TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Axisymmetric problems are sometimes referred to as radially symmetric problems.

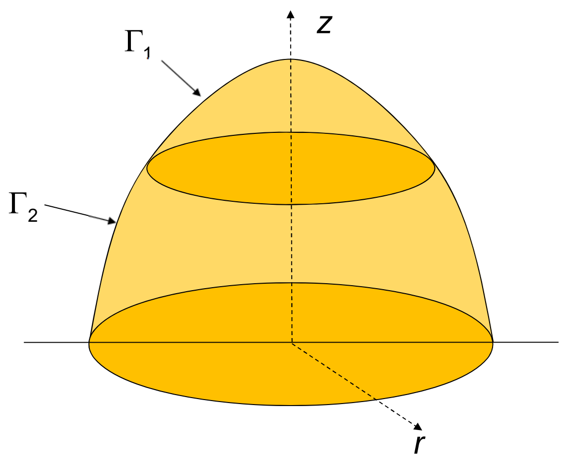
They are geometrically three-dimensional but mathematically only two-dimensional in the physics of the problem.

In other words, the dependent variable is a function of the coordinates *r* and *z* and not a function of the of the angle θ $u = u(r, z)$.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Axisymmetric problems are associated with bodies of revolution as indicated in the figure below:



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

The three-dimensional Laplacian operator in axisymmetric problems reduces to:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}$$

which may be written as:

$$\nabla^2 u = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(r\,\frac{\partial u}{\partial z}\right)\right]$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

The Poisson boundary value problem is:

$$\nabla^2 u(r,z) + f(r,z) = 0 \qquad \text{in } \Omega$$

$$u = g(s) \qquad \text{on } \Gamma_1$$

$$\frac{\partial u}{\partial n} + \alpha(s)u = h(s) \qquad \text{on } \Gamma_2$$

where the surface Γ_1 is the portion of the surface Γ where the Dirichlet type boundary conditions are defined and Γ_2 is the portions where the Neumann or Robin boundary conditions are prescribed.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

The corresponding energy functional that will serve as the basis for a Ritz finite element model is:

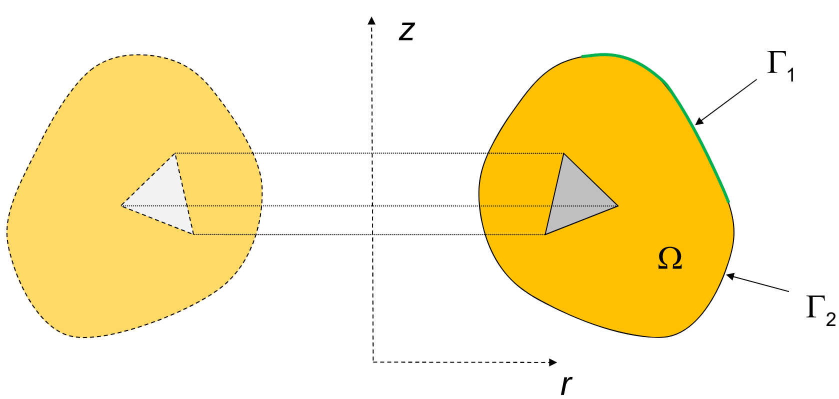
$$Z(u) = \frac{1}{2}\iint_{\Omega}\left[r\left(\frac{\partial u}{\partial r}\right)^2 + r\left(\frac{\partial u}{\partial z}\right)^2\right]dr\,dz - \iint_{\Omega} ur\,f\,dr\,dz$$
$$+\frac{1}{2}\int_{\Gamma_2}\alpha r u^2\,ds - \int_{\Gamma_2} urh\,ds = 0$$

Due to the mathematical nature of the problem, the analysis may be performed within a two-dimensional region in the *rz*-plane which is revolved about the *z*-axis.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

The revolving region defined the actual three-dimensional domain.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

