TIME-DEPENDENT PROBLEMS

The previous three chapters dealt exclusively with steady-state problems, that is, problems where time did not enter explicitly into the formulation or solution of the problem.

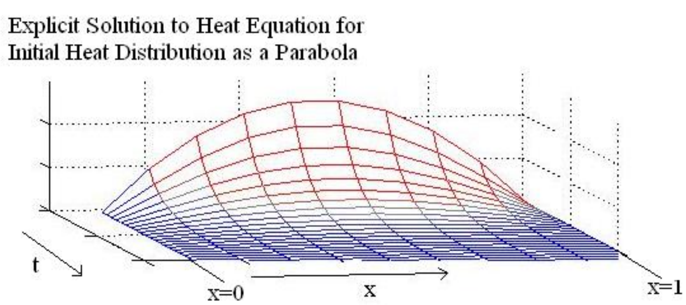
The types of problems considered in Chapters 2 and 3, respectively, were one- and two-dimensional elliptic boundary value problems.

In this chapter, finite element models for parabolic and hyperbolic equations, such as the one-dimensional transient heat conduction and the one-dimensional scalar wave equation, respectively, will be developed.

TIME-DEPENDENT PROBLEMS

The finite element models for these two types of initial-boundary value problems will turn out to be, respectively, first- and second-order systems of ordinary differential equations with time as the independent variable.

Analytical and numerical algorithms for the solution of these systems of equations will be presented and discussed.



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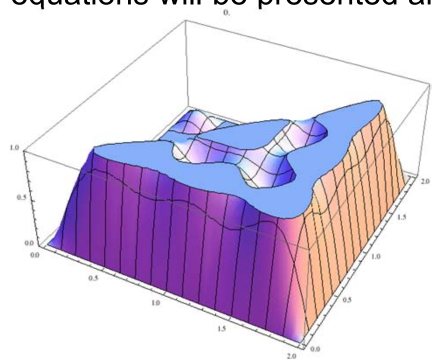

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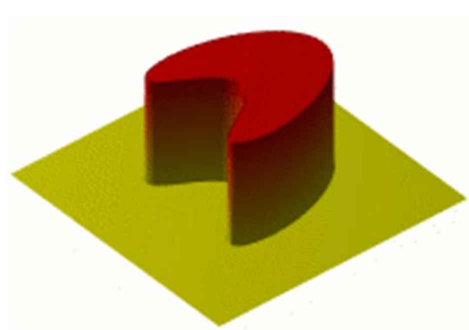

TIME-DEPENDENT PROBLEMS

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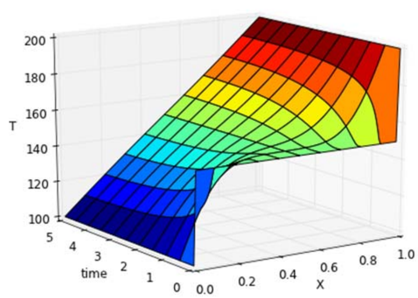
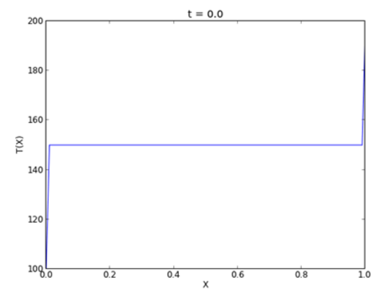


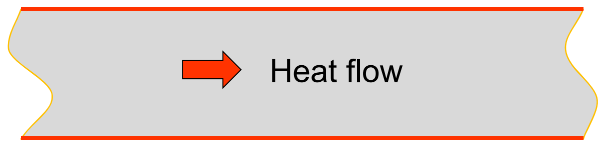


TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The example to be used to develop a model for one-dimensional diffusion processes is the classical heat conduction problem shown below:

Insulated



Insulated

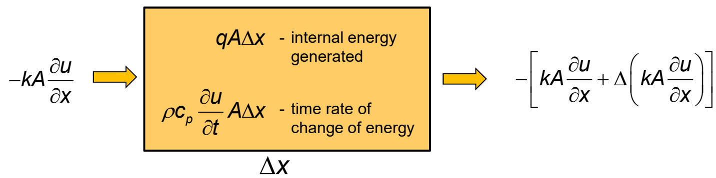
TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

We will assume that energy in the form of heat flows only in the *x*-direction, that is, that there is no flux perpendicular to the *x*-axis.

The basic physical principle for this type of problem is balance of energy.

A differential element of length Δx is isolated and an energy balance performed as:



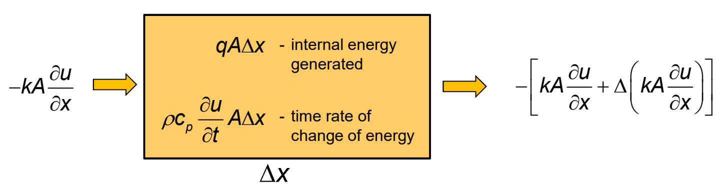
energy in - energy out + internal energy generated = time rate of change of energy within the element

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The coefficients in the energy balance are:

- k is the thermal conductivity [W/(m·K)], W = kg·m^2/s^3
- ρ is the mass density [kg/m^3],
- c_p is the specific heat capacity [kg·m^2/(K·s^2)], and
- $\alpha = k/(c_p\rho)$ is the thermal diffusivity [m^2/s]



energy in - energy out + internal energy generated = time rate of change of energy within the element

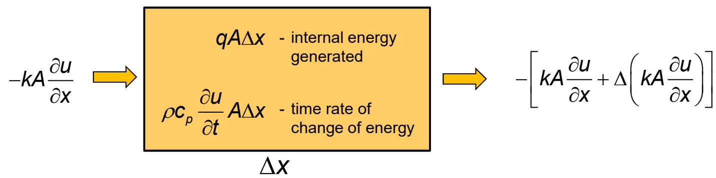
TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

With the energy terms as indicated below, the balance of energy statement becomes:

$$-\left[\cancel{kA}\frac{\partial u}{\partial x}\right]+\left[\cancel{kA}\frac{\partial u}{\partial x}+\Delta\left(kA\frac{\partial u}{\partial x}\right)\right]+qA\Delta x=\rho c_p A\Delta x\frac{\partial u}{\partial t}$$

$$\Delta\left(kA\frac{\partial u}{\partial x}\right)+qA\Delta x=\rho c_p A\Delta x\frac{\partial u}{\partial t}$$



energy in - energy out + internal energy generated = time rate of change of energy within the element

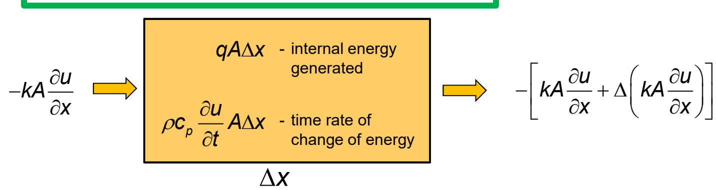
TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Divide by Δx and take the limit as $\Delta x \to 0$:

$$\lim_{\Delta x\to 0}\frac{\Delta\left(kA\frac{\partial u}{\partial x}\right)}{\Delta x}+qA=\rho c_p A\frac{\partial u}{\partial t}$$

$$\boxed{\frac{\partial}{\partial x}\left(kA\frac{\partial u}{\partial x}\right)+qA=\rho c_p A\frac{\partial u}{\partial t}}$$



energy in - energy out + internal energy generated = time rate of change of energy within the element

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

This a second-order, linear partial differential equation. The auxiliary conditions consist of two boundary conditions and one initial condition.

$$\frac{\partial}{\partial x}\left(kA\frac{\partial u}{\partial x}\right)+qA=\rho c_p A\frac{\partial u}{\partial t}$$

An appropriate boundary condition prescribes either:

1. The dependent variable u
2. The flux: $-kA\frac{\partial u}{\partial x}$
3. a linear combination of the flux and the dependent variable: $-kA\frac{\partial u}{\partial x}+hu$

TIME-DEPENDENT PROBLEMS

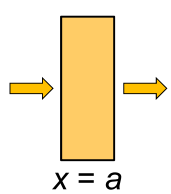
One-Dimensional Diffusion or Parabolic Equations

This third type of boundary condition is called a **convective boundary condition**

It is a local energy balance between the convection externally and the conduction internally.

At the left boundary $x = a,$ for instance, the external convective and internal conductive terms appear as:

$$h_L\left(u_L(t)-u(a,t)\right) \qquad -kA\frac{\partial u(a,t)}{\partial x}$$



TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The local energy balance produces:

$$-kA\frac{\partial u(a,t)}{\partial x}+h_L u(a,t)=h_L u_L(t)$$

A similar energy balance at the right end $x = b$ yields:

$$kA\frac{\partial u(b,t)}{\partial x}+h_R u(b,t)=h_R u_R(t)$$

For a time-dependent diffusion problem it is also necessary to specify an initial value for the dependent variable of the form:

$$u(x,0)=u_0(x)$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The complete statement of the initial-boundary value problem consists of the differential equation, two boundary conditions, and an initial condition:

$$\frac{\partial}{\partial x}\left(kA\frac{\partial u}{\partial x}\right)+qA=\rho c_p A\frac{\partial u}{\partial t} \qquad a\le x\le b,\quad t\ge 0$$

$$-kA\frac{\partial u(a,t)}{\partial x}+h_L u(a,t)=h_L u_L(t) \qquad t\ge 0$$

$$kA\frac{\partial u(b,t)}{\partial x}+h_R u(b,t)=h_R u_R(t) \qquad t\ge 0$$

$$u(x,0)=u_0(x) \qquad a\ge x\ge b$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

These equations represent a well-posed problem in partial differential equations.

$$\frac{\partial}{\partial x}\left(kA\frac{\partial u}{\partial x}\right) + qA = \rho c_p A\frac{\partial u}{\partial t} \qquad a \le x \le b, \quad t \ge 0$$

$$-kA\frac{\partial u(a,t)}{\partial x} + h_L u(a,t) = h_L u_L(t) \qquad t \ge 0$$

$$kA\frac{\partial u(b,t)}{\partial x} + h_R u(b,t) = h_R u_R(t) \qquad t \ge 0$$

$$u(x,0) = u_0(x) \qquad a \ge x \ge b$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

When there is no convection at a boundary the other two types of boundary conditions appropriate at $x = a$ are:

1. Where the temperature is specified:

$$u(x, 0) = u_0(x)$$

2. Where the energy flux is prescribed.

$$-kA\frac{\partial u(a, t)}{\partial x} = Q(t)$$

The general development of the finite element model will assume type 3 conditions at both boundaries.

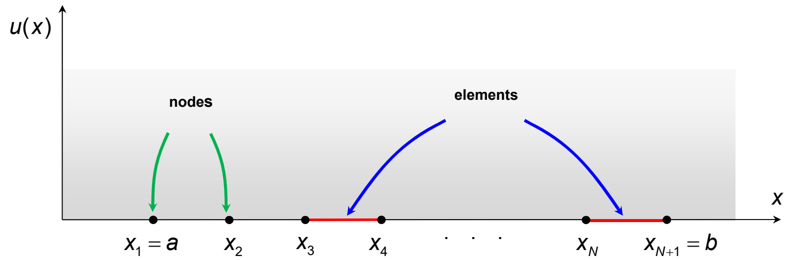
TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The Galerkin Finite Element Method

Consider the one-dimensional diffusion problem developed in this section.

Discretization. The first step in developing a finite element model is discretization. Nodes for the spatial domain $a \le x \le b$ are chosen as indicated below, with $a = x_1$ and $b = x_{N+1}$.

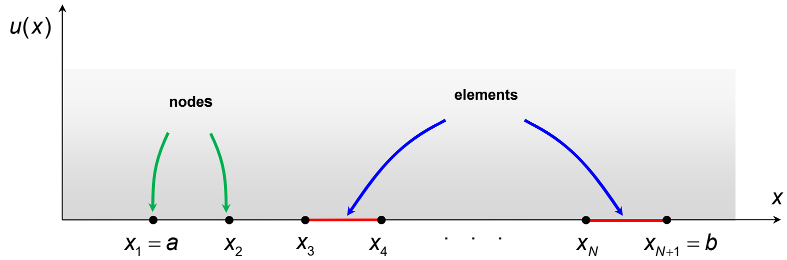


TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The Galerkin Finite Element Method

As was the case for steady-state problems considered in Chapter 2, the nodes are usually selected at equally spaced intervals, keeping in mind that it may be desirable in some problems to concentrate the nodes in regions of high gradients.



TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The Galerkin Finite Element Method

Interpolation. The interpolation functions are selected in exactly the same fashion as for the time-independent problem ***except*** that the nodal values are now taken to be functions of time rather than constants:

$$u(x,t) = \sum_{1}^{N+1} u_i(t) n_i(x)$$

The $n_i(x)$ are nodally based interpolation functions and can be linear, quadratic, or as otherwise desired.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The Galerkin Finite Element Method

Interpolation. The interpolation functions are selected in exactly the same fashion as for the time-independent problem ***except*** that the nodal values are now taken to be functions of time rather than constants:

$$u(x,t) = \sum_{1}^{N+1} u_i(t) n_i(x)$$

The representation above is referred to as ***semidiscretization*** in that the spatial variable x is discretized whereas the temporal variable t is not.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The Galerkin Finite Element Method

Interpolation. The interpolation functions are selected in exactly the same fashion as for the time-independent problem ***except*** that the nodal values are now taken to be functions of time rather than constants:

$$u(x,t) = \sum_{1}^{N+1} u_i(t) n_i(x)$$

A ***finite difference*** model of a time-dependent partial differential equation typically involves discretization of both the spatial and temporal variables**.**

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The Galerkin Finite Element Method

Elemental formulation. The elemental formulation for the diffusion problem is based on a corresponding weak statement.

The weak form is developed by multiplying the differential equation by a test function $v(x)$ satisfying any homogeneous essential boundary conditions, and integrating over the spatial region according to:

$$\int_a^b v\left(\frac{\partial}{\partial x}\left(kA\frac{\partial u}{\partial x}\right) + qA - \rho c_p A\frac{\partial u}{\partial t}\right)dx = 0$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The Galerkin Finite Element Method

Elemental formulation. The elemental formulation for the diffusion problem is based on a corresponding weak statement.

Integrating by parts and eliminating the derivative terms from the boundary conditions yields:

$$\int_a^b \left(v' \left(kA \frac{\partial u}{\partial x} \right) + \rho c_p A v \frac{\partial u}{\partial t} \right) dx + h_L v(a) u(a,t) + h_R v(b) u(b,t) = \int_a^b vAq \, dx + h_L v(a) u_L(t) + h_R v(b) u_R(t)$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The Galerkin Finite Element Method

Elemental formulation. The elemental formulation for the diffusion problem is based on a corresponding weak statement.

This is the required weak statement for the class of one-dimensional diffusion problems.

$$\int_a^b \left(v' \left(kA \frac{\partial u}{\partial x} \right) + \rho c_p A v \frac{\partial u}{\partial t} \right) dx + h_L v(a) u(a,t) + h_R v(b) u(b,t) = \int_a^b vAq \, dx + h_L v(a) u_L(t) + h_R v(b) u_R(t)$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The Galerkin Finite Element Method

Elemental formulation. The finite element model is obtained by substituting the approximate solution and $v = n_k$, $k = 1, 2, ..., N + 1$, successively, into the above expression to obtain:

$$\sum_{1}^{N+1} \int_a^b \left(n_k' kA n_i' u_i + n_k \rho c_p A n_i \dot{u} \right) dx + h_L \delta_{k1} u_1(t) + h_R \delta_{kN+1} u_{N+1}(t)$$

$$= \int_a^b n_k A q \, dx + h_L \delta_{k1} u_L(t) + h_R \delta_{kN+1} u_R(t)$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The Galerkin Finite Element Method

Elemental formulation. Which can be written as:

$$\sum_{1}^{N+1} \left[A_{ki} u_i(t) + B_{ki} \dot{u}_i(t) \right] = q_k(t) \qquad k = 1, 2, \ldots, N+1$$

$$A_{ki} = \int_a^b \left(n_k' kA n_i' \right) dx + h_L \delta_{k1} \delta_{ik} + h_R \delta_{kN+1} \delta_{ik}$$

$$B_{ki} = \int_a^b \left(n_k \rho c_p A n_i \right) dx$$

$$q_k = \int_a^b n_k A q \, dx + h_L \delta_{k1} u_L(t) + h_R \delta_{kN+1} u_R(t)$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The Galerkin Finite Element Method

Elemental formulation. In matrix notation, the above expression can be written as:

$$\mathbf{Au} + \mathbf{B\dot{u}} = \mathbf{q}$$

$$\mathbf{A} = \sum_e \mathbf{k_G} + \mathbf{BT} \qquad \mathbf{B} = \sum_e \mathbf{m_G} \qquad \mathbf{q} = \sum_e \mathbf{q_G} + \mathbf{bt}$$

$$\mathbf{k_e} = \int_{x_i}^{x_j} \left(\mathbf{N}' kA \mathbf{N}'^{\mathsf{T}}\right) dx \qquad \mathbf{m_e} = \int_{x_i}^{x_j} \left(\mathbf{N} \rho c_p A \mathbf{N}^{\mathsf{T}}\right) dx$$

$$\mathbf{q_e} = \int_{x_i}^{x_j} \left(qA\mathbf{N}\right) dx$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The Galerkin Finite Element Method

Elemental formulation. In matrix notation, the above expression can be written as:

$$\mathbf{Au} + \mathbf{B\dot{u}} = \mathbf{q}$$

$$\mathbf{A} = \sum_e \mathbf{k_G} + \mathbf{BT} \qquad \mathbf{B} = \sum_e \mathbf{m_G} \qquad \mathbf{q} = \sum_e \mathbf{q_G} + \mathbf{bt}$$

$$\mathbf{BT} = \begin{bmatrix} h_L & & & & & \\ & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & h_R \end{bmatrix} \qquad \mathbf{bt}^{\mathsf{T}} = \left\langle h_L u_L(t) \quad 0 \quad 0 \quad .... \quad 0 \quad h_R u_R(t) \right\rangle$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The Galerkin Finite Element Method

Elemental formulation. The original initial-boundary value problem has been converted into the initial value problem:

$$\mathbf{Au} + \mathbf{B\dot{u}} = \mathbf{q} \qquad \text{with} \qquad \mathbf{u}(0) = \mathbf{u}_0$$

The initial vector $\mathbf{u}_0$ is usually taken to be a vector consisting of the values of $u_0(x)$ at the nodes:

$$\mathbf{u}(0) = \mathbf{u}_0 = \left\langle u_0(a) \quad u_0(x_2) \quad u_0(x_3) \quad \ldots \quad u_0(x_N) \quad u_0(b) \right\rangle^T$$

Note that the assembly process has taken place implicitly during the process of carrying out the details of obtaining the governing equations using the Galerkin method.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The Galerkin Finite Element Method

Elemental formulation. The original initial-boundary value problem has been converted into the initial value problem:

$$\mathbf{Au} + \mathbf{B\dot{u}} = \mathbf{q} \qquad \text{with} \qquad \mathbf{u}(0) = \mathbf{u}_0$$

It is instructive to note that when time is not involved, the above equations are exactly what would result from the finite element model developed in Chapter 2 for the corresponding boundary value problem.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The Galerkin Finite Element Method

Elemental formulation. The original initial-boundary value problem has been converted into the initial value problem:

$$\mathbf{Au} + \mathbf{B\dot{u}} = \mathbf{q} \qquad \text{with} \qquad \mathbf{u}(0) = \mathbf{u}_0$$

Enforcement of constraints is necessary if either of the boundary conditions is essential, that is, if the dependent variable is prescribed at either boundary point.

The system equations must be altered to reflect these constraints.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The Galerkin Finite Element Method

Elemental formulation. Consider for example the case where the boundary condition at $x = a$ is $u(a, t) = u_a(t)$.

The h_L terms in both **BT** and **bt** would be taken as zero and the first equation would be replaced by the constraint resulting in:

$$\begin{aligned}
u_1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad &= u_a(t) \\
a_{21}u_1 + a_{22}u_2 + a_{23}u_3 + \cdots + b_{21}\dot{u}_1 + b_{22}\dot{u}_2 + b_{23}\dot{u}_3 + \cdots &= q_2(t) \\
a_{31}u_1 + a_{32}u_2 + a_{33}u_3 + \cdots + b_{31}\dot{u}_1 + b_{32}\dot{u}_2 + b_{33}\dot{u}_3 + \cdots &= q_3(t) \\
&\vdots
\end{aligned}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The Galerkin Finite Element Method

Elemental formulation. Consider for example the case where the boundary condition at $x = a$ is $u(a, t) = u_a(t)$.

The u_1 and $\mathring{u}_1$ terms in the remaining equations are transferred to the right-hand side to yield

$$
\begin{aligned}
u_1 \quad & = u_a(t) \\
a_{22}u_2 + a_{23}u_3 + \cdots + b_{22}\dot{u}_2 + b_{23}\dot{u}_3 + \cdots & = q_2(t) - a_{21}u_1 - b_{21}\dot{u}_1 \\
a_{22}u_2 + a_{23}u_3 + \cdots + b_{22}\dot{u}_2 + b_{23}\dot{u}_3 + \cdots & = q_3(t) - a_{31}u_1 - b_{31}\dot{u}_1 \\
& \vdots
\end{aligned}
$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The Galerkin Finite Element Method

Elemental formulation. For a linearly interpolated model the half bandwidth is two and only the terms involving u_1 and $\mathring{u}_1$ in the second equation need to be transferred to the right-hand side.

For a quadratically interpolated model the half bandwidth is three and terms from the first two equations need to be transferred.

If the constraint is at the right end, the N^{th}, $(N - 1)^{\text{st}}$, . . . equations would be similarly altered.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

The Galerkin Finite Element Method

Elemental formulation. For a linearly interpolated model the half bandwidth is two and only the terms involving u_1 and $\mathring{u}_1$ in the second equation need to be transferred to the right-hand side.

The constrained set of equations may be written as:

$$\mathbf{M}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f} \qquad \mathbf{u}(0) = \mathbf{u}_0$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

As a typical example consider the specific heat conduction problem:

$$\frac{\partial}{\partial x}\left(kA\frac{\partial u}{\partial x}\right) = \rho c_p A\frac{\partial u}{\partial t} \qquad 0 \le x \le L, \quad t \ge 0$$

$$u(0,t) = u_0 \quad \text{and} \quad u(L,t) = 0 \qquad t \ge 0$$

$$u(x,0) = 0 \qquad 0 \le x \le L$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

This corresponds to the idealized situation of a region initially at zero temperature and whose left end $x = 0$ is instantaneously forced to assume the value u_0 for all time greater than zero.

With A = constant and $\alpha = k/\rho c_p$ the initial boundary value problem can be written as:

$$\alpha \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad 0 \le x \le L, \quad t \ge 0$$

$$u(0,t) = u_0 \quad \text{and} \quad u(L,t) = 0 \qquad t \ge 0$$

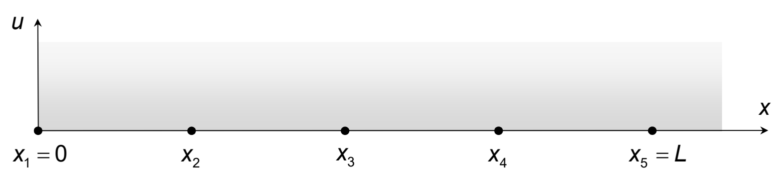
$$u(x,0) = 0 \qquad 0 \le x \le L$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

Discretization. For purposes of illustration, a four-element model will be investigated.



Assume all elements are of equal length $L/4$.

Interpolation. Linear interpolation will be used for the four elements.

