EIGENVALUE PROBLEMS

Many of the techniques for solving two-dimensional time-dependent problems such as the diffusion and wave equations are very closely related to a corresponding boundary value problem.

In this section, we will investigate the relationship between the time-dependent problem and its corresponding boundary value problem.

Also we investigate how the FEM can be used to extract information about the eigenvalue and eigenfunctions of the corresponding boundary value problems.

EIGENVALUE PROBLEMS

Time-dependent diffusion problems can frequently be stated as:

$$\nabla^2\phi(x,y) - \gamma\frac{\partial\phi}{\partial t} + f(x,y,t) = 0 \qquad \text{in } \Omega$$

$$\phi = g(s,t) \qquad \text{on } \Gamma_1$$

$$\frac{\partial\phi}{\partial n} + \alpha\phi = h(s,t) \qquad \text{on } \Gamma_2$$

$$\phi(x,y,0) = c(x,y)$$

The first two auxiliary conditions are boundary conditions and the third an initial condition.

Such problems are frequently referred to as ***initial-boundary value problems***.

EIGENVALUE PROBLEMS

The corresponding completely homogeneous boundary value problem, obtained by taking $f = g = h = 0$, is:

$$\nabla^2\phi(x,y) - \gamma\frac{\partial\phi}{\partial t} = 0 \qquad \text{in } \Omega$$

$$\phi = 0 \qquad \text{on } \Gamma_1$$

$$\frac{\partial\phi}{\partial n} + \alpha\phi = 0 \qquad \text{on } \Gamma_2$$

EIGENVALUE PROBLEMS

Solutions of this completely homogeneous boundary value problem have great utility in solving the original nonhomogeneous initial-boundary value problem.

For the diffusion problem, solutions of the homogeneous form of the time-depend diffusion equation are known generally to behave according to, $\phi(x,y,t) = \psi(x,y)\exp(-\beta t)$ leading to:

$$\nabla^2\psi - \lambda\psi = 0 \qquad \text{in } \Omega$$

$$\psi = 0 \qquad \text{on } \Gamma_1$$

$$\frac{\partial\psi}{\partial n} + \alpha\psi = 0 \qquad \text{on } \Gamma_2$$

where $\lambda = \gamma\beta$

EIGENVALUE PROBLEMS

The λ's and corresponding nontrivial ψ's that satisfy the above equation are known as eigenvalues and eigenfunctions respectively.

The words eigenvalue and eigenvector are derived from the German word "eigen" which means "proper" or "characteristic."

$$\nabla^2\psi - \lambda\psi = 0 \qquad \text{in } \Omega$$
$$\psi = 0 \qquad \text{on } \Gamma_1$$
$$\frac{\partial\psi}{\partial n} + \alpha\psi = 0 \qquad \text{on } \Gamma_2$$

These are the two-dimensional counterparts of the λ_n and u_n discussed in one-dimensional eigenvalue problems.

EIGENVALUE PROBLEMS

This differential equation is known as the ***Helmholtz Equation*** and occurs with remarkable frequency in mathematical models.

The problem to be solved is that of finding the eigenvalues and eigenfunctions satisfying the differential equation and boundary conditions for the two-dimensional region Ω.

$$\nabla^2\psi - \lambda\psi = 0 \qquad \text{in } \Omega$$
$$\psi = 0 \qquad \text{on } \Gamma_1$$
$$\frac{\partial\psi}{\partial n} + \alpha\psi = 0 \qquad \text{on } \Gamma_2$$

Exact solutions are known only for a few special regions Ω.

EIGENVALUE PROBLEMS

Hermann Ludwig Ferdinand von Helmholtz (August 31, 1821 – September 8, 1894) was a German physician and physicist who made significant contributions to several widely varied areas of modern science.

In physics, he is known for his theories on the conservation of energy, work in electrodynamics, chemical thermodynamics, and on a mechanical foundation of thermodynamics.

The largest German association of research institutions, the Helmholtz Association, is named after him.

EIGENVALUE PROBLEMS

In a completely similar fashion, the two-dimensional initial-boundary value problem associated with the scalar wave equation can be written as:

$$c^2\nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} + f(x,y,t) = 0 \qquad \text{in } \Omega$$

$$\phi = g(s,t) \qquad \text{on } \Gamma_1$$

$$\frac{\partial\phi}{\partial n} + \alpha(s)\phi = h(s,t) \qquad \text{on } \Gamma_2$$

$$\phi(x,y,0) = c(x,y)$$

$$\frac{\partial\phi(x,y,0)}{\partial t} = d(x,y)$$

Here, the first two conditions are boundary conditions and the second two are initial conditions.

EIGENVALUE PROBLEMS

Frequently, ϕ is a displacement or generalized displacement and hence $\partial\phi/\partial t$ is a velocity or generalized velocity.

For a physical problem associated with the wave equation it is usually appropriate to investigate solutions to the homogeneous differential equation and boundary conditions of the form, $\phi(x,y,t) = \psi(x,y)\exp(i\omega t)$ again leading to:

$$\nabla^2\psi - \lambda\psi = 0 \qquad \text{in } \Omega$$

$$\psi = 0 \qquad \text{on } \Gamma_1$$

$$\frac{\partial\psi}{\partial n} + \alpha\psi = 0 \qquad \text{on } \Gamma_2$$

where $\lambda = (\omega/c)^2$

EIGENVALUE PROBLEMS

This is precisely the same completely homogeneous boundary value problem for the diffusion problem.

This equation describes a two-dimensional vibrating membrane, with the edges clamped to be motionless.

$$\nabla^2\psi - \lambda\psi = 0 \qquad \text{in } \Omega$$

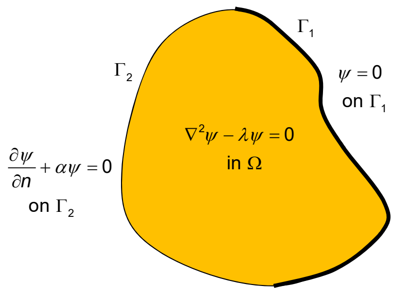
$$\psi = 0 \qquad \text{on } \Gamma_1$$

$$\frac{\partial\psi}{\partial n} + \alpha\psi = 0 \qquad \text{on } \Gamma_2$$

EIGENVALUE PROBLEMS

Finite Element Models for the Helmholtz Equation

The finite element model for the Helmholtz equation will be discussed in terms of the typical region shown below.



Discretization and interpolation are carried out in the same fashion as we have discussed earlier.

EIGENVALUE PROBLEMS

Finite Element Models for the Helmholtz Equation

The Galerkin finite element method will be used to generate the desired algebraic equations for determining the approximate eigenvalues and eigenfunctions.

The form of the solution is taken as:

$$\psi(x,y) = \sum_{i=1}^{N} \psi_i n_i(x,y)$$

where the $\boldsymbol{n}_i$ functions are appropriate nodally based interpolation functions and ψ_i are the unknown values of $\psi(x, y)$.

EIGENVALUE PROBLEMS

Finite Element Models for the Helmholtz Equation

With the test function designated as ζ, the appropriate weak form for this boundary value problem is:

$$\iint_{\Omega}\left(\frac{\partial \zeta}{\partial x}\frac{\partial \psi}{\partial x}+\frac{\partial \zeta}{\partial y}\frac{\partial \psi}{\partial y}\right)dA+\int_{\Gamma_2}\zeta\alpha\psi\,ds-\lambda\iint_{\Omega}\zeta\psi\,dA=0$$

Taking:

$$\psi(x,y)=\sum_{i=1}^{N}\psi_i n_i(x,y)$$

$$\zeta = n_k,\ k=1,\,2,\,\ldots,N \quad \text{yields:}$$

EIGENVALUE PROBLEMS

Finite Element Models for the Helmholtz Equation

With the test function designated as ζ, the appropriate weak form for this boundary value problem is:

$$\iint_{\Omega}\left(\frac{\partial \zeta}{\partial x}\frac{\partial \psi}{\partial x}+\frac{\partial \zeta}{\partial y}\frac{\partial \psi}{\partial y}\right)dA+\int_{\Gamma_2}\zeta\alpha\psi\,ds-\lambda\iint_{\Omega}\zeta\psi\,dA=0$$

Taking:

$$\psi(x,y)=\sum_{i=1}^{N}\psi_i n_i(x,y)$$

$$\sum_{i=1}^{N}\iint_{\Omega}\left\{\left(\frac{\partial n_k}{\partial x}\frac{\partial n_i}{\partial x}+\frac{\partial n_k}{\partial y}\frac{\partial n_i}{\partial y}\right)dA\right\}\psi_i$$