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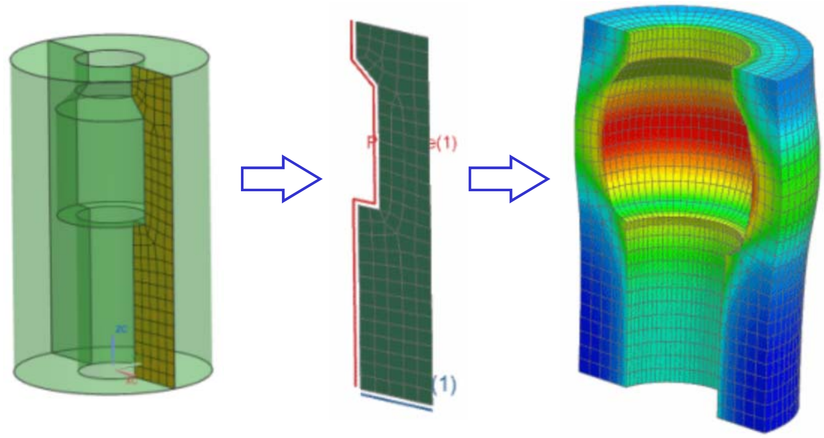
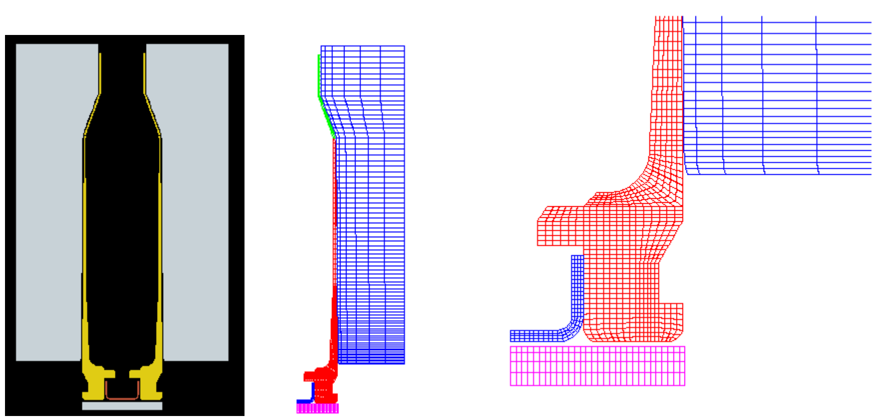


TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

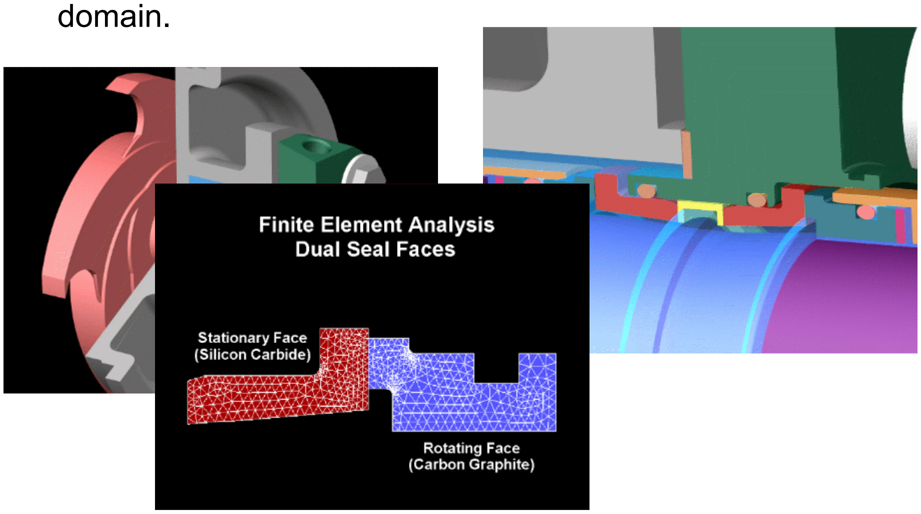
The revolving region defined the actual three-dimensional domain.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

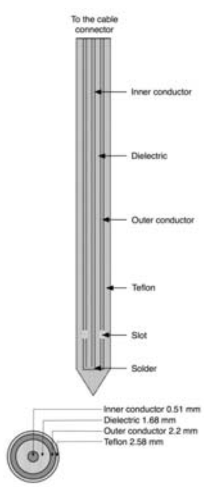
The revolving region defined the actual three-dimensional domain.

