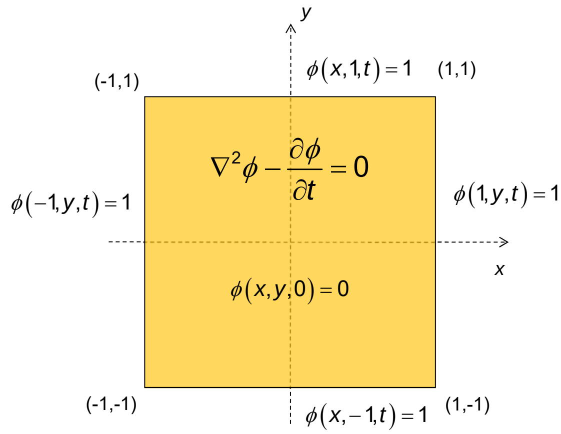
Derived variables - The derived variables will be time dependent, and depending on the particular problem being considered, may need to be computed at each time step.

The computations would be per element and would be carried out using the techniques described in Chapter 3.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

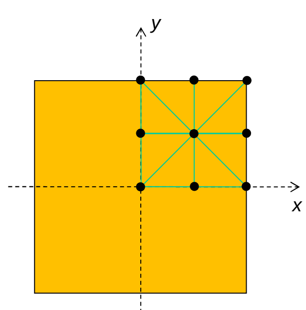
Consider the **Two-Dimensional Diffusion - Example 1** problem shown below.



TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

Discretization. Due to symmetry, we will model the top right-most quadrant of the membrane using eight equally-sized 3-node triangles.



$$\nabla^2\phi - \rho\frac{\partial \phi}{\partial t} = 0$$

$$\phi(\pm 1,y,t)=1$$

$$\phi(x,\pm 1,t)=1$$

$$\phi(x,y,0)=0$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

Elemental Formulation – This *N* x *N* set of linear algebraic equations can be written as:

$$\sum_{i=1}^{N}\left(A_{ki}\phi_i + B_{ki}\dot{\phi}_i\right) = 0 \qquad k = 1, 2, \ldots, N$$

$$A_{ki} = \iint_{\Omega^-}\left(\frac{\partial n_k}{\partial x}k\frac{\partial n_i}{\partial x} + \frac{\partial n_k}{\partial y}k\frac{\partial n_i}{\partial y}\right)d\Omega$$

$$B_{ki} = \iint_{\Omega^-} n_k \rho n_i \, d\Omega$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

Elemental Formulation – In terms of the corresponding elementally based interpolations

$$\phi_e(x,y) = \mathbf{N}^T\phi_e = \phi_e{}^T\mathbf{N}$$

The finite element model can be expressed as

$$\mathbf{k}_e = \iint_{A_e}\left(\frac{\partial \mathbf{N}}{\partial x}k\frac{\partial \mathbf{N}^T}{\partial x} + \frac{\partial \mathbf{N}}{\partial y}k\frac{\partial \mathbf{N}^T}{\partial y}\right)dA$$

$$\mathbf{r}_e = \iint_{A_e}\mathbf{N}\rho\mathbf{N}^T\,dA$$

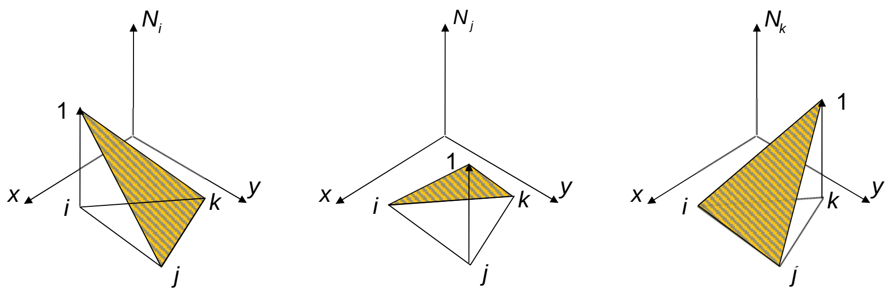
TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

Interpolation - In matrix notation, the distribution of the function over the element is:

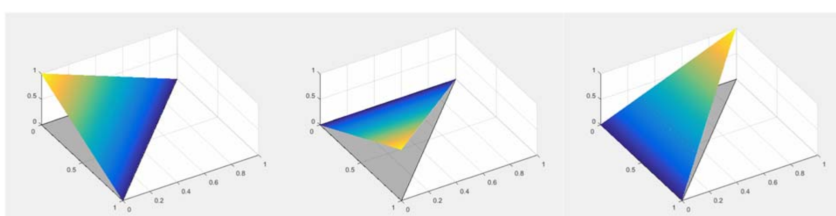
$$\phi_e(x,y) = \boldsymbol{\phi}_\mathbf{e}^\mathbf{T}\mathbf{N} = \mathbf{N}^\mathbf{T}\boldsymbol{\phi}_\mathbf{e}$$

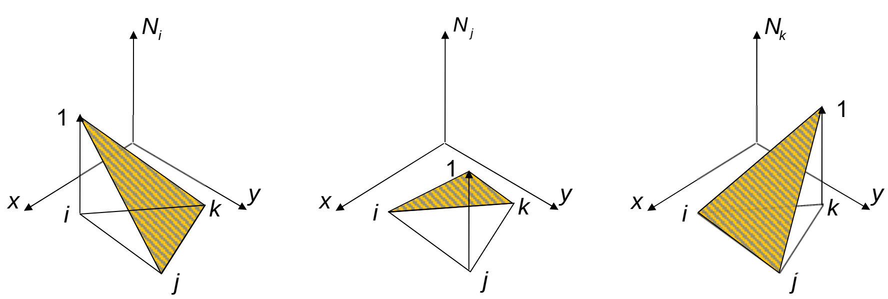
The linear triangular shape functions are illustrated below:



TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1





TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

Interpolation - The derivatives of *u* over the element with respect to both coordinates are:

$$\frac{\partial \phi_e(x,y)}{\partial x} = \boldsymbol{\phi}_\mathbf{e}^\mathbf{T} \frac{\partial \mathbf{N}}{\partial x} = \frac{\partial \mathbf{N}^\mathbf{T}}{\partial x} \boldsymbol{\phi}_\mathbf{e} \qquad \frac{\partial \phi_e(x,y)}{\partial y} = \boldsymbol{\phi}_\mathbf{e}^\mathbf{T} \frac{\partial \mathbf{N}}{\partial y} = \frac{\partial \mathbf{N}^\mathbf{T}}{\partial y} \boldsymbol{\phi}_\mathbf{e}$$

Calculating the derivatives of the shape functions gives:

$$\frac{\partial \mathbf{N}}{\partial x} = \frac{\mathbf{b_e}}{2A_e} \qquad \frac{\partial \mathbf{N}}{\partial y} = \frac{\mathbf{c_e}}{2A_e}$$

$$\mathbf{b_e}^\mathbf{T} = \langle b_i \quad b_j \quad b_k \rangle \qquad \mathbf{c_e}^\mathbf{T} = \langle c_i \quad c_j \quad c_k \rangle$$

$$b_i = y_j - y_k \qquad c_i = x_k - x_j$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

Elemental Formulation - The integrals defined in $\mathbf{k_e}$ are the elemental “stiffness” matrix.