EIGENVALUE PROBLEMS

Finite Element Models for the Helmholtz Equation

With the test function designated as ζ, the appropriate weak form for this boundary value problem is:

$$\iint_{\Omega}\left(\frac{\partial\zeta}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\zeta}{\partial y}\frac{\partial\psi}{\partial y}\right)dA+\int_{\Gamma_2}\zeta\alpha\psi\,ds-\lambda\iint_{\Omega}\zeta\psi\,dA=0$$

Taking:

$$\psi(x,y)=\sum_{i=1}^{N}\psi_i n_i(x,y)$$

$$\sum_{i=1}^{N}\left\{\int_{\Gamma_2}n_k\alpha n_i\,ds\right\}\psi_i$$

EIGENVALUE PROBLEMS

Finite Element Models for the Helmholtz Equation

With the test function designated as ζ, the appropriate weak form for this boundary value problem is:

$$\iint_{\Omega}\left(\frac{\partial\zeta}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\zeta}{\partial y}\frac{\partial\psi}{\partial y}\right)dA+\int_{\Gamma_2}\zeta\alpha\psi\,ds-\lambda\iint_{\Omega}\zeta\psi\,dA=0$$

Taking:

$$\psi(x,y)=\sum_{i=1}^{N}\psi_i n_i(x,y)$$

$$\lambda\sum_{i=1}^{N}\left\{\iint_{\Omega}n_k n_i\,dA\right\}\psi_i$$

EIGENVALUE PROBLEMS

Finite Element Models for the Helmholtz Equation

With the previous substitutions, the boundary value problems becomes:

$$\iint_{\Omega}\left(\frac{\partial \zeta}{\partial x}\frac{\partial \psi}{\partial x}+\frac{\partial \zeta}{\partial y}\frac{\partial \psi}{\partial y}\right)dA=\sum_{i=1}^{N}\iint_{\Omega}\left\{\left(\frac{\partial n_k}{\partial x}\frac{\partial n_i}{\partial x}+\frac{\partial n_k}{\partial y}\frac{\partial n_i}{\partial y}\right)dA\right\}\psi_i$$

$$\int_{\Gamma_2}\zeta\alpha\psi\, ds=\sum_{i=1}^{N}\left\{\int_{\Gamma_2} n_k\alpha n_i\, ds\right\}\psi_i$$

$$\lambda\iint_{\Omega}\zeta\psi\, dA=\lambda\sum_{i=1}^{N}\left\{\iint_{\Omega} n_k n_i\, dA\right\}\psi_i$$

$$k=1,2,\ldots,N$$

EIGENVALUE PROBLEMS

Finite Element Models for the Helmholtz Equation

With the previous substitutions, the boundary value problems becomes:

$$\sum_{i=1}^{N}\iint_{\Omega}\left\{\left(\frac{\partial n_k}{\partial x}\frac{\partial n_i}{\partial x}+\frac{\partial n_k}{\partial y}\frac{\partial n_i}{\partial y}\right)dA\right\}\psi_i$$

$$+\sum_{i=1}^{N}\left\{\int_{\Gamma_2} n_k\alpha n_i\, ds\right\}\psi_i-\lambda\sum_{i=1}^{N}\left\{\iint_{\Omega} n_k n_i\, dA\right\}\psi_i=0$$

$$k=1,2,\ldots,N$$

EIGENVALUE PROBLEMS

Finite Element Models for the Helmholtz Equation

These equations can be written in a more convenient form:

$$\sum_{i=1}^{N}\left(K_{ki} - \lambda M_{ki}\right)\psi_i = 0 \qquad k = 1, 2, \ldots, N$$

where:

$$\mathbf{K_G} = \sum_e \mathbf{k_G} + \sum_e{}' \mathbf{a_G} \qquad \mathbf{M_G} = \sum_e \mathbf{m_G}$$

$$\mathbf{k_e} = \iint_A \left[\frac{\partial \mathbf{N}}{\partial x}\frac{\partial \mathbf{N^T}}{\partial x} + \frac{\partial \mathbf{N}}{\partial y}\frac{\partial \mathbf{N^T}}{\partial y}\right] dA \qquad \mathbf{m_e} = \iint_{A_e} \mathbf{N}\mathbf{N^T}\, dA$$

$$\mathbf{a_e} = \int_{\gamma_{2e}} \mathbf{N}\alpha\mathbf{N^T}\, ds$$

EIGENVALUE PROBLEMS

Finite Element Models for the Helmholtz Equation

These equations can be written in a more convenient form:

$$\sum_{i=1}^{N}\left(K_{ki} - \lambda M_{ki}\right)\psi_i = 0 \qquad k = 1, 2, \ldots, N$$

The above equation is an example of the generalized linear algebraic eigenvalue problem.

The $\mathbf{m_e}$ is referred to as an **elemental consistent mass matrix**.

EIGENVALUE PROBLEMS

Finite Element Models for the Helmholtz Equation

The boundary conditions for the Helmholtz equation are entirely homogeneous.

In particular, the constraints associated with the essential boundary conditions on Γ_1 can be enforced by deleting the row and column corresponding to the degree of freedom in question.

This results in: $\mathbf{K}_G^* \mathbf{\Psi}_G^* - \lambda \mathbf{M}_G^* \mathbf{\Psi}_G^* = 0$

where the rows and columns associated with the constraints have been removed.

EIGENVALUE PROBLEMS

Finite Element Models for the Helmholtz Equation

The solution is obtained by determining the eigenvalues and corresponding eigenvectors.

For a problem with a small number (two to four) of constrained degrees of freedom, it is perhaps feasible to extract the eigen information by hand.

This results in: $\mathbf{K}_G^* \mathbf{\Psi}_G^* - \lambda \mathbf{M}_G^* \mathbf{\Psi}_G^* = 0$ $\qquad \mathbf{A} = \mathbf{M}_G^{*\,-1} \mathbf{K}_G^*$

where the rows and columns associated with the constraints have been removed.

EIGENVALUE PROBLEMS

Finite Element Models for the Helmholtz Equation

For larger problems, it is essential to have a robust computer code for this task.

Appendix C in your textbook contains FORTRAN source listings for several routines that may be used in this regard.

More conveniently, you can use the MATLAB function:

```
[V,D] = eig(A)
```

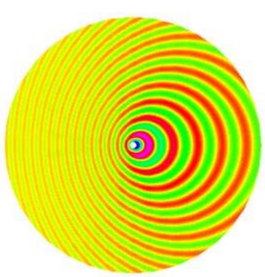
which returns diagonal matrix **D** of eigenvalues and matrix **V** whose columns are the corresponding right eigenvectors.

EIGENVALUE PROBLEMS

Finite Element Models for the Helmholtz Equation

The propagation of time harmonic sound waves is usually described by the Helmholtz equation.

When the pressure source is moving with respect to the fluid, or vice versa, the wave length is different in different directions - the phenomenon is referred to as the Doppler effect.



The figure represents the sound waves produced by a simple vibrating pressure source in the middle of the computational domain.

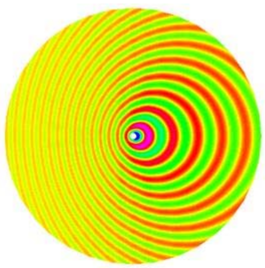
The fluid is moving with a constant velocity in the horizontal direction.

EIGENVALUE PROBLEMS

Finite Element Models for the Helmholtz Equation

The propagation of time harmonic sound waves is usually described by the Helmholtz equation.

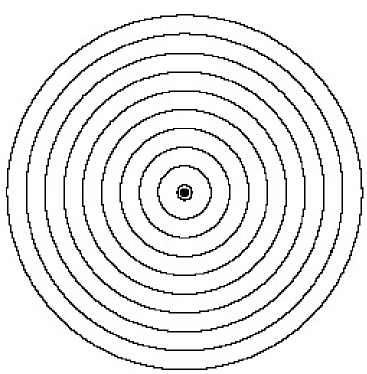
When the pressure source is moving with respect to the fluid, or vice versa, the wave length is different in different directions - the phenomenon is referred to as the Doppler effect.



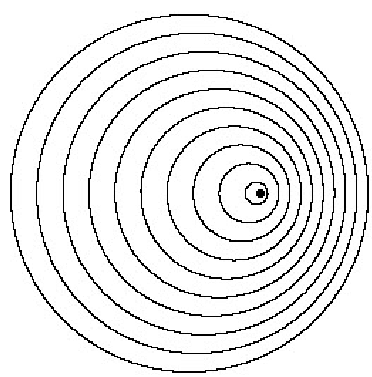
Because of the movement, the wave length is shorter and the amplitude smaller in the upwind direction.