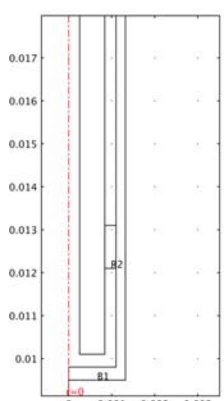


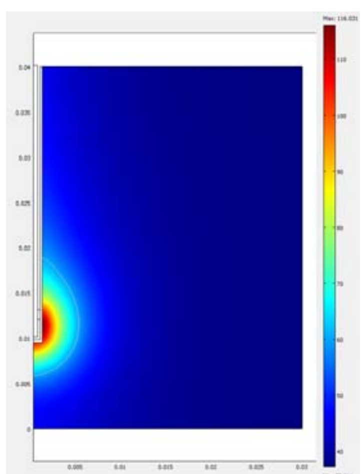
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Cross-section and axial schematic of the coaxial slot antenna



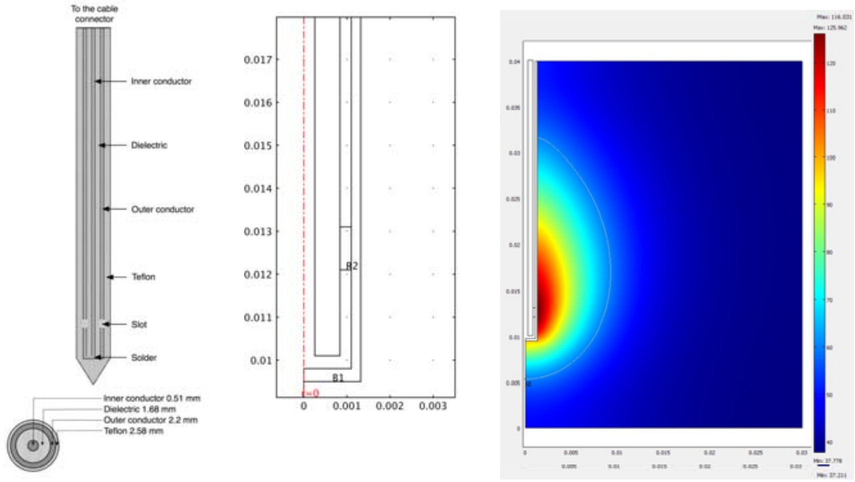




TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Cross-section and axial schematic of the coaxial slot antenna

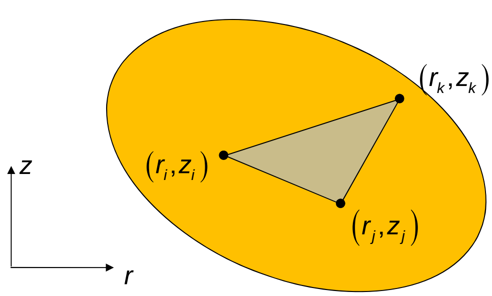


TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

<u>Discretization</u> - As usual the first step is developing a finite element model is the discretization of the problem geometry.

As an introduction, we will limit our discussion of the discretization and formulation of the axisymmetric problem to linear triangles.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

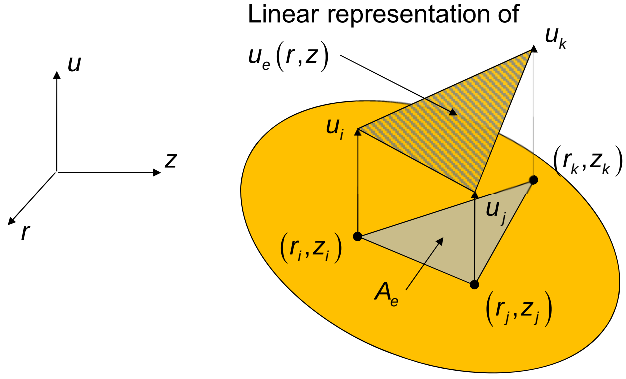
Discretization - In terms of the discretization, the functional *Z* is now a sum of the integrals over each element in the domain Ω, and the sum of surface integrals on the boundary segments along Γ_2:

$$Z(u) \approx \frac{1}{2}\left(\sum_e \iint_{A_e} \left[r\left(\frac{\partial u}{\partial r}\right)^2 + r\left(\frac{\partial u}{\partial z}\right)^2 \right] dr\, dz + \sum_e{}' \int_{\gamma_{2e}} \alpha r u^2\, ds \right) - \sum_e \iint_{A_e} urf\, dr\, dz - \sum_e{}' \int_{\gamma_{2e}} urh\, ds = 0$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

