TIME-DEPENDENT PROBLEMS

The previous three chapters dealt exclusively with steady-state problems, that is, problems where time did not enter explicitly into the formulation or solution of the problem.

The types of problems considered in Chapters 2 and 3, respectively, were one- and two-dimensional elliptic boundary value problems.

In this chapter, finite element models for parabolic and hyperbolic equations, such as the one-dimensional transient heat conduction and the one-dimensional scalar wave equation, respectively, will be developed.

TIME-DEPENDENT PROBLEMS

The wave equation is an important second-order linear partial differential equation for the description of waves – as they occur in physics – such as sound waves, light waves and water waves.

It arises in fields like acoustics, electromagnetics, and fluid dynamics.

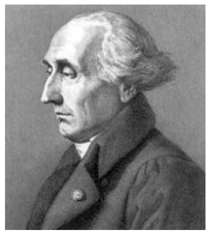
Historically, the problem of a vibrating string such as that of a musical instrument was studied by Jean le Rond d'Alembert, Leonhard Euler, Daniel Bernoulli, and Joseph-Louis Lagrange.

TIME-DEPENDENT PROBLEMS

Leonhard Euler

Jean le Rond d'Alembert

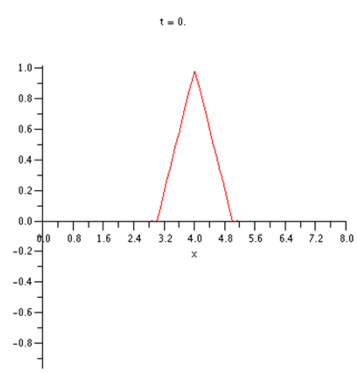
Daniel Bernoulli



Joseph-Louis Lagrange

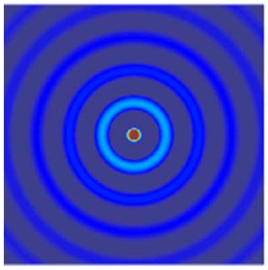
TIME-DEPENDENT PROBLEMS

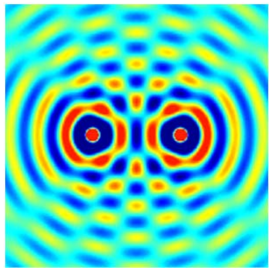
A pulse traveling through a string with fixed endpoints as modeled by the wave equation.



TIME-DEPENDENT PROBLEMS

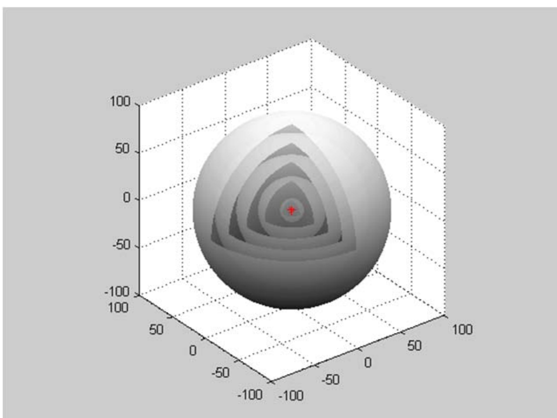
Spherical waves coming from a point source.





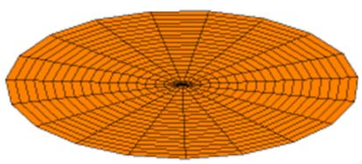
TIME-DEPENDENT PROBLEMS

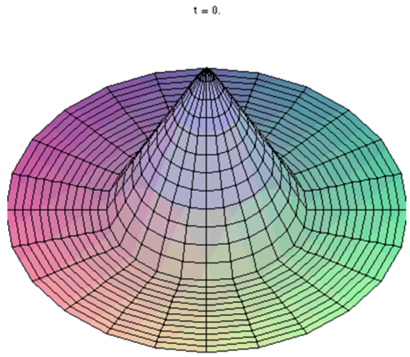
Cut-away of spherical wavefronts, with a wavelength of 10 units, propagating from a point source.



TIME-DEPENDENT PROBLEMS

A solution of the wave equation in two dimensions with a zero-displacement boundary condition along the entire outer edge.

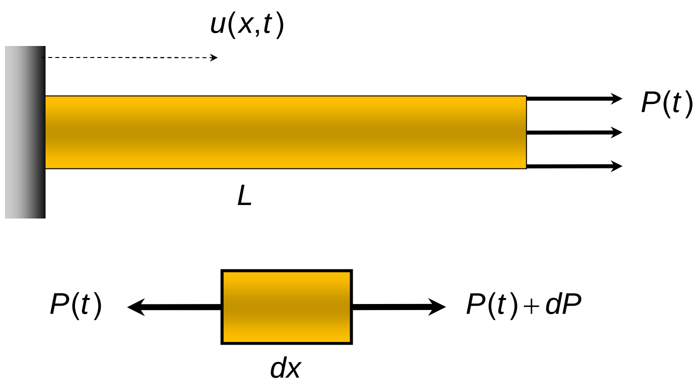




TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

An example of a physical problem whose behavior is described by the classical one-dimensional wave equation is the problem of the longitudinal or axial motion of a straight prismatic elastic bar as indicated below.



TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

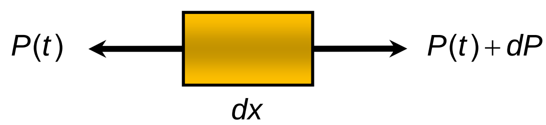
The basic physical principle governing the motion is Newton's second law which, when applied to a typical differential element as shown above, yields:

$$\sum F_x = -\cancel{P} + \left(\cancel{P} + dP\right) = \rho A dx \frac{d^2u}{dt^2}$$

with

$$P = A\sigma = AE\varepsilon = AE\frac{du}{dx}$$

$$\boxed{\frac{\partial}{\partial x}\left(AE\frac{\partial u}{\partial x}\right) = A\rho\frac{\partial^2 u}{\partial t^2}}$$



TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

The resulting equation: $\frac{\partial}{\partial x}\left(AE\frac{\partial u}{\partial x}\right) = A\rho\frac{\partial^2 u}{\partial t^2}$

where A is the area, E is Young's modulus, and ρ is the mass density.

This equation of motion is often referred to as the one-dimensional wave equation in that it is an example of the standard hyperbolic equation that predicts wave propagation in a one-dimensional setting.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

When *A* and *E* are constants, the equation is often written as:

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad \text{with} \quad c = \sqrt{E/\rho}$$

where *c* is the speed at which longitudinal waves propagate along the *x* -axis.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Appropriate boundary conditions are:

$$AE\frac{\partial u(L,t)}{\partial x} = P(t) \qquad u(0,\, t) = 0$$

stating that the displacement is zero for all time at $x = 0$ and that there is a force $P(t)$ applied at $x = L$.

Two initial conditions of the form:

$$\frac{\partial u(x,0)}{\partial t} = g(x) \qquad u(x,\, 0) = f(x)$$

are also prescribed, where $f(x)$ and $g(x)$ represent the initial axial displacement and axial velocity, respectively.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Thus a typical initial-boundary value problem associated with the wave equation can be stated as:

$$\frac{\partial}{\partial x}\left(AE\frac{\partial u}{\partial x}\right) = A\rho\frac{\partial^2 u}{\partial t^2} \qquad 0 \le x \le L, \quad t \ge 0$$

$$u(0,t) = 0 \qquad u(x,0) = f(x)$$

$$AE\frac{\partial u(L,t)}{\partial x} = P(t) \qquad \frac{\partial u(x,0)}{\partial t} = g(x)$$

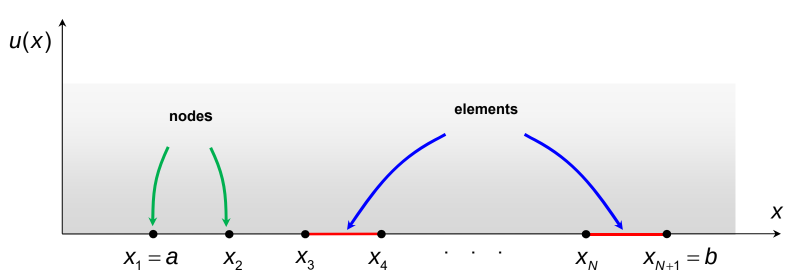
Many other physical situations such as the transverse motions of strings and membranes, propagation of sound, and dynamic disturbances in fluids and solids are governed by the wave equation.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

The Galerkin Finite Element Method

As has been indicated numerous times in the preceding material, the first steps in developing a finite element model are discretization and interpolation. These are carried out exactly as before.

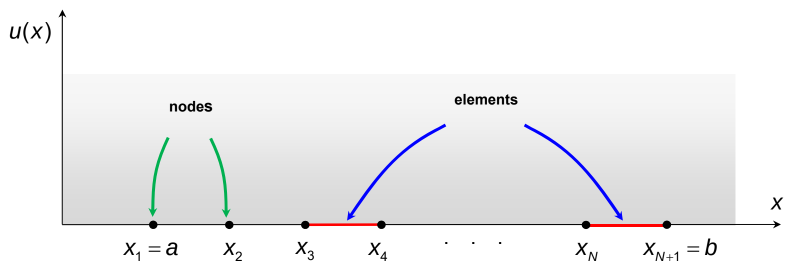


TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

The Galerkin Finite Element Method

Discretization. The first step in developing a finite element model is discretization. Nodes for the spatial domain $a \le x \le b$ are chosen as indicated below, with $a = x_1$ and $b = x_{N+1}$.



TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

The Galerkin Finite Element Method

Interpolation. Interpolation would again be semidiscrete, of the form:

$$u(x,t) = \sum_{1}^{N+1} u_i(t)n_i(x)$$

where the $n_i(x)$ are nodally based interpolation functions and can be linear, quadratic, or higher-order if desired.

The elements indicated above are specifically for linear interpolation.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

The Galerkin Finite Element Method

Elemental formulation. Consider again the initial-boundary value problem developed in the previous section:

$$\frac{\partial}{\partial x}\left(AE\frac{\partial u}{\partial x}\right) = A\rho\frac{\partial^2 u}{\partial t^2} \qquad 0 \le x \le L, \quad t \ge 0$$

$$u(0, t) = 0 \qquad u(x, 0) = f(x)$$

$$AE\frac{\partial u(L,t)}{\partial x} = P(t) \qquad \frac{\partial u(x,0)}{\partial t} = g(x)$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

The Galerkin Finite Element Method

Elemental formulation. The elemental formulation for the wave equation is based on a corresponding weak statement.

The weak form is developed by multiplying the differential equation by a test function *v*(*x*) satisfying any essential boundary conditions, with the result then integrated over the spatial region according to:

$$\int_a^b v\left(\frac{\partial}{\partial x}\left(AE\frac{\partial u}{\partial x}\right) - \rho A\frac{\partial^2 u}{\partial t^2}\right)dx = 0$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

The Galerkin Finite Element Method

Elemental formulation. The elemental formulation for the wave equation is based on a corresponding weak statement.

