TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Axisymmetric problems are sometimes referred to as radially symmetric problems.

They are geometrically three-dimensional but mathematically only two-dimensional in the physics of the problem.

In other words, the dependent variable is a function of the coordinates $r$ and $z$ and not a function of the angle $\theta$ $u = u(r, z)$.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Axisymmetric problems are associated with bodies of revolution as indicated in the figure below:
The three-dimensional Laplacian operator in axisymmetric problems reduces to:

\[ \nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} \]

which may be written as:

\[ \nabla^2 u = \frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial z} \left( r \frac{\partial u}{\partial z} \right) \right] \]

The Poisson boundary value problem is:

\[ \nabla^2 u(r, z) + f(r, z) = 0 \quad \text{in } \Omega \]

\[ u = g(s) \quad \text{on } \Gamma_1 \]

\[ \frac{\partial u}{\partial n} + \alpha(s) u = h(s) \quad \text{on } \Gamma_2 \]

where the surface \( \Gamma_1 \) is the portion of the surface \( \Gamma \) where the Dirichlet type boundary conditions are defined and \( \Gamma_2 \) is the portions where the Neumann or Robin boundary conditions are prescribed.
The corresponding energy functional that will serve as the basis for a Ritz finite element model is:

\[
Z(u) = \frac{1}{2} \int_{\Omega} \left[ r \left( \frac{\partial u}{\partial r} \right)^2 + r \left( \frac{\partial u}{\partial z} \right)^2 \right] dr \; dz - \int_{\Omega} urf \; dr \; dz \\
\quad + \frac{1}{2} \int_{\Gamma_r} \alpha ru^2 \; ds - \int_{\Gamma_r} urh \; ds = 0
\]

Due to the mathematical nature of the problem, the analysis may be performed within a two-dimensional region in the \( rz \)-plane which is revolved about the \( z \)-axis.

The revolving region defined the actual three-dimensional domain.
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

The revolving region defined the actual three-dimensional domain.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

The revolving region defined the actual three-dimensional domain.
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

The revolving region defined the actual three-dimensional domain.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Cross-section and axial schematic of the coaxial slot antenna
Cross-section and axial schematic of the coaxial slot antenna

**Discretization** - As usual the first step is developing a finite element model is the discretization of the problem geometry.

As an introduction, we will limit our discussion of the discretization and formulation of the axisymmetric problem to linear triangles.
**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Discretization** - In terms of the discretization, the functional \( Z \) is now a sum of the integrals over each element in the domain \( \Omega \), and the sum of surface integrals on the boundary segments along \( \Gamma_2 \):

\[
Z(u) \approx \frac{1}{2} \left( \sum_e \int_\Omega \left( r \left( \frac{\partial u}{\partial r} \right)^2 + r \left( \frac{\partial u}{\partial z} \right)^2 \right) dr \, dz + \sum_{e \in \Gamma_{2e}} \int_{\gamma_{2e}} \alpha ru^2 \, ds \right)
- \sum_e \int_\Omega \int \alpha r f \, dr \, dz - \sum_{e \in \Gamma_{2e}} \int_{\gamma_{2e}} \alpha rh \, ds = 0
\]

**Interpolation** - The simplest interpolation over a straight-sided three node triangular element is to assume the function \( u(r, z) \) is represented by a linear plane.

![Linear representation of u(r, z)](image)
**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Interpolation** - Linearly interpolated triangular elements represent the variation of the dependent variable $u$ over an element as:

$$u_e(r, z) = \alpha + \beta r + \gamma z$$

where $\alpha, \beta$, and $\gamma$ are constant determined by matching the function $u_e$ with the nodal values of the element:

$$u_e(r_i, z_i) = \alpha + \beta r_i + \gamma z_i$$

$$u_e(r_j, z_j) = \alpha + \beta r_j + \gamma z_j$$

$$u_e(r_k, z_k) = \alpha + \beta r_k + \gamma z_k$$
**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Interpolation** - Solving the three equations for $\alpha$, $\beta$, and $\gamma$ and substituting back into the expression representing the variation of $u$ over the element results in:

$$u_e(r, z) = N_i u_i + N_j u_j + N_k u_k$$

where:

$$N_i = \frac{a_i + b_i r + c_i z}{2A_e} \quad i = 1, 2, 3$$

$$a_i = r_j z_k - r_k z_j \quad b_i = z_j - z_k \quad c_i = r_k - r_j$$

where $i$, $j$, and $k$ are permuted cyclically.