EIGENVALUE PROBLEMS

Finite Element Models for the Helmholtz Equation



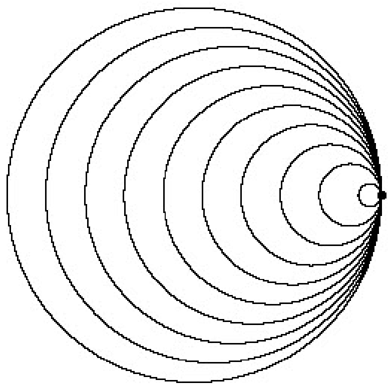
Stationary source



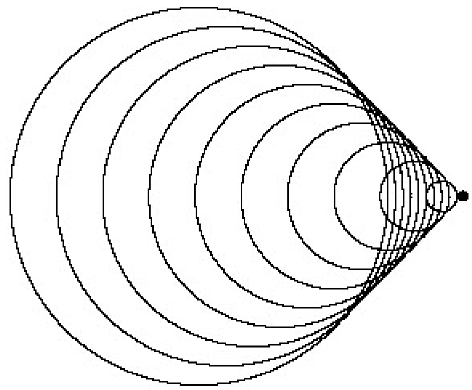
$V_{source} < V_{sound}$

EIGENVALUE PROBLEMS

Finite Element Models for the Helmholtz Equation



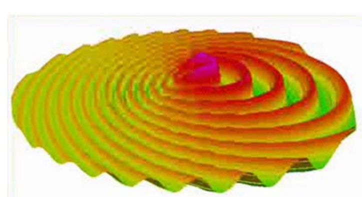
$V_{source} = V_{sound}$

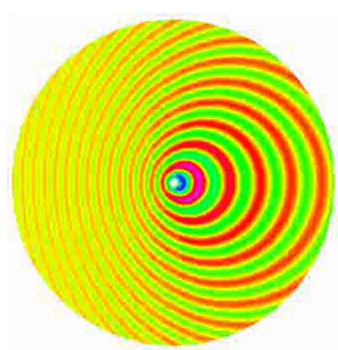


$V_{source} > V_{sound}$

EIGENVALUE PROBLEMS

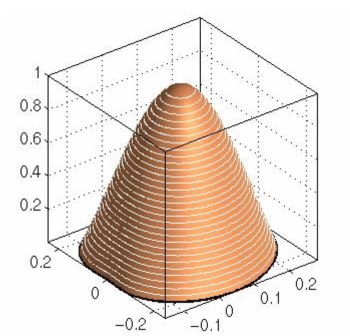
Finite Element Models for the Helmholtz Equation



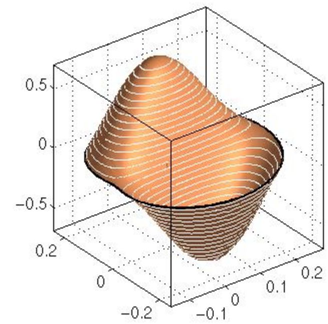


EIGENVALUE PROBLEMS

Finite Element Models for the Helmholtz Equation



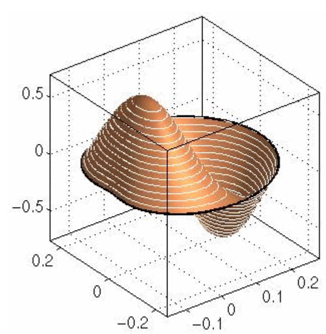
First eigenfunction



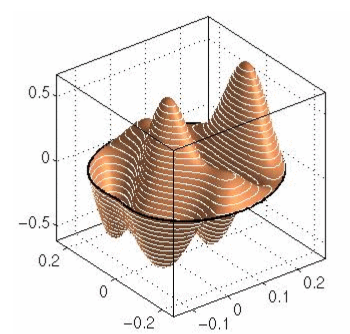
Second eigenfunction

EIGENVALUE PROBLEMS

Finite Element Models for the Helmholtz Equation



Third eigenfunction



Tenth eigenfunction

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 1

Consider the problem of a classical square vibrating membrane with all edges fixed against transverse displacement.

The differential equation of motion can be written as:

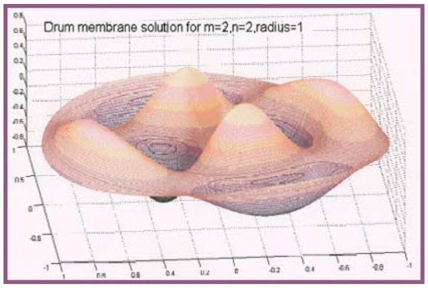
$$T\nabla^2 w = \rho \frac{\partial^2 w}{\partial t^2}$$

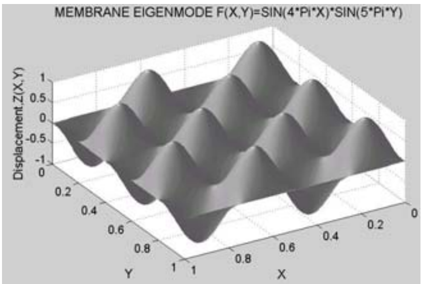
where ***T*** is the initial tension in the membrane and ρ the area density.

The boundary condition is that ***w*** equals zero on all the edges of the membrane.

EIGENVALUE PROBLEMS

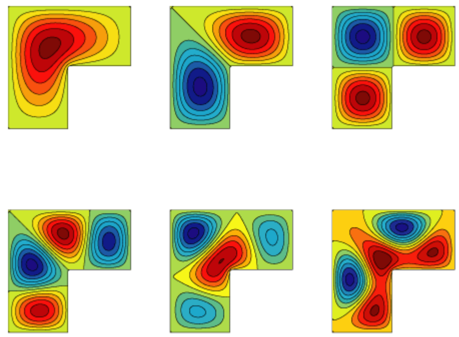
Examples for the Helmholtz Equation – Example 1





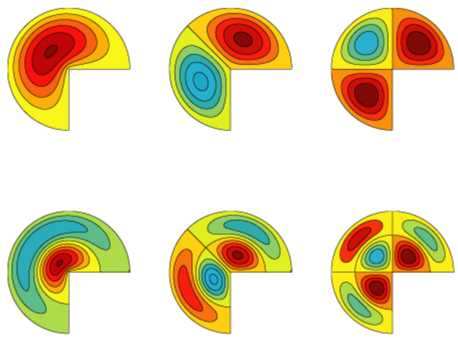
EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 1



EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 1



EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 1

Recall, the differential equation of motion can be written as:

$$T\nabla^2 w = \rho \frac{\partial^2 w}{\partial t^2}$$

Taking: $w(x,y,t) = \psi(x,y)\exp(i\omega t)$

$$\nabla^2\psi - \lambda\psi = 0 \qquad \text{in } \Omega \quad \text{with} \quad \psi = 0 \qquad \text{on } \Gamma_1$$

where $\lambda = \rho\omega^2/T$

This is the Helmholtz problem on the square, with the dependent variable ψ prescribed as zero everywhere on the boundary.

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 1

Recall, the differential equation of motion can be written as:

$$T\nabla^2 w = \rho \frac{\partial^2 w}{\partial t^2}$$

Taking: $w(x,y,t) = \psi(x,y)\exp(i\omega t)$

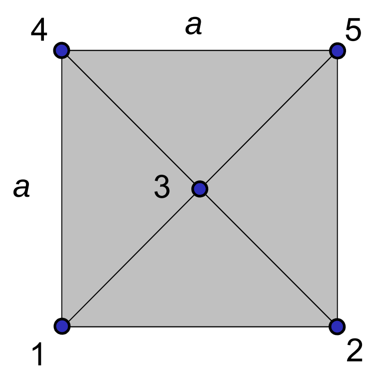
$$\nabla^2\psi - \lambda\psi = 0 \qquad \text{in } \Omega \quad \text{with} \quad \psi = 0 \qquad \text{on } \Gamma_1$$

The eigenvalues are related to the natural frequencies and the eigenfunctions to the mode shapes.

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 1

The simplest possible model using linearly interpolated triangular elements is shown below.



EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 1

The elemental $\mathbf{k_e}$ matrix using 3-noded triangular elements is:

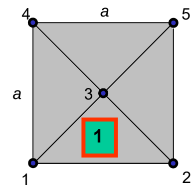
$$\mathbf{k_e} = \frac{\mathbf{b_e}\mathbf{b_e}^{\mathbf{T}} + \mathbf{c_e}\mathbf{c_e}^{\mathbf{T}}}{4A_e}$$

$$\mathbf{b_e} = \begin{Bmatrix} y_j - y_k \\ y_k - y_i \\ y_i - y_j \end{Bmatrix} \qquad \mathbf{c_e} = \begin{Bmatrix} x_k - x_j \\ x_i - x_k \\ x_j - x_i \end{Bmatrix}$$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 1

For element 1: node 1 is at (0, 0);
node 2 at (*a*, 0); and node 3 at (0.5*a*, 0.5*a*).

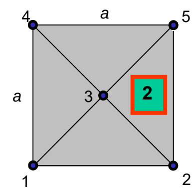


$$\mathbf{b}_1 = \frac{a}{2}\begin{Bmatrix} -1 \\ 1 \\ 0 \end{Bmatrix} \qquad \mathbf{c}_1 = \frac{a}{2}\begin{Bmatrix} -1 \\ 1 \\ 2 \end{Bmatrix} \qquad A_1 = \frac{a^2}{4} \qquad \mathbf{k}_1 = \frac{1}{2}\begin{array}{c} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \end{array}\begin{array}{c} \boxed{1} \\ \boxed{2} \\ \boxed{3} \end{array}$$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 1

For element 2: node 2 is at (*a*, 0);
node 5 at (*a*, *a*); and node 3 at (0.5*a*, 0.5*a*).

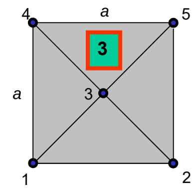


$$\mathbf{b}_2 = \frac{a}{2}\begin{Bmatrix} 2 \\ 2 \\ 1 \end{Bmatrix} \qquad \mathbf{c}_2 = \frac{a}{2}\begin{Bmatrix} 0 \\ 2 \\ 1 \end{Bmatrix} \qquad A_2 = \frac{a^2}{4} \qquad \mathbf{k}_2 = \frac{1}{2}\begin{array}{c} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \end{array}\begin{array}{c} \boxed{2} \\ \boxed{5} \\ \boxed{3} \end{array}$$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 1

For element 3: node 3 is at (0.5a, 0.5a); node 5 at (a, a); and node 4 at (0, a).

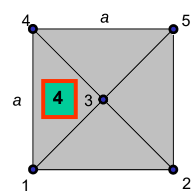


$$\mathbf{b}_3 = \frac{a}{2}\begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} \qquad \mathbf{c}_3 = \frac{a}{2}\begin{Bmatrix} -2 \\ 1 \\ 1 \end{Bmatrix} \qquad A_3 = \frac{a^2}{4} \qquad \mathbf{k}_3 = \frac{1}{2}\begin{array}{c} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \end{array}\begin{array}{c} \boxed{3} \\ \boxed{5} \\ \boxed{4} \end{array}$$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 1

For element 4: node 1 is at (0, 0); node 3 at (0.5a, 0.5a); and node 4 at (0, a).



$$\mathbf{b}_4 = \frac{a}{2}\begin{Bmatrix} -1 \\ 2 \\ -1 \end{Bmatrix} \qquad \mathbf{c}_4 = \frac{a}{2}\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \qquad A_4 = \frac{a^2}{4} \qquad \mathbf{k}_4 = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{array}{c} \boxed{1} \\ \boxed{3} \\ \boxed{4} \end{array}$$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 1

The elemental $\mathbf{m_e}$ matrices are:

$$\mathbf{m_e} = \iint_{A_e} \mathbf{N}\mathbf{N}^{\mathbf{T}}\, dA = \iint_{A_e} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} \langle N_1 \quad N_2 \quad N_3 \rangle\, dA$$

$$= \iint_{A_e} \begin{Bmatrix} N_1^2 & N_1 N_2 & N_1 N_3 \\ N_2 N_1 & N_2^2 & N_2 N_3 \\ N_3 N_1 & N_3 N_2 & N_3^2 \end{Bmatrix} dA$$

Recall, the following relationship:

$$\iint_{A_e} N_I^a N_J^b N_K^c\, dA = a!\,b!\,c!\,\frac{2A_e}{(a+b+c+2)!}$$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 1

The elemental $\mathbf{m_e}$ matrices are:

$$\mathbf{m_e} = \iint_{A_e} \mathbf{N}\mathbf{N}^{\mathbf{T}}\, dA = \iint_{A_e} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} \langle N_1 \quad N_2 \quad N_3 \rangle\, dA$$

$$= \frac{A_e}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Recall, the following relationship:

$$\iint_{A_e} N_I^a N_J^b N_K^c\, dA = a!\,b!\,c!\,\frac{2A_e}{(a+b+c+2)!}$$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 1

The elemental $\mathbf{m_e}$ matrices are:

$$\mathbf{m_e} = \iint_{A_e} \mathbf{N}\mathbf{N}^{\mathbf{T}}\, dA = \iint_{A_e} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} \langle N_1 \quad N_2 \quad N_3 \rangle\, dA$$

$$= \frac{A_e}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad A_e = \frac{a^2}{4}$$

$$\mathbf{m_1} = \mathbf{m_2} = \mathbf{m_3} = \mathbf{m_4} = \frac{a^2}{48}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 1

The assembled $\mathbf{K_G}$ and $\mathbf{M_G}$ matrices are determined to be:

$$\mathbf{K_G} = \frac{1}{2}\begin{bmatrix} 2 & 0 & -2 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ -2 & -2 & 8 & -2 & -2 \\ 0 & 0 & -2 & 2 & 0 \\ 0 & 0 & -2 & 0 & 2 \end{bmatrix}$$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 1

The assembled $\mathbf{K_G}$ and $\mathbf{M_G}$ matrices are determined to be:

$$\mathbf{M_G} = \frac{a^2}{48}\begin{bmatrix} 4 & 1 & 2 & 1 & 0 \\ 1 & 4 & 2 & 0 & 1 \\ 2 & 2 & 8 & 2 & 2 \\ 1 & 0 & 2 & 4 & 1 \\ 0 & 1 & 2 & 1 & 4 \end{bmatrix}$$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 1

Constraining ψ_1, ψ_2, ψ_4, and ψ_5 yields the single equation:

$$\left(\mathbf{K_G}\right)_{33} - \lambda\left(\mathbf{M_G}\right)_{33} = 0$$

$$4 - \lambda\frac{a^2}{6} = 0 \qquad \text{where } \lambda = \frac{24}{a^2}$$

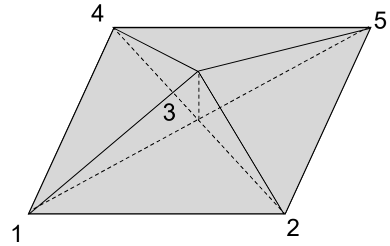
This approximate value is to be compared to the exact value of $2\pi^2/a^2$ ($19.74/a^2$), an error of approximately 22%.

This is quite reasonable for the very crude mesh.

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 1

When plotted, the corresponding eigenfunction is the pyramid-shaped function.

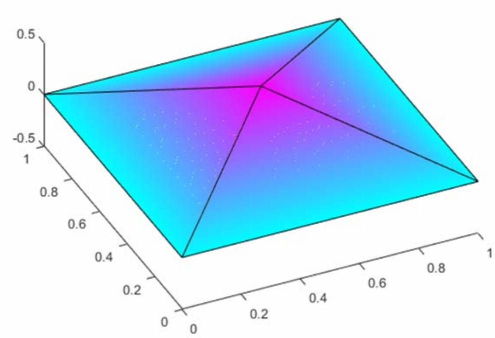


It can shown that a model using four square elements over the same region also yields $\lambda = 24/a^2$.

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 1

When plotted, the corresponding eigenfunction is the pyramid-shaped function.



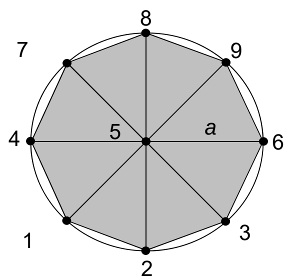
EIGENVALUE PROBLEMS

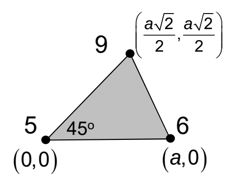
Examples for the Helmholtz Equation – Example 2

Consider the problem of the vibration of a circular membrane.

The Helmholtz problem for the circle is related to the vibration of the circular membrane.

