Axisymmetric problems are sometimes referred to as radially symmetric problems. They are geometrically three-dimensional but mathematically only two-dimensional in the physics of the problem. In other words, the dependent variable is a function of the coordinates $r$ and $z$ and not a function of the angle $\theta$ $u = u(r, z)$.

The three-dimensional Laplacian operator in axisymmetric problems reduces to:

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2}$$

which may be written as:

$$\nabla^2 u = \frac{1}{r} \left( \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial z} \left( r \frac{\partial u}{\partial z} \right) \right)$$

The corresponding energy functional that will serve as the basis for a Ritz finite element model is:

$$Z(u) = \frac{1}{2} \int_{\Omega} \left[ \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + f \left( r \frac{\partial u}{\partial r} \right) \right] dr dz - \int_{\Gamma} ur dr dz + \frac{1}{2} \int_{\Gamma} \frac{\partial u}{\partial n} \alpha(s) u + h(s) ds = 0$$

Due to the mathematical nature of the problem, the analysis may be performed within a two-dimensional region in the $rz$-plane which is revolved about the $z$-axis.
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS
AXISYMMETRIC PROBLEMS
The revolving region defined the actual three-dimensional domain.

Cross-section and axial schematic of the coaxial slot antenna

Discretization - As usual the first step is developing a finite element model is the discretization of the problem geometry.
As an introduction, we will limit our discussion of the discretization and formulation of the axisymmetric problem to linear triangles.

\[ Z(u) = \frac{1}{2} \sum_{e} \int_{\Omega} \left( \frac{\partial u}{\partial r} \right)^2 + r \left( \frac{\partial u}{\partial z} \right)^2 \, dr \, dz + \sum_{\Gamma_2} \int_{\Gamma_2} a \, u \, ds \]
\[ - \sum_{\Gamma_2} \int_{\Gamma_2} u \, t \, dr \, dz - \sum_{\Gamma_2} \int_{\Gamma_2} u \, h \, ds = 0 \]
Interpolation - The simplest interpolation over a straight-sided three node triangular element is to assume the function $u(r, z)$ is represented by a linear plane.

\[ u(r, z) = \alpha + \beta r + \gamma z \]

where $\alpha$, $\beta$, and $\gamma$ are constant determined by matching the function $u_e$ with the nodal values of the element:

\[ u_e(r, z) = \alpha + \beta r + \gamma z \]

\[ u_e(r_i, z_i) = \alpha + \beta r_i + \gamma z_i \]

\[ u_e(r_j, z_j) = \alpha + \beta r_j + \gamma z_j \]

\[ u_e(r_k, z_k) = \alpha + \beta r_k + \gamma z_k \]

Interpolation - Linearly interpolated triangular elements represent the variation of the dependent variable $u$ over an element as:

\[ u_e(r, z) = N_i u_i + N_j u_j + N_k u_k \]

where:

\[ N_i = \frac{a_i + b_i r + c_i z}{2A_e} \]

\[ a_i = r_j z_k - r_k z_j \]

\[ b_i = z_j - z_k \]

\[ c_i = r_k - r_j \]

where $i, j, k$ are permuted cyclically.

Interpolation - In matrix notation, the distribution of the function over the element is:

\[ u_e(r, z) = u_e^T N = N' u_e \]

The linear triangular shape functions are illustrated below.
**Two-Dimensional Boundary Value Problems**

**Interpolation** - The derivatives of $u$ over the element with respect to both coordinates are:

$$
\frac{\partial u_i(r,z)}{\partial r} = u_i^T \frac{\partial N}{\partial r} u_i + \frac{\partial u_i}{\partial z} \frac{\partial N}{\partial z}
$$

$$
\frac{\partial u_i(r,z)}{\partial z} = u_i^T \frac{\partial N}{\partial r} u_i + \frac{\partial u_i}{\partial r} \frac{\partial N}{\partial z}
$$