For the linear triangular element we have discussed the stiffness matrix reduces to:

$$\mathbf{k_e} = \iint_{A_e} \left[\frac{\mathbf{b_e b_e^T} + \mathbf{c_e c_e^T}}{4A_e^2} \right] dA$$

Since the integrand of $\mathbf{k_e}$ is a constant, the elemental stiffness matrix becomes:

$$\mathbf{k_e} = \frac{\mathbf{b_e b_e}^\mathbf{T} + \mathbf{c_e c_e}^\mathbf{T}}{4A_e}$$

The resulting is a 3x3 **elemental stiffness matrix.**

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

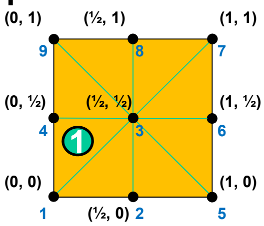
The elemental $\mathbf{k_e}$ matrix using 3-noded triangular elements is:

$$\mathbf{k_e} = \frac{\mathbf{b_e b_e}^{\mathbf{T}} + \mathbf{c_e c_e}^{\mathbf{T}}}{4A_e}$$

$$\mathbf{b_e} = \begin{Bmatrix} y_j - y_k \\ y_k - y_i \\ y_i - y_j \end{Bmatrix} \qquad \mathbf{c_e} = \begin{Bmatrix} x_k - x_j \\ x_i - x_k \\ x_j - x_i \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1



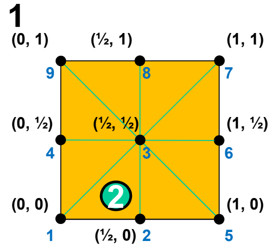
For element 1:
node 1 = (0, 0); node 3 = (½, ½); and node 4 = (0, ½).

$$\mathbf{b_1} = \frac{1}{2}\begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} \qquad \mathbf{c_1} = \frac{1}{2}\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \qquad A_1 = \frac{1}{8}$$

$$\mathbf{k}_1 = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{matrix} \boxed{1} \\ \boxed{3} \\ \boxed{4} \end{matrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1



For element 2:

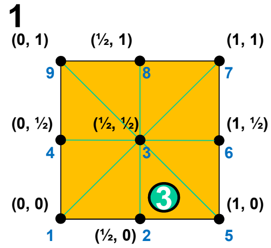
node 1 = (0, 0); node 2 = (½, 0); and node 3 = (½, ½).

$$\mathbf{b}_2 = \frac{1}{2}\begin{Bmatrix} -1 \\ 1 \\ 0 \end{Bmatrix} \qquad \mathbf{c}_2 = \frac{1}{2}\begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix} \qquad A_2 = \frac{1}{8}$$

$$\mathbf{k}_2 = \frac{1}{2}\begin{array}{c} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \boxed{1} \\ \boxed{2} \\ \boxed{3} \end{array}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1



For element 3:

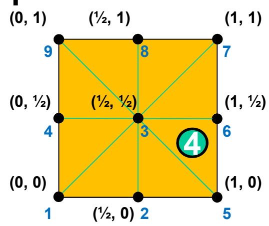
node 2 = (½, 0); node 5 = (1, 0); and node 3 = (½, ½).

$$\mathbf{b}_3 = \frac{1}{2}\begin{Bmatrix} -1 \\ 1 \\ 0 \end{Bmatrix} \qquad \mathbf{c}_3 = \frac{1}{2}\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \qquad A_3 = \frac{1}{8}$$

$$\mathbf{k}_3 = \frac{1}{2}\begin{array}{c} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \end{array}\begin{array}{c} \boxed{2} \\ \boxed{5} \\ \boxed{3} \end{array}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1



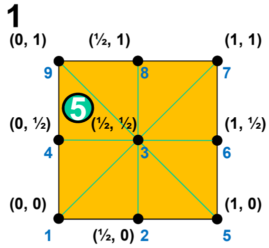
For element 4:
node 5 = (1, 0); node 6 = (1, ½); and node 3 = (½, ½).

$$\mathbf{b}_4 = \frac{1}{2}\begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} \qquad \mathbf{c}_4 = \frac{1}{2}\begin{Bmatrix} -1 \\ 1 \\ 0 \end{Bmatrix} \qquad A_4 = \frac{1}{8}$$

$$\mathbf{k}_4 = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{matrix} \boxed{5} \\ \boxed{6} \\ \boxed{3} \end{matrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1



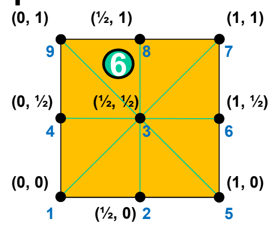
For element 5:
node 4 = (0, ½); node 3 = (½, ½); and node 9 = (0, 1).

$$\mathbf{b}_5 = \frac{1}{2}\begin{Bmatrix} -1 \\ 1 \\ 0 \end{Bmatrix} \qquad \mathbf{c}_5 = \frac{1}{2}\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \qquad A_5 = \frac{1}{8}$$

$$\mathbf{k}_5 = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{matrix} \boxed{4} \\ \boxed{3} \\ \boxed{9} \end{matrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1



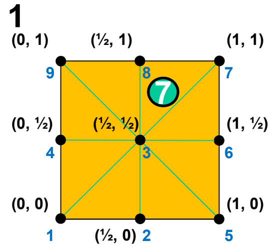
For element 6:
node 3 = (½, ½); node 8 = (½, 1); and node 9 = (0, 1).

$$\mathbf{b}_6 = \frac{1}{2}\begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} \qquad \mathbf{c}_6 = \frac{1}{2}\begin{Bmatrix} -1 \\ 1 \\ 0 \end{Bmatrix} \qquad A_6 = \frac{1}{8}$$

$$\mathbf{k}_6 = \frac{1}{2}\begin{array}{c} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \boxed{3} \\ \boxed{8} \\ \boxed{9} \end{array}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1



For element 7:
node 3 = (½, ½); node 7 = (1, 1); and node 8 = (½, 1).

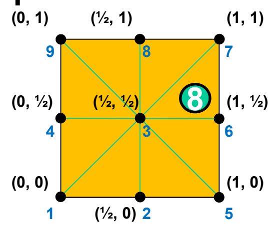
$$\mathbf{b}_7 = \frac{1}{2}\begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} \qquad \mathbf{c}_7 = \frac{1}{2}\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \qquad A_7 = \frac{1}{8}$$

$$\mathbf{k}_7 = \frac{1}{2}\begin{array}{c} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \end{array}\begin{array}{c} \boxed{3} \\ \boxed{7} \\ \boxed{8} \end{array}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

For element 8:
node 3 = (½, ½); node 6 = (1, ½); and node 7 = (1, 1).



$$\mathbf{b}_8 = \frac{1}{2}\begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \qquad \mathbf{c}_8 = \frac{1}{2}\begin{Bmatrix} -1 \\ 1 \\ 0 \end{Bmatrix} \qquad A_8 = \frac{1}{8}$$

$$\mathbf{k}_8 = \frac{1}{2}\begin{array}{c} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \end{array}\begin{array}{c} \boxed{3} \\ \boxed{6} \\ \boxed{7} \end{array}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

The elemental $\mathbf{m_e}$ matrices are:

$$\mathbf{m_e} = \iint_{A_e} \mathbf{N}\mathbf{N}^T\, dA = \iint_{A_e} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} \langle N_1 \quad N_2 \quad N_3 \rangle\, dA$$

$$= \frac{A_e}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad A_e = \frac{1}{8}$$

$$\mathbf{m_1} = \mathbf{m_2} = \cdots = \mathbf{m}_8 = \frac{1}{96}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

The assembled $\mathbf{K_G}$ and $\mathbf{M_G}$ matrices are:

$$\mathbf{K_G} = \frac{1}{2}\begin{array}{c} \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 8 & -2 & 0 & -2 & 0 & -2 & 0 \\ -1 & 0 & -2 & 4 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \end{array} \begin{array}{c} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{array}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