Integrating by parts and eliminating the derivative terms from the boundary conditions yields:

$$\int_a^b \left(v' \left(AE \frac{\partial u}{\partial x} \right) + \rho A v \frac{\partial^2 u}{\partial t^2} \right) dx - v(L) P(t) = 0$$

which is the required weak statement for the initial-boundary value problem associated with the one-dimensional wave equation.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

The Galerkin Finite Element Method

Elemental formulation. The finite element model is obtained by substituting the approximate solution and $v = n_k$, k = 1, 2, ..., N + 1, successively, into the above expression to obtain:

$$\sum_{1}^{N+1} \int_a^b \left(n_k' A E n_i' + n_k \rho A n_i \ddot{u} \right) dx + \delta_{kN+1} P(t)$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

The Galerkin Finite Element Method

Elemental formulation. Which can be written as:

$$\sum_{1}^{N+1}\left[A_{ki}u_i(t)+B_{ki}\ddot{u}_i(t)\right]=F_k(t) \qquad k=1,2,\ldots,N+1$$

$$A_{ki}=\int_a^b\left(n_k{}'AEn_i{}'\right)dx$$

$$B_{ki}=\int_a^b\left(n_k\rho An_i\right)dx$$

$$F_k=\delta_{kN+1}P(t)$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

The Galerkin Finite Element Method

Elemental formulation. In matrix notation, the above expression can be written as:

$$\mathbf{Au}+\mathbf{B\ddot{u}}=\mathbf{F}$$

$$\mathbf{A}=\sum_e\mathbf{k_G} \qquad \mathbf{B}=\sum_e\mathbf{m_G}$$

$$\mathbf{F}=\langle 0 \quad 0 \quad 0 \quad .... \quad 0 \quad P\rangle$$

$$\mathbf{k_e}=\int_{x_i}^{x_j}\left(\mathbf{N}'AE\mathbf{N}'^{\mathrm{T}}\right)dx \qquad \mathbf{m_e}=\int_{x_i}^{x_j}\left(\mathbf{N}\rho A\mathbf{N}^{\mathrm{T}}\right)dx$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

The Galerkin Finite Element Method

Elemental formulation. The original initial-boundary value problem has been converted into the initial value problem:

$$\mathbf{Au} + \mathbf{B\ddot{u}} = \mathbf{F} \qquad \text{with} \qquad \mathbf{u}(0) = \mathbf{u}_0 \quad \mathbf{\dot{u}}(0) = \mathbf{\dot{u}}_0$$

The vector $\mathbf{u}_0$ and $\mathring{\mathbf{u}}_0$, representing the discretized version of the initial conditions *f* and *g,* are usually taken to be respectively the vectors consisting of the values of *f*(*x*) and *g*(*x*) at the nodes, that is:

$$\mathbf{u}(0) = \mathbf{u}_0 = \left\langle f(0) \quad f(x_2) \quad f(x_3) \quad \ldots \quad f(x_N) \quad f(L) \right\rangle^T$$

$$\mathbf{\dot{u}}(0) = \mathbf{\dot{u}}_0 = \left\langle g(0) \quad g(x_2) \quad g(x_3) \quad \ldots \quad g(x_N) \quad g(L) \right\rangle^T$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

The Galerkin Finite Element Method

Elemental formulation. Note that the assembly process has taken place implicitly, while carrying out the details of obtaining the governing equations, using the Galerkin method in connection with the weak formulation.

Enforcement of constraints is necessary if either of the boundary conditions is essential, that is, if the dependent variable is prescribed at either boundary point.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

The Galerkin Finite Element Method

Elemental formulation. The system of equations must be altered to reflect these constraints.

Consider for example, the case where the boundary condition at $x = 0$ is $u(0, t) = \mathbf{u_0}(t)$. The first scalar equation of the set of equations would be replaced by the constraint so that there would result:

$$\begin{aligned} u_1 &= u_0(t) \\ a_{21}u_1 + a_{22}u_2 + a_{23}u_3 + \cdots + b_{21}\ddot{u}_1 + b_{22}\ddot{u}_2 + b_{23}\ddot{u}_3 + \cdots &= 0 \\ a_{31}u_1 + a_{32}u_2 + a_{33}u_3 + \cdots + b_{31}\ddot{u}_1 + b_{32}\ddot{u}_2 + b_{33}\ddot{u}_3 + \cdots &= 0 \\ &\vdots \end{aligned}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

The Galerkin Finite Element Method

Elemental formulation. The u_1 and terms in the remaining equations are transferred to the right-hand side to yield:

$$\begin{aligned} u_1 &= u_0(t) \\ a_{22}u_2 + a_{23}u_3 + \cdots + b_{22}\ddot{u}_2 + b_{23}\ddot{u}_3 + \cdots &= -a_{21}u_0 - b_{21}\ddot{u}_0 \\ a_{32}u_2 + a_{33}u_3 + \cdots + b_{32}\ddot{u}_2 + b_{33}\ddot{u}_3 + \cdots &= -a_{31}u_0 - b_{31}\ddot{u}_0 \\ &\vdots \end{aligned}$$

For a linearly interpolated model the half bandwidth is two, and only the u_1 and $\ddot{u}_1$ in terms in the second equation need be transferred to the right-hand side.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

The Galerkin Finite Element Method

Elemental formulation. The u_1 and terms in the remaining equations are transferred to the right-hand side to yield:

$$\begin{aligned}
u_1 &= u_0(t) \\
a_{22}u_2 + a_{23}u_3 + \cdots + b_{22}\ddot{u}_2 + b_{23}\ddot{u}_3 + \cdots &= -a_{21}u_0 - b_{21}\ddot{u}_0 \\
a_{32}u_2 + a_{33}u_3 + \cdots + b_{32}\ddot{u}_2 + b_{33}\ddot{u}_3 + \cdots &= -a_{31}u_0 - b_{31}\ddot{u}_0 \\
&\vdots
\end{aligned}$$

For a quadratically interpolated model the half bandwidth is three, and terms from the second and third equations need to be transferred. If the constraint is at the right end, the N^{th}, $(N - 1)^{st}$, . . . equations would be similarly altered.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

The Galerkin Finite Element Method

Elemental formulation. The constrained set of equations may be written as:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F} \qquad \mathbf{u}(0) = \mathbf{u}_0 \qquad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0$$

Note that if there were distributed inputs resulting in a more general nodal distribution of forces:

$$\mathbf{F} = \langle F_1(t) \quad F_2(t) \quad F_3(t) \quad \cdots \quad F_{N+1}(t) \rangle^T$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

The Galerkin Finite Element Method

Elemental formulation. The final set of equations would appear as:

$$\begin{aligned}
u_1 &= u_0(t) \\
a_{22}u_2 + a_{23}u_3 + \cdots + b_{22}\ddot{u}_2 + b_{23}\ddot{u}_3 + \cdots &= F_2(t) - a_{21}u_0 - b_{21}\ddot{u}_0 \\
a_{32}u_2 + a_{33}u_3 + \cdots + b_{32}\ddot{u}_2 + b_{33}\ddot{u}_3 + \cdots &= F_3(t) - a_{31}u_0 - b_{31}\ddot{u}_0 \\
&\vdots
\end{aligned}$$

In any case, algorithms for integrating these equations (the solution step) are studied in the following sections. The derived variables, which are now functions of time, are computed per element in exactly the same fashion as outlined for the one-dimensional problems in Chapter 2.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

Consider again the problem outlined below:

$$\frac{\partial}{\partial x}\left(AE\frac{\partial u}{\partial x}\right) = A\rho\frac{\partial^2 u}{\partial t^2} \qquad 0 \le x \le L, \quad t \ge 0$$

$$u(0,\, t) = 0 \qquad u(x,\, 0) = f(x)$$

$$AE\frac{\partial u(L,t)}{\partial x} = P(t) \qquad \frac{\partial u(x,0)}{\partial t} = g(x)$$

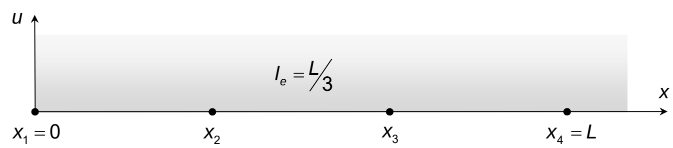
corresponding to a uniform bar initially at rest and undeformed, acted on suddenly by a constant force P_0 at the unsupported end.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

Discretization. A mesh for three equal-length, linearly interpolated elements is indicated below:



Interpolation. Linear interpolation will be used for the three elements.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

Elemental Formulation. The elemental matrices are:

$$\mathbf{k_e} = \int_{x_i}^{x_j} \left(\mathbf{N}' AE \mathbf{N}'^{\mathrm{T}}\right) dx \quad = \int_0^1 \left(\mathbf{N}' AE \mathbf{N}'^{\mathrm{T}}\right) l_e d\xi$$

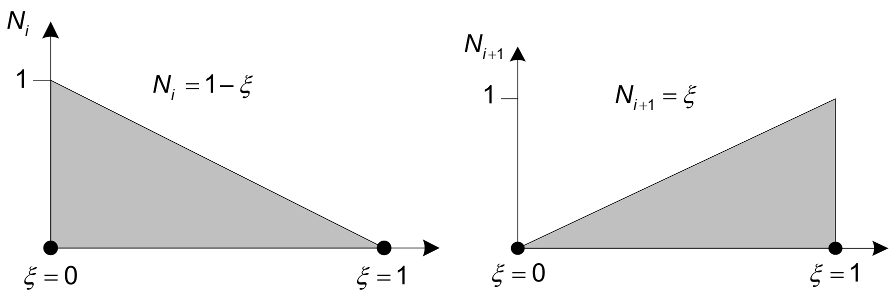
$$\mathbf{m_e} = \int_{x_i}^{x_j} \left(\mathbf{N} \rho A \mathbf{N}^{\mathrm{T}}\right) dx \quad = \int_0^1 \left(\mathbf{N} \rho A \mathbf{N}^{\mathrm{T}}\right) l_e d\xi$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

Elemental Formulation. The linear interpolation functions written in local space ξ are:

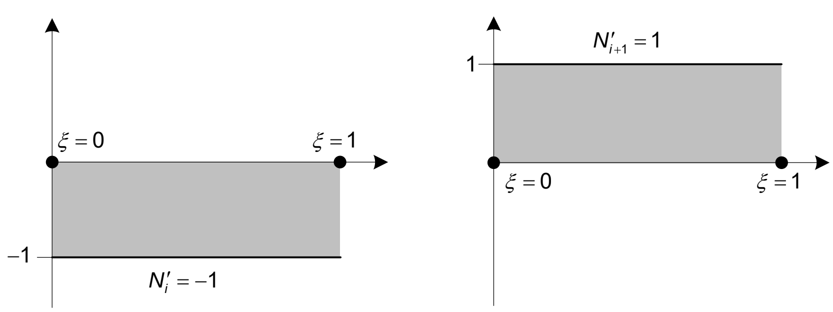


TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

Elemental Formulation. The derivative shape functions are:



TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

Elemental Formulation. The $\mathbf{k_e}$ elemental matrix is:

$$\mathbf{k_e} = \int_{x_i}^{x_j} \left(\mathbf{N}' AE \mathbf{N}'^{\mathrm{T}}\right) dx = \int_0^1 \left(\mathbf{N}' AE \mathbf{N}'^{\mathrm{T}}\right) l_e d\xi$$

$$= \frac{1}{(l_e)^2} \int_0^1 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} AE \langle -1 \quad 1 \rangle l_e d\xi$$

$$= \frac{AE}{l_e} \int_0^1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\xi = \frac{AE}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{3AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

Elemental Formulation. The $\mathbf{m_e}$ elemental matrix is:

$$\mathbf{m_e} = \int_{x_i}^{x_{i+1}} \mathbf{N} \rho A \mathbf{N}^{\mathrm{T}} \, dx$$

$$\mathbf{m_{e1}} = \int_{x_i}^{x_{i+1}} \mathbf{N} \rho A \mathbf{N}^{\mathrm{T}} \, dx = \int_0^1 \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} \rho A \langle 1-\xi \quad \xi \rangle l_e \, d\xi$$

$$= \rho A \int_0^1 \begin{Bmatrix} (1-\xi)^2 & \xi(1-\xi) \\ \xi(1-\xi) & \xi^2 \end{Bmatrix} l_e \, d\xi$$

$$= \frac{\rho AL}{18} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

Elemental Formulation. The elemental matrices are:

$$\mathbf{k_e} = \int_0^1 \left(\mathbf{N}' AE \mathbf{N}'^{\mathrm{T}}\right) l_e \, dx = \frac{3AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{m_e} = \int_0^1 \left(\mathbf{N} \rho A \mathbf{N}^{\mathrm{T}}\right) l_e dx = \frac{\rho AL}{18}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