Calculating the derivatives of the shape functions gives:

$$
\frac{\partial N}{\partial r} = \frac{b_s}{2A_e}
$$

$$
\frac{\partial N}{\partial z} = \frac{c_s}{2A_e}
$$

$$
b_s^T = (b_1, b_2, b_3)
$$

$$
c_s^T = (c_1, c_2, c_3)
$$

$$
b_i = Z_j - Z_s
$$

$$
c_i = r_j - r_s
$$

**Elemental Formulation** - The functional for the Poisson equation is:

$$
Z(u) = \sum_s \frac{1}{2} \int_{\Delta e_s} \left[ \left( \frac{\partial u}{\partial r} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2 \right] dA
$$

$$
- \frac{1}{2} \sum_s \int_{\Delta e_s} \left[ \alpha u^2 ds + \sum_s \int_{\Delta e_s} urh ds \right] = 0
$$

We can write the functional in the following form:

$$
Z(u) = \sum_s \frac{Z_{s1}}{2} \sum_s \frac{Z_{s2}}{2} \sum_s Z_{s3} - \sum_s Z_{s4}
$$

**Interpolation** - Observing the form of the derivative it is apparent that the partial derivatives of the function $u$ will be constant over a linear triangular element.

There are many problems associated with accuracy and convergence for this type of element.

In elasticity analysis, stress and strain are related by a partial differential equation, using a linear triangular element to describe stress will result in a constant approximation for strain over the element.

Therefore, elements of this type are called **constant strain elements**.

**Elemental Formulation - Evaluation of $Z_{s1}$**:

$$
Z_{s1} = \int_{A_e} \left[ \frac{\partial u}{\partial r} \frac{\partial u}{\partial r} + r \frac{\partial u}{\partial z} \frac{\partial u}{\partial z} \right] dA
$$

Recall the first derivatives of $u$ with respect to $r$ and $z$ are:

$$
\frac{\partial u_i(r,z)}{\partial r} = u_i^T \frac{\partial N}{\partial r} u_i + \frac{\partial u_i}{\partial z} \frac{\partial N}{\partial z}
$$

$$
\frac{\partial u_i(r,z)}{\partial z} = u_i^T \frac{\partial N}{\partial r} u_i + \frac{\partial u_i}{\partial r} \frac{\partial N}{\partial z}
$$

**Elemental Formulation - Evaluation of $Z_{s1}$**: Replacing the derivatives with the above approximations gives:

$$
Z_{s1} = \int_{A_e} \left[ u_i^T \frac{\partial N}{\partial r} + \frac{\partial u_i}{\partial z} \frac{\partial N}{\partial z} \right] u_i dA
$$

$$
k_s = \int_{A_e} \left[ \frac{\partial N}{\partial r} \frac{\partial N}{\partial r} + \frac{\partial N}{\partial z} \frac{\partial N}{\partial z} \right] dA
$$
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Elemental Formulation - Evaluation of $Z_e$:
The integrals defined in $k_e$ are the elemental “stiffness” matrix.

For the linear triangular element we have discussed the
stiffness matrix reduces to:

$$k_e = \iint_A \left[ \frac{b \, r \, b_r + c \, r \, c_r}{4 \, A_e} \right] \, dr \, dz$$

The resulting $3 \times 3$ elemental stiffness matrix contributes to
the global system equations at locations corresponding to
the element nodes.

Elemental Formulation - Evaluation of $Z_{e2}$:
In this case, the interpolation of $u$ with respect to $r$ and $z$ is
used to describe the behavior along the boundary:

$$Z_{e2} = \int_{\gamma_e} u_i N_i N_e ds = u_i^s \left( \int_{\gamma_e} N_i N_e ds \right) u_e = u_i^s a_e u_e$$

$$a_e = \left( \int_{\gamma_e} N_i N_e ds \right)$$

The resulting is a $2 \times 2$ elemental stiffness matrix.

Elemental Formulation - Evaluation of $Z_{e3}$:
Substituting the approximation for $u$ into the integral results in:

$$Z_{e3} = u_i^s \left( \int_A N_r f dA \right) = u_i^s f_e = \left( \int_A N_r f dA \right)$$

The resulting is a $3 \times 1$ elemental load vector.

Elemental Formulation - Evaluation of $Z_{e4}$:
Substituting the approximation for $u$ into the integral results in:

$$Z_{e4} = u_i^s \left( \int_{\gamma_e} N_r h ds \right) = u_i^s h_e = \left( \int_{\gamma_e} N_r h ds \right)$$

The resulting is a $2 \times 1$ elemental load vector.