The assembled $\mathbf{K_G}$ and $\mathbf{M_G}$ matrices are:

$$\mathbf{M_G} = \frac{1}{96}\begin{bmatrix} 4 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 16 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & 2 & 4 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{c} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{array}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

Applying nonhomogeneous boundary conditions that $\{\phi_5, \ldots, \phi_9\} = \{1, 1, 1, 1, 1\}$; the constrained $\mathbf{K_G}$ is:

$$\mathbf{K_G}\phi + \mathbf{M_G}\dot{\phi} = 0$$

$$\frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -2 & 0 & \boxed{-1} & 0 & 0 & 0 & 0 \\ 0 & -2 & 8 & -2 & 0 & -2 & 0 & -2 & 0 \\ -1 & 0 & -2 & 4 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \boxed{1} \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} + \frac{1}{96}\begin{bmatrix} 4 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 16 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & 2 & 4 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}\begin{Bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \\ \dot{\phi}_4 \\ \dot{\phi}_5 \\ \dot{\phi}_6 \\ \dot{\phi}_7 \\ \dot{\phi}_8 \\ \dot{\phi}_9 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \boxed{0.5} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{\dot{u}_5, \cdots, \dot{u}_9\} = 0$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

Applying nonhomogeneous boundary conditions that $\{\phi_5, \ldots, \phi_9\} = \{1, 1, 1, 1, 1\}$; the constrained $\mathbf{K_G}$ is:

$$\mathbf{K_G}\phi + \mathbf{M_G}\dot{\phi} = 0$$

$$\frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -2 & 0 & \boxed{-1} & 0 & 0 & 0 & 0 \\ 0 & -2 & 8 & -2 & 0 & \boxed{-2} & 0 & \boxed{-2} & 0 \\ -1 & 0 & -2 & 4 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \boxed{1} \\ \boxed{1} \\ 1 \\ \boxed{1} \\ 1 \end{Bmatrix} + \frac{1}{96}\begin{bmatrix} 4 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 16 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & 2 & 4 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}\begin{Bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \\ \dot{\phi}_4 \\ \dot{\phi}_5 \\ \dot{\phi}_6 \\ \dot{\phi}_7 \\ \dot{\phi}_8 \\ \dot{\phi}_9 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \boxed{0.5} \\ \boxed{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{\dot{u}_5, \cdots, \dot{u}_9\} = 0$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

Applying nonhomogeneous boundary conditions that $\{\phi_5, \dots, \phi_9\} = \{1, 1, 1, 1, 1\}$; the constrained $\mathbf{K_G}$ is:

$$\mathbf{K_G}\phi + \mathbf{M_G}\dot{\phi} = 0$$

$$\frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -2 & 0 & \boxed{-1} & 0 & 0 & 0 & 0 \\ 0 & -2 & 8 & -2 & 0 & \boxed{-2} & 0 & \boxed{-2} & 0 \\ -1 & 0 & -2 & 4 & 0 & 0 & 0 & 0 & \boxed{-1} \\ 0 & -1 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \boxed{1} \\ \boxed{1} \\ \boxed{1} \\ \boxed{1} \\ \boxed{1} \end{Bmatrix} + \frac{1}{96}\begin{bmatrix} 4 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 16 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & 2 & 4 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{Bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \\ \dot{\phi}_4 \\ \dot{\phi}_5 \\ \dot{\phi}_6 \\ \dot{\phi}_7 \\ \dot{\phi}_8 \\ \dot{\phi}_9 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \boxed{0.5} \\ \boxed{2} \\ \boxed{0.5} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{\dot{u}_5, \cdots, \dot{u}_9\} = 0$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

Assembly. With both the boundary conditions essential, **BT**=0 and **bt**=0. It follows that the assembled equations are:

$$\mathbf{K_G}\phi + \mathbf{M_G}\dot{\phi} = 0$$

$$\frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 4 & -2 & 0 \\ 0 & -2 & 8 & -2 \\ -1 & 0 & -2 & 4 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} + \frac{1}{96}\begin{bmatrix} 4 & 1 & 2 & 1 \\ 1 & 4 & 2 & 0 \\ 2 & 2 & 16 & 2 \\ 1 & 0 & 2 & 4 \end{bmatrix} \begin{Bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \\ \dot{\phi}_4 \end{Bmatrix} = \frac{1}{2}\begin{Bmatrix} 0 \\ 1 \\ 4 \\ 1 \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

The θ algorithm for the time integration is:

$$\left(\mathbf{M} + h\theta\mathbf{K}\right)\phi_{n+1} = \left(\mathbf{M} - h(1-\theta)\mathbf{K}\right)\phi_n + h\left[\theta\mathbf{f}_{n+1} + \left(1-\theta\right)\mathbf{f}_n\right]$$

It is easily seen that: $\begin{cases} \theta = 0 & \text{Euler method} \\ \theta = \frac{1}{2} & \text{Crank-Nicolson method} \end{cases}$

The value $\theta = 1$ corresponds to what is referred to as the *modified* Euler method and corresponds to using a **backward difference scheme** obtained by evaluating the differential equation at t_{n+1} and taking:

$$\dot{\phi}_{n+1} = \frac{\phi_{n+1} - \phi_n}{h}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

For the θ algorithm, recall the largest eigenvalue of the $\mathbf{K} - \lambda\mathbf{M}$ help define the time step:

$$h < \frac{2}{\lambda_{\max}}$$

The eigenvalue problem is:

$$\left(\mathbf{K_G} + \lambda\mathbf{M_G}\right)\phi = 0$$

$$\left(\frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 4 & -2 & 0 \\ 0 & -2 & 8 & -2 \\ -1 & 0 & -2 & 4 \end{bmatrix} - \frac{\lambda}{96}\begin{bmatrix} 4 & 1 & 2 & 1 \\ 1 & 4 & 2 & 0 \\ 2 & 2 & 16 & 2 \\ 1 & 0 & 2 & 4 \end{bmatrix}\right)\phi = 0$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

The eigenvalues can be found using a variety of available solution techniques. In Matlab, use **`[V,D] = eig(A).`**

For example, **`[V,D] = eig(`$M_G^{-1}K_G$`)`** gives eigenvalues **`D`**:

$$\lambda_1 = 5.6265 \quad \lambda_2 = 32 \quad \lambda_3 = 48 \quad \lambda_4 = 102.3735$$

The eigenvectors **`V`** are:

$$V_1 = \begin{Bmatrix} -0.69829 \\ -0.44187 \\ -0.34914 \\ -0.44187 \end{Bmatrix} \quad V_2 = \begin{Bmatrix} 0.89443 \\ 0.00000 \\ -0.44721 \\ 0.00000 \end{Bmatrix} \quad V_3 = \begin{Bmatrix} 0.00000 \\ -0.70711 \\ 0.00000 \\ 0.70711 \end{Bmatrix} \quad V_4 = \begin{Bmatrix} -0.44393 \\ 0.61386 \\ -0.22197 \\ 0.61386 \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

For the example the largest eigenvalue is 102.3735, the critical time step is:

$$h < \frac{2}{\lambda_{\max}} < \frac{2}{102.3735} < 0.0195 \text{ sec.}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

The θ algorithm for the time integration is:

$$\left(\mathbf{M}+h\theta\mathbf{K}\right)\phi_{n+1}=\left(\mathbf{M}-h(1-\theta)\mathbf{K}\right)\phi_n+h\left[\theta\mathbf{f}_{n+1}+\left(1-\theta\right)\mathbf{f}_n\right]$$