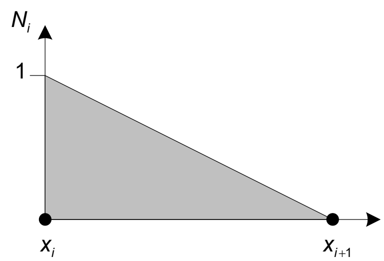
TIME-DEPENDENT PROBLEMS

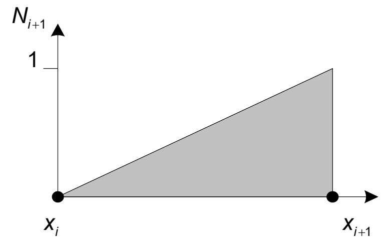
One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

Elemental Formulation.

$$N_i = \frac{x_{i+1} - x}{x_{i+1} - x_i} \qquad N_{i+1} = \frac{x - x_i}{x_{i+1} - x_i}$$





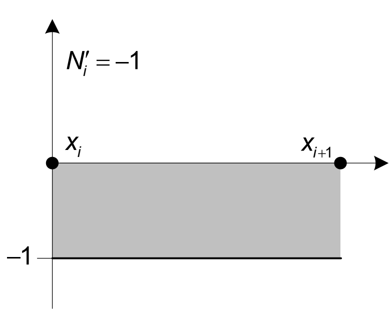
TIME-DEPENDENT PROBLEMS

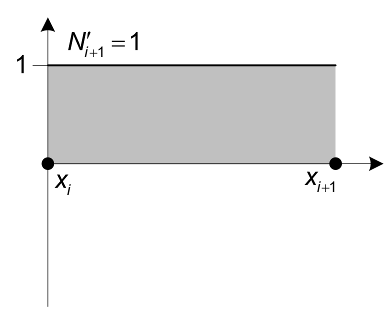
One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

Elemental Formulation. The elemental matrices are:

$$N'_i = -\frac{1}{x_{i+1} - x_i} \qquad N'_{i+1} = \frac{1}{x_{i+1} - x_i}$$





TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

Elemental Formulation. The elemental matrices are:

$$\mathbf{k_e} = \int_{x_i}^{x_{i+1}} \left(\mathbf{N}'\alpha\mathbf{N}'^{\mathsf{T}}\right) dx$$

$$= \int_{x_i}^{x_{i+1}} \begin{Bmatrix} -\dfrac{1}{x_{i+1}-x_i} \\ \dfrac{1}{x_{i+1}-x_i} \end{Bmatrix} \alpha \left\langle -\frac{1}{x_{i+1}-x_i} \quad \frac{1}{x_{i+1}-x_i} \right\rangle dx$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

Elemental Formulation. The elemental matrices are:

$$\mathbf{k_e} = \int_{x_i}^{x_{i+1}} \left(\mathbf{N}'\alpha\mathbf{N}'^{\mathsf{T}}\right) dx$$

$$= \frac{1}{\left(x_{i+1}-x_i\right)^2} \int_{x_i}^{x_{i+1}} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \alpha \left\langle -1 \quad 1 \right\rangle dx$$

$$= \frac{\alpha}{l_e^{\,2}} \int_{x_i}^{x_{i+1}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx = \frac{\alpha}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{4\alpha}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

Elemental Formulation. The elemental matrices are:

$$\mathbf{m_e} = \int_{x_i}^{x_{i+1}} \mathbf{N}\mathbf{N}^{\mathbf{T}}\, dx$$

$$\mathbf{m_{e1}} = \int_{x_i}^{x_{i+1}} \mathbf{N}\mathbf{N}^{\mathbf{T}}\, dx \;= \int_{x_i}^{x_{i+1}} \begin{Bmatrix} \dfrac{x_{i+1}-x}{x_{i+1}-x_i} \\ \dfrac{x-x_i}{x_{i+1}-x_i} \end{Bmatrix} \left\langle \dfrac{x_{i+1}-x}{x_{i+1}-x_i} \quad \dfrac{x-x_i}{x_{i+1}-x_i} \right\rangle dx$$

At $x_i = 0$ and $x_{i+1} = L/4$, then

$$\mathbf{m_{e1}} = \int_{0}^{L/4} \mathbf{N}\mathbf{N}^{\mathbf{T}}\, dx \;= \int_{0}^{L/4} \begin{Bmatrix} 1-\frac{4x}{L} \\ \frac{4x}{L} \end{Bmatrix} \left\langle 1-\frac{4x}{L} \quad \frac{4x}{L} \right\rangle dx \;= \frac{L}{24}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

Elemental Formulation. The elemental matrices are:

$$\mathbf{m_e} = \int_{x_i}^{x_{i+1}} \mathbf{N}\mathbf{N}^{\mathbf{T}}\, dx$$

$$\mathbf{m_{e1}} = \int_{0}^{1} \mathbf{N}\mathbf{N}^{\mathbf{T}}\, dx \;= \int_{0}^{1} \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} \left\langle 1-\xi \quad \xi \right\rangle l_e\, d\xi$$

$$= \int_{0}^{1} \begin{Bmatrix} (1-\xi)^2 & \xi(1-\xi) \\ \xi(1-\xi) & \xi^2 \end{Bmatrix} l_e\, d\xi \;= \frac{L}{24}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

Elemental Formulation. The elemental matrices are:

$$\mathbf{k_e} = \int_{x_i}^{x_{i+1}} \left(\mathbf{N}'\alpha\mathbf{N}'^{\mathrm{T}}\right)dx = \frac{4\alpha}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{m_e} = \int_{x_i}^{x_{i+1}} \left(\mathbf{N}\mathbf{N}^{\mathrm{T}}\right)dx = \frac{L}{24}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{q_e} = \int_{x_i}^{x_{i+1}} \left(q\mathbf{N}\right)dx = 0$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

Assembly. With both the boundary conditions essential, **BT**=0 and **bt**=0. It follows that the assembled equations are:

$$\mathbf{Au} + \mathbf{B\dot{u}} = 0 \qquad \mathbf{u} = \langle u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \rangle^{\mathrm{T}}$$

$$\mathbf{A} = \sum_e k_G = \frac{4\alpha}{L}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{B} = \sum_e m_G = \frac{L}{24}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The initial condition is homogeneous so that: $\mathbf{u}(0) = 0$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

Assembly. With both the boundary conditions essential, **BT**=0 and **bt**=0. It follows that the assembled equations are:

$$\mathbf{Au} + \mathbf{B\dot{u}} = 0$$

$$\frac{4\alpha}{L}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} + \frac{L}{24}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \dot{u}_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

Assembly. With both the boundary conditions essential, **BT**=0 and **bt**=0. It follows that the assembled equations are:

$$\mathbf{Au} + \mathbf{B\dot{u}} = 0 \qquad u_1 = u_0 \quad \text{and} \quad u_5 = 0$$

$$\frac{4\alpha}{L}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} + \frac{L}{24}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \dot{u}_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -4\alpha/L \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$u_1 \to u_0$, $u_5 \to 0$, $\dot{u}_1 = 0$, $\dot{u}_5 = 0$, $0 \to u_0$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

Constraints. The constraints equations are:

$$\psi\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} + \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}\begin{Bmatrix} \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \end{Bmatrix} = \begin{Bmatrix} \psi u_0 \\ 0 \\ 0 \end{Bmatrix}$$

Subject to the initial condition: $\mathbf{u}(0,t) = u_0$ $\qquad \psi = \dfrac{96\alpha}{L^2}$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

These approximate equations must now be integrated for an estimate of the time-dependent solution.

Appropriate analytical and numerical methods of integration are presented and discussed in the following sections.

$$\psi\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} + \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}\begin{Bmatrix} \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \end{Bmatrix} = \begin{Bmatrix} \psi u_0 \\ 0 \\ 0 \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

The elemental mass matrices $\mathbf{m_e}$ are referred to as **consistent mass matrices** in that they are determined on the basis of the same interpolation functions as were used for the corresponding stiffnesses $\mathbf{k_e}$.

$$\mathbf{m_e} = \int_{x_i}^{x_{i+1}} \left(\mathbf{N}\mathbf{N}^{\mathrm{T}}\right) dx$$

$$= \int_0^1 \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} \langle 1-\xi \quad \xi \rangle l_e d\xi = \frac{l_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

Another approach to generating mass matrices is referred to as *lumping* with the results referred to as **lumped mass matrices**. The idea is simply that the total mass associated with the consistent mass matrix:

$$\mathbf{m}_{ii}^{lumped} = \sum_j \mathbf{m}_{ij}$$

For a two-noded linear element the lumped mass matrix $\mathbf{m_{le}}$:

$$\mathbf{m_{le}} = \sum_j \mathbf{m}_{ij} = \frac{l_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{l_e}{6}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

This lumped mass matrix has advantages in certain of the time integration algorithms to be discussed in later sections.

Also, it has the interesting property that the resulting eigenvalues are generally smaller than the exact values.

Eigenvalues are generally overestimated when using the consistent mass matrices.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

This suggests that a third possibility for treating the mass is to consider a weighted average of the consistent mass matrix $\mathbf{m_{ce}}$ and lumped mass matrix $\mathbf{m_{le}}$ according to:

$$\mathbf{m_w} = \beta \mathbf{m_{ce}} + (1 - \beta)\mathbf{m_{le}}$$

If $\beta = ½$ then:

$$\mathbf{m_w} = \frac{1}{2}\left[\frac{l_e}{6}\left\{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right\}\right] = \frac{l_e}{12}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Example of One-Dimensional Diffusion

This is one of the so-called **higher-order accurate mass matrices.**

Its use results in improved estimates for the eigenvalues as compared with estimates using either of the consistent or lumped mass matrix formulations.

The corresponding result for the quadratically interpolated element turns out to be:

$$\mathbf{m_w} = \frac{l_e}{60}\begin{bmatrix} 9 & 2 & -1 \\ 2 & 36 & 2 \\ -1 & 2 & 9 \end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

An analytical approach to the integration of the set of equations decomposes the solution of:

$$\mathbf{Ku} + \mathbf{M\dot{u}} = \mathbf{f} \qquad \text{into} \qquad \mathbf{u} = \mathbf{u_h} + \mathbf{u_p}$$

where $\mathbf{u_h}$ is the homogeneous solution satisfying:

$$\mathbf{Ku_h} + \mathbf{M\dot{u}_h} = 0$$

and $\mathbf{u_p}$ is any particular solution satisfying:

$$\mathbf{Ku_p} + \mathbf{M\dot{u}_p} = \mathbf{f}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

Homogenous Solution. For the case where **K** and **M** are matrices of constants:

$$\mathbf{K}\mathbf{u}_{\mathbf{h}}(t) + \mathbf{M}\dot{\mathbf{u}}_{\mathbf{h}} = 0$$

is a set of linear constant-coefficient, ordinary differential equations.

When **K** and **M** are not constant matrices, it is necessary to use techniques such as discussed in the next section.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

Homogenous Solution. The standard approach to the solution of such a constant coefficient system is to assume:

$$\mathbf{u}_{\mathbf{h}}(t) = \mathbf{v}e^{-\lambda t}$$

where **v** is a vector of constants.

The negative sign in the exponential function is a matter of anticipating the decaying character of the solution of diffusion problems.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

Homogenous Solution. Substitute: $\mathbf{u_h}(t) = \mathbf{v}e^{-\lambda t}$

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{v}e^{-\lambda t} = 0 \qquad\qquad (\mathbf{K} - \lambda\mathbf{M})\mathbf{v} = 0$$

Thus the homogeneous solutions are obtained by solving the generalized linear algebraic eigenvalue problem.

Nontrivial solutions of this expression require:

$$\det(\mathbf{K} - \lambda\mathbf{M}) = 0$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

Homogenous Solution. The eigenvalues are obtained λ_1, λ_2, … and the corresponding eigenvectors v_1, v_2, … are then determined by back-substituting the eigenvalues one at a time.

The analytical approach is quite valuable from the stand-point that the results can be immediately interpreted in terms of the decay rates [exp($-\lambda_i t$) as determined by the eigenvalues] that will be present in the solution regardless of whether an analytical or a numerical approach is being used.