Assembly. It follows that the assembled equations are:

$$\frac{\rho AL}{18}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\ddot{\mathbf{u}} + \frac{3AE}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\mathbf{u} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_0 \end{Bmatrix}$$

Dividing by $3AE/L$ gives:

$$\frac{\rho L^2}{54E}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\ddot{\mathbf{u}} + \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\mathbf{u} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{P_0 L}{3AE} \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

Assembly. The unconstrained equations are:

$$\phi\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\ddot{\mathbf{u}} + \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\mathbf{u} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \Lambda/3 \end{Bmatrix} \qquad u_1 = 0$$

$$\phi = \frac{\rho L^2}{54E} \qquad \Lambda = \frac{P_0 L}{AE}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

Constraints. The constraints follow from the boundary conditions as:

$$u_1(t) = 0 \quad \text{and} \quad u(0, t) = 0$$

The constrained equations become:

$$\phi\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\ddot{\mathbf{u}} + \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\mathbf{u} += \begin{Bmatrix} 0 \\ 0 \\ \Lambda/3 \end{Bmatrix}$$

Subject to the initial condition:

$$\mathbf{u}(0) = u_0 = 0$$

$$\dot{\mathbf{u}}(0) = \dot{u}_0 = 0$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

The comments made regarding the different approaches available for handling the mass matrices in connection with one-dimensional diffusion equations are equally applicable for the wave equation. The forms of the mass matrices are identical, so that:

$$\mathbf{m_{ce}} = \frac{\rho A l_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{m_{le}} = \frac{\rho A l_e}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{m_{w}} = \frac{\rho A l_e}{12}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Analytical Integration Techniques

Generally, for a one-dimensional wave equation the constrained system of ordinary differential equations resulting from the application of the finite element method is of the form:

$$\mathbf{Ku} + \mathbf{M\ddot{u}} = \mathbf{F}(t)$$

that is, a coupled system of linear second-order ordinary differential equations.

This system of differential equations will be treated analytically by decomposing the general solution u into a homogeneous solution $\mathbf{u}_h$ and a particular solution $\mathbf{u_p}$ according to: $\mathbf{u} = \mathbf{u_h} + \mathbf{u_p}$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Analytical Integration Techniques

The homogeneous equations are satisfied by $\mathbf{u_h}$

$$\mathbf{K}\mathbf{u_h} + \mathbf{M}\ddot{\mathbf{u}}_\mathbf{h} = 0$$

and $\mathbf{u_p}$ is any particular solution satisfying:

$$\mathbf{K}\mathbf{u_p} + \mathbf{M}\ddot{\mathbf{u}}_\mathbf{p} = \mathbf{F}(t)$$

This procedure is essentially the well-known **superposition principle**, valid for linear systems.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Analytical Integration Techniques

Homogenous Solution. For a system of second-order ordinary differential equations representing an undamped physical model, the homogeneous solution is taken to be of the form:

$$\mathbf{u_h}(t) = \mathbf{v}e^{-i\omega t}$$

a solution that is harmonic or periodic in time.

Substituting into the governing equation yields:

$$\left(\mathbf{K} - \omega^2\mathbf{M}\right)\mathbf{v} = 0$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Analytical Integration Techniques

This equation is the generalized linear algebraic eigenvalue problem discussed several times in previous sections.

When **K** and **M** are symmetric and positive definite, as is the case for the one-dimensional problems currently being considered, all the eigenvalues ω^2_j are positive and real with the eigenvectors $\mathbf{v_j}$ also real and M-orthogonal.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Analytical Integration Techniques

The corresponding homogeneous solution is written as:

$$\mathbf{u_h}(t) = \sum c_j \mathbf{v_j} e^{-i\omega_j t}$$

where the c_j are complex constants. Expressed in real form:

$$\mathbf{u_h}(t) = \sum \mathbf{v_j}\left[a_j \cos(\omega_j t) + b_j \sin(\omega_j t)\right]$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Analytical Integration Techniques

Particular solution. The particular solution is any solution of:

$$\mathbf{K}\mathbf{u}_\mathbf{p} + \mathbf{M}\ddot{\mathbf{u}}_\mathbf{p} = \mathbf{F}(t)$$

and, depending on the specific form of **F**, can be determined by using:

1. Undetermined coefficients (*intelligent guessing*)
2. Variation of parameters
3. Laplace transform methods

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Analytical Integration Techniques

After determining the particular solution using one of these approaches, the general solution of can be written as:

$$\mathbf{u}(t) = \sum \mathbf{v}_\mathbf{j}\left[a_j \cos(\omega_j t) + b_j \sin(\omega_j t)\right] + \mathbf{u}_\mathbf{p}(t)$$

The initial conditions are used to determine the *2N* constants a_j and b_j, j = 1,2, ... , *N,* according to:

$$\mathbf{u}(0) = \mathbf{u}_\mathbf{0} = \sum \mathbf{v}_\mathbf{j} a_j + \mathbf{u}_\mathbf{p}(0) \qquad \mathbf{V}\mathbf{a} = \mathbf{u}_\mathbf{0} - \mathbf{u}_\mathbf{p}(0)$$

where **V** is the *N* x *N* matrix consisting of the eigenvectors as columns and **a** is a vector of containing the a_j values.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Analytical Integration Techniques

After determining the particular solution using one of these approaches, the general solution of can be written as:

$$\mathbf{u}(t) = \sum \mathbf{v_j}\left[a_j \cos(\omega_j t) + b_j \sin(\omega_j t)\right] + \mathbf{u_p}(t)$$

Similarly

$$\dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_\mathbf{0} = \sum \mathbf{v}_\mathrm{j}\omega_j b_j + \dot{\mathbf{u}}_\mathbf{p}(0) \qquad \mathbf{V\omega b} = \dot{\mathbf{u}}_\mathbf{0} - \dot{\mathbf{u}}_\mathbf{p}(0)$$

$$\mathbf{\omega b} = \langle \omega_1 b_1 \quad \omega_2 b_2 \quad \omega_3 b_3 \quad \cdots \quad \omega_N b_N \rangle^T$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

A unique solution to each of the sets of equations is guaranteed on the basis of the linearly independent character of the $\mathbf{v}_\mathrm{j}$ for the case where **M** and **K** are symmetric and positive definite.

For the particular example developed here:

$$\phi\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\ddot{\mathbf{u}} + \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\mathbf{u} + = \begin{Bmatrix} 0 \\ 0 \\ \Lambda/3 \end{Bmatrix} \qquad \phi = \frac{\rho L^2}{54E} \qquad \Lambda = \frac{P_0 L}{AE}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

The eigenvalues and eigenvectors are determined from:

$$\begin{bmatrix} 2-4\lambda & -(1-\lambda) & 0 \\ -(1-\lambda) & 2-4\lambda & -(1-\lambda) \\ 0 & -(1-\lambda) & 2-4\lambda \end{bmatrix} \mathbf{v} = 0 \qquad \lambda = \frac{\rho\omega^2 L^2}{54E}$$

$$\begin{vmatrix} 2-4\lambda & -(1-\lambda) & 0 \\ -(1-\lambda) & 2-4\lambda & -(1-\lambda) \\ 0 & -(1-\lambda) & 2-4\lambda \end{vmatrix} = 0$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

The roots of the corresponding characteristic equation:

$$(1-2\lambda)\left[4(1-2\lambda)^2 - 3(1+\lambda)^2\right] = 0$$

are:

$$\lambda_1 = 0.0467 \qquad \lambda_2 = 0.5000 \qquad \lambda_3 = 1.6456$$

$$\lambda = \frac{\rho\omega^2 L^2}{54E} \qquad \omega = \sqrt{\frac{\lambda 54E}{\rho L^2}} = c\sqrt{\frac{\lambda 54}{L^2}}$$

$$c = \sqrt{E/\rho}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

From which:

$$\omega_1 = \left(\frac{1.5887}{L}\right)c \quad \omega_2 = \left(\frac{5.1962}{L}\right)c \quad \omega_3 = \left(\frac{9.4266}{L}\right)c$$

where c is the speed of waves propagating along the bar

$$c = \sqrt{E/\rho}$$

The corresponding exact values are:

$$\beta_1 = \left(\frac{1.5708}{L}\right)c \quad \beta_2 = \left(\frac{4.7124}{L}\right)c \quad \beta_3 = \left(\frac{7.8540}{L}\right)c$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

The corresponding eigenvectors are:

$$\mathbf{v_1} = \langle 0.5000 \quad 0.8660 \quad 1.0000 \rangle^T$$

$$\mathbf{v_2} = \langle 1.0000 \quad 0.0000 \quad -1.0000 \rangle^T$$

$$\mathbf{v}_3 = \langle 0.5000 \quad -0.8660 \quad 1.0000 \rangle^T$$

The homogeneous solution is:

$$\mathbf{u_h}(t) = \mathbf{v}_1\left[a_1\cos(\omega_1 t) + b_1\sin(\omega_1 t)\right] + \mathbf{v}_2\left[a_2\cos(\omega_2 t) + b_2\sin(\omega_2 t)\right] + \mathbf{v}_3\left[a_3\cos(\omega_3 t) + b_3\sin(\omega_3 t)\right]$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

Note that there are six arbitrary constants to be determined from the six scalar equations represented by:

$$\mathbf{u}(0) = \mathbf{u}_0 \qquad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

Observe that in:

$$\mathbf{K}\mathbf{u}_{\mathbf{p}} + \mathbf{M}\ddot{\mathbf{u}}_{\mathbf{p}} = \left\langle 0 \quad 0 \quad \Lambda/3 \right\rangle^T$$

a particular solution is easily obtained by taking $\mathbf{u}_{\mathbf{p}} = \mathbf{d}$, a constant, resulting in

$$\mathbf{K}\mathbf{d} = \left\langle 0 \quad 0 \quad \Lambda/3 \right\rangle^T$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & \Lambda/3 \end{array}\right] \Rightarrow \quad \mathbf{d} = \Lambda/3 \left\langle 1 \quad 2 \quad 3 \right\rangle^T$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

Applying the initial condition **u**(0) = 0 yields:

$$\mathbf{u}(0) = 0 = \mathbf{v}_1 a_1 + \mathbf{v}_2 a_2 + \mathbf{v}_3 a_3 + \mathbf{d}$$

$$\underbrace{\begin{bmatrix} 0.5000 \\ 0.8660 \\ 1.0000 \end{bmatrix}}_{V_1}\underbrace{\begin{matrix} 1.0000 \\ 0.0000 \\ -1.0000 \end{matrix}}_{V_2}\underbrace{\begin{matrix} 0.5000 \\ -0.8660 \\ 1.0000 \end{matrix}}_{V_3} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} -\Lambda/3 \\ -2\Lambda/3 \\ -3\Lambda/3 \end{Bmatrix}$$

$$\mathbf{a} = \langle a_1 \quad a_2 \quad a_3 \rangle^T = \langle -0.8294\Lambda \quad 0.1111\Lambda \quad -0.0595\Lambda \rangle^T$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