**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Interpolation** - The determinant of the coefficients is:

$$2A_e = \begin{vmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_k & z_k \end{vmatrix}$$

where $A_e$ is the area of the element.

Any numbering scheme that proceeds counterclockwise around the element is valid, for example $(i, j, k)$, $(j, k, i)$, or $(k, i, j)$.

This numbering convention is important and necessary in order to compute a positive area for $A_e$. 
**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Interpolation** - In matrix notation, the distribution of the function over the element is:

\[ u_e(r, z) = u_e^T N = N^T u_e \]

The linear triangular shape functions are illustrated below:
**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Interpolation** - The derivatives of $u$ over the element with respect to both coordinates are:

$$\frac{\partial u_e}{\partial r} = u_e^T \frac{\partial N}{\partial r} = u_e^T \frac{\partial N^T}{\partial r} u_e$$

$$\frac{\partial u_e}{\partial z} = u_e^T \frac{\partial N}{\partial z} = u_e^T \frac{\partial N^T}{\partial z} u_e$$

Calculating the derivatives of the shape functions gives:

$$\frac{\partial N}{\partial r} = \frac{b_e}{2A_e}$$

$$\frac{\partial N}{\partial z} = \frac{c_e}{2A_e}$$

$$b_e^T = \langle b_1 \quad b_2 \quad b_3 \rangle$$

$$c_e^T = \langle c_1 \quad c_2 \quad c_3 \rangle$$

$$b_i = z_j - z_k$$

$$c_i = r_k - r_j$$

**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Interpolation** - Observing the form of the derivative it is apparent that the partial derivatives of the function $u$ will be constant over a linear triangular element.

There are many problems associated with accuracy and convergence for this type of element.

In elasticity analysis, stress and strain are related by a partial differential equation, using a linear triangular element to described stress will result in a constant approximation for strain over the element.

Therefore, elements of this type are called **constant strain elements**.
**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Elemental Formulation** - The functional for the Poisson equation is:

\[
Z(u) = \frac{1}{2} \sum_{e} \int_{A_e} \left[ r \left( \frac{\partial u}{\partial r} \right)^2 + r \left( \frac{\partial u}{\partial z} \right)^2 \right] dA
\]

\[+ \frac{1}{2} \sum_{e} \int_{\gamma_{2e}} \alpha r u^2 ds - \sum_{e} \int_{A_e} urf dA - \sum_{e} \int_{\gamma_{2e}} urh ds = 0\]

We can write the functional in the following form:

\[
Z(u) \approx \sum_{e} Z_{e1} + \sum_{e} Z_{e2} - \sum_{e} Z_{e3} - \sum_{e} Z_{e4}
\]

**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Elemental Formulation** – Where the components are defined as:

\[
Z_{e1} = \int_{A_e} \left[ r \left( \frac{\partial u}{\partial r} \right)^2 + r \left( \frac{\partial u}{\partial z} \right)^2 \right] dr dz
\]

\[Z_{e2} = \int_{\gamma_{2e}} \alpha r u^2 ds\]

\[Z_{e3} = \int_{A_e} urf dr dz\]

\[Z_{e4} = \int_{\gamma_{2e}} urh ds\]
**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Elemental Formulation - Evaluation of $Z_{e1}$:**

\[
Z_{e1} = \int_A \left[ r \frac{\partial u}{\partial r} \frac{\partial u}{\partial r} + r \frac{\partial u}{\partial z} \frac{\partial u}{\partial z} \right] dA
\]

Recall the first derivatives of $u$ with respect to $r$ and $z$ are:

\[
\frac{\partial u_e(r,z)}{\partial r} = u_e^T \frac{\partial N}{\partial r} = \frac{\partial N^T}{\partial r} u_e
\]

\[
\frac{\partial u_e(r,z)}{\partial z} = u_e^T \frac{\partial N}{\partial z} = \frac{\partial N^T}{\partial z} u_e
\]
**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Elemental Formulation - Evaluation of $Z_{e1}$:** The integrals defined in $k_e$ are the elemental “stiffness” matrix. For the linear triangular element we have discussed the stiffness matrix reduces to:

$$k_e = \iint_{A_e} \left[ \frac{b_e r b_e^T + c_e r c_e^T}{4A_e^2} \right] dr dz$$

The resulting 3 x 3 elemental stiffness matrix contributes to the global system equations at locations corresponding to the element nodes.