Elemental Formulation - In terms of the matrix definitions, the
functional may be written in the following form:

$$Z(\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}_u) = \sum_{e} \left[ \frac{1}{2} (\mathbf{u}_e - \mathbf{u}_j)^T \mathbf{K}_e (\mathbf{u}_e - \mathbf{u}_j) + \sum_{e} \frac{1}{2} (\mathbf{u}_e^a - \mathbf{u}_j^a)^T \mathbf{h}_e \right]$$

where the first sum is over the each element of area $A_e$
describing the domain $\Omega$ and the second sum is over every
element that has a segment along the $\Gamma_e$ portion of the boundary.

Assembly - The assembly is denoted by the summation in
the matrix equation. The global matrix form of the
formulation is:

$$Z = \mathbf{u}_0^T \mathbf{K}_a \mathbf{u}_0 - \mathbf{u}_0^T \mathbf{F}_0 = Z(\mathbf{u}_0)$$

$$\mathbf{K}_a = \sum_0 \mathbf{K}_0 + \sum_0 \mathbf{a}_0 \quad \mathbf{F}_0 = \sum_0 \mathbf{f}_0 + \sum_0 \mathbf{h}_0$$

$$\frac{\partial Z}{\partial u_i} = 0 \quad \frac{\partial Z}{\partial u_i} \left( \mathbf{K}_a + \frac{\partial \mathbf{K}_0}{\partial u_i} \right) u_i - \mathbf{F}_0 \rightarrow \mathbf{K}_a \mathbf{u}_0 = \mathbf{F}_0$$
**Constraints** - The constraints on the system equations are the forced boundary conditions \( u = g(s) \) on the surface \( \Gamma_1 \).

These conditions are applied to the system equations in a manner similar to that discussed for one- and two-dimensional problems.

**Solution** - Details of the solution of the simultaneous equations resulting from axisymmetric boundary value problems are presented in the two-dimensional section of the notes.

**Computation of Derived Variables** - In this case, the values of the function \( u \) are the primary variables and \( \frac{\partial u}{\partial r} \) and \( \frac{\partial u}{\partial z} \) are considered a secondary variable.

The partial derivatives are determined by the following expressions:

\[
\frac{\partial u(r,z)}{\partial r} = u_{r}(r,z) = 2A_{1} A_{2} \int_{\Omega} b_{r} b_{r}^{T} + c_{r} c_{r}^{T} dr dz
\]

\[
\frac{\partial u(r,z)}{\partial z} = u_{z}(r,z) = 2A_{1} A_{2} \int_{\Omega} b_{z} b_{z}^{T} + c_{z} c_{z}^{T} dr dz
\]

**Evaluation of Matrices - Linear Triangular Elements**

Recall the elemental matrices have the following form:

\[
k_{e} = \int_{\Omega} b_{r} b_{r}^{T} + c_{r} c_{r}^{T} dr dz
\]

\[
a_{e} = \int_{\Gamma} \mathbf{N}_{r} \mathbf{N}^{T} d\Gamma
\]

\[
h_{e} = \int_{\Gamma} \mathbf{N}_{r} \mathbf{N}^{T} d\Gamma
\]

These integrals are essentially the same as the terms developed for two-dimensional Poisson’s equations using linear triangular element.

The obvious difference is the \( r \) coordinate which appears in each integral.

**Evaluation of \( k_{e} \)** - Substituting all the pieces of the transformation in the \( k_{e} \) terms gives:

\[
k_{e} = \int_{\Omega} b_{r} b_{r}^{T} + c_{r} c_{r}^{T} dr dz
\]

\[
= \frac{b_{r} b_{r}^{T} + c_{r} c_{r}^{T}}{4A_{e}} \int_{\Omega} r dr dz
\]

\[
R = r_{e} + r_{i} + r_{c} \frac{1}{3}
\]

The value of \( R \) is the \( r \)-coordinate of the centroid of the linear triangular element.