With $\mathring{\mathbf{u}}(0) = \mathring{\mathbf{u}}_0 = \mathbf{u}_\mathbf{p}(0) = 0$, it follows that:

$$\dot{\mathbf{u}}(0) = 0 = \mathbf{v}_1 b_1 + \mathbf{v}_2 b_2 + \mathbf{v}_3 b_3$$

$$\underbrace{\begin{bmatrix} 0.5000 \\ 0.8660 \\ 1.0000 \end{bmatrix}}_{V_1}\underbrace{\begin{matrix} 1.0000 \\ 0.0000 \\ -1.0000 \end{matrix}}_{V_2}\underbrace{\begin{matrix} 0.5000 \\ -0.8660 \\ 1.0000 \end{matrix}}_{V_3} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{b} = \langle b_1 \quad b_2 \quad b_3 \rangle^T = 0$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

The solution can then be written as:

$$\begin{aligned}\mathbf{u}(t) = \mathbf{v}_1 &\left[a_1 \cos(\omega_1 t) + b_1 \sin(\omega_1 t)\right] \\ &+ \mathbf{v}_2 \left[a_2 \cos(\omega_2 t) + b_2 \sin(\omega_2 t)\right] \\ &+ \mathbf{v}_3 \left[a_3 \cos(\omega_3 t) + b_3 \sin(\omega_3 t)\right] + \mathbf{u_p}\end{aligned}$$

$$\begin{aligned}\frac{\mathbf{u}}{\Lambda} = -0.8294\mathbf{v}_1 \cos(\omega_1 t) + 0.1111\mathbf{v}_2 \cos(\omega_2 t) \\ -0.0595\mathbf{v}_3 \cos(\omega_3 t) + \tfrac{1}{3}\langle 1 \quad 2 \quad 3\rangle^T\end{aligned}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

Recall the corresponding eigenvectors are:

$$\mathbf{v_1} = \langle 0.5000 \quad 0.8660 \quad 1.0000\rangle^T$$

$$\mathbf{v_2} = \langle 1.0000 \quad 0.0000 \quad -1.0000\rangle^T$$

$$\mathbf{v}_3 = \langle 0.5000 \quad -0.8660 \quad 1.0000\rangle^T$$

$$\begin{aligned}\frac{\mathbf{u}}{\Lambda} = -0.8294\mathbf{v}_1 \cos(\omega_1 t) + 0.1111\mathbf{v}_2 \cos(\omega_2 t) \\ -0.0595\mathbf{v}_3 \cos(\omega_3 t) + \tfrac{1}{3}\langle 1 \quad 2 \quad 3\rangle^T\end{aligned}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

Substituting the eigenvectors gives:

$$\frac{\mathbf{u}}{\Lambda} = -0.8294\underbrace{\begin{Bmatrix} 0.5000 \\ 0.8660 \\ 1.0000 \end{Bmatrix}}_{V_1}\cos(\omega_1 t) + 0.1111\underbrace{\begin{Bmatrix} 1.0000 \\ 0.0000 \\ -1.0000 \end{Bmatrix}}_{V_2}\cos(\omega_2 t)$$

$$-0.0595\underbrace{\begin{Bmatrix} 0.5000 \\ -0.8660 \\ 1.0000 \end{Bmatrix}}_{V_3}\cos(\omega_3 t) + \frac{1}{3}\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

$$\omega_1 = \left(\frac{1.5887}{L}\right)c \quad \omega_2 = \left(\frac{5.1962}{L}\right)c \quad \omega_3 = \left(\frac{9.4266}{L}\right)c$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

In an expanded form:

$$\frac{u_2}{\Lambda} = 0.3333 - 0.4147\cos(\omega_1 t) + 0.1111\cos(\omega_2 t) - 0.0298\cos(\omega_3 t)$$

$$\frac{u_3}{\Lambda} = 0.6667 - 0.7183\cos(\omega_1 t) + 0.0516\cos(\omega_3 t)$$

$$\frac{u_4}{\Lambda} = 1.0000 - 0.8294\cos(\omega_1 t) - 0.1111\cos(\omega_2 t) - 0.0595\cos(\omega_3 t)$$

The constant term represents, in a sense, the steady-state or static solution

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

In an expanded form:

$$\frac{u_2}{\Lambda} = 0.3333 - 0.4147\cos(\omega_1 t) + 0.1111\cos(\omega_2 t) - 0.0298\cos(\omega_3 t)$$

$$\frac{u_3}{\Lambda} = 0.6667 - 0.7183\cos(\omega_1 t) + 0.0516\cos(\omega_3 t)$$

$$\frac{u_4}{\Lambda} = 1.0000 - 0.8294\cos(\omega_1 t) - 0.1111\cos(\omega_2 t) - 0.0595\cos(\omega_3 t)$$

If damping were included in the physical model, the terms in the homogeneous solution corresponding to the present cosine terms would eventually damp out.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

The corresponding exact solution can be represented in terms of the infinite series:

$$\frac{u(x,t)}{\Lambda} = 2\sum \frac{\phi_n(\alpha_n L)\phi_n(\alpha_n x)\left[1-\cos(\alpha_n ct)\right]}{(\alpha_n L)^2}$$

$$\alpha_n L = \frac{(2n+1)\pi}{2} \qquad \phi_n(x) = \sin(\alpha_n x)$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

Retaining the first three terms of the series solution at *x = L/3, 2L/3,* and *L* yields:

$$\frac{u(L/3,\, t)}{\Lambda} = 0.3314 - 0.4053\cos(\beta_1 t) + 0.0901\cos(\beta_2 t) - 0.0162\cos(\beta_3 t)$$

$$\frac{u(2L/3,\, t)}{\Lambda} = 0.7639 - 0.7020\cos(\beta_1 t) + 0.0901\cos(\beta_2 t) + 0.0281\cos(\beta_3 t)$$

$$\frac{u(L,\, t)}{\Lambda} = 0.9330 - 0.8106\cos(\beta_1 t) - 0.0901\cos(\beta_2 t) - 0.0324\cos(\beta_3 t)$$

where $\beta_n = \alpha_n c$. Note the general similarity between the three-term expansion of the exact solution and the approximate solution from the three-element finite element model.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

The approximate lowest frequency ω_1 is quite close to the exact lowest frequency β_1, with:

$$\frac{\omega_1}{\beta_1} = 1.0114$$

The other two ratios:

$$\frac{\omega_2}{\beta_2} = 1.1027 \qquad \frac{\omega_3}{\beta_3} = 1.2002$$

are not quite as accurate.

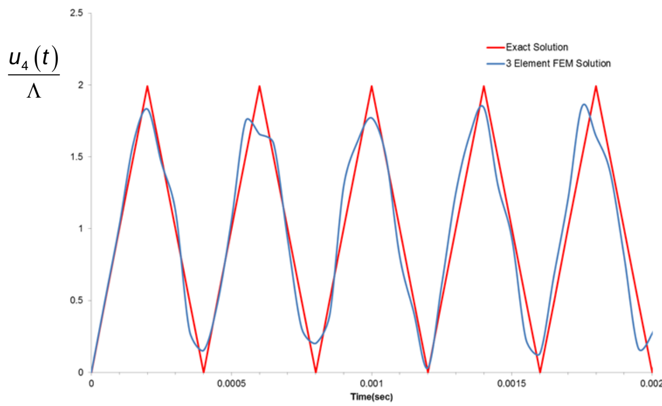
Recall, the general rule stating that approximately *2N* unconstrained degrees of freedom are necessary in order that the first *N* frequencies be determined accurately. In this instance, the first frequency should be quite accurate, which is certainly the case.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

The exact solutions $u(L, t)$ and $u_4(t)$ are indicated for the first few oscillations below:

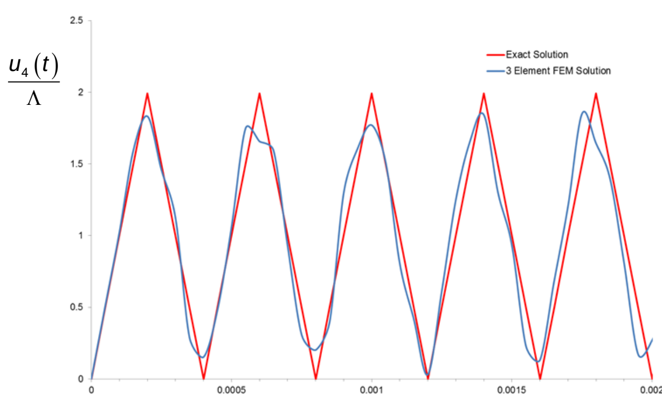


TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

The results are for E = 3 X 10^7 psi, ρ = 7.5 X 10^{-4} lbf-s^2/in^4, and L = 20 in.

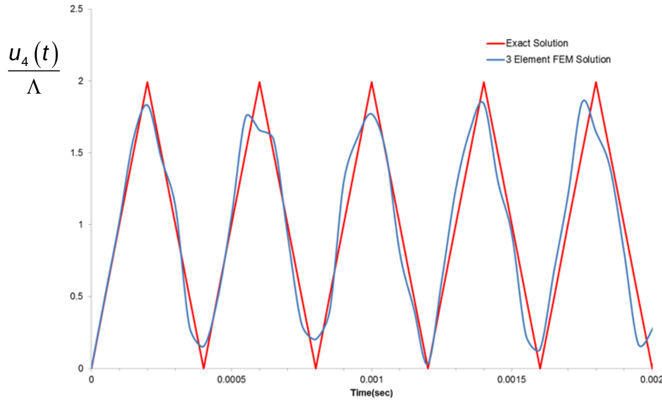


TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

The agreement is quite reasonable with the approximate solution beginning to peak early due to the fact that all the approximate frequencies exceed the exact values.

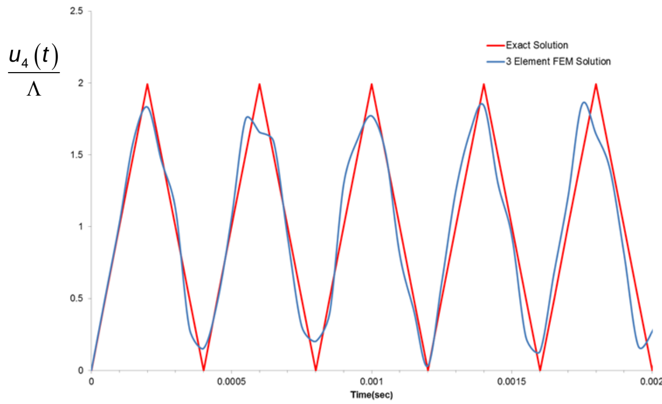


TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example

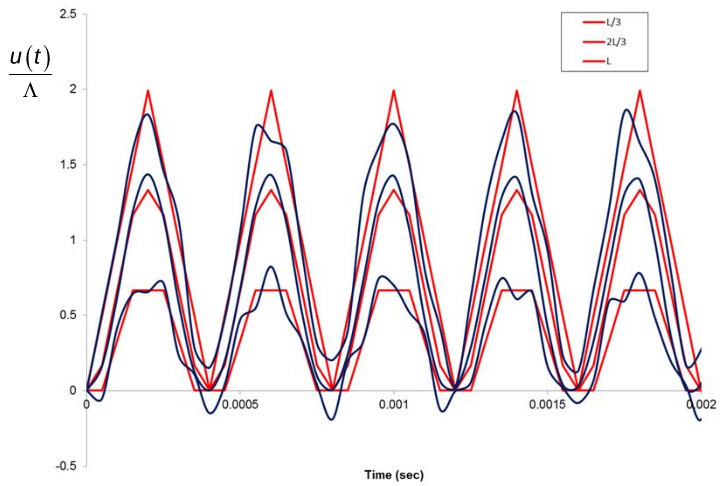
A finer mesh would result in better agreement.



TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

One-Dimensional Wave Example



TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

As was the case for the systems of first-order equations, there may be situations where **M** and **K** are time-dependent or where **F**(t) is such that an analytical approach is not an intelligent way to proceed.

Numerical integration techniques, which are appropriate in such situations, are presented and discussed in the next sections.

TIME-DEPENDENT PROBLEMS

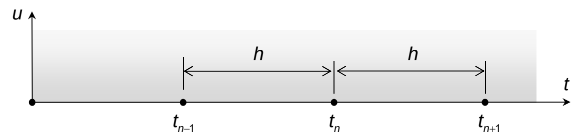
One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

The Central Difference Method - The system of second-order linear ordinary differential equations in question is restated as:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F} \qquad \mathbf{u}(0) = \mathbf{u}_0 \qquad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0$$

A discretization of the time variable with $t_n - t_{n-1} = t_{n+1} - t_n = h$, the time step.



TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

The differential equation is evaluated at $t = t_n$ to yield:

$$\mathbf{M}\ddot{\mathbf{u}}_n + \mathbf{K}\mathbf{u}_n = \mathbf{F}_n$$

where $\mathbf{u}_n = \mathbf{u}(t_n) = \mathbf{u}(nh)$, and $\mathbf{F}_n = \mathbf{F}(t_n) = \mathbf{F}(nh)$.

Central difference representations are used for the velocity and acceleration vectors, namely,

$$\dot{\mathbf{u}}_n = \frac{\mathbf{u}_{n+1} - \mathbf{u}_{n-1}}{2h} \qquad \ddot{\mathbf{u}}_n = \frac{\mathbf{u}_{n+1} - 2\mathbf{u}_n + \mathbf{u}_{n-1}}{h^2}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Each term is accurate to order h^2. Substituting the acceleration approximation into the original equation and multiplying through by h^2 and gives:

$$\mathbf{M}\mathbf{u}_{n+1} = \left(2\mathbf{M} - h^2\mathbf{K}\right)\mathbf{u}_n - \mathbf{M}\mathbf{u}_{n-1} + h^2\mathbf{F}_n$$

This gives a three-term recurrence relation to be used for stepping ahead in time.

A special starting procedure is necessary in that two successive **u**'s are required in order to accomplish the solution.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

The procedure used is as follows: the vector function **u** is expanded in a Taylor's series about t = 0 according to

$$\mathbf{u}(-h) = \mathbf{u}(0) - h\dot{\mathbf{u}}(0) + \frac{h^2\ddot{\mathbf{u}}(0)}{2} + \cdots$$

with $\ddot{\mathbf{u}}(0)$ computed from the differential equation evaluated at t = 0,

$$\ddot{\mathbf{u}}(0) = \mathbf{M}^{-1}\left[\mathbf{F}(0) - \mathbf{K}\mathbf{u}(0)\right]$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Usually $\mathbf{M}^{-1}$ is not computed; rather, the system of equations

$$\mathbf{M}\ddot{\mathbf{u}}(0) = \mathbf{F}(0) - \mathbf{K}\mathbf{u}(0)$$

is solved for **u**(0) using an **LU** decomposition.

The special starting value **u**(-*h)* is then given formally by

$$\mathbf{u}(-h) = \mathbf{u}(0) - h\dot{\mathbf{u}}(0) + \frac{h^2\left[\mathbf{M}^{-1}\left[\mathbf{F}(0) - \mathbf{K}\mathbf{u}(0)\right]\right]}{2}$$

$$\mathbf{u}_{-1} = \mathbf{u}_0 - h\dot{\mathbf{u}}_0 + \frac{h^2\left[\mathbf{M}^{-1}\left[\mathbf{F}_0 - \mathbf{K}\mathbf{u}_0\right]\right]}{2}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

The recurrence relation is then evaluated for n = 0 to yield

$$\mathbf{M}\mathbf{u}_1 = \left(2\mathbf{M} - h^2\mathbf{K}\right)\mathbf{u}_0 - \mathbf{M}\mathbf{u}_{-1} + h^2\mathbf{F}_0$$

from which $\mathbf{u}_1$ is determined using $\mathbf{u}_{-1}$ and $\mathbf{u}_0$ from the initial conditions.

The recurrence relation is then used successively for n = 1, 2, ... until the desired time range is included.

After determining $\mathbf{u}_{n+1}$, the velocity $\dot{\mathbf{u}}_n$ and the acceleration $\ddot{\mathbf{u}}_n$ at t_n are computed at each time step.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

The entire process is summarized as:

Given: The initial conditions $\mathbf{u}(0)$ and $\mathring{\mathbf{u}}_n(0)$,

Compute:
$$\ddot{\mathbf{u}}_0 = \mathbf{M}^{-1}\left[\mathbf{F}_0 - \mathbf{K}\mathbf{u}_0\right]$$

$$\mathbf{u}_{-1} = \mathbf{u}_0 - h\dot{\mathbf{u}}_0 + \frac{h^2\left[\mathbf{M}^{-1}\left[\mathbf{F}_0 - \mathbf{K}\mathbf{u}_0\right]\right]}{2}$$

$$\mathbf{u}_1 = \left(2 - h^2\mathbf{M}^{-1}\mathbf{K}\right)\mathbf{u}_0 - \mathbf{u}_{-1} + h^2\mathbf{M}^{-1}\mathbf{F}_0$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Then for n = 1, 2, …. Compute $\mathbf{u}_{n+1}$ using

$$\mathbf{u}_{n+1} = \left(2 - h^2\mathbf{M}^{-1}\mathbf{K}\right)\mathbf{u}_n - \mathbf{u}_{n-1} + h^2\mathbf{M}^{-1}\mathbf{F}_n$$

$$\dot{\mathbf{u}}_n = \frac{\mathbf{u}_{n+1} - \mathbf{u}_{n-1}}{2h}$$

$$\ddot{\mathbf{u}}_n = \frac{\mathbf{u}_{n+1} - 2\mathbf{u}_n + \mathbf{u}_{n-1}}{h^2}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

As will be indicated later, this method is conditionally stable with the critical step size given by

$$h_{cr} = \frac{2}{\omega_{\max}}$$

where $(\omega_{\max})^2$ is the largest eigenvalue of the algebraic eigenvalue problem

$$\left(\mathbf{K} - \omega^2\mathbf{M}\right)\mathbf{x} = 0$$

Just as for the first-order system, values of $h > h_{cr}$ result in an unbounded oscillation of the numerical solution.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example: Consider the one-dimensional problem

$$m\ddot{x} + kx = f_0 \qquad x(0) = 0 \qquad \dot{x}(0) = 0$$

Define the dimensionless displacement $z = kx/f_0$ and rewrite the differential equation as:

$$\ddot{z} + \omega^2 z = \omega^2 \qquad z(0) = 0 \qquad \dot{z}(0) = 0$$

where $\omega^2 = k/m$.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example: It can be shown, that the recurrence relation for this one-dimensional problem is:

$$z_{n+1} = \left(2 - (\omega h)^2\right) z_n - z_{n-1} + (\omega h)^2$$

with: $\ddot{z}(0) = \omega^2$

$$z_{-1} = z_0 - h\dot{z}_0 + \frac{h^2\left[\mathbf{M}^{-1}\left[\omega^2 - \mathbf{K}z_0\right]\right]}{2} = \frac{(\omega h)^2}{2}$$

$$z_1 = \left(2 - h^2\mathbf{M}^{-1}\mathbf{K}\right) z_0 - z_{-1} + h^2\mathbf{M}^{-1}\omega^2 = \frac{(\omega h)^2}{2}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example: Then as outlined previously for *n* = 1, 2, ...,

$$z_{n+1} = \left(2 - (\omega h)^2\right) z_n - z_{n-1} + (\omega h)^2$$

with:

$$\dot{z}_n = \frac{z_{n+1} - z_{n-1}}{2h}$$

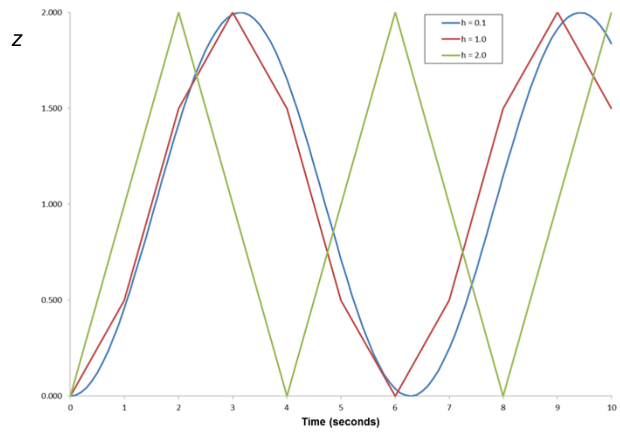
$$\ddot{z}_n = \frac{z_{n+1} - 2z_n + z_{n-1}}{h^2}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example: For $\omega = 1$, the following graph presents the results for the *displacement* as a function of time.

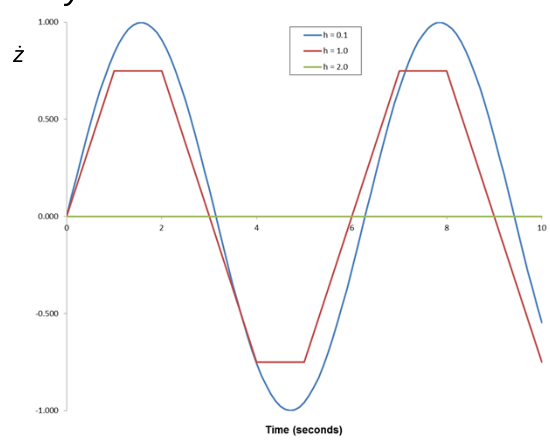


TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example: For $\omega = 1$, the following graph presents the results for the *velocity* as a function of time.

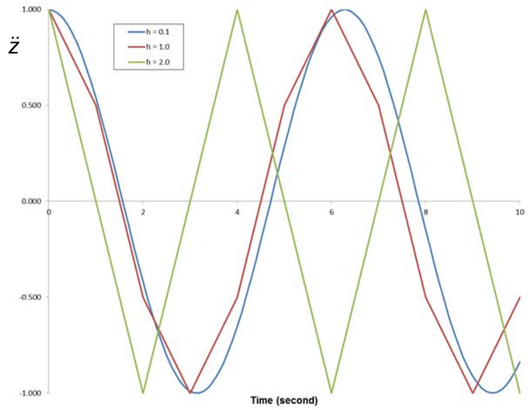


TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example: For $\omega = 1$, the following graph presents the results for the *acceleration*, as a function of time.



TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example: The graphs present results for h = 0.1, h = 1.0, and h = 2.0.

The results for h = 0.1 are essentially the same as the exact results.

The critical step size is represented by h = 2.0 and is thus the upper limit for stability.

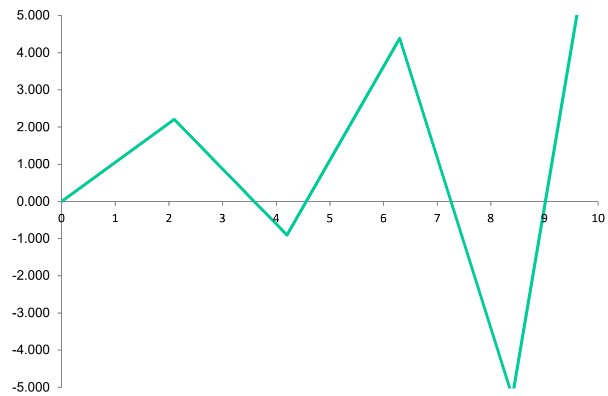
Values above h = 2.0 would result in unbounded oscillations.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example: For $\omega = 1$, the following graph presents the results for the *displacement* as a function of time (h = 2.1).