In this example, let's assume h = 0.001 sec. and θ = ½, therefore:

$$\left(\mathbf{M}+h\theta\mathbf{K}\right)=\frac{1}{96}\begin{bmatrix}4 & 1 & 2 & 1\\ 1 & 4 & 2 & 0\\ 2 & 2 & 16 & 2\\ 1 & 0 & 2 & 4\end{bmatrix}+(0.001)\frac{1}{2}\left(\frac{1}{2}\right)\begin{bmatrix}2 & -1 & 0 & -1\\ -1 & 4 & -2 & 0\\ 0 & -2 & 8 & -2\\ -1 & 0 & -2 & 4\end{bmatrix}$$

$$=\begin{bmatrix}0.04217 & 0.01017 & 0.02083 & 0.01017\\ 0.01017 & 0.04267 & 0.02033 & 0.00000\\ 0.02083 & 0.02033 & 0.16867 & 0.02033\\ 0.01017 & 0.00000 & 0.02033 & 0.04267\end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

The θ algorithm for the time integration is:

$$\left(\mathbf{M}+h\theta\mathbf{K}\right)\phi_{n+1}=\left(\mathbf{M}-h(1-\theta)\mathbf{K}\right)\phi_n+h\left[\theta\mathbf{f}_{n+1}+\left(1-\theta\right)\mathbf{f}_n\right]$$

In this example, let's assume h = 0.001 sec. and θ = ½, therefore:

$$\left(\mathbf{M}+h\theta\mathbf{K}\right)^{-1}=\begin{bmatrix}27.40488 & -5.55522 & -2.04559 & -5.55522\\ -5.55522 & 26.08491 & -2.77761 & 2.64741\\ -2.04559 & -2.77761 & 6.85122 & -2.77761\\ -5.55522 & 2.64741 & -2.77761 & 26.08491\end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

The θ algorithm for the time integration is:

$$\left(\mathbf{M}+h\theta\mathbf{K}\right)\phi_{n+1}=\left(\mathbf{M}-h(1-\theta)\mathbf{K}\right)\phi_n+h\left[\theta\mathbf{f}_{n+1}+\left(1-\theta\right)\mathbf{f}_n\right]$$

In this example, let's assume h = 0.001 sec. and θ = ½, therefore:

$$\left(\mathbf{M}-h\theta\mathbf{K}\right)=\frac{1}{96}\begin{bmatrix}4&1&2&1\\1&4&2&0\\2&2&16&2\\1&0&2&4\end{bmatrix}-(0.001)\frac{1}{2}\left(\frac{1}{2}\right)\begin{bmatrix}2&-1&0&-1\\-1&4&-2&0\\0&-2&8&-2\\-1&0&-2&4\end{bmatrix}$$

$$=\begin{bmatrix}0.04117&0.01067&0.02083&0.01067\\0.01067&0.04067&0.02133&0.00000\\0.02083&0.02133&0.16467&0.02133\\0.01067&0.00000&0.02133&0.04067\end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

The θ algorithm for the time integration is:

$$\left(\mathbf{M}+h\theta\mathbf{K}\right)\phi_{n+1}=\left(\mathbf{M}-h(1-\theta)\mathbf{K}\right)\phi_n+h\left[\theta\mathbf{f}_{n+1}+\left(1-\theta\right)\mathbf{f}_n\right]$$

In this example, let's assume h = 0.001 sec. and θ = ½, therefore:

$$\mathbf{f}=\frac{1}{2}\begin{Bmatrix}0\\1\\4\\1\end{Bmatrix}\quad\Rightarrow\quad h\left[\theta\mathbf{f}_{n+1}+\left(1-\theta\right)\mathbf{f}_n\right]=\frac{(0.001)}{2}\begin{Bmatrix}0\\1\\4\\1\end{Bmatrix}=\begin{Bmatrix}0.00000\\0.00050\\0.00200\\0.00050\end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

The θ algorithm for the time integration is:

$$\left(\mathbf{M}+h\theta\mathbf{K}\right)\phi_{n+1}=\left(\mathbf{M}-h(1-\theta)\mathbf{K}\right)\phi_{n}+h\left[\theta\mathbf{f}_{n+1}+\left(1-\theta\right)\mathbf{f}_{n}\right]$$

In this example, let's assume h = 0.001 sec. and θ = ½, therefore:

$$\mathbf{u}_{n+1}=\left(\mathbf{M}+h\theta\mathbf{K}\right)^{-1}\left[\left(\mathbf{M}-h(1-\theta)\mathbf{K}\right)\mathbf{u}_{n}+h\left[\theta\mathbf{f}_{n+1}+\left(1-\theta\right)\mathbf{f}_{n}\right]\right]$$

$$\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}_{n+1}=\begin{bmatrix}0.96704 & 0.02277 & -0.00293 & 0.02277\\0.01992 & 0.94227 & 0.03984 & -0.01085\\-0.00073 & 0.01138 & 0.96704 & 0.01138\\0.01992 & -0.01085 & 0.03984 & 0.94227\end{bmatrix}\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}_{n}+\begin{Bmatrix}-0.00965\\0.00881\\0.01092\\0.00881\end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

The θ algorithm for the time integration is:

$$\left(\mathbf{M}+h\theta\mathbf{K}\right)\phi_{n+1}=\left(\mathbf{M}-h(1-\theta)\mathbf{K}\right)\phi_{n}+h\left[\theta\mathbf{f}_{n+1}+\left(1-\theta\right)\mathbf{f}_{n}\right]$$

For n = 1, with $u(x, y, 0) = 0$, then:

$$\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}_{1}=\begin{bmatrix}0.96704 & 0.02277 & -0.00293 & 0.02277\\0.01992 & 0.94227 & 0.03984 & -0.01085\\-0.00073 & 0.01138 & 0.96704 & 0.01138\\0.01992 & -0.01085 & 0.03984 & 0.94227\end{bmatrix}\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}_{0}+\begin{Bmatrix}-0.00965\\0.00881\\0.01092\\0.00881\end{Bmatrix}$$

$$\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}_{1}=\begin{Bmatrix}-0.00965\\0.00881\\0.01092\\0.00881\end{Bmatrix}\qquad \dot{\phi}_1=\frac{\phi_1-\phi_0}{h}=\begin{Bmatrix}-9.64640\\8.81094\\10.92483\\8.81094\end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

The θ algorithm for the time integration is:

$$\left(\mathbf{M}+h\theta\mathbf{K}\right)\phi_{n+1}=\left(\mathbf{M}-h(1-\theta)\mathbf{K}\right)\phi_n+h\left[\theta\mathbf{f}_{n+1}+\left(1-\theta\right)\mathbf{f}_n\right]$$

For n = 2, with $u(x, y, 0)$ = 0, then:

$$\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}_2=\begin{bmatrix}0.96704 & 0.02277 & -0.00293 & 0.02277\\0.01992 & 0.94227 & 0.03984 & -0.01085\\-0.00073 & 0.01138 & 0.96704 & 0.01138\\0.01992 & -0.01085 & 0.03984 & 0.94227\end{bmatrix}\begin{Bmatrix}-0.00965\\0.00881\\0.01092\\0.00881\end{Bmatrix}_1+\begin{Bmatrix}-0.00965\\0.00881\\0.01092\\0.00881\end{Bmatrix}$$

$$\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}_2=\begin{Bmatrix}-0.01861\\0.01726\\0.02170\\0.01726\end{Bmatrix}\qquad\dot{\phi}_2=\frac{\phi_2-\phi_1}{h}=\begin{Bmatrix}-8.95924\\8.44984\\10.77241\\8.44984\end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

The θ algorithm for the time integration is:

$$\left(\mathbf{M}+h\theta\mathbf{K}\right)\phi_{n+1}=\left(\mathbf{M}-h(1-\theta)\mathbf{K}\right)\phi_n+h\left[\theta\mathbf{f}_{n+1}+\left(1-\theta\right)\mathbf{f}_n\right]$$