These eigenvalues are an important part of the discussions of convergence and stability covered in later sections.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

Homogenous Solution. For the particular example, the equations can be written as:

$$\det(\mathbf{K} - \lambda\mathbf{M}) = 0$$

$$\left| \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} - \frac{\lambda}{\psi} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \right| = 0 \qquad \psi = \frac{96\alpha}{L^2}$$

$$\left| \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} - \phi \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \right| = 0 \qquad \phi = \frac{\lambda L^2}{96\alpha}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

Homogenous Solution. For the particular example, the equations can be written as:

$$\det(\mathbf{K} - \lambda\mathbf{M}) = \begin{vmatrix} 2-4\phi & -(1-\phi) & 0 \\ -(1-\phi) & 2-4\phi & -(1-\phi) \\ 0 & -(1-\phi) & 2-4\phi \end{vmatrix} = 0$$

$$= -56\phi^3 + 76\phi^2 - 32\phi + 4 = 0$$

Solving for ϕ gives:

$$\phi_1 = 0.1082 \qquad \phi_2 = 0.5000 \qquad \phi_3 = 1.3204$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

Homogenous Solution. The roots of the corresponding characteristic equation:

$$\phi_1 = 0.1082 \qquad \phi_2 = 0.5000 \qquad \phi_3 = 1.3204$$

$$\lambda_1 = \frac{10.387\alpha}{L^2} \qquad \lambda_2 = \frac{48.000\alpha}{L^2} \qquad \lambda_3 = \frac{126.76\alpha}{L^2}$$

Which compares to the exact eigenvalues of:

$$\lambda_1 = \frac{\pi^2\alpha}{L^2} \qquad \lambda_2 = \frac{4\pi^2\alpha}{L^2} \qquad \lambda_3 = \frac{9\pi^2\alpha}{L^2}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

Homogenous Solution. The roots of the corresponding characteristic equation:

$$\phi_1 = 0.1082 \qquad \phi_2 = 0.5000 \qquad \phi_3 = 1.3204$$

$$\lambda_1 = \frac{10.387\alpha}{L^2} \qquad \lambda_2 = \frac{48.000\alpha}{L^2} \qquad \lambda_3 = \frac{126.76\alpha}{L^2}$$

Which compares to the exact eigenvalues of:

$$\lambda_1 = \frac{9.8696\alpha}{L^2} \qquad \lambda_2 = \frac{39.478\alpha}{L^2} \qquad \lambda_3 = \frac{88.826\alpha}{L^2}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

Homogenous Solution. It can easily be seen that, roughly, the solutions will decay to steady state too rapidly in view of the fact that the λ's predicted by the finite element solution are larger than the corresponding exact values.

$$\lambda_1 = \frac{10.387\alpha}{L^2} \qquad \lambda_2 = \frac{48.000\alpha}{L^2} \qquad \lambda_3 = \frac{126.76\alpha}{L^2}$$

Which compares to the exact eigenvalues of:

$$\lambda_1 = \frac{9.8696\alpha}{L^2} \qquad \lambda_2 = \frac{39.478\alpha}{L^2} \qquad \lambda_3 = \frac{88.826\alpha}{L^2}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

Homogenous Solution. The corresponding eigenvectors obtained by back-substitution:

$$v_1 = \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix} \qquad v_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \qquad v_3 = \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix}$$

And the homogeneous solution can then be written as:

$$\mathbf{u_h}(t) = c_1 v_1 e^{-\lambda_1 t} + c_2 v_2 e^{-\lambda_2 t} + c_3 v_3 e^{-\lambda_3 t}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

Particular Solution. For the particular solution, consider:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \mathbf{u_p} + \psi^{-1} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \dot{\mathbf{u}}_\mathbf{p} = \begin{Bmatrix} u_0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\psi = \frac{96\alpha}{L^2}$$

By inspection (Method of Intelligent Guessing!) it can be see that by taking $\mathbf{u_p}=\mathbf{d}$, a constant, there results:

$$\mathbf{d} = \frac{u_0}{4} \langle 3 \quad 2 \quad 1 \rangle^T$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

The general solution can then be written as:

$$\mathbf{u} = c_1 v_1 e^{-\lambda_1 t} + c_2 v_2 e^{-\lambda_2 t} + c_3 v_3 e^{-\lambda_3 t} + \mathbf{u_p}$$

Satisfying the initial conditions $\mathbf{u}(0) = 0$ leads to the set of linear algebraic equations:

$$\begin{matrix} u_2 \\ u_3 \\ u_4 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -1 & 1 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} -\frac{3u_0}{4} \\ -\frac{2u_0}{4} \\ -\frac{u_0}{4} \end{Bmatrix}$$

$$\quad v_1 \qquad v_2 \qquad v_3$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

Solving the set of linear algebraic equations yields:

$$c_1 = -0.4268u_0 \qquad c_2 = -0.2500u_0 \qquad c_3 = -0.0732u_0$$

The solution can finally be expressed as:

$$v_1 = \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix} \qquad v_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \qquad v_3 = \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix}$$

$$\mathbf{u} = c_1 v_1 e^{-\lambda_1 t} + c_2 v_2 e^{-\lambda_2 t} + c_3 v_3 e^{-\lambda_3 t} + \mathbf{u_p}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

Solving the set of linear algebraic equations yields:

$$c_1 = -0.4268u_0 \qquad c_2 = -0.2500u_0 \qquad c_3 = -0.0732u_0$$

The solution can finally be expressed as:

$$\frac{u_2(t)}{u_0} = 0.7500 + c_1(\ 1\ )e^{-\lambda_1 t} + c_2(\ 1\ )e^{-\lambda_2 t} + c_3(\ 1\ )e^{-\lambda_3 t}$$

$$\frac{u_3(t)}{u_0} = 0.5000 + c_1(\sqrt{2})e^{-\lambda_1 t} + c_2(\ 0)e^{-\lambda_2 t} + c_3(-\sqrt{2})e^{-\lambda_3 t}$$

$$\frac{u_4(t)}{u_0} = 0.2500 + c_1(\ 1\ )e^{-\lambda_1 t} + c_2(-1)e^{-\lambda_2 t} + c_3(\ 1\ )e^{-\lambda_3 t}$$

$\mathbf{u_p}$ $\quad v_1$ $\quad v_2$ $\quad v_3$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

Solving the set of linear algebraic equations yields:

$$c_1 = -0.4268u_0 \qquad c_2 = -0.2500u_0 \qquad c_3 = -0.0732u_0$$

The solution can finally be expressed as:

$$\frac{u_2(t)}{u_0} = 0.7500 - 0.4268e^{-\lambda_1 t} + 0.2500e^{-\lambda_2 t} - 0.0732e^{-\lambda_3 t}$$

$$\frac{u_3(t)}{u_0} = 0.5000 - 0.6036e^{-\lambda_1 t} + 0.1036e^{-\lambda_3 t}$$

$$\frac{u_4(t)}{u_0} = 0.2500 - 0.4268e^{-\lambda_1 t} + 0.2500e^{-\lambda_2 t} - 0.0732e^{-\lambda_3 t}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

Note that as:

$$t \to \infty,\ u_2(t) \to 0.7500u_0,\ u_3(t) \to 0.5000u_0,\ u_4(t) \to 0.2500u_0,$$

This is the correct steady state solution.

The solution can finally be expressed as:

$$\frac{u_2(t)}{u_0} = 0.7500 - 0.4268e^{-\lambda_1 t} + 0.2500e^{-\lambda_2 t} - 0.0732e^{-\lambda_3 t}$$

$$\frac{u_3(t)}{u_0} = 0.5000 - 0.6036e^{-\lambda_1 t} + 0.1036e^{-\lambda_3 t}$$

$$\frac{u_4(t)}{u_0} = 0.2500 - 0.4268e^{-\lambda_1 t} + 0.2500e^{-\lambda_2 t} - 0.0732e^{-\lambda_3 t}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

As mentioned previously, the finite element model predicts that these steady-state values are reached too quickly.

For a specific case assume that the bar is 0.2 m in length and is composed of an aluminum alloy for which $\alpha = 8.4 \times 10^{-4}$ m^2/s, from which $\alpha/L^2 = 0.021$ s^{-1}.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

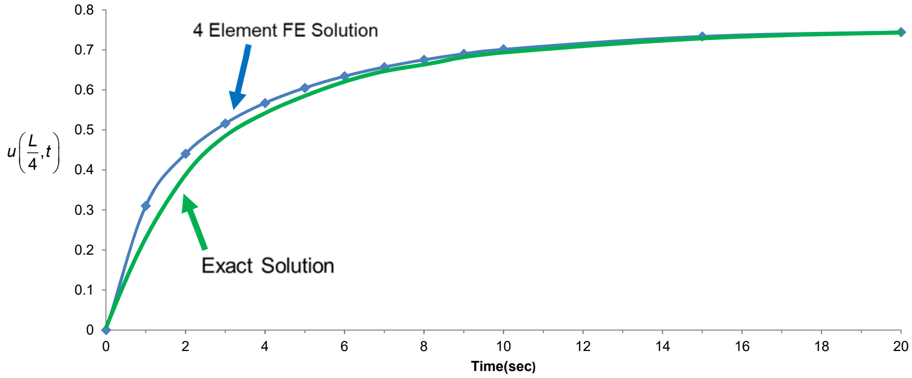
Four-element analytical versus exact solution

t (sec)	(u_2/u_0)	$(u_2/u_0)_{exact}$	(u_3/u_0)	$(u_3/u_0)_{exact}$	(u_4/u_0)	$(u_4/u_0)_{exact}$
1.0	0.3105	0.2225	0.0219	0.0147	-0.0070	0.0003
2.0	0.4404	0.3884	0.1103	0.0845	0.0070	0.0096
3.0	0.5106	0.4812	0.1863	0.1589	0.0403	0.0342
4.0	0.5672	0.5419	0.2478	0.2223	0.0761	0.0650
5.0	0.6050	0.5852	0.2972	0.2742	0.1082	0.0953
6.0	0.6341	0.6180	0.3369	0.3164	0.1353	0.1224
7.0	0.6571	0.6435	0.3689	0.3508	0.1575	0.1455
8.0	0.6754	0.6638	0.3946	0.3787	0.1755	0.1647
9.0	0.6900	0.6801	0.4152	0.4014	0.1901	0.1805
10.0	0.7018	0.6933	0.4319	0.4199	0.2018	0.1934
15.0	0.7338	0.7299	0.4771	0.4716	0.2338	0.2299
20.0	0.7446	0.7429	0.4923	0.4899	0.2446	0.2429

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

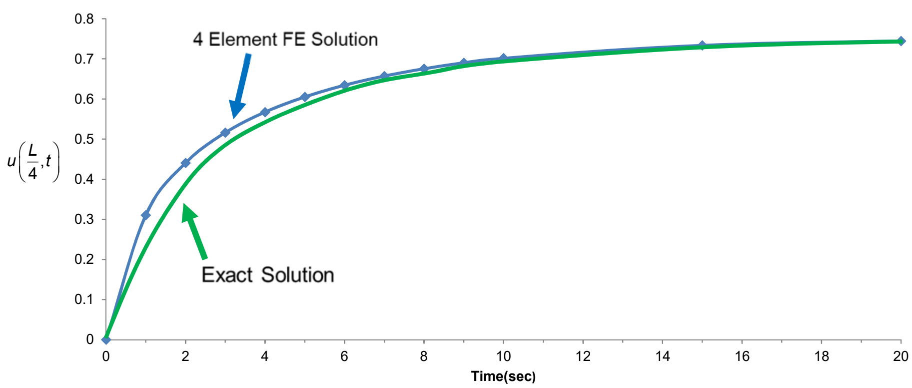


As mentioned previously and as can easily be seen from these data, the finite element solution tends towards steady state too rapidly.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

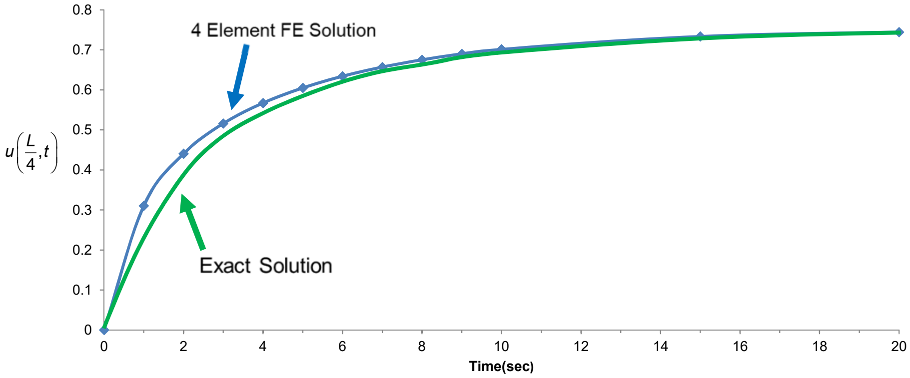


The trend is of course helped by taking more elements, in which case the approximate eigenvalues arising from the finite element model approach the exact.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

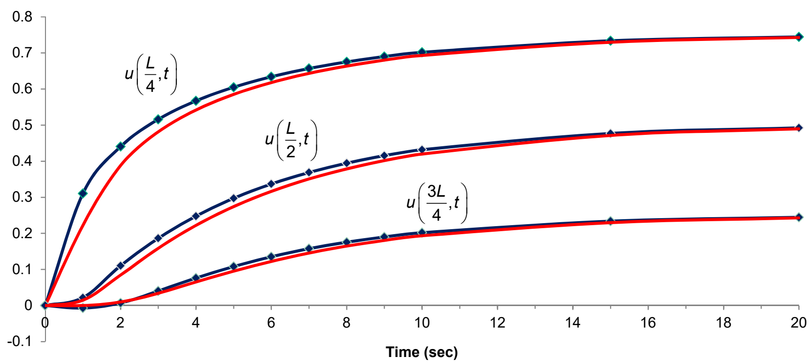


With more elements the results are correspondingly closer to those given by the exact solution.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques



With more elements the results are correspondingly closer to those given by the exact solution.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

Repeat the solution using the lumped mass matrix.

$$\det(\mathbf{K} - \lambda\mathbf{M}) = 0$$

$$\left| \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} - \phi \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \right| = 0 \qquad \phi = \frac{\lambda L^2}{96\alpha}$$

$$\det(\mathbf{K} - \lambda\mathbf{M}) = \begin{vmatrix} 2-6\phi & -1 & 0 \\ -1 & 2-6\phi & -1 \\ 0 & -1 & 2-6\phi \end{vmatrix} = 0$$

TIME-DEPENDENT PROBLEMS

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Repeat the solution using the lumped mass matrix.

$$\det(\mathbf{K} - \lambda\mathbf{M}) = -216\phi^3 + 216\phi^2 - 60\phi + 4 = 0$$

Solving for ϕ gives: $\phi_1 = 0.0976$; $\phi_2 = 0.3333$; $\phi_3 = 0.5690$

$$\lambda_1 = \frac{9.3726\alpha}{L^2} \qquad \lambda_2 = \frac{32.000\alpha}{L^2} \qquad \lambda_3 = \frac{54.627\alpha}{L^2}$$

Which compares to the exact eigenvalues of:

$$\lambda_1 = \frac{9.8696\alpha}{L^2} \qquad \lambda_2 = \frac{39.478\alpha}{L^2} \qquad \lambda_3 = \frac{88.826\alpha}{L^2}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

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The general solution can then be written as:

$$\mathbf{u} = c_1 v_1 e^{-\lambda_1 t} + c_2 v_2 e^{-\lambda_2 t} + c_3 v_3 e^{-\lambda_3 t} + \mathbf{u_p}$$

Satisfying the initial conditions **u**(0) = 0 leads to the set of linear algebraic equations:

$$\begin{matrix} u_2 \\ u_3 \\ u_4 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -1 & 1 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} -\frac{3u_0}{4} \\ -\frac{2u_0}{4} \\ -\frac{u_0}{4} \end{Bmatrix}$$

$$\quad v_1 \quad v_2 \quad v_3$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

Solving the set of linear algebraic equations yields:

$$c_1 = -0.4268u_0 \qquad c_2 = -0.2500u_0 \qquad c_3 = -0.0732u_0$$

The solution can finally be expressed as:

$$v_1 = \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix} \qquad v_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \qquad v_3 = \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix}$$

$$\mathbf{u} = c_1 v_1 e^{-\lambda_1 t} + c_2 v_2 e^{-\lambda_2 t} + c_3 v_3 e^{-\lambda_3 t} + \mathbf{u_p}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

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$$c_1 = -0.4268u_0 \qquad c_2 = -0.2500u_0 \qquad c_3 = -0.0732u_0$$

The solution can finally be expressed as:

$$\frac{u_2(t)}{u_0} = 0.7500 + c_1(\ 1\ )e^{-\lambda_1 t} + c_2(\ 1\ )e^{-\lambda_2 t} + c_3(\ 1\ )e^{-\lambda_3 t}$$
$$\frac{u_3(t)}{u_0} = 0.5000 + c_1(\sqrt{2})e^{-\lambda_1 t} + c_2(\ 0)e^{-\lambda_2 t} + c_3(-\sqrt{2})e^{-\lambda_3 t}$$
$$\frac{u_4(t)}{u_0} = 0.2500 + c_1(\ 1\ )e^{-\lambda_1 t} + c_2(-1)e^{-\lambda_2 t} + c_3(\ 1\ )e^{-\lambda_3 t}$$

$\mathbf{u_p}$ $\quad v_1$ $\quad v_2$ $\quad v_3$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

Solving the set of linear algebraic equations yields:

$$c_1 = -0.4268u_0 \qquad c_2 = -0.2500u_0 \qquad c_3 = -0.0732u_0$$

The solution can finally be expressed as:

$$\frac{u_2(t)}{u_0} = 0.7500 - 0.4268e^{-\lambda_1 t} + 0.2500e^{-\lambda_2 t} - 0.0732e^{-\lambda_3 t}$$
$$\frac{u_3(t)}{u_0} = 0.5000 - 0.6036e^{-\lambda_1 t} + 0.1036e^{-\lambda_3 t}$$
$$\frac{u_4(t)}{u_0} = 0.2500 - 0.4268e^{-\lambda_1 t} + 0.2500e^{-\lambda_2 t} - 0.0732e^{-\lambda_3 t}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

As mentioned previously, the finite element model predicts that these steady-state values are reached too quickly.

For a specific case assume that the bar is 0.2 m in length and is composed of an aluminum alloy for which $\alpha = 8.4 \times 10^{-4}$ m^2/s, from which $\alpha/L^2 = 0.021$ s^{-1}.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

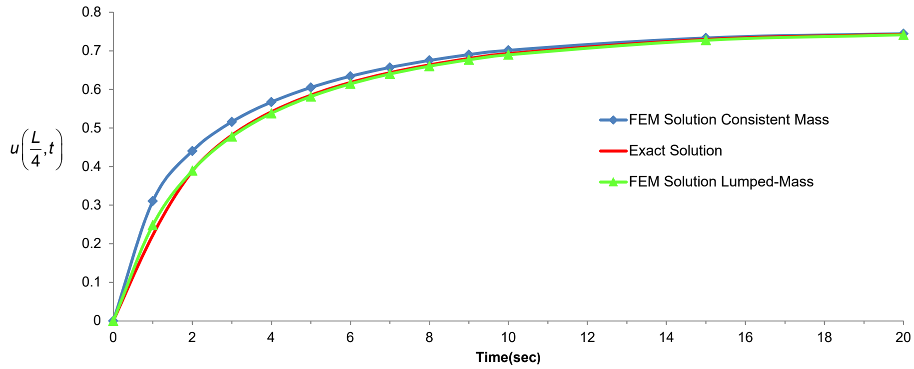
Four-element analytical versus exact solution

t (sec)	(u_2/u_0)	$(u_2/u_0)_{exact}$	(u_3/u_0)	$(u_3/u_0)_{exact}$	(u_4/u_0)	$(u_4/u_0)_{exact}$
1.0	0.2485	0.2225	0.0371	0.0147	0.0039	0.0003
2.0	0.3895	0.3884	0.1033	0.0845	0.0199	0.0096
3.0	0.4779	0.4812	0.1689	0.1589	0.0445	0.0342
4.0	0.5380	0.5419	0.2264	0.2223	0.0720	0.065
5.0	0.5816	0.5852	0.2747	0.2742	0.0989	0.0953
6.0	0.6145	0.6180	0.3148	0.3164	0.1233	0.1224
7.0	0.6401	0.6435	0.3478	0.3508	0.1446	0.1455
8.0	0.6604	0.6638	0.3750	0.3787	0.1628	0.1647
9.0	0.6768	0.6801	0.3973	0.4014	0.1780	0.1805
10.0	0.6901	0.6933	0.4157	0.4199	0.1907	0.1934
15.0	0.7277	0.7299	0.4685	0.4716	0.2277	0.2299
20.0	0.7417	0.7429	0.4882	0.4899	0.2417	0.2429

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques

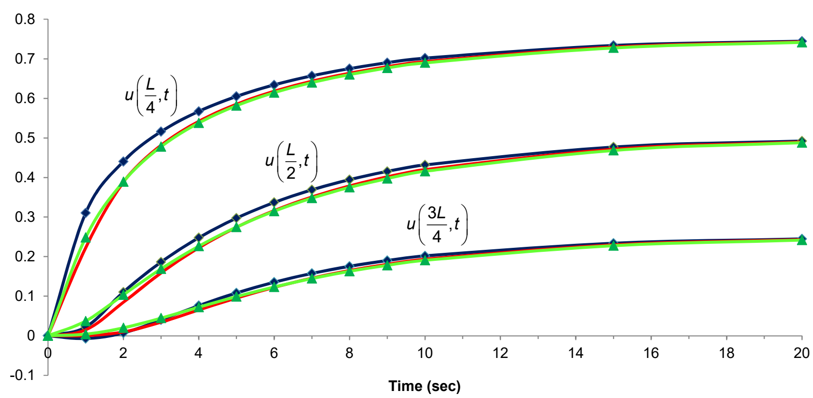


The trend is of course helped by taking more elements, in which case the approximate eigenvalues arising from the finite element model approach the exact.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Analytical Integration Techniques



As you can see, the lumped-mass solution is better than the consistent mass solution, even with just a few elements.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

In situations where **M** and/or **K** in:

$$\mathbf{M}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f} \qquad \text{with} \qquad \mathbf{u}(0) = \mathbf{u}_0$$

are functions of *t* or where **f** is such that a particular solution by analytic means is difficult or impossible, the analytical technique discussed may be practically impossible to carry through.

Depending on the particulars, numerical techniques are attractive or even necessary to carry out the integration.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

The Euler Method - Recall that for a first-order initial value problem of the form:

$$y' = f(x, y) \qquad \text{with} \qquad y(x_0) = y_0$$

it is possible to develop numerical integration schemes for the approximate integration of the initial value problem.

The problem is to determine a function $y(x)$ passing through the initial point (x_0, y_0) and satisfying the differential equation $y' = f$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

The Euler Method - Recall that for a first-order initial value problem of the form:

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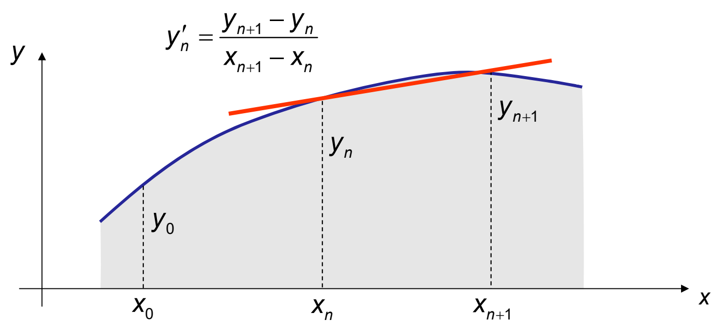


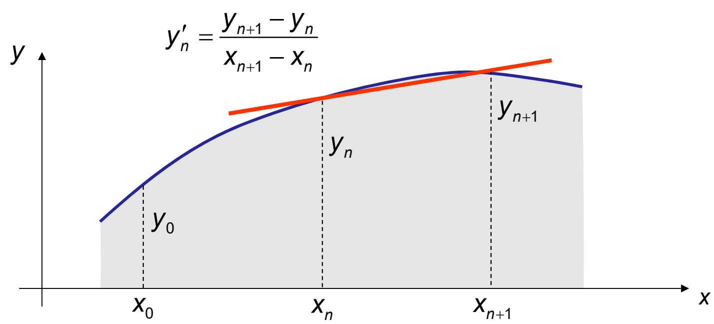
TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

In the **Euler Method**, the derivative y′ is represented as a **forward difference** according to:

$$y' = \frac{y_{n+1} - y_n}{x_{n+1} - x_n}$$



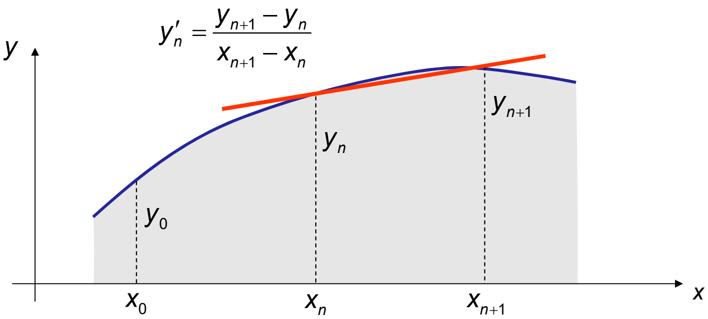
TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

Giving a linear approximation to the derivative at x_n with $h = x_{n+1} - x_n$. The differential equation at x_n is:

$$y' = \frac{y_{n+1} - y_n}{h} = f\left(x_n, y_n\right)$$

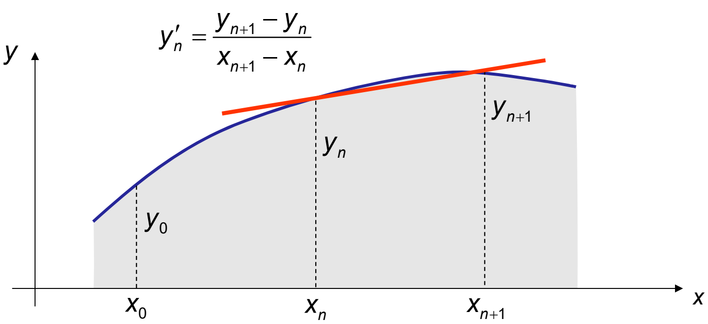


TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

Solving for y_{n+1} results in: $y_{n+1} = y_n + hf\left(x_n, y_n\right)$



TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

Starting with $y(x_0) = y_0$, this algorithm can be used to step ahead in the independent variable x to determine an approximate solution.