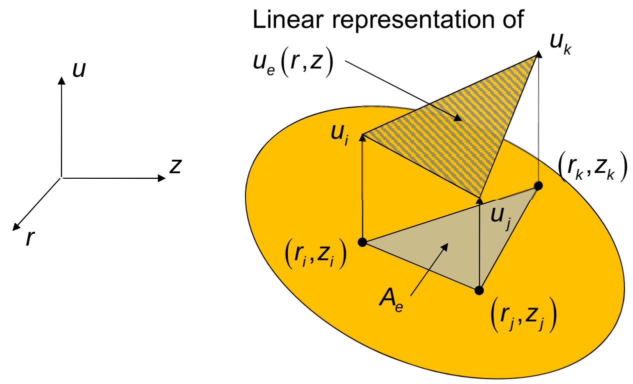
Interpolation - The simplest interpolation over a straight-sided three node triangular element is to assume the function *u*(*r, z*) is represented by a linear plane.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Interpolation - Linearly interpolated triangular elements represent the variation of the dependent variable *u* over an element as: $u_e(r,z) = \alpha + \beta r + \gamma z$



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Interpolation - Linearly interpolated triangular elements represent the variation of the dependent variable *u* over an element as: $u_e(r,z) = \alpha + \beta r + \gamma z$

where α, β, and γ are constant determined by matching the function u_e with the nodal values of the element:

$$u_e(r_i, z_i) = \alpha + \beta r_i + \gamma z_i$$

$$u_e(r_j, z_j) = \alpha + \beta r_j + \gamma z_j$$

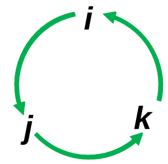
$$u_e(r_k, z_k) = \alpha + \beta r_k + \gamma z_k$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Interpolation - Solving the three equations for α, β, and γ and substituting back into the expression representing the variation of *u* over the element results in:

$$u_e(r,z) = N_i u_i + N_j u_j + N_k u_k$$



where :

$$N_i = \frac{a_i + b_i r + c_i z}{2A_e} \qquad i = 1, 2, 3$$

$$a_i = r_j z_k - r_k z_j \qquad b_i = z_j - z_k \qquad c_i = r_k - r_j$$

where *i*, *j*, and *k* are permuted cyclically

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Interpolation - The determinant of the coefficients is:

$$2A_e = \begin{vmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_k & z_k \end{vmatrix}$$

where A_e is the area of the element.

Any numbering scheme that proceeds counterclockwise around the element is valid, for example (*i, j, k*), (*j, k, i*), or (*k, i, j*).

This numbering convention is important and necessary in order to compute a positive area for A_e.

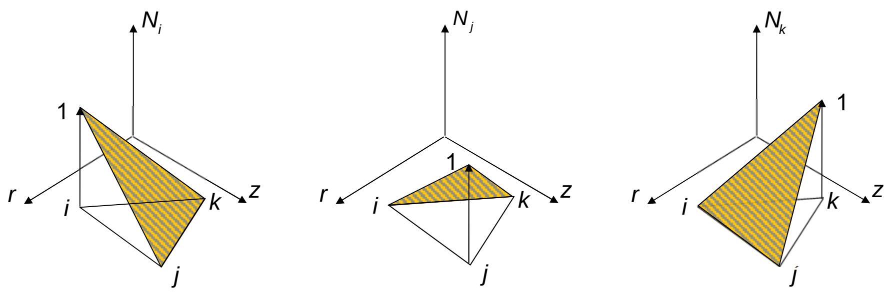
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

<u>Interpolation</u> - In matrix notation, the distribution of the function over the element is:

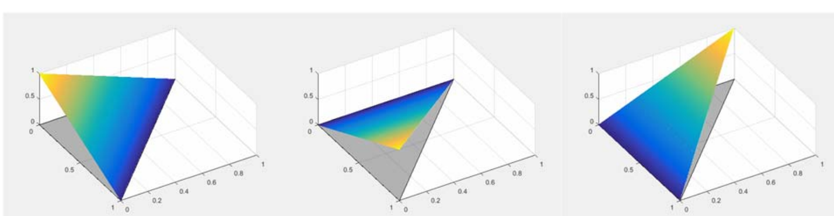
$$u_e(r,z) = \mathbf{u_e}^{\mathbf{T}}\mathbf{N} = \mathbf{N}^{\mathbf{T}}\mathbf{u_e}$$

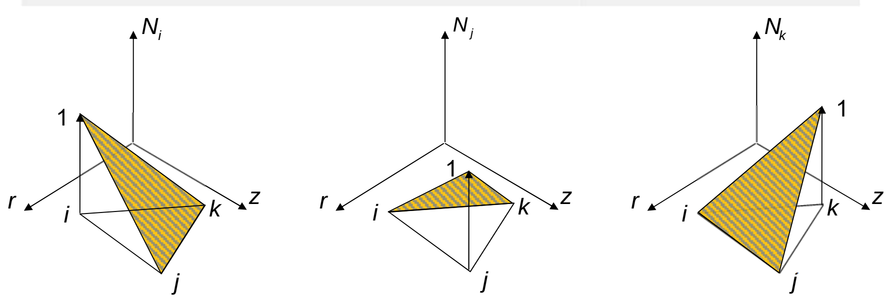
The linear triangular shape functions are illustrated below:



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS





TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Interpolation - The derivatives of *u* over the element with respect to both coordinates are:

$$\frac{\partial u_e(r,z)}{\partial r} = \mathbf{u_e}^{\mathbf{T}} \frac{\partial \mathbf{N}}{\partial r} = \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial r} \mathbf{u_e} \qquad \frac{\partial u_e(r,z)}{\partial z} = \mathbf{u_e}^{\mathbf{T}} \frac{\partial \mathbf{N}}{\partial z} = \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial z} \mathbf{u_e}$$

Calculating the derivatives of the shape functions gives:

$$\frac{\partial \mathbf{N}}{\partial r} = \frac{\mathbf{b_e}}{2A_e} \qquad \frac{\partial \mathbf{N}}{\partial z} = \frac{\mathbf{c_e}}{2A_e}$$

$$\mathbf{b_e}^{\mathbf{T}} = \langle b_1 \quad b_2 \quad b_3 \rangle \qquad \mathbf{c_e}^{\mathbf{T}} = \langle c_1 \quad c_2 \quad c_3 \rangle$$

$$b_i = z_j - z_k \qquad c_i = r_k - r_j$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Interpolation - Observing the form of the derivative it is apparent that the partial derivatives of the function *u* will be constant over a linear triangular element.

There are many problems associated with accuracy and convergence for this type of element.

In elasticity analysis, stress and strain are related by a partial differential equation, using a linear triangular element to described stress will result in a constant approximation for strain over the element.

Therefore, elements of this type are called **constant strain elements**.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Elemental Formulation - The functional for the Poisson equation is:

$$Z(u) \approx \frac{1}{2}\sum_{e} \iint_{A_e} \left[r\left(\frac{\partial u}{\partial r}\right)^2 + r\left(\frac{\partial u}{\partial z}\right)^2 \right] dA$$

$$+\frac{1}{2}\sum_{e}{}' \int_{\gamma_{2e}} \alpha r u^2\, ds - \sum_{e} \iint_{A_e} urf\, dA - \sum_{e}{}' \int_{\gamma_{2e}} urh\, ds = 0$$

We can write the functional in the following form:

$$Z(u) \approx \sum_{e} \frac{Z_{e1}}{2} + \sum_{e}{}' \frac{Z_{e2}}{2} - \sum_{e} Z_{e3} - \sum_{e}{}' Z_{e4}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Elemental Formulation – Where the components are defined as:

$$Z_{e1} = \iint_{A_e} \left[r\left(\frac{\partial u}{\partial r}\right)^2 + r\left(\frac{\partial u}{\partial z}\right)^2 \right] dr\, dz \qquad Z_{e2} = \int_{\gamma_{2e}} \alpha r\, u^2\, ds$$

$$Z_{e3} = \iint_{A_e} ur\, f\, dr\, dz \qquad Z_{e4} = \int_{\gamma_{2e}} ur\, h\, ds$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