---

**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Elemental Formulation - Evaluation of $Z_{e2}$:**

$$Z_{e2} = \int_{\gamma_{2e}} \alpha r u^2 \, ds$$

In this case, the interpolation of $u$ with respect to $r$ and $z$ is used to describe the behavior along the boundary:

$$Z_{e2} = \int_{\gamma_{2e}} u_e^T N \alpha r N^T u_e \, ds = u_e^T \left( \int_{\gamma_{2e}} N \alpha r N^T \, ds \right) u_e = u_e^T a_e u_e$$

$$a_e = \int_{\gamma_{2e}} N \alpha r N^T \, ds$$

The resulting is a 2 x 2 elemental stiffness matrix.
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Elemental Formulation - Evaluation of $Z_{e3}$:

$$Z_{e3} = \int_{A_e} ur f dA$$

Substituting the approximation for $u$ into the integral results in:

$$Z_{e3} = u_e^T \left( \int_{\gamma_{e2}} N r f ds \right) = u_e^T f_e \quad f_e = \int_{A_e} N r f dA$$

The resulting is a $3 \times 1$ **elemental load vector**

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TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Elemental Formulation - Evaluation of $Z_{e4}$:

$$Z_{e4} = \int_{\gamma_{e2}} ur h ds$$

Substituting the approximation for $u$ into the integral results in:

$$Z_{e4} = u_e^T \left( \int_{\gamma_{e2}} N r h ds \right) = u_e^T h_e \quad h_e = \int_{\gamma_{e2}} N r h ds$$

The resulting is a $2 \times 1$ **elemental load vector**
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Elemental Formulation - In terms of the matrix definitions, the functional may be written in the following form:

\[ Z(u_1, u_2, u_3, \ldots, u_N) = \sum_{e} \left( \frac{u_{e}^T k_{e} u_{e}}{2} - u_{e}^T f_{e} \right) + \sum_{e} \left( \frac{u_{e}^T a_{e} u_{e}}{2} - u_{e}^T h_{e} \right) \]

where the first sum is over the each element of area \( A_e \) describing the domain \( \Omega \) and the second sum is over every element that has a segment along the \( \Gamma_2 \) portion of the boundary.

Assembly - The assembly is denoted by the summation in the matrix equation. The global matrix form of the formulation is:

\[ Z \approx \frac{u_G^T K_G u_G}{2} - u_G^T F_G = Z(u_G) \]

\[ K_G = \sum_{e} k_{e} + \sum_{e} 'a_{e} \quad F_G = \sum_{e} f_{e} + \sum_{e} 'h_{e} \]

\[ \frac{\partial Z}{\partial u_i} = 0 \quad \frac{\partial Z}{\partial u_i} = \frac{(K_G + K_G^T) u_e}{2} - F_G \rightarrow K_G u_G = F_G \]
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Constraints - The constraints on the system equations are the forced boundary conditions $u = g(s)$ on the surface $\Gamma_1$.

These conditions are applied to the system equations in a manner similar to that discussed for one- and two-dimensional problems.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Solution - Details of the solution of the simultaneous equations resulting from axisymmetric boundary value problems are presented in the two-dimensional section of the notes.
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Computation of Derived Variables - In this case, the values of the function \( u \) are the primary variables and \( \frac{\partial u}{\partial r} \) and \( \frac{\partial u}{\partial z} \) are considered a secondary variable.

The partial derivatives are determined by the following expressions:

\[
\frac{\partial u_e(r,z)}{\partial r} = u_e^T \frac{\partial N}{\partial r} = \frac{\partial N^T}{\partial r} u_e = \frac{b_e^T u_e}{2A_e}
\]

\[
\frac{\partial u_e(r,z)}{\partial z} = u_e^T \frac{\partial N}{\partial z} = \frac{\partial N^T}{\partial z} u_e = \frac{c_e^T u_e}{2A_e}
\]

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Recall the elemental matrices have the following form:

\[
k_e = \int_{\gamma} \left[ b_e^T b_e + c_e^T c_e \right] dr dz
\]

\[
f_e = \int_{\gamma} \left[ N f \right] dr dz
\]

\[
a_e = \int_{\gamma} \left[ N \alpha r N^T \right] ds
\]

\[
h_e = \int_{\gamma} \left[ N r h \right] ds
\]

These integrals are essentially the same as the terms developed for two-dimensional Poisson’s equations using linear triangular element.