Note that the Euler Method can also be viewed in the following manner.

Integrate the differential equation between the limits of x_n and x_{n+1} to obtain:

$$\int_{x_n}^{x_{n+1}} y'\,dx = \frac{y_{n+1} - y_n}{x_{n+1} - x_n} x \Big|_{x_n}^{x_{n+1}} = y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} f(x, y)\,dx$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

Approximate the remaining integral by $hf(x_n, y_n)$ to obtain:

$$y_{n+1} = y_n + hf(x_n, y_n)$$

The integral has been approximated by evaluating $\boldsymbol{f}$ at the left end of the interval over which the integral is evaluated and multiplying by the interval h.

This is again clearly the Euler algorithm.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

The analogue of the Euler Method for the system of equations:

$$\mathbf{M}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f} \qquad \text{with} \qquad \mathbf{u}(0) = \mathbf{u}_0$$

is obtained by again representing the derivative term at a particular value of the time t_n as a forward difference according to:

$$\dot{\mathbf{u}}_n = \frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{h} \qquad h = t_{n+1} - t_n$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

And evaluating the differential equation at $t = t_n$ to obtain:

$$\mathbf{M}\dot{\mathbf{u}}_n + \mathbf{K}\mathbf{u}_n = \mathbf{f}_n \; = \; \mathbf{M}\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{h} + \mathbf{K}\mathbf{u}_n$$

From which the Euler algorithm is obtained:

$$\mathbf{M}\mathbf{u}_{n+1} = \left(\mathbf{M} - h\mathbf{K}\right)\mathbf{u}_n + h\mathbf{f}_n \qquad \mathbf{u}(0) = u_0$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

And evaluating the differential equation at $t = t_n$ to obtain:

$$n = 0 \qquad \mathbf{M}\mathbf{u}_1 = (\mathbf{M} - h\mathbf{K})\mathbf{u}_0 + h\mathbf{f}_0 \rightarrow \boxed{\mathbf{u}_1}$$

$$n = 1 \qquad \mathbf{M}\mathbf{u}_2 = (\mathbf{M} - h\mathbf{K})\mathbf{u}_1 + h\mathbf{f}_1 \rightarrow \boxed{\mathbf{u}_2}$$

$$n = 2 \qquad \mathbf{M}\mathbf{u}_3 = (\mathbf{M} - h\mathbf{K})\mathbf{u}_2 + h\mathbf{f}_2 \rightarrow \boxed{\mathbf{u}_3}$$

$$\vdots$$

The utility and effectiveness of the algorithm is affected by its ***stability****,* that is, by whether for large time the solution predicted by the algorithm remains finite, independent of the step size *h.*

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

And evaluating the differential equation at $t = t_n$ to obtain:

$$n = 0 \qquad \mathbf{M}\mathbf{u}_1 = (\mathbf{M} - h\mathbf{K})\mathbf{u}_0 + h\mathbf{f}_0 \rightarrow \boxed{\mathbf{u}_1}$$

$$n = 1 \qquad \mathbf{M}\mathbf{u}_2 = (\mathbf{M} - h\mathbf{K})\mathbf{u}_1 + h\mathbf{f}_1 \rightarrow \boxed{\mathbf{u}_2}$$

$$n = 2 \qquad \mathbf{M}\mathbf{u}_3 = (\mathbf{M} - h\mathbf{K})\mathbf{u}_2 + h\mathbf{f}_2 \rightarrow \boxed{\mathbf{u}_3}$$

$$\vdots$$

The Euler algorithm is ***conditionally stable***: there is a critical step size h_{cr} such that when $h > h_{cr}$ the solution oscillates with ever increasing amplitude, obviously negating the results of the algorithm.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

The use of the Euler algorithm results in a local discretization error *e,* which depends upon the step size *h* according to:

$$e = O\left(h^2\right)$$

indicating that when the time step *h* is halved the local discretization error is reduced approximately by 1/4.

The accumulated discretization error *E* is given by:

$$E = O\left(h\right)$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

The accumulated discretization error is approximately halved by halving the integration time step.

One might assume on the basis of these results that by decreasing the step size sufficiently the error could be decreased indefinitely.

This is not the case in that the roundoff error begins to dominate the process for *h* too small.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

Example 1 – Consider the application of the Euler algorithm to the set of equations previously developed for the four-element problem:

$$\mathbf{Ku} + \mathbf{M\dot{u}} = \mathbf{f} \qquad \mathbf{w} = \frac{\mathbf{u}}{u_0} \qquad \phi = \frac{96\alpha}{L^2}$$

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \dot{\mathbf{w}} + \phi \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \mathbf{w} = \phi \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{Mw}_{n+1} = \left(\mathbf{M} - h\mathbf{K}\right)\mathbf{w}_n + h\mathbf{f}_n$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

These equations become:

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \mathbf{w}_{n+1} = \left\{ \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} - \phi h \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \right\} \mathbf{w}_n + \phi h \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

For this set of equation, the inverse of **M** can be determined, so that the equations may be written as:

$$\mathbf{w}_{n+1} = \left(\mathbf{I} - h\mathbf{M}^{-1}\mathbf{K}\right)\mathbf{w}_n + h\mathbf{M}^{-1}\mathbf{f}_n$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

These equations become:

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \mathbf{w}_{n+1} = \left\{ \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} - \phi h \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \right\} \mathbf{w}_n + \phi h \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

For this set of equation, the inverse of **M** can be determined, so that the equations may be written as:

$$\mathbf{w}_{n+1} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{\phi h}{56} \begin{bmatrix} 34 & -24 & 6 \\ -24 & 40 & -24 \\ -6 & -24 & 34 \end{bmatrix} \right\} \mathbf{w}_n + \frac{\phi h}{56} \begin{Bmatrix} 15 \\ -4 \\ 1 \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

With α/L^2 = 0.021 s^{-1} and h = 0.1 s the equations become:

$$\mathbf{w}_{n+1} = \begin{bmatrix} 0.8776 & 0.0864 & -0.0216 \\ 0.0864 & 0.8560 & 0.0864 \\ -0.0216 & 0.0864 & 0.8776 \end{bmatrix} \mathbf{w}_n + \begin{Bmatrix} 0.0540 \\ -0.0144 \\ 0.0036 \end{Bmatrix}$$

$$\mathbf{w}_{n+1} = \mathbf{S}\mathbf{w}_n + \mathbf{b}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

With w(0) = 0, successive iterations of the equations give:

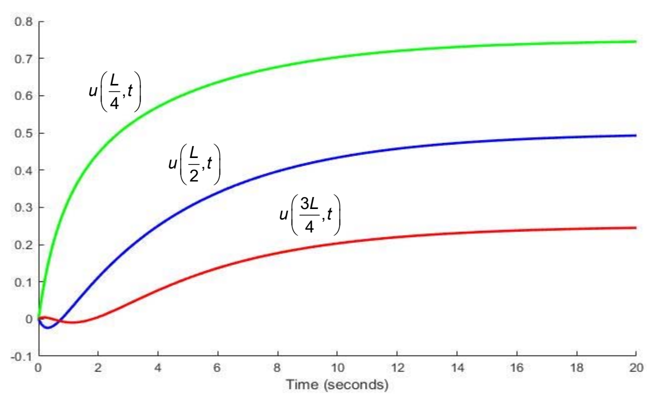
$$\mathbf{w}_1 = \mathbf{b} = \langle 0.0540 \quad -0.0144 \quad 0.0036 \rangle^T$$
$$\mathbf{w}_2 = \mathbf{S}\mathbf{w}_1 + \mathbf{b} = \langle 0.1001 \quad -0.0217 \quad 0.0043 \rangle^T$$
$$\mathbf{w}_3 = \mathbf{S}\mathbf{w}_2 + \mathbf{b} = \langle 0.1398 \quad -0.0240 \quad 0.0034 \rangle^T$$
$$\mathbf{w}_4 = \mathbf{S}\mathbf{w}_3 + \mathbf{b} = \langle 0.1746 \quad -0.0226 \quad 0.0015 \rangle^T$$
$$\vdots$$

The algorithm is repeatedly applied until the range of times of interest is covered.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

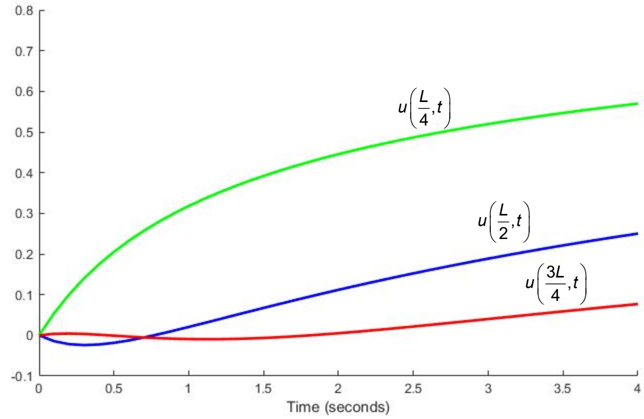


$h = 0.1s$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

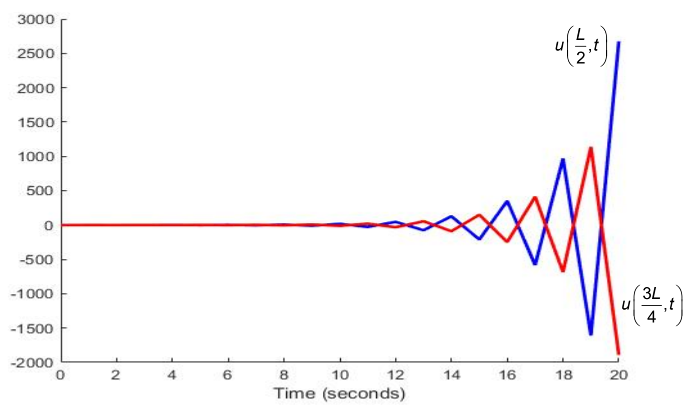


$h = 0.1s$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

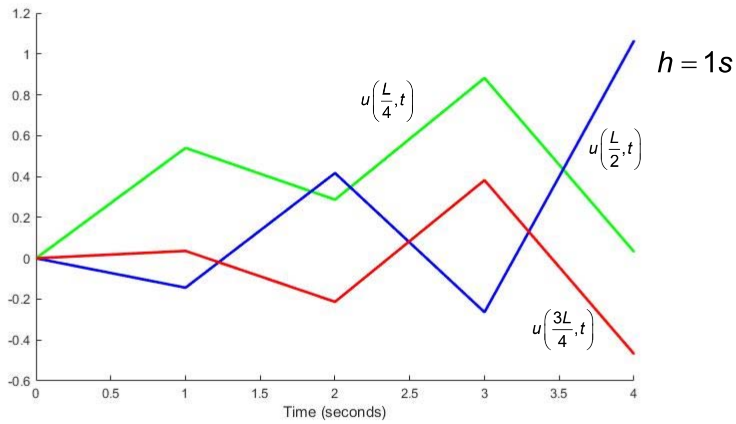


$h = 1s$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems



TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

The algorithm in the example is used with several values of the step size $h = \Delta t$.