Consider a circular region modeled with eight linearly interpolated triangular elements.





EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

For element 1: node 5 is at (0, 0); node 6 at (a, 0); and node 9 at (0.70710a, 0.70710a).

$$\mathbf{b}_e = a\begin{Bmatrix} -0.70710 \\ 0.70710 \\ 0 \end{Bmatrix} \qquad \mathbf{c}_e = a\begin{Bmatrix} -0.29290 \\ -0.70710 \\ 1 \end{Bmatrix} \qquad A_e = 0.35355\,a^2$$

The elemental stiffness matrix for the single triangular element is:

$$\mathbf{k}_e = \begin{array}{c} \begin{bmatrix} 0.41421 & -0.20710 & -0.20710 \\ -0.20710 & 0.70709 & -0.50000 \\ -0.20710 & -0.50000 & 0.70710 \end{bmatrix} \end{array} \begin{array}{c} \boxed{5} \\ \boxed{6} \\ \boxed{9} \end{array}$$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

The elemental mass matrix is:

$$\mathbf{m_e} = \iint_{A_e} \mathbf{N}\mathbf{N}^{\mathbf{T}}\, dA = \frac{A_e}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad A_e = 0.35355\,a^2$$

$$\mathbf{m_e} = \frac{0.35355a^2}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{array}{c} \boxed{5} \\ \boxed{6} \\ \boxed{9} \end{array}$$

where a is the radius of the circle.

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

With a little forward planning, it can be seen that all of nodes 1-4 and 6-9 will be constrained with only node 5 remaining.

Thus only $(\mathbf{K_G})_{55}$ and $(\mathbf{M_G})_{55}$ need to be computed.

The contribution to $(\mathbf{K_G})_{55}$ and $(\mathbf{M_G})_{55}$ from each element will be the (5, 5) element of the $\mathbf{k_e}$ and $\mathbf{m_e}$, yielding:

$$\left(\mathbf{K_G}\right)_{55} = 8\left(0.41422\right) \qquad \left(\mathbf{M_G}\right)_{55} = \frac{8\left(0.35355\,a^2\right)}{6}$$

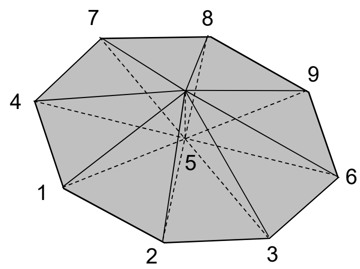
The resulting solution is: $$\lambda = \frac{\left(\mathbf{K_G}\right)_{55}}{\left(\mathbf{M_G}\right)_{55}} = \frac{7.02961}{a^2}$$

The exact value is $5.78/a^2$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

The eigenfunction plots as the pyramidal cone.



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EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

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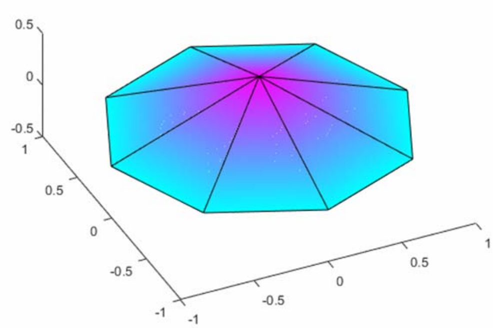
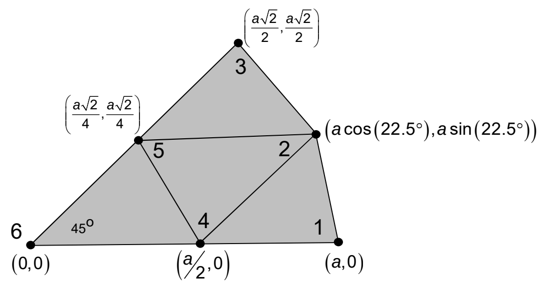


EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

For the circle, a better approximation to the fundamental eigenvalue can be obtained without having to solve an excessively large problem by considering the mesh shown below.

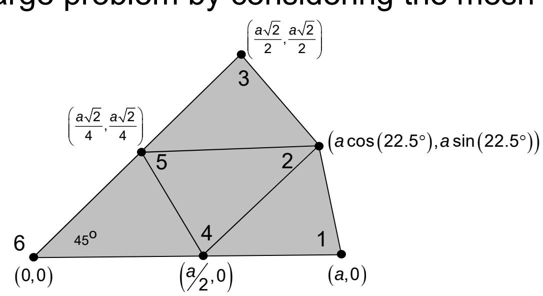


For this model, we will enforce constraints at nodes 1, 2, and 3 with nodes 4, 5, and 6 unconstrained.

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

For the circle, a better approximation to the fundamental eigenvalue can be obtained without having to solve an excessively large problem by considering the mesh shown below.

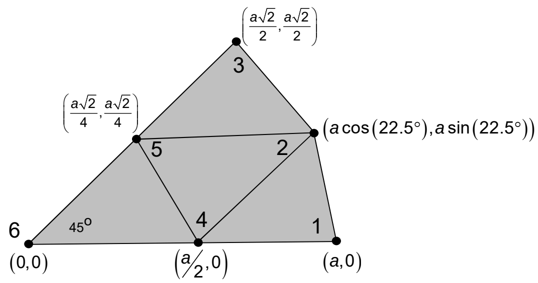


The portion of the boundary containing nodes 1-2-3 is Γ_1 and the two straight portions 1-4-6 and 3-5-6 must be considered as Γ_2.

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

For the circle, a better approximation to the fundamental eigenvalue can be obtained without having to solve an excessively large problem by considering the mesh shown below.

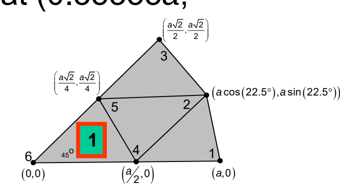


In the limit as the mesh is refined, the natural boundary conditions ($\partial\psi/\partial n = 0$) will be satisfied on Γ_2.

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

For element 1: node 4 is at (a/2, 0); 5 at (0.35355a, 0.35355a); and node 6 at (0, 0).



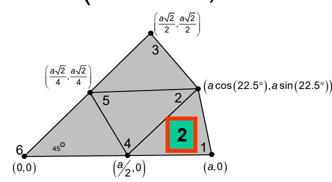
$$\mathbf{b}_1 = a\begin{Bmatrix} 0.35355 \\ 0 \\ -0.35355 \end{Bmatrix} \qquad \mathbf{c}_1 = a\begin{Bmatrix} -0.35355 \\ 0.50000 \\ -0.14645 \end{Bmatrix} \qquad A_1 = 0.08839a^2$$

$$\mathbf{k}_1 = \begin{array}{c} \begin{bmatrix} 0.70709 & -0.50000 & -0.20710 \\ -0.50000 & 0.70709 & -0.20711 \\ -0.20710 & -0.20711 & 0.41421 \end{bmatrix} \end{array} \begin{array}{c} \boxed{4} \\ \boxed{5} \\ \boxed{6} \end{array}$$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

For element 2: node 1 is at (a, 0); node 2 at (0.9289a, 0.3827a); and node 4 at (a/2, 0).



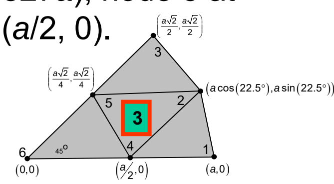
$$\mathbf{b}_2 = a\begin{Bmatrix} 0.38296 \\ 0.00000 \\ -0.38296 \end{Bmatrix} \qquad \mathbf{c}_2 = a\begin{Bmatrix} -0.42388 \\ 0.50000 \\ -0.07612 \end{Bmatrix} \qquad A_2 = 0.09574a^2$$

$$\mathbf{k}_2 = \begin{bmatrix} 0.85186 & -0.55327 & -0.29860 \\ -0.55327 & 0.65281 & -0.09954 \\ -0.29860 & -0.09954 & 0.39813 \end{bmatrix} \begin{matrix} \boxed{1} \\ \boxed{2} \\ \boxed{4} \end{matrix}$$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

For element 3: node 2 is (0.9289a, 0.3827a); node 5 at (0.35355a, 0.35355a); and node 4 at (a/2, 0).