For n = 3, with $u(x, y, 0)$ = 0, then:

$$\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}_3=\begin{bmatrix}0.96704 & 0.02277 & -0.00293 & 0.02277\\0.01992 & 0.94227 & 0.03984 & -0.01085\\-0.00073 & 0.01138 & 0.96704 & 0.01138\\0.01992 & -0.01085 & 0.03984 & 0.94227\end{bmatrix}\begin{Bmatrix}-0.01861\\0.01726\\0.02170\\0.01726\end{Bmatrix}_2+\begin{Bmatrix}-0.00965\\0.00881\\0.01092\\0.00881\end{Bmatrix}$$

$$\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}_3=\begin{Bmatrix}-0.02692\\0.02538\\0.03231\\0.02538\end{Bmatrix}\qquad\dot{\phi}_3=\frac{\phi_3-\phi_2}{h}=\begin{Bmatrix}-8.31072\\8.12111\\10.61629\\8.12111\end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

The θ algorithm for the time integration is:

$$\left(\mathbf{M}+h\theta\mathbf{K}\right)\phi_{n+1}=\left(\mathbf{M}-h(1-\theta)\mathbf{K}\right)\phi_n+h\left[\theta\mathbf{f}_{n+1}+\left(1-\theta\right)\mathbf{f}_n\right]$$

For n = 100, with $u(x, y, 0) = 0$, then:

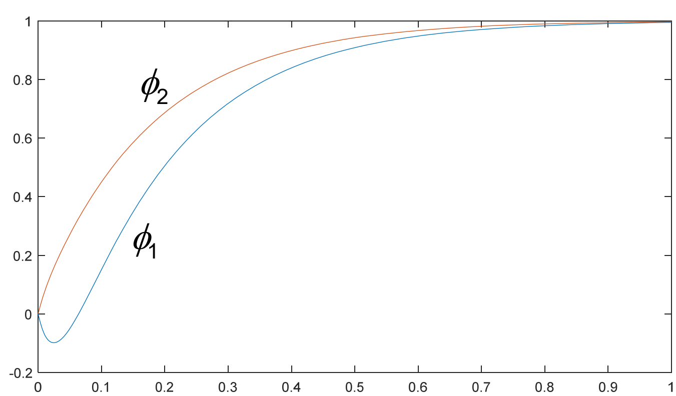
$$\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}_{100}=\begin{bmatrix}0.96704 & 0.02277 & -0.00293 & 0.02277\\ 0.01992 & 0.94227 & 0.03984 & -0.01085\\ -0.00073 & 0.01138 & 0.96704 & 0.01138\\ 0.01992 & -0.01085 & 0.03984 & 0.94227\end{bmatrix}\begin{Bmatrix}0.14298\\0.44394\\0.54977\\0.44394\end{Bmatrix}_{99}+\begin{Bmatrix}-0.00965\\0.00881\\0.01092\\0.00881\end{Bmatrix}$$

$$\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}_{100}=\begin{Bmatrix}0.14722\\0.44706\\0.55257\\0.44706\end{Bmatrix}\qquad \dot{\phi}_{100}=\frac{\phi_{100}-\phi_{99}}{h}=\begin{Bmatrix}4.24609\\3.11999\\2.80721\\3.11999\end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2

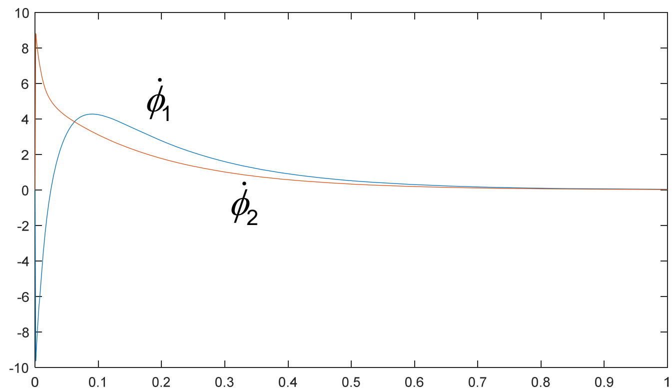
The values of ϕ_1 and ϕ_2 for 0 < t < 1 sec. are show below:



TIME-DEPENDENT PROBLEMS

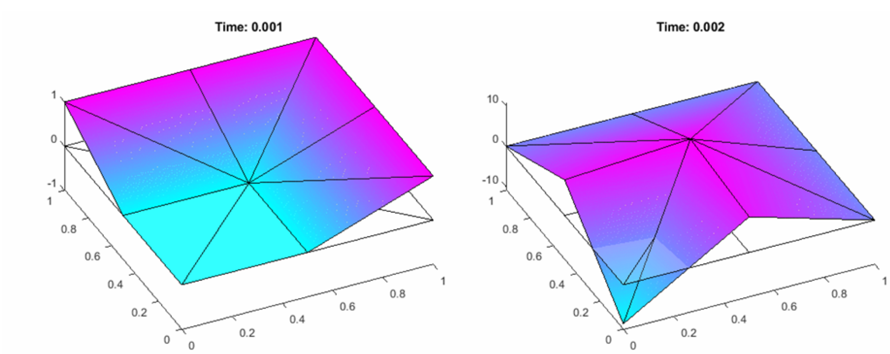
Two-Dimensional Diffusion - Example 2

The derivative values of ϕ_1 and ϕ_2 for 0 < t < 1 sec. are:



TIME-DEPENDENT PROBLEMS

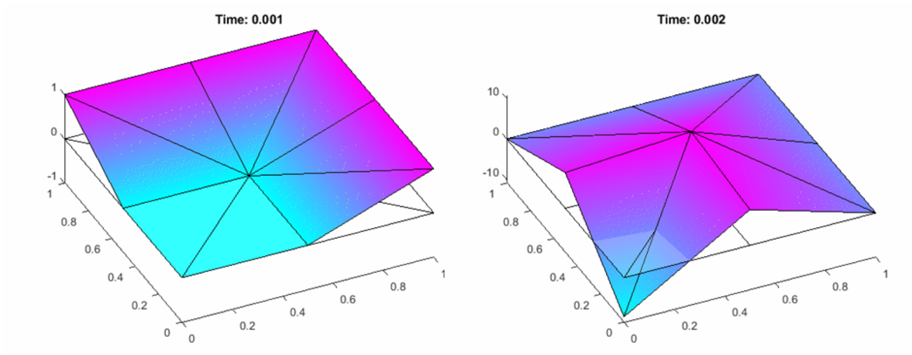
Two-Dimensional Diffusion - Example 1



$\phi(x,y,t)$ $\qquad$ $\dfrac{\partial \phi(x,y,t)}{\partial t}$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 1

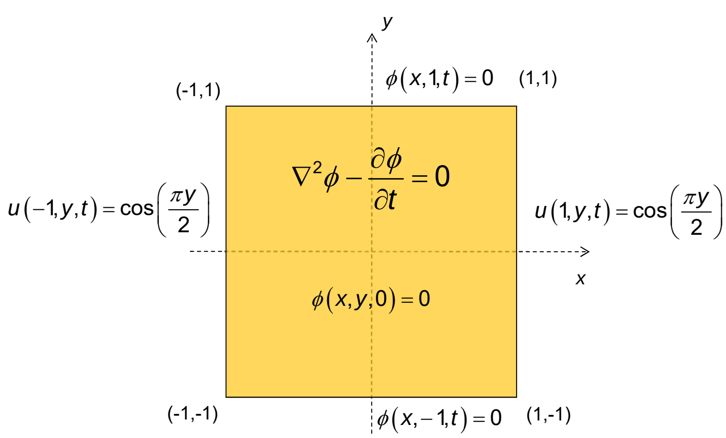


$\phi(x,y,t)$ $\qquad\qquad$ $\dfrac{\partial \phi(x,y,t)}{\partial t}$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2

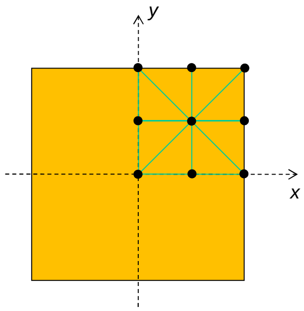
Consider the two-dimensional diffusion problem shown below.



TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2

Discretization. Due to symmetry, we will model the top right-most quadrant of the membrane using eight equally-sized 3-node triangles.



$$\nabla^2\phi - \rho\frac{\partial \phi}{\partial t} = 0$$

$$\phi(\pm 1, y, t) = \cos\left(\frac{\pi y}{2}\right)$$

$$\phi(x, \pm 1, t) = 0$$

$$\phi(x, y, 0) = 0$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2

Applying homogeneous boundary conditions that u_7, u_8, and $u_9 = 0$; then the assembled $\mathbf{K_G}$ and $\mathbf{M_G}$ matrices are:

$$\mathbf{K_G} = \frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -2 & 0 & -1 & 0 \\ 0 & -2 & 8 & -2 & 0 & -2 \\ -1 & 0 & -2 & 4 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & -2 & 0 & -1 & 4 \end{bmatrix} \begin{array}{c} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{array}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2

Applying homogeneous boundary conditions that u_7, u_8, and u_9 = 0; then the assembled $\mathbf{K_G}$ and $\mathbf{M_G}$ matrices are:

$$\mathbf{M_G} = \frac{1}{96}\begin{bmatrix} 4 & 1 & 2 & 1 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 & 0 \\ 2 & 2 & 16 & 2 & 2 & 2 \\ 1 & 0 & 2 & 4 & 0 & 0 \\ 0 & 1 & 2 & 0 & 4 & 1 \\ 0 & 0 & 2 & 0 & 1 & 4 \end{bmatrix} \begin{matrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \\ \dot{\phi}_4 \\ \dot{\phi}_5 \\ \dot{\phi}_6 \end{matrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2

Assembly. With both the boundary conditions essential, **BT**=0 and **bt**=0. It follows that the assembled equations are:

$$\mathbf{K_G u} + \mathbf{M_G \dot{u}} = 0 \qquad u_5 = 1 \text{ and } u_6 = 0.7071$$

$$\frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -2 & 0 & \boxed{-1} & 0 \\ 0 & -2 & 8 & -2 & 0 & -2 \\ -1 & 0 & -2 & 4 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & -2 & 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \boxed{1} \\ 0.7071 \end{Bmatrix} + \frac{1}{96}\begin{bmatrix} 4 & 1 & 2 & 1 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 & 0 \\ 2 & 2 & 16 & 2 & 2 & 2 \\ 1 & 0 & 2 & 4 & 0 & 0 \\ 0 & 1 & 2 & 0 & 4 & 1 \\ 0 & 0 & 2 & 0 & 1 & 4 \end{bmatrix} \begin{Bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \\ \dot{\phi}_4 \\ \dot{\phi}_5 \\ \dot{\phi}_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \boxed{0.5} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\dot{u}_5 = 0$$

$$\dot{u}_6 = 0$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2

Assembly. With both the boundary conditions essential, **BT**=0 and **bt**=0. It follows that the assembled equations are:

$$\mathbf{K_G u} + \mathbf{M_G \dot{u}} = 0 \qquad u_5 = 1 \text{ and } u_6 = 0.7071$$

$$\frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -2 & 0 & \boxed{-1} & 0 \\ 0 & -2 & 8 & -2 & 0 & \boxed{-2} \\ -1 & 0 & -2 & 4 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & -2 & 0 & -1 & 4 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \boxed{1} \\ \boxed{0.7071} \end{Bmatrix} + \frac{1}{96}\begin{bmatrix} 4 & 1 & 2 & 1 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 & 0 \\ 2 & 2 & 16 & 2 & 2 & 2 \\ 1 & 0 & 2 & 4 & 0 & 0 \\ 0 & 1 & 2 & 0 & 4 & 1 \\ 0 & 0 & 2 & 0 & 1 & 4 \end{bmatrix}\begin{Bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \\ \dot{\phi}_4 \\ \dot{\phi}_5 \\ \dot{\phi}_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \boxed{0.5} \\ \boxed{0.7071} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\dot{u}_5 = 0$$
$$\dot{u}_6 = 0$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2

Assembly. With both the boundary conditions essential, **BT**=0 and **bt**=0. It follows that the assembled equations are:

$$\mathbf{K_G}\phi + \mathbf{M_G}\dot{\phi} = 0$$

$$\frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 4 & -2 & 0 \\ 0 & -2 & 8 & -2 \\ -1 & 0 & -2 & 4 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} + \frac{1}{96}\begin{bmatrix} 4 & 1 & 2 & 1 \\ 1 & 4 & 2 & 0 \\ 2 & 2 & 16 & 2 \\ 1 & 0 & 2 & 4 \end{bmatrix}\begin{Bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \\ \dot{\phi}_4 \end{Bmatrix} = \frac{1}{2}\begin{Bmatrix} 0 \\ 1 \\ \sqrt{2} \\ 0 \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2

The θ algorithm for the time integration is:

$$\left(\mathbf{M} + h\theta\mathbf{K}\right)\phi_{n+1} = \left(\mathbf{M} - h(1-\theta)\mathbf{K}\right)\phi_n + h\left[\theta\mathbf{f}_{n+1} + \left(1-\theta\right)\mathbf{f}_n\right]$$

It is easily seen that: $\begin{cases} \theta = 0 & \text{Euler method} \\ \theta = \frac{1}{2} & \text{Crank-Nicolson method} \end{cases}$

The value θ = 1 corresponds to what is referred to as the *modified* Euler method and corresponds to using a **backward difference scheme** obtained by evaluating the differential equation at t_{n+1} and taking:

$$\dot{\phi}_{n+1} = \frac{\phi_{n+1} - \phi_n}{h}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2

The θ algorithm for the time integration is:

$$\left(\mathbf{M} + h\theta\mathbf{K}\right)\phi_{n+1} = \left(\mathbf{M} - h(1-\theta)\mathbf{K}\right)\phi_n + h\left[\theta\mathbf{f}_{n+1} + \left(1-\theta\right)\mathbf{f}_n\right]$$

In this example, let's assume h = 0.002 sec. and θ = ½, therefore:

$$\left(\mathbf{M} + h\theta\mathbf{K}\right) = \frac{1}{96}\begin{bmatrix} 4 & 1 & 2 & 1 \\ 1 & 4 & 2 & 0 \\ 2 & 2 & 16 & 2 \\ 1 & 0 & 2 & 4 \end{bmatrix} + \left(0.002\right)\frac{1}{2}\left(\frac{1}{2}\right)\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 4 & -2 & 0 \\ 0 & -2 & 8 & -2 \\ -1 & 0 & -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.04267 & 0.00992 & 0.02083 & 0.00992 \\ 0.00992 & 0.04367 & 0.01983 & 0.00000 \\ 0.02083 & 0.01983 & 0.17067 & 0.01983 \\ 0.00992 & 0.00000 & 0.01983 & 0.04367 \end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2

The θ algorithm for the time integration is:

$$\left(\mathbf{M}+h\theta\mathbf{K}\right)\phi_{n+1}=\left(\mathbf{M}-h(1-\theta)\mathbf{K}\right)\phi_{n}+h\left[\theta\mathbf{f}_{n+1}+\left(1-\theta\right)\mathbf{f}_{n}\right]$$