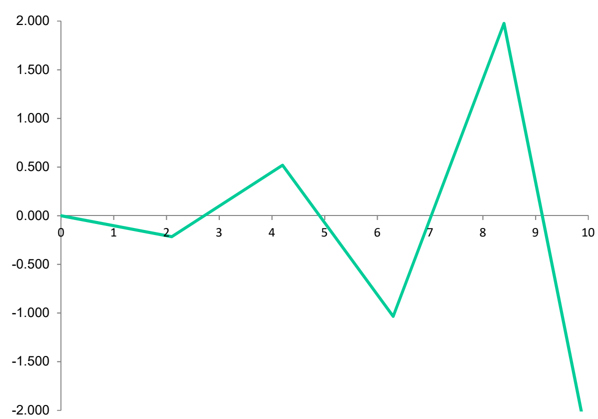


TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example: For $\omega = 1$, the following graph presents the results for the *velocity* as a function of time (h = 2.1).

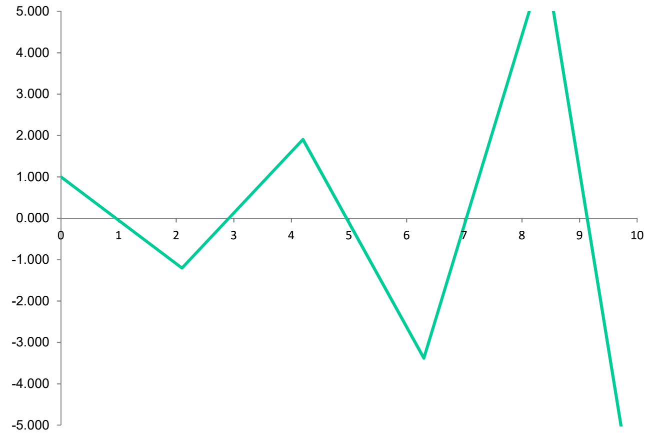


TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example: For $\omega = 1$, the following graph presents the results for the *acceleration*, as a function of time (h = 2.1).



TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example – Consider again the example of the one dimensional wave equation previously developed for the three-element problem:

$$\phi m\ddot{\mathbf{v}} + k\mathbf{v} = \mathbf{F} \qquad \mathbf{v}(0) = 0 \qquad \dot{\mathbf{v}}(0) = 0$$

$$\phi = \frac{\rho L^2}{54E} \qquad \mathbf{v} = \frac{\mathbf{u}}{\Lambda} \qquad \Lambda = \frac{P_0 L}{AE}$$

$$\phi\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\ddot{\mathbf{v}} + \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\mathbf{v} = \begin{Bmatrix} 0 \\ 0 \\ 1/3 \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example – Numerical results will be based on the values
E = 3 X 10^7 psi, ρ = 7.5 X 10^{-4} lbf-s^2/in^4, L = 20 in.,
A = 1 in^2, and P = 1,000 lbf.

Evaluating the differential equation at t = 0 yields:

$$\phi\begin{bmatrix}4 & 1 & 0\\ 1 & 4 & 1\\ 0 & 1 & 2\end{bmatrix}\ddot{\mathbf{v}}_0 + \begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}\mathbf{v}_0 = \begin{Bmatrix}0\\ 0\\ \frac{1}{3}\end{Bmatrix}$$

$\mathbf{v}_0(0) = 0$

$$\phi\begin{bmatrix}4 & 1 & 0\\ 1 & 4 & 1\\ 0 & 1 & 2\end{bmatrix}\ddot{\mathbf{v}}_0 = \begin{Bmatrix}0\\ 0\\ \frac{1}{3}\end{Bmatrix} \qquad \ddot{\mathbf{v}}_0 = \frac{1}{78\phi}\begin{Bmatrix}1\\ -4\\ 15\end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example – Recall, the general form of the $\mathbf{v}_{-1}$ is:

$$\ddot{\mathbf{v}}_0 = \mathbf{M}^{-1}\left[\mathbf{F}_0 - \mathbf{K}\mathbf{v}_0\right]$$

$$\mathbf{v}_{-1} = \mathbf{v}_0 - h\dot{\mathbf{v}}_0 + \frac{h^2\ddot{\mathbf{v}}_0}{2}$$

$$\mathbf{v}_{n+1} = \left(2 - h^2\mathbf{M}^{-1}\mathbf{K}\right)\mathbf{v}_n - \mathbf{v}_{n-1} + h^2\mathbf{M}^{-1}\mathbf{F}_n$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example – The $\mathbf{v}_{-1}$ is determined:

$\mathbf{v}_0(0) = 0$ $\quad$ $\dot{\mathbf{v}}_0(0) = 0$

$$\mathbf{v}_{-1} = \mathbf{v}_0 - h\dot{\mathbf{v}}_0 + \frac{h^2\ddot{\mathbf{v}}_0}{2} = \frac{h^2\ddot{\mathbf{v}}_0}{2} = \frac{h^2}{156\phi}\begin{Bmatrix} 1 \\ -4 \\ 15 \end{Bmatrix}$$

The basic algorithm can be expressed as

$$\mathbf{m}\mathbf{v}_{n+1} = \left(2\mathbf{m} - \psi\mathbf{k}\right)\mathbf{v}_n - \mathbf{m}\mathbf{v}_{n-1} + \psi\mathbf{F}_n$$

$$\psi = \frac{h^2}{\phi} = \frac{54Eh^2}{\rho L^2}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example – The first iteration yields

$\mathbf{v}_0(0) = 0$

$$\mathbf{m}\mathbf{v}_1 = \left(2\mathbf{m} - \psi\mathbf{k}\right)\mathbf{v}_0 - \mathbf{m}\mathbf{v}_{-1} + \psi\mathbf{F}_0$$

$$\mathbf{v}_1 = -\mathbf{v}_{-1} + \psi\mathbf{m}^{-1}\mathbf{F}_0$$

$$= -\frac{h^2}{156\phi}\begin{Bmatrix} 1 \\ -4 \\ 15 \end{Bmatrix} + \frac{h^2}{26\phi}\begin{bmatrix} 7 & -2 & 1 \\ -2 & 8 & -4 \\ 1 & -4 & 15 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 1/3 \end{Bmatrix}$$

$$= -\frac{h^2}{156\phi}\begin{Bmatrix} 1 \\ -4 \\ 15 \end{Bmatrix} + \frac{h^2}{156\phi}\begin{Bmatrix} 2 \\ -8 \\ 30 \end{Bmatrix} = \frac{\psi}{156}\begin{Bmatrix} 1 \\ -4 \\ 15 \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example – In order to further carry out the numerical integration for this example, a step size $\Delta t = h$ must be chosen.

Recall that the largest eigenvalue is so that the critical step size is given by:

$$(\omega_{\max})^2 = 1.6456\left(\frac{54E}{\rho L^2}\right)$$

The critical step size is given by:

$$h_{cr} = \frac{2}{\omega_{\max}} = 0.2121\left(\frac{\rho L^2}{E}\right)^{1/2}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example – In terms of the parameter ψ appearing in the differential equation:

$$\psi_{cr} = \frac{h_{cr}^{\;2}}{\phi} = \frac{54Eh_{cr}^{\;2}}{\rho L^2} = 54(0.2121)^2 = 2.4307$$

For values ψ of below ψ_{cr} the solution will remain bounded for large *t.*

Whereas for $\psi > \psi_{cr}$ the solution as given by the numerical procedure will oscillate with ever-increasing amplitude; that is, the algorithm is not stable when $\psi > \psi_{cr}$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example – As was seen from the analytical solution presented previously, all of the frequencies determined from |**K** - ω^2**M**| = 0 are contained in the solution.

In order to obtain numerical results that accurately contain the effects of all the frequency components, it is necessary to choose a step size that is relatively small compared with the period of the largest frequency.

A general rule is to break half the period of the largest frequency into 10 equal intervals; that is, take:

$$h^* = \frac{\pi}{10\omega_{\max}}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example – For the present example

$$h^* = \frac{\pi}{10\omega_{\max}} = \frac{\pi L}{94.267c}$$

with the parameter ψ given by

$$\psi^* = \frac{h^2}{\phi} = \frac{\left(\frac{\pi L}{94.267c}\right)^2}{\phi} = \left(\frac{\pi L}{94.267c}\right)^2 \left(\frac{54E}{\rho L^2}\right)$$

$$= 0.05998$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example – Results for this example for $h_1 = 4.2433(10^{-6})$ sec and $h_2 = 2.1216(10^{-5})$ sec.

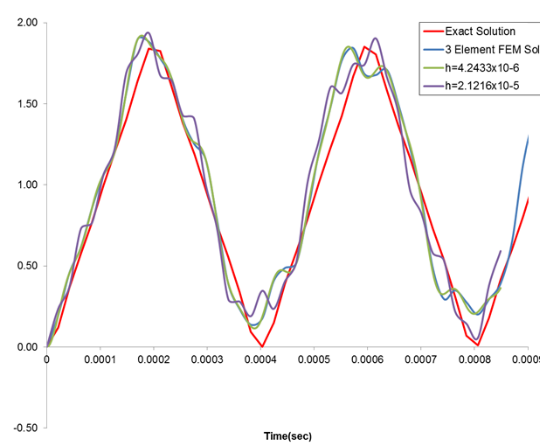
The critical step size is h_2 and $h_1 = h_2/5$ is a value somewhat larger than the one corresponding to dividing the half period of the maximum frequency into 10 equal segments.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

The displacement at $x = L$, that is, $u_4(t)$ is shown below

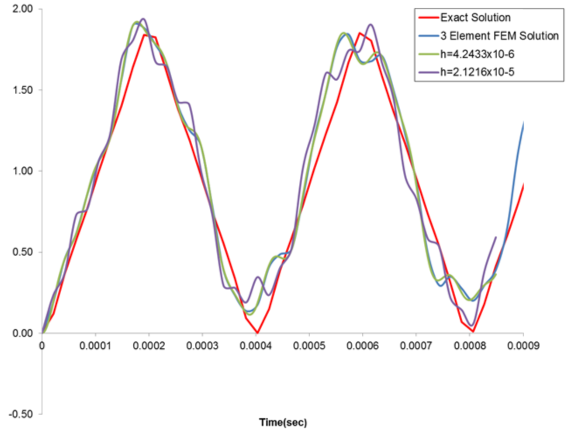


TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

The analytical solution and the central difference numerical solution for $\Delta t = h = 4.2433(10^{-6})$ sec. agree well.