<u>Elemental Formulation</u> - **Evaluation of Z_{e1}**:

$$Z_{e1} = \iint_A \left[r \frac{\partial u}{\partial r}\frac{\partial u}{\partial r} + r \frac{\partial u}{\partial z}\frac{\partial u}{\partial z} \right] dA$$

Recall the first derivatives of *u* with respect to *r* and *z* are:

$$\frac{\partial u_e(r,z)}{\partial r} = \mathbf{u_e}^{\mathbf{T}} \frac{\partial \mathbf{N}}{\partial r} = \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial r}\mathbf{u_e}$$

$$\frac{\partial u_e(r,z)}{\partial z} = \mathbf{u_e}^{\mathbf{T}} \frac{\partial \mathbf{N}}{\partial z} = \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial z}\mathbf{u_e}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

<u>Elemental Formulation</u> - **Evaluation of Z_{e1}**: Replacing the derivatives with the above approximations gives:

$$Z_{e1} = \iint_A \left[\mathbf{u_e^T} \frac{\partial \mathbf{N}}{\partial r} r \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial r}\mathbf{u_e} + \mathbf{u_e^T} \frac{\partial \mathbf{N}}{\partial z} r \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial z}\mathbf{u_e} \right] dr\, dz$$

$$= \mathbf{u_e^T} \left(\iint_A \left[\frac{\partial \mathbf{N}}{\partial r} r \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial r} + \frac{\partial \mathbf{N}}{\partial z} r \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial z} \right] dr\, dz \right) \mathbf{u_e} = \mathbf{u_e^T k_e u_e}$$

$$\mathbf{k_e} = \iint_A \left[\frac{\partial \mathbf{N}}{\partial r} r \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial r} + \frac{\partial \mathbf{N}}{\partial z} r \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial z} \right] dA$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Elemental Formulation - **Evaluation of Z_{e1}**:The integrals defined in $\mathbf{k_e}$ are the elemental "stiffness" matrix.

For the linear triangular element we have discussed the stiffness matrix reduces to:

$$\mathbf{k_e} = \iint_{A_e} \left[\frac{\mathbf{b_e} r\, \mathbf{b_e^T} + \mathbf{c_e} r\, \mathbf{c_e^T}}{4A_e^2} \right] dr\, dz$$

The resulting 3 x 3 elemental stiffness matrix contributes to the global system equations at locations corresponding to the element nodes.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Elemental Formulation - **Evaluation of Z_{e2}**:

$$Z_{e2} = \int_{\gamma_{2e}} \alpha r\, u^2\, ds$$

In this case, the interpolation of u with respect to r and z is used to describe the behavior along the boundary:

$$Z_{e2} = \int_{\gamma_{2e}} \mathbf{u_e^T} \mathbf{N} \alpha r\, \mathbf{N^T} \mathbf{u_e}\, ds = \mathbf{u_e^T} \left(\int_{\gamma_{2e}} \mathbf{N} \alpha r\, \mathbf{N^T}\, ds \right) \mathbf{u_e} = \mathbf{u_e^T} \mathbf{a_e} \mathbf{u_e}$$

$$\mathbf{a_e} = \int_{\gamma_{2e}} \mathbf{N} \alpha r\, \mathbf{N^T}\, ds$$

The resulting is a 2 x 2 **elemental stiffness matrix**

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

<u>Elemental Formulation</u> - Evaluation of Z_{e3}:

$$Z_{e3} = \iint_{A_e} ur\, f\, dA$$

Substituting the approximation for u into the integral results in:

$$Z_{e3} = \mathbf{u_e^T}\left(\iint_{A_e} \mathbf{N}r\, f\, ds\right) = \mathbf{u_e^T f_e} \qquad \mathbf{f_e} = \iint_{A_e} \mathbf{N}r\, f\, dA$$