In this example, let's assume h = 0.002 sec. and θ = ½, therefore:

$$\left(\mathbf{M}+h\theta\mathbf{K}\right)^{-1}=\begin{bmatrix} 26.84816 & -5.15256 & -2.07980 & -5.15256 \\ -5.15256 & 25.24105 & -2.57628 & 2.34028 \\ -2.07980 & -2.57628 & 6.71204 & -2.57628 \\ -5.15256 & 2.34028 & -2.57628 & 25.24105 \end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2

The θ algorithm for the time integration is:

$$\left(\mathbf{M}+h\theta\mathbf{K}\right)\phi_{n+1}=\left(\mathbf{M}-h(1-\theta)\mathbf{K}\right)\phi_{n}+h\left[\theta\mathbf{f}_{n+1}+\left(1-\theta\right)\mathbf{f}_{n}\right]$$

In this example, let's assume h = 0.002 sec. and θ = ½, therefore:

$$\left(\mathbf{M}-h\theta\mathbf{K}\right)=\frac{1}{96}\begin{bmatrix} 4 & 1 & 2 & 1 \\ 1 & 4 & 2 & 0 \\ 2 & 2 & 16 & 2 \\ 1 & 0 & 2 & 4 \end{bmatrix}-(0.002)\frac{1}{2}\left(\frac{1}{2}\right)\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 4 & -2 & 0 \\ 0 & -2 & 8 & -2 \\ -1 & 0 & -2 & 4 \end{bmatrix}$$

$$=\begin{bmatrix} 0.04067 & 0.01092 & 0.02083 & 0.01092 \\ 0.01092 & 0.03967 & 0.02183 & 0.00000 \\ 0.02083 & 0.02183 & 0.16267 & 0.02183 \\ 0.01092 & 0.00000 & 0.02183 & 0.03967 \end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2

The θ algorithm for the time integration is:

$$\left(\mathbf{M}+h\theta\mathbf{K}\right)\phi_{n+1}=\left(\mathbf{M}-h(1-\theta)\mathbf{K}\right)\phi_n+h\left[\theta\mathbf{f}_{n+1}+\left(1-\theta\right)\mathbf{f}_n\right]$$

In this example, let's assume h = 0.002 sec. and θ = ½, therefore:

$$\mathbf{f}=\frac{1}{2}\begin{Bmatrix}0\\1\\\sqrt{2}\\0\end{Bmatrix}\quad\Rightarrow\quad h\left[\theta\mathbf{f}_{n+1}+\left(1-\theta\right)\mathbf{f}_n\right]=\frac{(0.002)}{2}\begin{Bmatrix}0\\1\\\sqrt{2}\\0\end{Bmatrix}=\begin{Bmatrix}0.00000\\0.00100\\0.00141\\0.00000\end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2

The θ algorithm for the time integration is:

$$\left(\mathbf{M}+h\theta\mathbf{K}\right)\phi_{n+1}=\left(\mathbf{M}-h(1-\theta)\mathbf{K}\right)\phi_n+h\left[\theta\mathbf{f}_{n+1}+\left(1-\theta\right)\mathbf{f}_n\right]$$

In this example, let's assume h = 0.002 sec. and θ = ½, therefore:

$$\mathbf{u}_{n+1}=\left(\mathbf{M}+h\theta\mathbf{K}\right)^{-1}\left[\left(\mathbf{M}-h(1-\theta)\mathbf{K}\right)\mathbf{u}_n+h\left[\theta\mathbf{f}_{n+1}+\left(1-\theta\right)\mathbf{f}_n\right]\right]$$

$$\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}_{n+1}=\begin{bmatrix}0.93600 & 0.04330 & -0.00397 & 0.04330\\0.03789 & 0.88873 & 0.07577 & -0.01967\\-0.00099 & 0.02165 & 0.93600 & 0.02165\\0.03789 & -0.01967 & 0.07577 & 0.88873\end{bmatrix}\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}_n+\begin{Bmatrix}-0.00809\\0.02160\\0.00692\\-0.00130\end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2

The θ algorithm for the time integration is:

$$\left(\mathbf{M}+h\theta\mathbf{K}\right)\phi_{n+1}=\left(\mathbf{M}-h(1-\theta)\mathbf{K}\right)\phi_n+h\left[\theta\mathbf{f}_{n+1}+\left(1-\theta\right)\mathbf{f}_n\right]$$

For n = 1, with $u(x, y, 0) = 0$, then:

$$\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}_1=\begin{bmatrix}0.93600 & 0.04330 & -0.00397 & 0.04330\\0.03789 & 0.88873 & 0.07577 & -0.01967\\-0.00099 & 0.02165 & 0.93600 & 0.02165\\0.03789 & -0.01967 & 0.07577 & 0.88873\end{bmatrix}\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}_0+\begin{Bmatrix}-0.00809\\0.02160\\0.00692\\-0.00130\end{Bmatrix}$$

$$\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}_1=\begin{Bmatrix}-0.00809\\0.02160\\0.00692\\-0.00130\end{Bmatrix} \qquad \dot{\phi}_1=\frac{\phi_1-\phi_0}{h}=\begin{Bmatrix}-4.04692\\10.79882\\3.45799\\-0.65156\end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2

The θ algorithm for the time integration is:

$$\left(\mathbf{M}+h\theta\mathbf{K}\right)\phi_{n+1}=\left(\mathbf{M}-h(1-\theta)\mathbf{K}\right)\phi_n+h\left[\theta\mathbf{f}_{n+1}+\left(1-\theta\right)\mathbf{f}_n\right]$$

For n = 2, with $u(x, y, 0) = 0$, then:

$$\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}_2=\begin{bmatrix}0.93600 & 0.04330 & -0.00397 & 0.04330\\0.03789 & 0.88873 & 0.07577 & -0.01967\\-0.00099 & 0.02165 & 0.93600 & 0.02165\\0.03789 & -0.01967 & 0.07577 & 0.88873\end{bmatrix}\begin{Bmatrix}-0.00809\\0.02160\\0.00692\\-0.00130\end{Bmatrix}_1+\begin{Bmatrix}-0.00809\\0.02160\\0.00692\\-0.00130\end{Bmatrix}$$

$$\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{Bmatrix}_2=\begin{Bmatrix}-0.01482\\0.04104\\0.01384\\-0.00267\end{Bmatrix} \qquad \dot{\phi}_2=\frac{\phi_2-\phi_1}{h}=\begin{Bmatrix}-3.36228\\9.71875\\3.46037\\-0.68274\end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2

The θ algorithm for the time integration is:

$$\left(\mathbf{M} + h\theta\mathbf{K}\right)\phi_{n+1} = \left(\mathbf{M} - h(1-\theta)\mathbf{K}\right)\phi_n + h\left[\theta\mathbf{f}_{n+1} + \left(1-\theta\right)\mathbf{f}_n\right]$$

For n = 3, with $u(x, y, 0) = 0$, then:

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix}_3 = \begin{bmatrix} 0.93600 & 0.04330 & -0.00397 & 0.04330 \\ 0.03789 & 0.88873 & 0.07577 & -0.01967 \\ -0.00099 & 0.02165 & 0.93600 & 0.02165 \\ 0.03789 & -0.01967 & 0.07577 & 0.88873 \end{bmatrix} \begin{Bmatrix} -0.01482 \\ 0.04104 \\ 0.01384 \\ -0.00267 \end{Bmatrix}_2 + \begin{Bmatrix} -0.00809 \\ 0.02160 \\ 0.00692 \\ -0.00130 \end{Bmatrix}$$