Approximate solution $w_3(t)$ for different h using the Euler algorithm

t (sec)	h = 0.083	h = 0.167	h = 0.333	h = 1.0	u_{exact}
1	0.0208	0.0198	0.0189	-0.1440	0.0147
2	0.1116	0.1131	0.1163	0.4170	0.0845
3	0.1881	0.1901	0.1940	-0.2683	0.1589
4	0.2497	0.2518	0.2560	1.0643	0.2223
5	0.2991	0.3013	0.3055	0.9891	0.2742
6	0.3388	0.3409	0.3449	2.5436	0.3164
7	0.3706	0.3726	0.3763	-3.2333	0.3508
8	0.3962	0.3980	0.4014	6.4408	0.3787
9	0.4167	0.4183	0.4214	-9.5789	0.4014
10	0.4331	0.4346	0.4373	17.0888	0.4199
20	0.4926	0.4929	0.4935	2674.49	0.4899

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

These results show clearly several aspects of the numerical solution:

1. The approximate solution tends toward steady state more rapidly than the exact solution. This property is primarily attributable to the eigenvalues of **M** - λ**K** being larger than the exact eigenvalues.
2. There is clearly a step size h above which the approximate solution is unstable as indicated by the h = 1.0 results.
3. When the step size is not exceeded, the approximate solution approaches the correct steady-state values for large t.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

These results show clearly several aspects of the numerical solution:

4. If lumping is used, the assembled mass matrix **M** is diagonal and easily inverted. The integration procedure is reduced at each step to a simple matrix multiplication and vector addition:

$$\mathbf{u}_{n+1} = \mathbf{S}\mathbf{u}_n + h\mathbf{M}^{-1}\mathbf{f}_n$$

where if **M** and **K** are constant matrices, $\mathbf{S} = \mathbf{I} - h\mathbf{M}^{-1}\mathbf{K}$ can be computed at the first time step and used thereafter for the subsequent applications of the basic algorithm.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

Example 1 – Consider the Euler algorithm for the four-element problem using the lumped mass matrix:

$$\mathbf{Ku} + \mathbf{M}\dot{\mathbf{u}} = \mathbf{f} \qquad \mathbf{w} = \frac{\mathbf{u}}{u_0} \qquad \phi = \frac{96\alpha}{L^2}$$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \dot{\mathbf{w}} + \phi \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \mathbf{w} = \phi \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{Mw}_{n+1} = \left(\mathbf{M} - h\mathbf{K}\right)\mathbf{w}_n + h\mathbf{f}_n$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

These equations become:

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \mathbf{w}_{n+1} = \left\{ \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} - \phi h \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \right\} \mathbf{w}_n + \phi h \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

For this set of equation, the inverse of **M** can be determined, so that the equations may be written as:

$$\mathbf{w}_{n+1} = \left(\mathbf{I} - h\mathbf{M}^{-1}\mathbf{K}\right)\mathbf{w}_n + h\mathbf{M}^{-1}\mathbf{f}_n$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

These equations become:

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \mathbf{w}_{n+1} = \left\{ \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} - \phi h \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \right\} \mathbf{w}_n + \phi h \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

For this set of equation, the inverse of **M** can be determined, so that the equations may be written as:

$$\mathbf{w}_{n+1} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{\phi h}{6} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ -6 & -1 & 2 \end{bmatrix} \right\} \mathbf{w}_n + \frac{\phi h}{6} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

With α/L^2 = 0.021 s^{-1} and h = 0.1 s the equations become:

$$\mathbf{w}_{n+1} = \begin{bmatrix} 0.9328 & 0.0336 & 0 \\ 0.0336 & 0.9328 & 0.0336 \\ 0 & 0.0336 & 0.9328 \end{bmatrix} \mathbf{w}_n + \begin{Bmatrix} 0.0336 \\ 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{w}_{n+1} = \mathbf{S}\mathbf{w}_n + \mathbf{b}$$

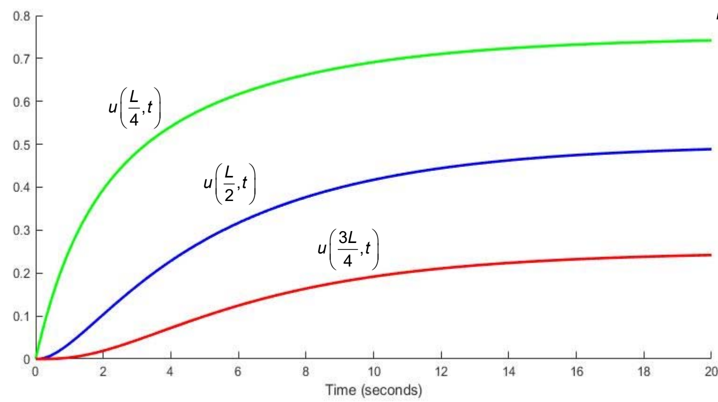
TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

With lumped mass matrix the solution become:

$h = 0.1s$



TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

With lumped mass matrix the solution become:

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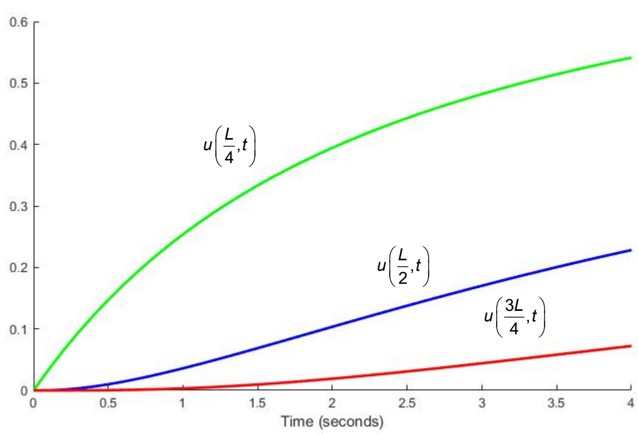


TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

The Improved Euler or Crank-Nicolson Method - The *improved* Euler or Crank-Nicolson algorithm can be thought of is the following way.

$$y' = f(x,y) \qquad \text{with} \qquad y(x_0) = y_0$$

The improved Euler algorithm is developed according to:

$$y_{n+1} = y_n + \frac{h\left[f\left(x_n, y_n\right) + f\left(x_{n+1}, y_{n+1}\right)\right]}{2}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

That is, the integral:

$$\int_{x_n}^{x_{n+1}} f\left(x, y\right) dx$$

has been evaluated by taking the average of *f* at the ends of the interval (x_n, x_{n+1}).

For the vector equation in question, the corresponding expression is:

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \frac{h\left(\dot{\mathbf{u}}_n + \dot{\mathbf{u}}_{n+1}\right)}{2}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

This is equivalent to a central difference representation for the derivative $\dot{\mathbf{u}}$. Multiplying through by **M** yields:

$$\mathbf{M}\mathbf{u}_{n+1} = \mathbf{M}\mathbf{u}_n + \frac{h\left(\mathbf{M}\dot{\mathbf{u}}_n + \mathbf{M}\dot{\mathbf{u}}_{n+1}\right)}{2}$$

which, on using the differential equation $\mathbf{M}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}$, can be written as:

$$\mathbf{M}\dot{\mathbf{u}} = \mathbf{f} - \mathbf{K}\mathbf{u} \rightarrow \begin{cases} \mathbf{M}\dot{\mathbf{u}}_n = \mathbf{f}_n - \mathbf{K}\mathbf{u}_n \\ \mathbf{M}\dot{\mathbf{u}}_{n+1} = \mathbf{f}_{n+1} - \mathbf{K}\mathbf{u}_{n+1} \end{cases}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

This is equivalent to a central difference representation for the derivative $\dot{\mathbf{u}}$. Multiplying through by **M** yields:

$$\mathbf{M}\mathbf{u}_{n+1} = \mathbf{M}\mathbf{u}_n + \frac{h\left(\mathbf{f}_n - \mathbf{K}\mathbf{u}_n + \mathbf{f}_{n+1} - \mathbf{K}\mathbf{u}_{n+1}\right)}{2}$$

which, on using the differential equation $\mathbf{M}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}$, can be written as:

$$\mathbf{M}\dot{\mathbf{u}} = \mathbf{f} - \mathbf{K}\mathbf{u} \rightarrow \begin{cases} \mathbf{M}\dot{\mathbf{u}}_n = \mathbf{f}_n - \mathbf{K}\mathbf{u}_n \\ \mathbf{M}\dot{\mathbf{u}}_{n+1} = \mathbf{f}_{n+1} - \mathbf{K}\mathbf{u}_{n+1} \end{cases}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

This is equivalent to a central difference representation for the derivative $\mathring{\mathbf{u}}$. Multiplying through by **M** yields:

$$\mathbf{M}\mathbf{u}_{n+1} = \mathbf{M}\mathbf{u}_n + \frac{h\left(\mathbf{f}_n - \mathbf{K}\mathbf{u}_n + \mathbf{f}_{n+1} - \mathbf{K}\mathbf{u}_{n+1}\right)}{2}$$

$$\left(\mathbf{M} + \frac{h\mathbf{K}}{2}\right)\mathbf{u}_{n+1} = \left(\mathbf{M} - \frac{h\mathbf{K}}{2}\right)\mathbf{u}_n + \frac{h\left(\mathbf{f}_n + \mathbf{f}_{n+1}\right)}{2}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

This is the improved Euler or Crank-Nicolson algorithm.

As opposed to the algorithm studied in the last section, this algorithm is ***unconditionally stable***, that is, although the accuracy may suffer considerably and oscillations occur for a large step size *h,* the oscillations never become unbounded.

In addition to being unconditionally stable, the improved Euler or Crank-Nicolson algorithm is one order more accurate than the previously developed Euler algorithm in that the accumulated discretization error E is given approximately by $E = O(h^2)$.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

To implement the method, write the equations as:

$$\mathbf{A}\mathbf{u}_{n+1} = \mathbf{b}_{n+1}$$

$$\mathbf{A} = \mathbf{M} + \frac{h\mathbf{K}}{2} \qquad \mathbf{b}_{n+1} = \frac{h\left(\mathbf{f}_n + \mathbf{f}_{n+1}\right)}{2} + \left(\mathbf{M} - \frac{h\mathbf{K}}{2}\right)\mathbf{u}_n$$

This set of linear algebraic equations must be solved at each time step.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

Given u(0) = u_0,

$$\mathbf{A}\mathbf{u}_1 = \mathbf{b}_1 = \frac{h\left[\mathbf{f}(0) + \mathbf{f}(h)\right]}{2} + \left(\mathbf{M} - \frac{h\mathbf{K}}{2}\right)\mathbf{u}_0$$

is to be solved for u_1, after which:

$$\mathbf{A}\mathbf{u}_2 = \mathbf{b}_2 = \frac{h\left[\mathbf{f}(h) + \mathbf{f}(2h)\right]}{2} + \left(\mathbf{M} - \frac{h\mathbf{K}}{2}\right)\mathbf{u}_1$$

is to be solved for u_2, and so forth.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

As long as h = constant, the same coefficient matrix **M**+h**K**/2 is involved at each step and it is economical to use an equation solver that decomposes **A** at the first step according to **A** = **LU**.

This decomposition is then saved so that at each succeeding step, two triangular systems can be solved.

This is substantially more economical than to solve **Ax** = **b** at each step.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

In the event that **M** and **K** are functions of t, then **M**, **K**, **M** + h**K**/2, and **M** - h**K**/2 must potentially be recomputed at each step.

If the variation of these matrices with time is small, recalculation of the necessary matrices and decomposition of **M** + h**K**/2 can be done at suitable regular intervals rather than at each time step.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

Example 2 - Consider the previous example with the following set of equations

$$\mathbf{Ku} + \mathbf{M\dot{u}} = \mathbf{f} \qquad \mathbf{w} = \frac{\mathbf{u}}{u_0} \qquad \phi = \frac{96\alpha}{L^2}$$

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \mathbf{\dot{w}} + \phi \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \mathbf{w} = \phi \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

Generally,

$$\mathbf{M} \pm \frac{h\mathbf{K}}{2} = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \pm \frac{48\alpha h}{L^2} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Taking $\alpha = 8.4 \times 10^{-4}$ m^2/s, $L = 0.2$ m, and $h = 0.1$s results in:

$$\mathbf{M} + \frac{h\mathbf{K}}{2} = \begin{bmatrix} 4.2016 & 0.8992 & 0.0000 \\ 0.8992 & 4.2016 & 0.8992 \\ 0.0000 & 0.8992 & 4.2016 \end{bmatrix}$$

$$\mathbf{M} - \frac{h\mathbf{K}}{2} = \begin{bmatrix} 3.7984 & 1.1008 & 0.0000 \\ 1.1008 & 3.7984 & 1.1008 \\ 0.0000 & 1.1008 & 3.7984 \end{bmatrix}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

The 3 x 3 matrix **M** + *h***K**/2 can be inverted so that there results specifically:

$$\mathbf{w}_{n+1} = \begin{bmatrix} 0.8879 & 0.0754 & -0.0161 \\ 0.0754 & 0.8718 & 0.0754 \\ -0.0161 & 0.0754 & 0.8879 \end{bmatrix} \mathbf{w}_n + \begin{Bmatrix} 0.0504 \\ -0.0113 \\ 0.0024 \end{Bmatrix}$$

$$\mathbf{w}_{n+1} = \mathbf{S}\mathbf{w}_n + \mathbf{b} \qquad \mathbf{w}(0) = 0$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

The first couple of iterations yield:

$$\mathbf{w}_1 = \mathbf{b} = \langle 0.0540 \quad -0.0113 \quad 0.0024 \rangle^T$$

$$\mathbf{w}_2 = \mathbf{S}\mathbf{w}_1 + \mathbf{b} = \langle 0.1011 \quad -0.0169 \quad 0.0028 \rangle^T$$

$$\mathbf{w}_3 = \mathbf{S}\mathbf{w}_2 + \mathbf{b} = \langle 0.1424 \quad -0.0182 \quad 0.0020 \rangle^T$$

$$\mathbf{w}_4 = \mathbf{S}\mathbf{w}_3 + \mathbf{b} = \langle 0.1790 \quad -0.0163 \quad 0.0005 \rangle^T$$

$$\vdots$$

with iteration being continued until the time interval of interest is covered.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

For this example, results are given for several values of *h* for $u_3(t)$ along with those as given by the exact solution.