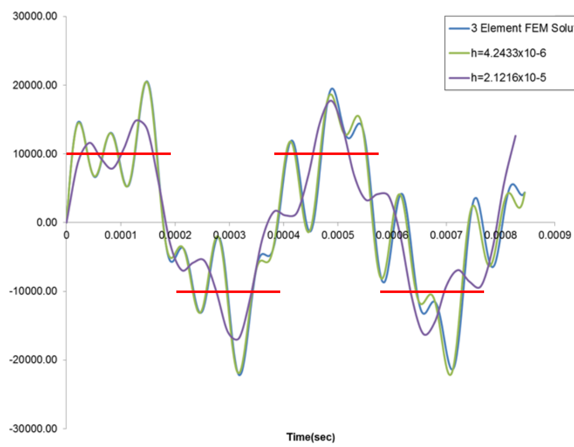


TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example – The *velocity* at $x = L$ is:

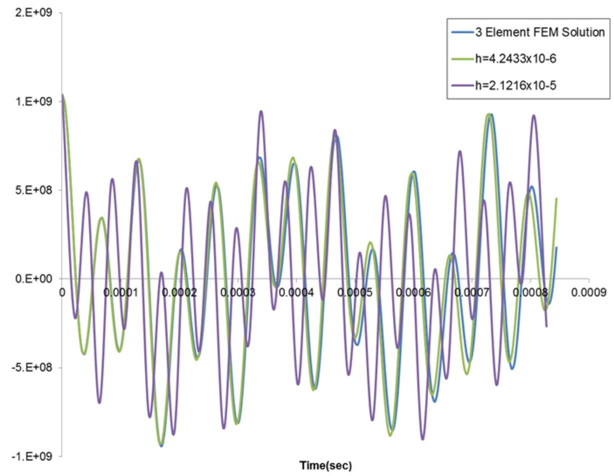


TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example – The *acceleration* at $x = L$ is:



TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Generally, the accuracy of the results improves with an increase in the number of elements used.

This can be traced to the fact that more of the approximate eigenvalues corresponding to the exact solution are more accurately determined using more elements.

The use of higher-order interpolations may also result in some improvement in accuracy, although not to the same extent as increasing the number of linearly interpolated elements.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

As is apparent from the results of the example, that all three of the frequencies contribute to the solution.

This means that the combined requirements of not exceeding the critical time step and integrating the effects of the higher modes accurately can lead to a very small h, and hence an expensive algorithm.

Fortunately for large systems the higher modes do not contribute significantly to the solution so that an unconditionally stable algorithm with a larger time step can be used satisfactorily.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Finally, it is easily seen that if lumped mass matrices are used, **M** is a diagonal matrix and the computations involved in the central difference algorithm reduce at each step to a matrix multiplication and vector additions, that is, no solution of a set of algebraic equations is required at each step.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD - Newmark's method is based on an extension of the average acceleration method, which is conditionally stable.

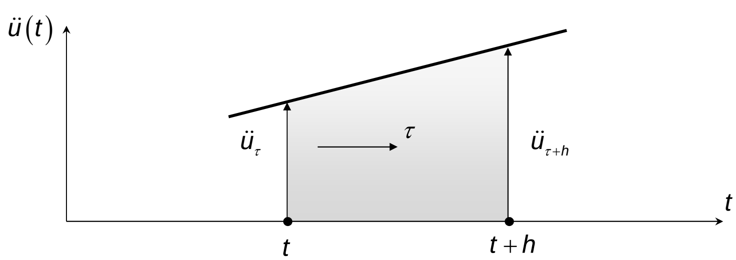
Newmark was able to generalize the algorithm so as to retain its simple form, yet produce an unconditionally stable algorithm.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD - The average acceleration method is based on the assumption that over a small time increment any nodal acceleration can be considered to be a linear function of time.



TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD - For the interval $0 < \tau < h$, the interval corresponding to the time step, the acceleration is expressed as

$$\ddot{\mathbf{u}}_{t+\tau} = \ddot{\mathbf{u}}_t\left(1-\frac{\tau}{h}\right) + \ddot{\mathbf{u}}_{t+h}\left(\frac{\tau}{h}\right)$$

Linear function in τ

Integrating yields

$$\dot{\mathbf{u}}_{t+\tau} = \dot{\mathbf{u}}_t + \ddot{\mathbf{u}}_t\left(\tau - \frac{\tau^2}{2h}\right) + \ddot{\mathbf{u}}_{t+h}\left(\frac{\tau^2}{2h}\right)$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – If $\tau = h$, then

$$\dot{\mathbf{u}}_{t+\tau} = \dot{\mathbf{u}}_t + \frac{\left(\ddot{\mathbf{u}}_t + \ddot{\mathbf{u}}_{t+h}\right)h}{2} = \dot{\mathbf{u}}_t + h\ddot{\mathbf{u}}_{average}$$

That is, the increment in the velocity is based on the approximate average acceleration on the interval (0, h).

Integrating $\mathbf{u}_{t+\tau}$ yields:

$$\mathbf{u}_{t+\tau} = \mathbf{u}_t + \tau\dot{\mathbf{u}}_t + \ddot{\mathbf{u}}_t\left(\frac{\tau^2}{2} - \frac{\tau^3}{6h}\right) + \ddot{\mathbf{u}}_{t+h}\left(\frac{\tau^3}{6h}\right)$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – These expressions are employed with the differential equations

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}$$

to yield the conditionally stable average acceleration algorithm. Newmark generalized equations as:

$$\dot{\mathbf{u}}_{t+h} = \dot{\mathbf{u}}_t + \left[(1-\delta)\ddot{\mathbf{u}}_t + \delta\ddot{\mathbf{u}}_{t+h}\right]h$$

$$\mathbf{u}_{t+h} = \mathbf{u}_t + h\dot{\mathbf{u}}_t + \left[\left(\frac{1}{2}-\alpha\right)\ddot{\mathbf{u}}_t + \alpha\ddot{\mathbf{u}}_{t+h}\right]h^2$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – Newmark generalized equations

$$\dot{\mathbf{u}}_{t+h} = \dot{\mathbf{u}}_t + \left[(1-\delta)\ddot{\mathbf{u}}_t + \delta\ddot{\mathbf{u}}_{t+h}\right]h$$

$$\mathbf{u}_{t+h} = \mathbf{u}_t + h\dot{\mathbf{u}}_t + \left[\left(\frac{1}{2}-\alpha\right)\ddot{\mathbf{u}}_t + \alpha\ddot{\mathbf{u}}_{t+h}\right]h^2$$

The method is ***unconditionally stable*** as long as the parameters δ and α are chosen to satisfy $\delta \geq 0.5$ and $\alpha \geq 0.25(\delta + 0.5)^2$.

Note that $\delta = ½$ and $\alpha = ¼$ corresponds to the average acceleration method.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – The equation for $\mathbf{u}_{t+h}$ is solved for $\ddot{\mathbf{u}}_{t+h}$ and substituted into the equation for $\dot{\mathbf{u}}_{t+h}$ to yield

$$\dot{\mathbf{u}}_{t+h} = \dot{\mathbf{u}}_t + \frac{\delta\left(\mathbf{u}_{t+h} - \mathbf{u}_t - h\dot{\mathbf{u}}_t\right)}{\alpha h} + c_2 h\ddot{\mathbf{u}}_t$$

where $c_2 = 1 - \delta/(2\alpha)$ and then into the differential equation evaluated at $t + h$ to yield

$$\left(\mathbf{M} + \alpha h^2\mathbf{K}\right)\mathbf{u}_{t+h} = \mathbf{M}\left(\mathbf{u}_t + h\dot{\mathbf{u}}_t + c_1 h^2\ddot{\mathbf{u}}_t\right) + \alpha h^2\mathbf{F}_{t+h}$$

where $c_1 = 1/2 - \alpha$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – This equation, together with the two equations for the velocity and acceleration at $t + h$, can be used to step ahead in time to determine the solution

$$\left(\mathbf{M} + \alpha h^2\mathbf{K}\right)\mathbf{u}_{t+h} = \mathbf{M}\left(\mathbf{u}_t + h\dot{\mathbf{u}}_t + c_1 h^2\ddot{\mathbf{u}}_t\right) + \alpha h^2\mathbf{F}_{t+h}$$

$$\dot{\mathbf{u}}_{t+h} = \dot{\mathbf{u}}_t + \frac{\delta\left(\mathbf{u}_{t+h} - \mathbf{u}_t - h\dot{\mathbf{u}}_t\right)}{\alpha h} + c_2 h\ddot{\mathbf{u}}_t$$

$$\ddot{\mathbf{u}}_{t+h} = \frac{\mathbf{u}_{t+h} - \mathbf{u}_t - h\dot{\mathbf{u}}_t}{\alpha h^2} - \frac{c_1\ddot{\mathbf{u}}_t}{\alpha}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – In order to start the process, the acceleration at t = 0 is needed and is determined by solving the governing equations evaluated at t = 0,

$$\mathbf{M}\ddot{\mathbf{u}}(0) = \mathbf{F}(0) - \mathbf{K}\mathbf{u}(0)$$

for acceleration $\ddot{\mathbf{u}}(0)$, the previous equations are then used to step ahead using the unconditionally stable Newmark algorithm.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – The algorithm consists of:

Given: The initial conditions $\mathbf{u}(0)$ and $\mathring{\mathbf{u}}_n(0)$,

Compute: $\ddot{\mathbf{u}}(0)$, then $\mathbf{u}_n$, $\mathring{\mathbf{u}}_n$, and $\ddot{\mathbf{u}}_n$, for n = 1, 2, …..

$$\left(\mathbf{M} + \alpha h^2 \mathbf{K}\right)\mathbf{u}_{n+1} = \mathbf{M}\left(\mathbf{u}_n + h\dot{\mathbf{u}}_n + c_1 h^2 \ddot{\mathbf{u}}_n\right) + \alpha h^2 \mathbf{F}_{n+1}$$

$$\dot{\mathbf{u}}_{n+1} = \dot{\mathbf{u}}_n + \frac{\delta\left(\mathbf{u}_{n+1} - \mathbf{u}_n - h\dot{\mathbf{u}}_n\right)}{\alpha h} + c_2 h \ddot{\mathbf{u}}_n$$

$$\ddot{\mathbf{u}}_{n+1} = \frac{\mathbf{u}_{n+1} - \mathbf{u}_n - h\dot{\mathbf{u}}_n}{\alpha h^2} - \frac{c_1 \ddot{\mathbf{u}}_n}{\alpha}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – The algorithm consists of:

Specifically, with $\mathbf{u}_0$, $\dot{\mathbf{u}}_0$, and $\ddot{\mathbf{u}}_0$ known

$$\left(\mathbf{M} + \alpha h^2 \mathbf{K}\right)\mathbf{u}_1 = \mathbf{M}\left(\mathbf{u}_0 + h\dot{\mathbf{u}}_0 + c_1 h^2 \ddot{\mathbf{u}}_0\right) + \alpha h^2 \mathbf{F}_1$$

$$\dot{\mathbf{u}}_1 = \dot{\mathbf{u}}_0 + \frac{\delta\left(\mathbf{u}_1 - \mathbf{u}_0 - h\dot{\mathbf{u}}_0\right)}{\alpha h} + c_2 h \ddot{\mathbf{u}}_0$$

$$\ddot{\mathbf{u}}_1 = \frac{\mathbf{u}_1 - \mathbf{u}_0 - h\dot{\mathbf{u}}_0}{\alpha h^2} - \frac{c_1 \ddot{\mathbf{u}}_0}{\alpha}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – The algorithm consists of:

Then with $\mathbf{u}_1$, $\dot{\mathbf{u}}_1$, and $\ddot{\mathbf{u}}_1$ known

$$\left(\mathbf{M} + \alpha h^2 \mathbf{K}\right)\mathbf{u}_2 = \mathbf{M}\left(\mathbf{u}_1 + h\dot{\mathbf{u}}_1 + c_1 h^2 \ddot{\mathbf{u}}_1\right) + \alpha h^2 \mathbf{F}_2$$

$$\dot{\mathbf{u}}_2 = \dot{\mathbf{u}}_1 + \frac{\delta\left(\mathbf{u}_2 - \mathbf{u}_1 - h\dot{\mathbf{u}}_1\right)}{\alpha h} + c_2 h \ddot{\mathbf{u}}_1$$

$$\ddot{\mathbf{u}}_2 = \frac{\mathbf{u}_2 - \mathbf{u}_1 - h\dot{\mathbf{u}}_1}{\alpha h^2} - \frac{c_1 \ddot{\mathbf{u}}_1}{\alpha}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – The algorithm is continued until the time range of interest is covered.