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix}_3 = \begin{Bmatrix} -0.02036 \\ 0.05861 \\ 0.02071 \\ -0.00399 \end{Bmatrix} \qquad \dot{\phi}_3 = \frac{\phi_3 - \phi_2}{h} = \begin{Bmatrix} -2.76958 \\ 8.78560 \\ 3.43787 \\ -0.66308 \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2

The θ algorithm for the time integration is:

$$\left(\mathbf{M} + h\theta\mathbf{K}\right)\phi_{n+1} = \left(\mathbf{M} - h(1-\theta)\mathbf{K}\right)\phi_n + h\left[\theta\mathbf{f}_{n+1} + \left(1-\theta\right)\mathbf{f}_n\right]$$

For n = 100, with $u(x, y, 0) = 0$, then:

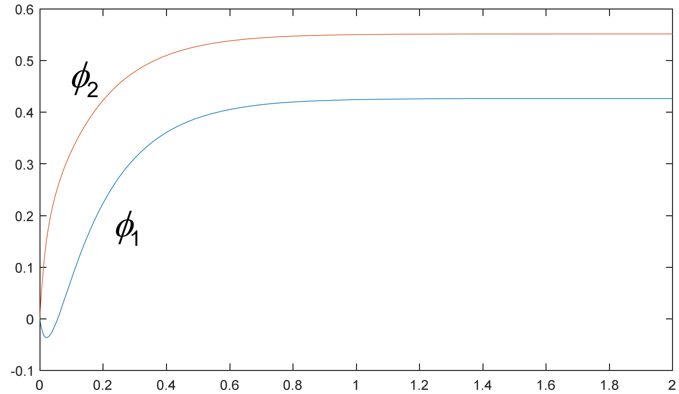
$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix}_{100} = \begin{bmatrix} 0.93600 & 0.04330 & -0.00397 & 0.04330 \\ 0.03789 & 0.88873 & 0.07577 & -0.01967 \\ -0.00099 & 0.02165 & 0.93600 & 0.02165 \\ 0.03789 & -0.01967 & 0.07577 & 0.88873 \end{bmatrix} \begin{Bmatrix} 0.21938 \\ 0.42032 \\ 0.28613 \\ 0.17034 \end{Bmatrix}_{99} + \begin{Bmatrix} -0.00809 \\ 0.02160 \\ 0.00692 \\ -0.00130 \end{Bmatrix}$$

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix}_{100} = \begin{Bmatrix} 0.22168 \\ 0.42179 \\ 0.28730 \\ 0.17181 \end{Bmatrix} \qquad \dot{\phi}_{100} = \frac{\phi_{100} - \phi_{99}}{h} = \begin{Bmatrix} 1.15193 \\ 0.73592 \\ 0.58629 \\ 0.73499 \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2

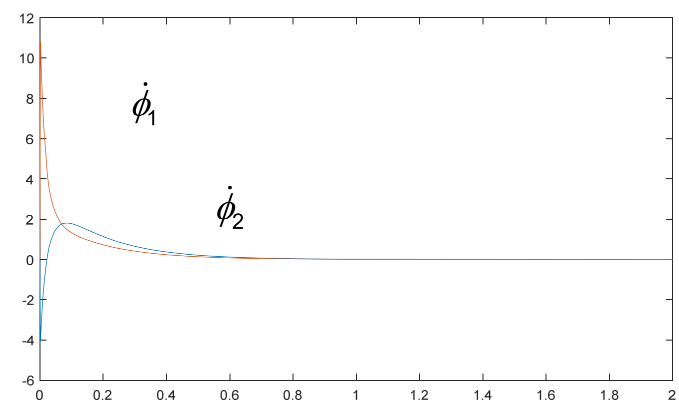
The values of ϕ_1 and ϕ_2 for 0 < *t* < 2 sec. are show below:



TIME-DEPENDENT PROBLEMS

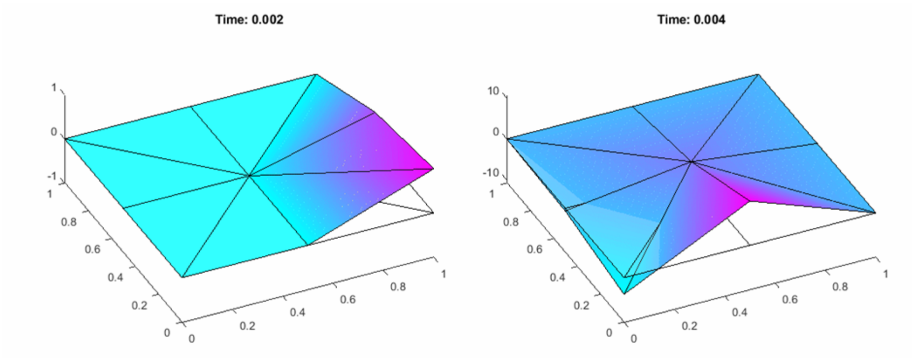
Two-Dimensional Diffusion - Example 2

The derivative values of ϕ_1 and ϕ_2 for 0 < *t* < 2 sec. are:



TIME-DEPENDENT PROBLEMS

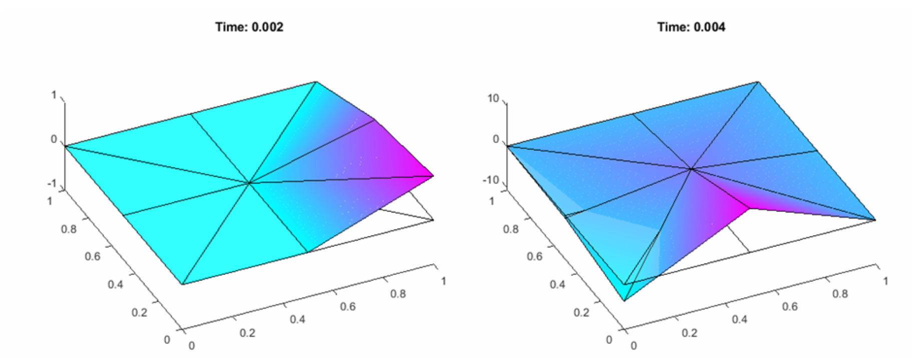
Two-Dimensional Diffusion - Example 2



$\phi(x,y,t)$ $\qquad$ $\dfrac{\partial \phi(x,y,t)}{\partial t}$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion - Example 2



$\phi(x,y,t)$ $\qquad$ $\dfrac{\partial \phi(x,y,t)}{\partial t}$

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

Time-dependent problems are inherently more difficult and expensive to solve than their corresponding steady-state counterparts.

The expense of generating the global matrices is higher for the time-dependent problems because of the necessity of computing the mass matrices.

The main extra expense, however, is in solving the resulting time-dependent global equations.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

For an analytical approach to the solution, additional expense is incurred in terms of having to determine eigenvalues and eigenvectors.

The actual amount of expense depends on the specific form of the stiffness and mass matrices and the algorithm used, but in any case it is significantly in excess of the expense of solving the single set of linear algebraic equations associated with the steady-state problem.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

For a time domain integration technique, the additional expense is clearly related to the number of time steps necessary to trace out the desired time history.

In addition to several matrix multiplications and additions, each step can involve the solution of a set of linear algebraic equations.

In some instances this expense can be minimized by using a decomposition that can be reused for the computation of the solution at each new time.

TIME-DEPENDENT PROBLEMS

Two-Dimensional Diffusion

In this regard recall that the Euler and central difference algorithms require that the size of the time step not exceed a value proportional to the inverse of the largest eigenvalue.

For large systems this critical step size can be very small resulting in many applications of the algorithm to trace out the time history.

The unconditionally stable Crank-Nicolson and Newmark algorithms, can be used with arbitrary step size that has been chosen so as to accurately integrate the lower modes, with significant improvement in the expense relative to the conditionally stable algorithms.

End of Chapter 4c