The obvious difference is the \( r \) coordinate which appears in each integral.
**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Evaluation of** $k_e$ - Substituting all the pieces of the transformation in the $k_e$ terms gives:

$$
\begin{align*}
  k_e &= \iint_{A_e} \left[ \frac{b_e^r b_e^T + c_e^r c_e^T}{4A_e^4} \right] dr \, dz \\
  &= \frac{b_e^r b_e^T + c_e^r c_e^T}{4A_e^4} \iint_{A_e} r \, dr \, dz \\
  &= \frac{b_e^r b_e^T + c_e^r c_e^T}{4A_e^4} R \\
  R &= \frac{r_i + r_j + r_k}{3}
\end{align*}
$$

The value of $R$ is the $r$-coordinate of the centroid of the linear triangular element.

**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Evaluation of** $f_e$ - In general, the integral $f_e$ is:

$$
  f_e = \iint_{A_e} Nr f(r, z) dA
$$

Replacing $r$ by the exact representation $r_e^T N$, and assuming that the function $f$ may be approximated by the linear interpolation $N^T f$, the element matrix $f_e$ becomes:

$$
  f_e \approx \iint_{A_e} N r_e^T N N^T f \, dr \, dz
$$
**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Evaluation of \( f_\theta \)** - The formula for integrations of the type is given without proof as:

\[
\int_\mathcal{A}_e \int \left[N_i^a N_j^b N_K^c\right] dA = \frac{2A_e}{(a + b + c + 2)!} a!b!c!
\]

Therefore:

\[
f_\theta = \frac{A_e}{60} \begin{bmatrix}
6r_i + 2r_j + 2r_k & 2r_i + 2r_j + r_k & 2r_i + r_j + 2r_k \\
2r_i + 6r_j + 2r_k & r_i + 2r_j + 2r_k & 2r_i + 2r_j + r_k \\
symmetric & 2r_i + 2r_j + 6r_k
\end{bmatrix}
\begin{bmatrix}
f_i \\
f_j \\
f_k
\end{bmatrix}
\]

**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Evaluation of \( f_\theta \)** - The formula for integrations of the type is given without proof as:

\[
\int_\mathcal{A}_e \int \left[N_i^a N_j^b N_K^c\right] dA = \frac{2A_e}{(a + b + c + 2)!} a!b!c!
\]

If the function \( f \) is a constant, \( f_0 \), the above matrix reduces to:

\[
f_\theta = \frac{A_e f_0}{12} \begin{bmatrix}
2r_i + r_j + r_k \\
r_i + 2r_j + r_k \\
r_i + r_j + 2r_k
\end{bmatrix}
\]

The resulting is a 3 x 1 **elemental load vector**
**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

Evaluation of $a_e$: \[ a_e = \int_{\gamma_{2e}} N \alpha r N^T \, ds \]

Since $a_e$ is evaluated along a segment of the boundary $\gamma_{2e}$, the interpolation functions reduce to their one-dimensional counterparts.

\[
\begin{align*}
N_i &= 0 \\
N_j &= 1 - \xi \\
N_k &= \xi
\end{align*}
\]

As before, $r$ may be replaced by the exact expression $r_e^TN$ and assuming that $\alpha$ may be approximated by a linear interpolation as $N^T\alpha$, therefore, $a_e$ becomes:

\[
\begin{align*}
N_i &= 0 \\
N_j &= 1 - \xi \\
N_k &= \xi
\end{align*}
\]

\[ a_e = \int_{0}^{1} N r_e^T N N^T \alpha N^T \, l_e \, d\xi \]
**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

Evaluation of $a_e$: 

$$a_e = \int_{\gamma_{2e}} N \alpha r N^T ds$$

$$a_e = \int_0^1 N e r^T N \alpha N^T l_e d\xi$$

$$= \int_0^1 \left[ \begin{array}{c} 1 - \xi \\ \xi \end{array} \right]_{2x1} \left[ \begin{array}{c} r_j \\ r_k \end{array} \right]_{1x2} \left[ \begin{array}{c} 1 - \xi \\ \xi \end{array} \right]_{2x1} \left[ \begin{array}{c} \alpha_j \\ \alpha_k \end{array} \right]_{2x1} \left[ \begin{array}{c} 1 - \xi \\ \xi \end{array} \right]_{1x2} l_e d\xi$$

$$= \int_0^1 \left[ \begin{array}{c} 1 - \xi \\ \xi \end{array} \right]_{2x1} \left( \alpha_j \left[ r_j (\xi - 1) - r_k \xi \right] - \alpha_k \xi \left[ r_j (\xi - 1) - r_k \xi \right] \right)_{1x1} \left[ \begin{array}{c} 1 - \xi \\ \xi \end{array} \right]_{1x2} l_e d\xi$$