**Evaluation of \( f_{e} \)** - In general, the integral \( f_{e} \) is:

\[
f_{e} = \int_{\Omega} \mathbf{N}_{r} f(x,y) d\Omega
\]

Replacing \( r \) by the exact representation \( r_{e} \mathbf{N} \) and assuming that the function \( f \) may be approximated by the linear interpolation \( \mathbf{N}^{T} \), the element matrix \( f_{e} \) becomes:

\[
f_{e} \approx \int_{\Omega} \mathbf{N}_{e} \mathbf{N}^{T} f dr dz
\]
**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

Evaluation of $f_1$ - The formula for integrations of the type is given without proof as:

$$
\int_{\gamma_0} N_1^a N_1^b \, ds = a! b! \frac{I_a}{(a + b + 1)!}
$$

Therefore:

$$
f_1 = \frac{A_{e1}}{60} \left[ \begin{array}{cccc}
6r_1 + 2r_2 + r_3 & 2r_2 + r_1 + r_3 & 2r_2 + r_1 + 2r_3 & \frac{\xi}{\xi} \\
2r_3 + 6r_2 + r_1 & r_1 + 2r_3 & 2r_1 + 2r_2 + 6r_3 & \frac{\xi}{\xi} \\
\end{array} \right]
$$

**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

Evaluation of $f_2$ - The formula for integrations of the type is given without proof as:

$$
\int_{\gamma_0} N_2^a N_2^b \, ds = a! b! \frac{I_a}{(a + b + 1)!}
$$

If the function $f$ is a constant, $f_0$, the above matrix reduces to:

$$
f_0 = \frac{A_{e2}}{12} \left[ \begin{array}{cccc}
2r_1 + r_2 + r_3 & 2r_2 + r_3 & 2r_1 + 2r_2 + 2r_3 & \frac{\xi}{\xi} \\
r_1 + r_2 + r_3 & r_1 + r_2 + 2r_3 & r_1 + 2r_3 & \frac{\xi}{\xi} \\
\end{array} \right]
$$

The resulting is a 3 x 1 **elemental load vector**.

---

**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

Evaluation of $a_i$ - The integration formula for the type of integrals is:

$$
\int_{\gamma_0} N_i^a N_i^b \, ds = a! b! \frac{I_a}{(a + b + 1)!}
$$

$$
[a_i]_{11} = \frac{I_a}{60} \left( 12\alpha_j \gamma + 3(\alpha_j \gamma + \alpha_i \gamma) + 2\alpha_i \gamma \right)
$$

$$
[a_i]_{12} = \frac{I_a}{60} \left( 3\alpha_j \gamma + 2(\alpha_j \gamma + \alpha_i \gamma) + 3\alpha_i \gamma \right)
$$

---

**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

Evaluation of $a_i$ - The integration formula for the type of integrals is:

$$
\int_{\gamma_0} N_i^a N_i^b \, ds = a! b! \frac{I_a}{(a + b + 1)!}
$$

$$
[a_i]_{11} = \frac{I_a}{60} \left( 3\alpha_j \gamma + 2(\alpha_j \gamma + \alpha_i \gamma) + 3\alpha_i \gamma \right)
$$

$$
[a_i]_{12} = \frac{I_a}{60} \left( 2\alpha_j \gamma + 3(\alpha_j \gamma + \alpha_i \gamma) + 12\alpha_i \gamma \right)
$$
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Evaluation of \( h_\gamma \) - Consider the integral: 

\[
\gamma = \int_{r_s} r h(d) \, d\gamma
\]

where the integration is along a boundary segment of the element.

Since, the integration is computed along a single side of the triangular element, the original shape functions reduce to:

\[
h_\gamma = \int \left(1 - \xi\right) r h(\xi) \, d\xi
\]

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Evaluation of \( h_\gamma \) - Replacing \( r \) by the exact representation \( r_sN \), and assuming that the function \( h \) may be approximated by the linear interpolation \( N^T h \), the element matrix \( h_\gamma \) becomes:

\[
h_\gamma = \int_{r_s} r_s N^T N h \, d\gamma
\]

If the function \( h \) is a constant, \( h_\gamma \), the above matrix reduces to:

\[
h_\gamma = \frac{l h}{12} \left[ r_j + r_s \right]
\]

The resulting is a 2 x 1 elemental load vector

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Recall the elemental matrices have the following form:

\[
K_e = \int r_s r_t d\tau \quad T_e = \int N^T f \, d\tau
\]

If it should be clear that any of the elements we have discussed, (quadratic triangles, quadrilaterals, etc.) may be used in connection with the axisymmetric functional to develop a finite element model.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

End of Axisymmetric Problems