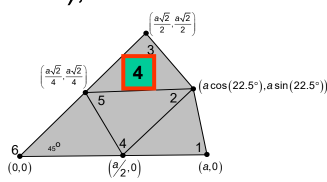
$$\mathbf{b}_3 = a\begin{Bmatrix} -0.02913 \\ -0.38268 \\ 0.02913 \end{Bmatrix} \qquad \mathbf{c}_3 = a\begin{Bmatrix} 0.14645 \\ 0.42388 \\ -0.57033 \end{Bmatrix} \qquad A_3 = 0.10295a^2$$

$$\mathbf{k}_3 = \begin{bmatrix} 0.05413 & 0.17781 & -0.20488 \\ 0.17781 & 0.79191 & -0.61410 \\ -0.20488 & -0.61410 & 0.79191 \end{bmatrix} \begin{matrix} \boxed{2} \\ \boxed{5} \\ \boxed{4} \end{matrix}$$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

For element 4: node 2 is (0.9289*a*, 0.3827*a*); node 3 at (0.70711*a*, 0.70711*a*); and node 5 at (0.35355*a*, 0.35355*a*).



$$\mathbf{b}_4 = a\begin{Bmatrix} 0.32442 \\ 0.02913 \\ -0.32442 \end{Bmatrix} \qquad \mathbf{c}_4 = a\begin{Bmatrix} -0.35355 \\ 0.57032 \\ -0.21677 \end{Bmatrix} \qquad A_4 = 0.09567a^2$$

$$\mathbf{k}_4 = \begin{array}{c} \begin{bmatrix} 0.60167 & -0.55161 & -0.07476 \\ -0.55161 & 0.85219 & -0.29837 \\ -0.07476 & -0.29837 & 0.39782 \end{bmatrix} \end{array} \begin{array}{c} \boxed{2} \\ \boxed{3} \\ \boxed{5} \end{array}$$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

The elemental $\mathbf{m_e}$ matrices are:

$$\mathbf{m_e} = \iint_{A_e} \mathbf{N}\mathbf{N}^{\mathrm{T}}\, dA = \frac{A_e}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\mathbf{m}_1 = \frac{0.08839a^2}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{array}{c} \boxed{4} \\ \boxed{5} \\ \boxed{6} \end{array} \qquad \mathbf{m}_2 = \frac{0.095671a^2}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{array}{c} \boxed{1} \\ \boxed{2} \\ \boxed{4} \end{array}$$

$$\mathbf{m}_3 = \frac{0.10295a^2}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{array}{c} \boxed{2} \\ \boxed{5} \\ \boxed{4} \end{array} \qquad \mathbf{m}_4 = \frac{0.09567a^2}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{array}{c} \boxed{2} \\ \boxed{3} \\ \boxed{5} \end{array}$$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

For the subregion shown in the mesh above, the *constrained* $\mathbf{K}^*_\mathbf{G}$ and $\mathbf{M}^*_\mathbf{G}$ matrices respectively are:

$$\mathbf{K}^*_\mathbf{G} = \begin{array}{c} \begin{bmatrix} 1.89713 & -1.11411 & -0.20710 \\ -1.11411 & 1.89682 & -0.20710 \\ -0.20710 & -0.20710 & 0.41421 \end{bmatrix} \end{array} \begin{array}{c} \boxed{4} \\ \boxed{5} \\ \boxed{6} \end{array}$$

$$\mathbf{M}^*_\mathbf{G} = \frac{a^2}{12} \begin{bmatrix} 0.57416 & 0.19134 & 0.08839 \\ 0.19134 & 0.57388 & 0.08839 \\ 0.08839 & 0.08839 & 0.17678 \end{bmatrix} \begin{array}{c} \boxed{4} \\ \boxed{5} \\ \boxed{6} \end{array}$$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

The eigenvalues can be found using a variety of available solution techniques. In Matlab, use **`[V,D] = eig(A).`**

For example, **`[V,D] = eig(`**$\mathbf{M_G}^{-1}\mathbf{K_G}$**`)`** gives eigenvalues **D**:

$$\lambda_1 = \frac{6.1185}{a^2} \qquad \lambda_2 = \frac{46.8869}{a^2} \qquad \lambda_3 = \frac{94.4155}{a^2}$$

The eigenvectors **V** are:

$$\psi_1^* = \begin{Bmatrix} 0.4756 \\ 0.4756 \\ 0.7401 \end{Bmatrix} \qquad \psi_2^* = \begin{Bmatrix} -0.2360 \\ -0.2358 \\ 0.9427 \end{Bmatrix} \qquad \psi_3^* = \begin{Bmatrix} -0.7069 \\ 0.7073 \\ -0.0004 \end{Bmatrix}$$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

The eigenvectors as given by Bickford (1994) are:

$$\psi_1^* = \begin{Bmatrix} 0.643 \\ 0.643 \\ 1.000 \end{Bmatrix} \qquad \psi_2^* = \begin{Bmatrix} 0.250 \\ 0.250 \\ -1.000 \end{Bmatrix} \qquad \psi_3^* = \begin{Bmatrix} -1.000 \\ 1.000 \\ 0.000 \end{Bmatrix}$$

If the Matlab eigenvectors are scaled they match Bickford's:

$$\psi_1^* = \begin{Bmatrix} 0.6426 \\ 0.6426 \\ 1.0000 \end{Bmatrix} \quad \psi_2^* = \begin{Bmatrix} -0.2503 \\ -0.2502 \\ 1.0000 \end{Bmatrix} \quad \psi_3^* = \begin{Bmatrix} -0.9994 \\ 1.0000 \\ -0.0006 \end{Bmatrix}$$

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

As the results indicate, the lowest eigenvalue is associated with an eigenfunction that is symmetric, that is, $\psi_4 = \psi_5$ with ψ_4, ψ_5, and ψ_6 all positive, so that an improved approximation for the lowest eigenvalue for the entire circle will be obtained.

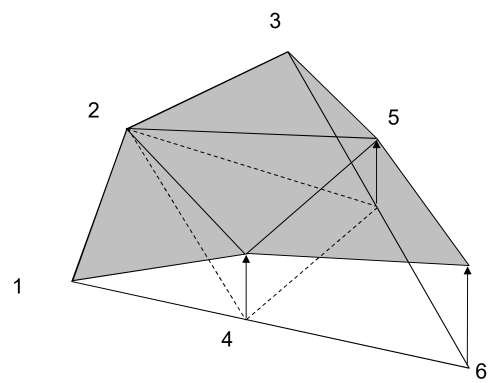
The other λ's and ψ's must be interpreted very carefully for a model consisting of such a subregion, in that correctly identifying conditions of symmetry or antisymmetry and the character of the corresponding modes can require considerable insight into the mathematics of the problem.

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

The corresponding approximate eigenfunctions are shown below:

$$\psi_1^* = \begin{Bmatrix} 0.643 \\ 0.643 \\ 1.000 \end{Bmatrix}$$



EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

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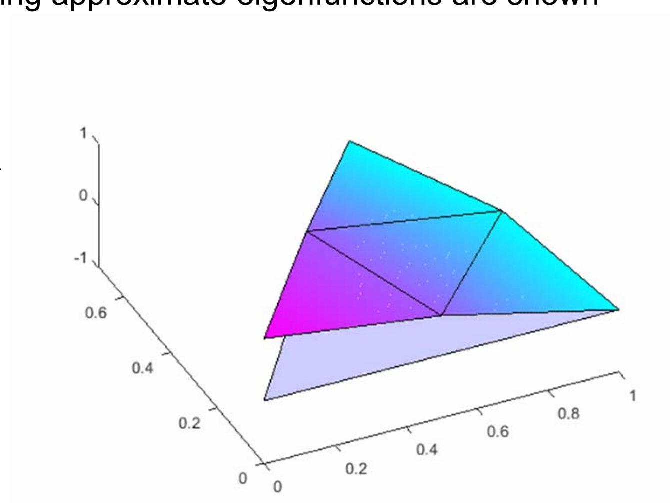
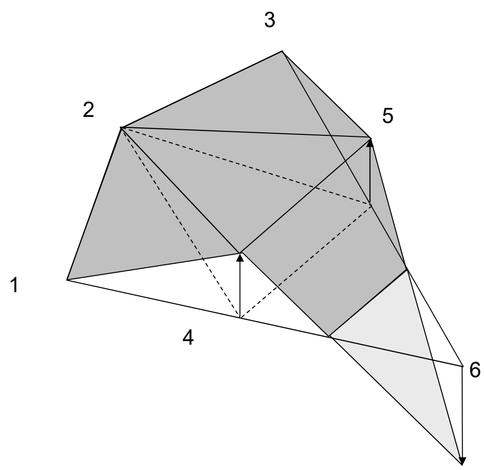


EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

The corresponding approximate eigenfunctions are shown below:

$$\psi_2^* = \begin{Bmatrix} 0.250 \\ 0.250 \\ -1.000 \end{Bmatrix}$$



EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

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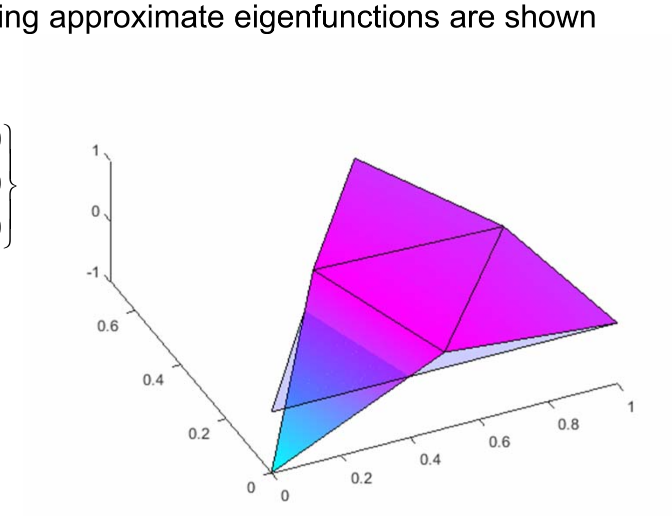

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

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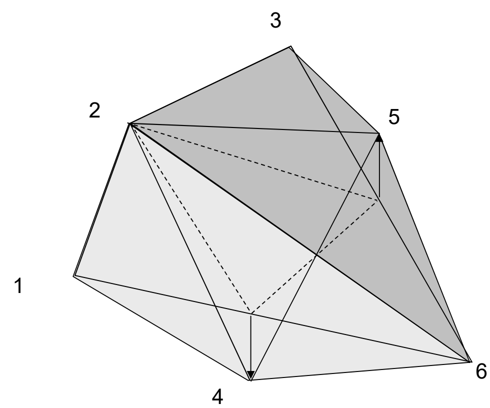

EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

The corresponding approximate eigenfunctions are shown below:

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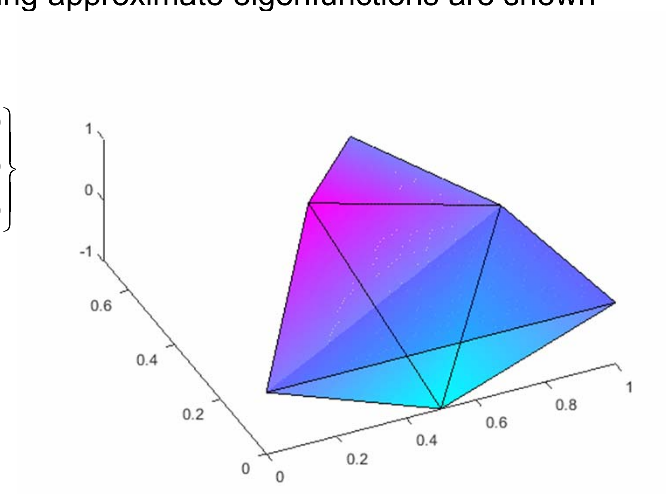
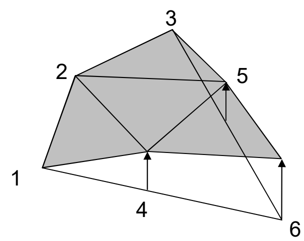


EIGENVALUE PROBLEMS

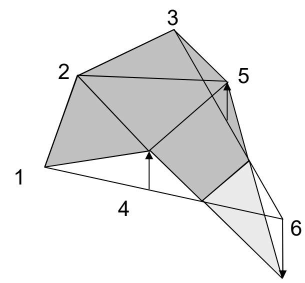
Examples for the Helmholtz Equation – Example 2

The first two eigenvectors clearly have a symmetric character in the $\psi_4 = \psi_5$.

$$\psi_1^* = \begin{Bmatrix} 0.643 \\ 0.643 \\ 1.000 \end{Bmatrix}$$



$$\psi_2^* = \begin{Bmatrix} 0.250 \\ 0.250 \\ -1.000 \end{Bmatrix}$$



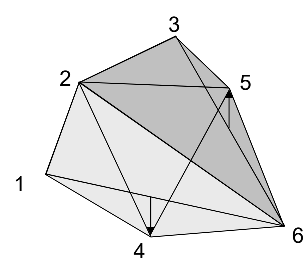
EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

One would suspect that these eigenvalues-eigenfunction pairs correspond to radially symmetric modes for the circle, with the lowest eigenvalue for this model representing an improvement in the single eigenvalue obtained from the first model using eight elements.

The third eigenvalue-eigenfunction demonstrates asymmetry with respect to the bisector of the angle subtended by the element.

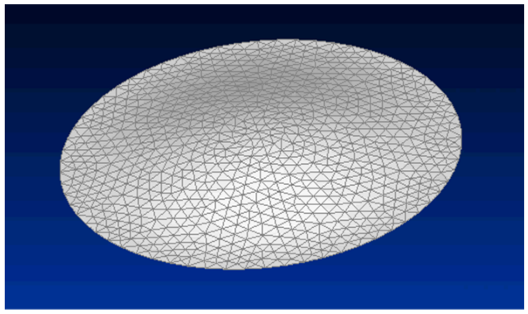
$$\psi_3^* = \begin{Bmatrix} -1.000 \\ 1.000 \\ 0.000 \end{Bmatrix}$$



EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

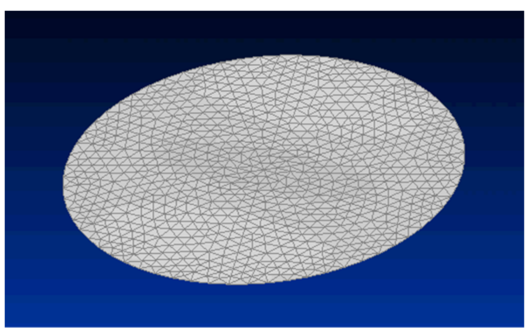
Bending modes of a fixed circular plate - mode 1



EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

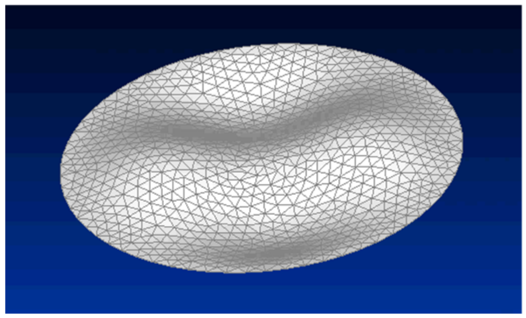
Bending modes of a fixed circular plate – mode 2



EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

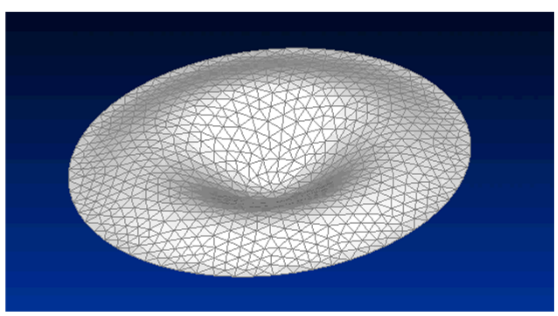
Bending modes of a fixed circular plate – mode 3



EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

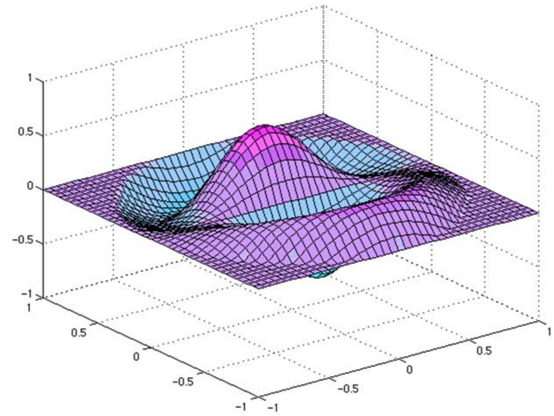
Bending modes of a fixed circular plate – mode 4



EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

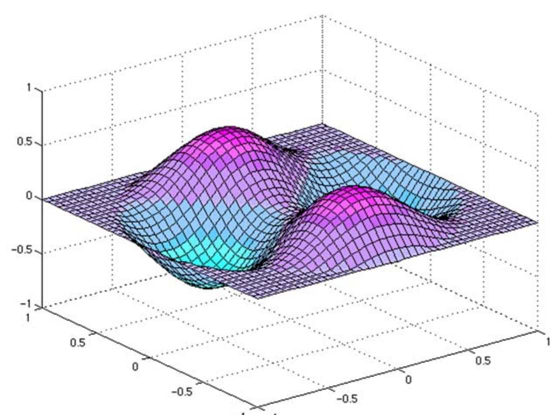
Normal vibration modes of a circular membrane



EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

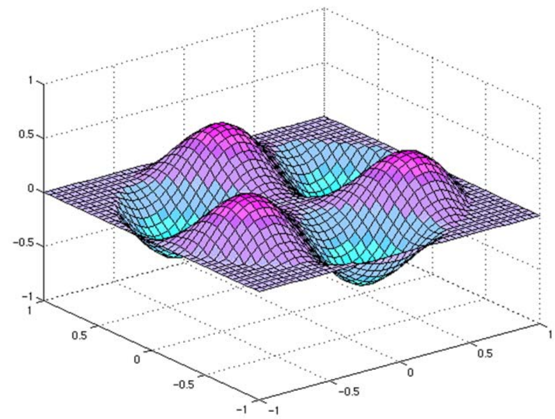
Normal vibration modes of a circular membrane



EIGENVALUE PROBLEMS

Examples for the Helmholtz Equation – Example 2

Normal vibration modes of a circular membrane



EIGENVALUE PROBLEMS

PROBLEM #26 - Consider the problem of a classical square vibrating membrane with all edges fixed against transverse displacement. The differential equation of motion can be written as:

$$T\nabla^2 w = \rho \frac{\partial^2 w}{\partial t^2}$$

where ***T*** is the initial tension in the membrane and $\boldsymbol{\rho}$ the area density. The boundary condition is that ***w*** vanishes on all the edges of the membrane. Taking:

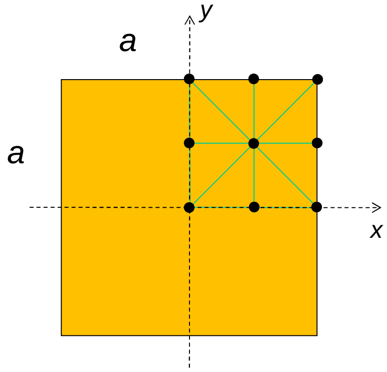
$$w(x,y,t) = \psi(x,y)\exp(i\omega t)$$

$$\nabla^2\psi - \lambda\psi = 0 \qquad \text{in } \Omega \quad \text{with} \quad \psi = 0 \qquad \text{on } \Gamma_1$$

where $\lambda = \rho\omega^2/T$

EIGENVALUE PROBLEMS

PROBLEM #26 - Model the top right-most quadrant of the membrane using eight equally-sized 3-node triangles and compute the associated eigenvalues and the eigenfunctions.



Note: Use a node numbering scheme that will make assembly of the stiffness and mass matrices as simple as possible.

EIGENVALUE PROBLEMS

PROBLEM #26

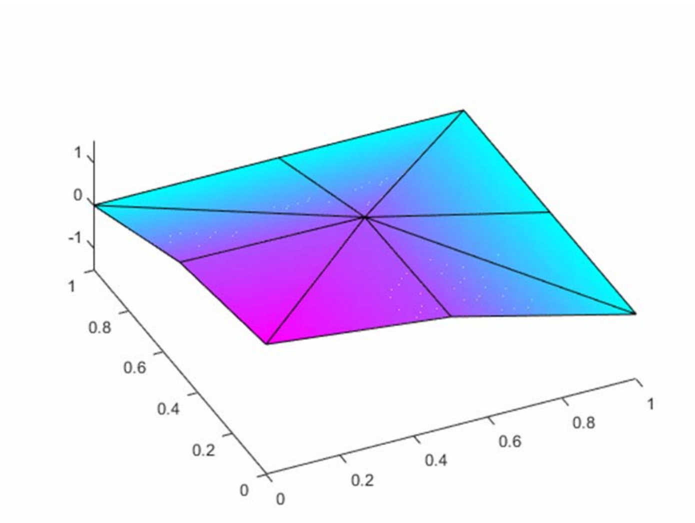
Using **`[V,D] = eig(`**$\mathbf{M_G^{-1}K_G}$**`)`** gives eigenvalues **`D`**:

$$\lambda_1 = \frac{4.3889}{a^2} \qquad \lambda_2 = \frac{33.6174}{a^2} x$$

$$\lambda_3 = \frac{48.0000}{a^2} \qquad \lambda_4 = \frac{110.2290}{a^2}$$

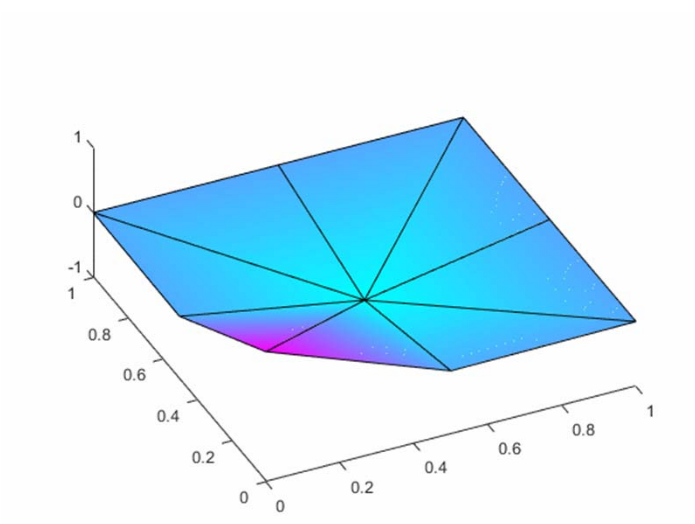
EIGENVALUE PROBLEMS

PROBLEM #26 – The results for **mode 1** are:



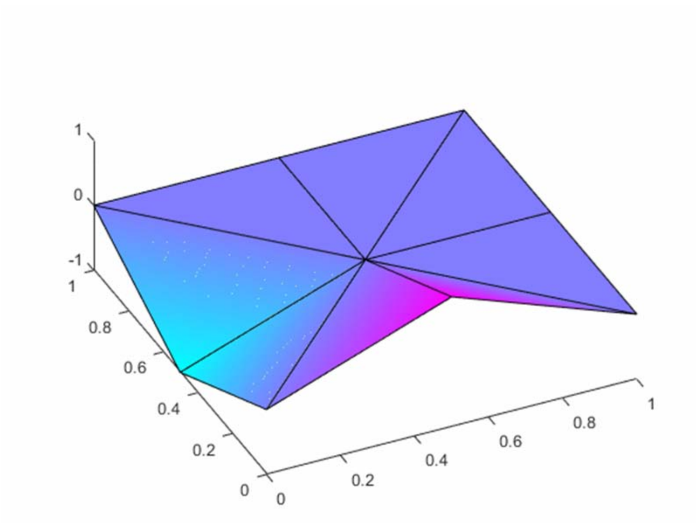
EIGENVALUE PROBLEMS

PROBLEM #26 – The results for **mode 2** are:



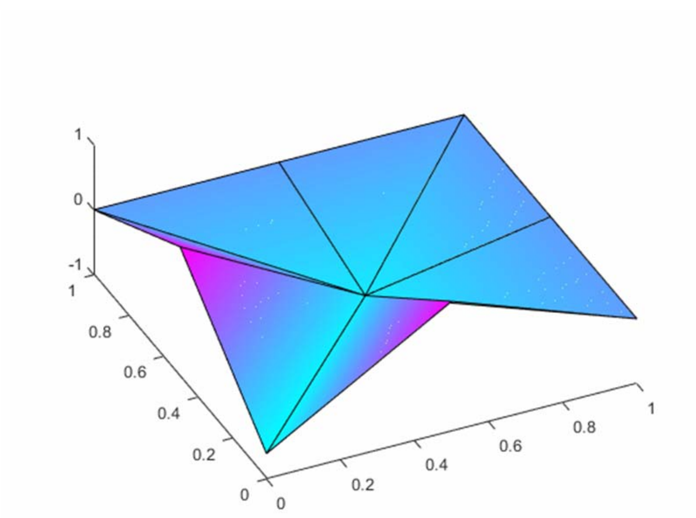
EIGENVALUE PROBLEMS

PROBLEM #26 – The results for **mode 3** are:



EIGENVALUE PROBLEMS

PROBLEM #26 – The results for **mode 4** are:



End of
2-D Eigenvalues