Note that for the Newmark algorithm, lumping of the mass matrix results in no computational advantage.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

Example – Consider again the example of the one dimensional wave equation previously developed for the three-element problem:

$$\phi \mathbf{m}\ddot{\mathbf{v}} + \mathbf{k}\mathbf{v} = \mathbf{F} \qquad \mathbf{v}(0) = 0 \qquad \dot{\mathbf{v}}(0) = 0$$

$$\phi = \frac{\rho L^2}{9E} \qquad \mathbf{v} = \frac{\mathbf{u}}{\Lambda} \qquad \Lambda = \frac{P_0 L}{AE}$$

$$\frac{\phi}{6}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\ddot{\mathbf{v}} + \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\mathbf{v} = \begin{Bmatrix} 0 \\ 0 \\ 1/3 \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – The equations to be solved at the first step can be written as

$$(\mathbf{m} + \alpha\psi\mathbf{k})\mathbf{v}_1 = \mathbf{m}\left(\mathbf{v}_0 + h\dot{\mathbf{v}}_0 + c_1 h^2 \ddot{\mathbf{v}}_0\right) + \alpha\psi\mathbf{f}_1$$

$$\psi = \frac{h^2}{\phi}$$

$$\dot{\mathbf{v}}_1 = \dot{\mathbf{v}}_0 + \frac{\delta\left(\mathbf{v}_1 - \mathbf{v}_0 - h\dot{\mathbf{v}}_0\right)}{\alpha h} + c_2 h\ddot{\mathbf{v}}_0$$

$$\ddot{\mathbf{v}}_1 = \frac{\mathbf{v}_1 - \mathbf{v}_0 - h\dot{\mathbf{v}}_0}{\alpha h^2} - \frac{c_1\ddot{\mathbf{v}}_0}{\alpha}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – Evaluating the differential equation at t = 0 yields

$$\phi\mathbf{m}\ddot{\mathbf{v}}_0 = \mathbf{f} - \mathbf{k}\mathbf{v}_0 \qquad \Rightarrow \qquad \ddot{\mathbf{v}}_0 = \frac{\mathbf{m}^{-1}}{\phi}\left(\mathbf{f} - \mathbf{k}\mathbf{v}_0\right)$$

$\mathbf{v}_0(0) = 0$

$$\ddot{\mathbf{v}}_0 = \frac{1}{26\phi}\begin{bmatrix} 7 & -2 & 1 \\ -2 & 8 & -4 \\ 1 & -4 & 15 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 1/3 \end{Bmatrix} = \frac{1}{78\phi}\begin{Bmatrix} 1 \\ -4 \\ 15 \end{Bmatrix} = 10^8\begin{Bmatrix} 0.6923 \\ -2.7692 \\ 10.3846 \end{Bmatrix}$$

Numerical results will be based on the values:
$E = 3 \times 10^7$ psi, $\rho = 7.5 \times 10^{-4}$ lbf-s^2/in^4, and L = 20 in.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – Taking α = 0.25, δ = 0.5, and h = 4.2422 x 10^{-6} seconds yield

$$(\mathbf{m}+\alpha\psi\mathbf{k}) = \begin{bmatrix} 0.6748 & 0.1626 & 0.0000 \\ 0.1626 & 0.6748 & 0.1626 \\ 0.0000 & 0.1626 & 0.3374 \end{bmatrix} \qquad \alpha\psi\mathbf{f} = \begin{Bmatrix} 0.0000 \\ 0.0000 \\ 0.0081 \end{Bmatrix}$$

At step 1:

$$\mathbf{v}_0(0)=0 \quad \dot{\mathbf{v}}_0(0)=0$$

$$(\mathbf{m}+\alpha\psi\mathbf{k})\mathbf{v}_1 = \mathbf{m}\left(\mathbf{v}_0 + h\dot{\mathbf{v}}_0 + c_1 h^2\ddot{\mathbf{v}}_0\right) + \alpha\psi\mathbf{f}_1$$

$$\begin{bmatrix} 0.6748 & 0.1626 & 0.0000 \\ 0.1626 & 0.6748 & 0.1626 \\ 0.0000 & 0.1626 & 0.3374 \end{bmatrix}\mathbf{v}_1 = \mathbf{m}\left(0.25h^2\ddot{\mathbf{v}}_0\right) + \begin{Bmatrix} 0.0000 \\ 0.0000 \\ 0.0014 \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – The solution for $\mathbf{v}_1$ is

$$\mathbf{v}_1 = \langle 0.000563 \quad -0.00234 \quad 0.009131 \rangle^T$$

Now solve for the velocity at t = 0

$$\dot{\mathbf{v}}_0(0)=0 \qquad \dot{\mathbf{v}}_0(0)=0$$

$$\dot{\mathbf{v}}_1 = \dot{\mathbf{v}}_0 + \frac{\delta(\mathbf{v}_1 - \mathbf{v}_0 - h\dot{\mathbf{v}}_0)}{\alpha h} + c_2 h\ddot{\mathbf{v}}_0$$

$$\mathbf{v}_0(0)=0$$

$$\dot{\mathbf{v}}_1 = \frac{\delta(\mathbf{v}_1)}{\alpha h} + c_2 h\ddot{\mathbf{v}}_0 \quad = \frac{\delta}{\alpha h}\begin{Bmatrix} 0.00056 \\ -0.00234 \\ 0.00913 \end{Bmatrix} + c_2 h \begin{Bmatrix} 0.6923 \\ -2.7692 \\ 10.3846 \end{Bmatrix} 10^8$$

$$= 10^3 \langle 0.265373 \quad -1.10116 \quad 4.303857 \rangle^T$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – The equation for acceleration at $t = 0$ is

$$\ddot{\mathbf{v}}_1 = \frac{\mathbf{v}_1 - \mathbf{v}_0 - h\dot{\mathbf{v}}_0}{\alpha h^2} - \frac{c_1\ddot{\mathbf{v}}_0}{\alpha} = \frac{\mathbf{v}_1}{\alpha h^2} - \frac{c_1\ddot{\mathbf{v}}_0}{\alpha}$$

$\dot{\mathbf{v}}_0(0) = 0$

$\dot{\mathbf{v}}_0(0) = 0$

Now solve for the acceleration at $t = h$

$$\ddot{\mathbf{v}}_1 = \frac{\delta}{\alpha h}\begin{Bmatrix} 0.2654 \\ -1.1012 \\ 4.3039 \end{Bmatrix} 10^3 - c_2 h \begin{Bmatrix} 0.6923 \\ -2.7692 \\ 10.3846 \end{Bmatrix} 10^8$$

$$\ddot{\mathbf{v}}_1 = 10^8 \langle 0.5585 \quad -2.4209 \quad 9.9008 \rangle^T$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – For n = 2:

$$\mathbf{v}_2 = \langle 0.0020 \quad -0.0087 \quad 0.0357 \rangle^T$$

$$\dot{\mathbf{v}}_2 = 10^3 \langle 0.4231 \quad -1.9182 \quad 8.2112 \rangle^T$$

$$\ddot{\mathbf{v}}_2 = 10^8 \langle 0.1848 \quad -1.4300 \quad 8.5155 \rangle^T$$

For n = 3:

$$\mathbf{v}_3 = \langle 0.0037 \quad -0.0175 \quad 0.0772 \rangle^T$$

$$\dot{\mathbf{v}}_3 = 10^3 \langle 0.3881 \quad -2.2111 \quad 11.3793 \rangle^T$$

$$\ddot{\mathbf{v}}_3 = 10^8 \langle -0.3497 \quad 0.0491 \quad 6.4172 \rangle^T$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – The results for further integration are presented in following figures. The step size h_1 = 4.2433 x 10^{-6} sec indicated above is the same as the smaller of the two values used for the central difference algorithm in the previous section.

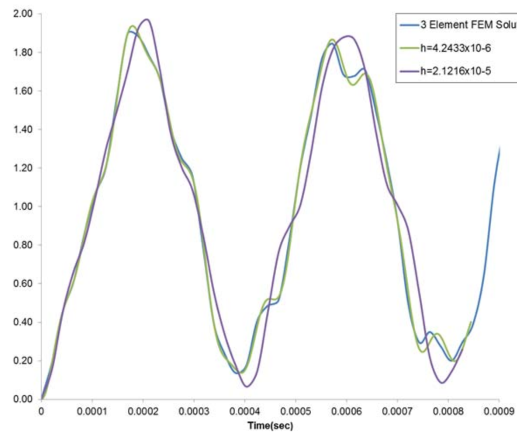
Integrations are also carried out for h_2 = 4.2433 x 10^{-5} sec = $10h_1$, a value twice that of the critical value for the central difference algorithm of the previous section. In all the figures, the abscissa *n* represents the number of time steps of length h_1.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – Displacement at $x = L$

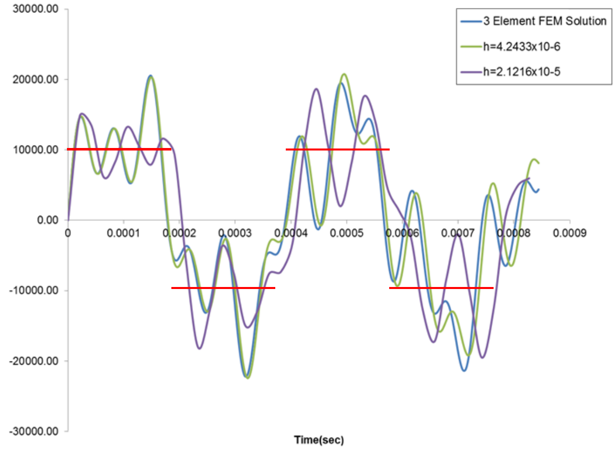


TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – Velocity at $x = L$

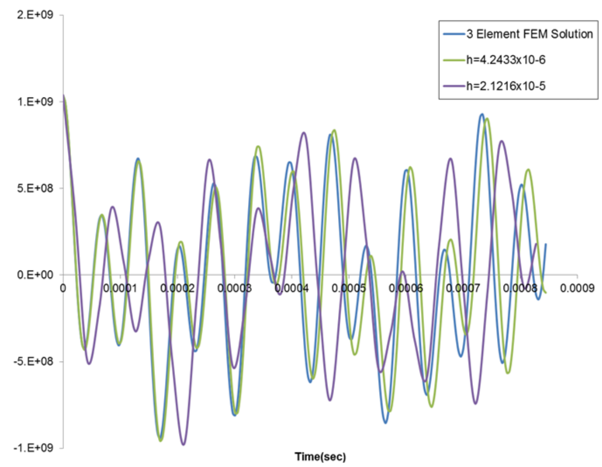


TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – Acceleration at $x = L$



TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – The results for displacement indicate that for $h = h_1$, there is very good agreement between the numerical solution and the corresponding analytical solution, both comparing favorably with the exact solution.

For $h = h_2$, the unconditionally stable Newmark algorithm is unable to predict the part of the response arising from the higher frequencies, but is able to predict the essential character of the displacement at the end $x = L$.

TIME-DEPENDENT PROBLEMS

One-Dimensional Wave or Hyperbolic Equations

Time Integration Techniques – Second-Order Systems

NEWMARK'S METHOD – The results velocity at $x = L$ indicate a rough similarity between the analytical solution and the Newmark solution for $h = h_1$.

Similarly, the numerical results for $h = h_2$ bear some resemblance to the analytical and exact solutions, but are neither qualitatively nor quantitatively satisfactory.

The results for the accelerations, as was the case for the central difference algorithm, are completely unsatisfactory.

End of 1-D Time Dependent Problems – Part b