---

**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

Evaluation of $a_e$: - The integration formula for the type of integrals is:

$$\int_{\gamma_{2e}} N_i^a N_j^b ds = a! b! \frac{l_e}{(a + b + 1)!}$$

$$[a_e]_{11} = \frac{l_e}{60} \left( 12 \alpha_j r_j + 3 \left( \alpha_j r_k + \alpha_k r_j \right) + 2 \alpha_k r_k \right)$$

$$[a_e]_{12} = \frac{l_e}{60} \left( 3 \alpha_j r_j + 2 \left( \alpha_j r_k + \alpha_k r_j \right) + 3 \alpha_k r_k \right)$$
**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Evaluation of $a_e$** - The integration formula for the type of integrals is:

\[
\int_{\gamma_{2e}} N_i^a N_j^b \, ds = a! \, b! \frac{l_{ae}}{(a + b + 1)!}
\]

\[
[a_e]_{21} = \frac{l_{ae}}{60} \left(3\alpha_j r_j + 2\left(\alpha_j r_k + \alpha_k r_j\right) + 3\alpha_k r_k\right)
\]

\[
[a_e]_{22} = \frac{l_{ae}}{60} \left(2\alpha_j r_j + 3\left(\alpha_j r_k + \alpha_k r_j\right) + 12\alpha_k r_k\right)
\]

**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Evaluation of $h_e$** - Consider the integral: \( h_e = \int_{\gamma_{2e}} N r \, h \, ds \)

where the integration is along a boundary segment of the element.

Since, the integration is computed along a single side of the triangular element, the original shape functions reduce to:

\( N_i = 0 \quad N_j = 1 - \xi \quad N_k = \xi \)

\[
h_e = \int_{\xi=0}^{\xi=1} \left(1 - \frac{\xi}{\xi}\right) r \, h(\xi) \, l_{ae} \, d\xi
\]
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Evaluation of $h_e$ - Replacing $r$ by the exact representation $r_e^T N$, and assuming that the function $h$ may be approximated by the linear interpolation $N^T h$, the element matrix $h_e$ becomes:

$$h_e = \int_{\gamma e} N r_e^T N N^T h d\gamma = \int_{\gamma e} \left\{ \frac{1 - \xi}{\xi} \right\} \left\{ \begin{array}{c} r_j \\ r_k \end{array} \right\} \left\{ \begin{array}{c} 1 - \xi \\ \xi \end{array} \right\} \left\{ \begin{array}{c} h_j \\ h_k \end{array} \right\} l_e d\xi$$

$$N_i = 0 \quad N_j = 1 - \xi \quad N_k = \xi$$

Recall that $h_e$ in $x$ and $y$ is:

$$h_e = \int_{\gamma e} \left\{ \frac{1 - \xi}{\xi} \right\} r h(\xi) l_e d\xi$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Evaluation of $h_e$ - With this assumption the integral becomes:

$$h_e = \frac{l_e}{12} \begin{bmatrix} 3r_j + r_k & r_j + r_k \\ r_j + r_k & r_j + 3r_k \end{bmatrix} \begin{bmatrix} h_j \\ h_k \end{bmatrix}$$

Recall that $h_e$ in $x$ and $y$ is:

$$h_e = \frac{l_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} h_j \\ h_k \end{bmatrix} \quad r_j = r_k = 1$$
**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Evaluation of** $h_e$ **-** If the function $h$ is a constant, $h_0$, the above matrix reduces to:

$$h_e = \frac{l_e h_0}{6} \begin{bmatrix} 2r_j + r_k \\ r_j + 2r_k \end{bmatrix}$$

Recall that $h_e$ in $x$ and $y$ is:

$$h_e = \frac{l_e h_0}{6} \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad r_j = r_k = 1$$

The resulting is a $2 \times 1$ elemental load vector

**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**AXISYMMETRIC PROBLEMS**

**Evaluation of Matrices - Linear Triangular Elements**

Recall the elemental matrices have the following form:

$$k_e = \iint_A \left[ b_e r b_e^T + c_e r c_e^T \right] dr \, dz$$

$$f_e = \iint_{A_e} N r f \, dr \, dz$$

$$a_e = \int_{r_2} N \alpha r N^T \, ds$$

$$h_e = \int_{r_2} N r h \, ds$$

It should be clear that any of the elements we have discussed, (quadratic triangles, quadrilaterals, etc.) may be used in connection with the axisymmetric functional to develop a finite element model.
PROBLEM #25 - Use the axisymmetric form of POIS36 to find the temperature distribution in the problem shown below. Justify your discrimination of the problem and present your solution as a plot of isothermal lines at 10 °C intervals.

End of Axisymmetric Problems