Approximate solution $u_3(t)$ for different *h* using the improved Euler algorithm

t (sec)	*h* = 0.083	*h* = 0.167	*h* = 0.333	*h* = 1.0	u_{exact}
1	0.0219	0.0216	0.0207	0.0004	0.0147
2	0.1103	0.1103	0.1102	0.1126	0.0845
3	0.1863	0.1863	0.1864	0.1868	0.1589
4	0.2478	0.2478	0.2479	0.2487	0.2223
5	0.2972	0.2972	0.2973	0.2981	0.2742
6	0.3369	0.3370	0.3370	0.3378	0.3164
7	0.3689	0.3689	0.3690	0.3697	0.3508
8	0.3946	0.3946	0.3947	0.3953	0.3787
9	0.4153	0.4153	0.4153	0.4159	0.4014
10	0.4319	0.4319	0.4319	0.4324	0.4199
20	0.4923	0.4923	0.4923	0.4924	0.4899

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

Time Integration Techniques – First-Order Systems

Comparing these results with the results presented using the Euler method; the following observations can be made:

1. The results using the improved Euler algorithm are more accurate than those from the Euler algorithm, with steady state not being approached as rapidly using the improved Euler algorithm.
2. For the values of *h* investigated, the improved Euler algorithm is stable.
3. Lumping does not result in any computational advantage for the improved Euler algorithm since the coefficient matrix is $\mathbf{M} + h\mathbf{K}/2$ rather than $\mathbf{M}$ as for the Euler algorithm.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

ANALYSIS OF ALGORITHMS

Both the Euler and improved Euler or Crank-Nicolson algorithms presented in the preceding sections can be considered as special cases of the so-called θ algorithm that assumes:

$$\mathbf{u}_{n+1} = \mathbf{u}_n + h\left[\theta\dot{\mathbf{u}}_{n+1} + (1-\theta)\dot{\mathbf{u}}_n\right]$$

that is, a weighted sum of the derivatives at the beginning and end of the interval (t_n, t_{n+1}) is used to evaluate the integral:

$$\int_{t_n}^{t_{n+1}} \dot{\mathbf{u}}_n dt$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

ANALYSIS OF ALGORITHMS

Multiplying the θ algorithm by **M** and subsequently using the differential equation to eliminate the **Mu** terms yields:

$$(\mathbf{M} + h\theta\mathbf{K})\mathbf{u}_{n+1} = (\mathbf{M} - h(1-\theta)\mathbf{K})\mathbf{u}_n + h\left[\theta\mathbf{f}_{n+1} + (1-\theta)\mathbf{f}_n\right]$$

It is easily seen that: $\begin{cases} \theta = 0 & \text{Euler method} \\ \theta = \tfrac{1}{2} & \text{Crank-Nicolson method} \end{cases}$

The value $\theta = 1$ corresponds to what is referred to as the *modified* Euler method and corresponds to using a **backward difference scheme** obtained by evaluating the differential equation at t_{n+1} and taking:

$$\dot{\mathbf{u}}_{n+1} = \frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{h}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

ANALYSIS OF ALGORITHMS

Example - As a very simple demonstration, consider a two-element model for the previous problem.

$$\dot{w} + \phi w = \frac{\phi}{2} \qquad \text{with} \qquad w(0) = 0$$

where $w = u_2/u_0$ and $\phi = 12\alpha/L^2$.

The eigenvalue is determined by taking $w = c\exp(-\lambda t)$ in the homogeneous equation leading to $\lambda = \phi = 12\alpha/L^2$.

The θ method yields:

$$w_{n+1} = \frac{\left(1-(1-\theta)h\lambda\right)w_n + h\lambda/2}{(1-\theta h\lambda)}$$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

ANALYSIS OF ALGORITHMS

Example – The results from the Euler, the Crank-Nicolson, and the modified Euler algorithms, along with the analytical solution, for $h\lambda$ = 0.1, 1.0, 2.0, and 2.2, respectively.

Comparison of numerical and analytical solutions for $p = h\lambda = 0.1$

Step #	Euler	C-N	M-Euler	Analytical
1	0.0500	0.0476	0.0455	0.0476
2	0.0950	0.0907	0.0868	0.0906
3	0.1355	0.1297	0.1243	0.1296
4	0.1720	0.1650	0.1585	0.1648
5	0.2048	0.1969	0.1895	0.1967
6	0.2343	0.2257	0.2178	0.2256
7	0.2609	0.2519	0.2434	0.2517
8	0.2848	0.2755	0.2667	0.2753
9	0.3063	0.2969	0.2880	0.2967
10	0.3257	0.3162	0.3072	0.3161

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

ANALYSIS OF ALGORITHMS

Example – The results from the Euler, the Crank-Nicolson, and the modified Euler algorithms, along with the analytical solution, for $h\lambda$ = 0.1, 1.0, 2.0, and 2.2, respectively.

Comparison of numerical and analytical solutions for $p = h\lambda$ = 1.0

Step #	Euler	C-N	M-Euler	Analytical
1	0.5000	0.3333	0.2500	0.3161
2	0.5000	0.4444	0.3750	0.4323
3	0.5000	0.4815	0.4375	0.4751
4	0.5000	0.4938	0.4688	0.4908
5	0.5000	0.4979	0.4844	0.4966
6	0.5000	0.4993	0.4922	0.4988
7	0.5000	0.4998	0.4961	0.4995
8	0.5000	0.4999	0.4980	0.4998
9	0.5000	0.5000	0.4990	0.4999
10	0.5000	0.5000	0.4995	0.5000

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

ANALYSIS OF ALGORITHMS

Example – The results from the Euler, the Crank-Nicolson, and the modified Euler algorithms, along with the analytical solution, for $h\lambda$ = 0.1, 1.0, 2.0, and 2.2, respectively.

Comparison of numerical and analytical solutions for $p = h\lambda$ = 2.0

Step #	Euler	C-N	M-Euler	Analytical
1	1.0000	0.5000	0.3333	0.4323
2	0.0000	0.5000	0.4444	0.4908
3	1.0000	0.5000	0.4815	0.4988
4	0.0000	0.5000	0.4938	0.4998
5	1.0000	0.5000	0.4979	0.5000
6	0.0000	0.5000	0.4993	0.5000
7	1.0000	0.5000	0.4998	0.5000
8	0.0000	0.5000	0.4999	0.5000
9	1.0000	0.5000	0.5000	0.5000
10	0.0000	0.5000	0.5000	0.5000

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

ANALYSIS OF ALGORITHMS

Example – The results from the Euler, the Crank-Nicolson, and the modified Euler algorithms, along with the analytical solution, for $h\lambda$ = 0.1, 1.0, 2.0, and 2.2, respectively.

Comparison of numerical and analytical solutions for $p = h\lambda = 2.2$

Step #	Euler	C-N	M-Euler	Analytical
1	1.1000	0.5238	0.3438	0.4446
2	-0.2200	0.4989	0.4512	0.4939
3	1.3640	0.5001	0.4847	0.4993
4	-0.5368	0.5000	0.4952	0.4999
5	1.7442	0.5000	0.4985	0.5000
6	-0.9930	0.5000	0.4995	0.5000
7	2.2916	0.5000	0.4999	0.5000
8	-1.6499	0.5000	0.5000	0.5000
9	3.0799	0.5000	0.5000	0.5000
10	-2.5959	0.5000	0.5000	0.5000

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

ANALYSIS OF ALGORITHMS

The Euler and Crank-Nicolson algorithms both tend toward steady state too rapidly, with the modified Euler lagging consistently behind steady state.

For small h ($h\lambda$ = 0.1), the Crank-Nicolson algorithm essentially reproduces the analytical solution with the Euler and modified Euler algorithms above and below the analytical solutions, respectively.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

ANALYSIS OF ALGORITHMS

For the step size *h* equal to the critical value for the Euler algorithm, the Euler algorithm diverges by oscillation between the values of 0 and 1.

The Crank-Nicolson and modified Euler solutions tend toward steady state for large *t.*

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

ANALYSIS OF ALGORITHMS

For $h > h_{cr}$ the oscillations of the Euler algorithm become unbounded, whereas the Crank-Nicolson oscillates about and converges to the steady-state solution.

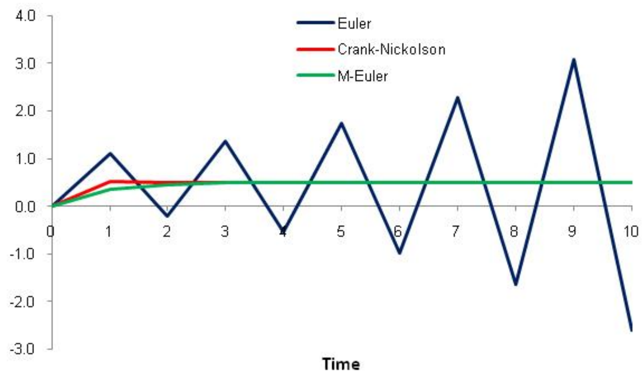
The corresponding modified Euler solution converges from below to the steady-state solution. These tendencies, which are well known for large values of *h* ($h > h_{cr}$), are shown below.

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

ANALYSIS OF ALGORITHMS

These tendencies, which are well known for large values of h ($h > h_{cr}$), are shown below.

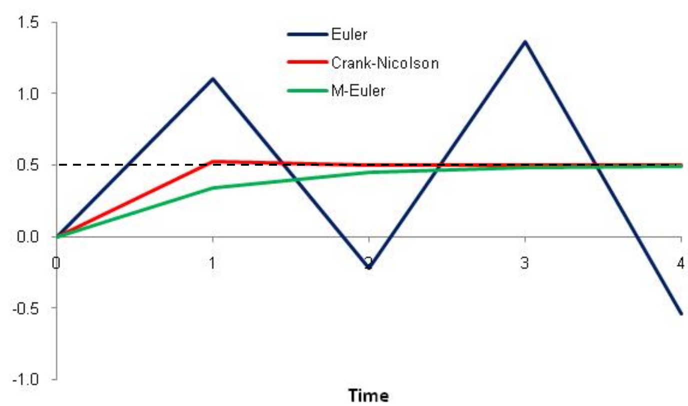


TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

ANALYSIS OF ALGORITHMS

These tendencies, which are well known for large values of h ($h > h_{cr}$), are shown below.



TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

ANALYSIS OF ALGORITHMS

It can shown that for the θ algorithm, the time step *h* should satisfies the following ratio:

$$-1 < \frac{1 - h(1-\theta)\lambda_i}{1 + h\theta\lambda_i} < 1$$

For $\theta < 0.5$ this is: $h\lambda_i < \dfrac{2}{1-2\theta}$

For $\theta = 0$: $h < \dfrac{2}{\lambda_i}$

TIME-DEPENDENT PROBLEMS

One-Dimensional Diffusion or Parabolic Equations

ANALYSIS OF ALGORITHMS

These inequalities are governed by the largest eigenvalue of the **K** – λ**M** so that:

$$h < \frac{2}{\lambda_{\max}}$$

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End of
1-D Time Dependent
Problems – Part a