The resulting is a 3 x 1 **elemental load vector**

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

<u>Elemental Formulation</u> - Evaluation of Z_{e4}:

$$Z_{e4} = \int_{\gamma_{2e}} ur\, h\, ds$$

Substituting the approximation for u into the integral results in:

$$Z_{e4} = \mathbf{u_e^T}\left(\int_{\gamma_{2e}} \mathbf{N}r\, h\, ds\right) = \mathbf{u_e^T h_e} \qquad \mathbf{h_e} = \int_{\gamma_{2e}} \mathbf{N}r\, h\, ds$$

The resulting is a 2 x 1 **elemental load vector**

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Elemental Formulation - In terms of the matrix definitions, the functional may be written in the following form:

$$Z(u_1, u_2, u_3, \ldots, u_N) \approx \sum_e \left(\frac{\mathbf{u_e^T k_e u_e}}{2} - \mathbf{u_e^T f_e} \right) + \sum_e{}' \left(\frac{\mathbf{u_e^T a_e u_e}}{2} - \mathbf{u_e^T h_e} \right)$$

where the first sum is over the each element of area A_e describing the domain Ω and the second sum is over every element that has a segment along the Γ_2 portion of the boundary.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Assembly - The assembly is denoted by the summation in the matrix equation. The global matrix form of the formulation is:

$$Z \approx \frac{\mathbf{u_G}^{\mathbf{T}} \mathbf{K_G u_G}}{2} - \mathbf{u_G}^{\mathbf{T}} \mathbf{F_G} = Z(\mathbf{u_G})$$

$$\mathbf{K_G} = \sum_e \mathbf{k_G} + \sum_e{}' \mathbf{a_G} \qquad \mathbf{F_G} = \sum_e \mathbf{f_G} + \sum_e{}' \mathbf{h_G}$$

$$\frac{\partial Z}{\partial u_i} = 0 \qquad \frac{\partial Z}{\partial u_i} = \frac{\left(\mathbf{K_G} + \mathbf{K_G}^{\mathbf{T}}\right)\mathbf{u_e}}{2} - \mathbf{F_G} \quad \rightarrow \quad \boxed{\mathbf{K_G u_G} = \mathbf{F_G}}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Constraints - The constraints on the system equations are the forced boundary conditions $u = g(s)$ on the surface Γ_1.

These conditions are applied to the system equations in a manner similar to that discussed for one- and two-dimensional problems.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Solution - Details of the solution of the simultaneous equations resulting from axisymmetric boundary value problems are presented in the two-dimensional section of the notes.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

<u>Computation of Derived Variables</u> - In this case, the values of the function *u* are the primary variables and *∂u/∂r* and *∂u/∂z* are considered a secondary variable.

The partial derivatives are determined by the following expressions:

$$\frac{\partial u_e(r,z)}{\partial r} = \mathbf{u_e}^{\mathbf{T}} \frac{\partial \mathbf{N}}{\partial r} = \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial r}\mathbf{u_e} = \frac{\mathbf{b_e}^{\mathbf{T}}\mathbf{u_e}}{2A_e}$$

$$\frac{\partial u_e(r,z)}{\partial z} = \mathbf{u_e}^{\mathbf{T}} \frac{\partial \mathbf{N}}{\partial z} = \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial z}\mathbf{u_e} = \frac{\mathbf{c_e}^{\mathbf{T}}\mathbf{u_e}}{2A_e}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Recall the elemental matrices have the following form:

$$\mathbf{k_e} = \iint_A \left[\frac{\mathbf{b_e} r\, \mathbf{b_e}^{\mathbf{T}} + \mathbf{c_e} r\, \mathbf{c_e}^{\mathbf{T}}}{4A_e}\right] dr\, dz \qquad \mathbf{f_e} = \iint_{A_e} \mathbf{N} r\, f\, dr\, dz$$

$$\mathbf{a_e} = \int_{\gamma_{2e}} \mathbf{N} \alpha r\, \mathbf{N}^{\mathbf{T}}\, ds \qquad \mathbf{h_e} = \int_{\gamma_{2e}} \mathbf{N} r\, h\, ds$$

These integrals are essentially the same as the terms developed for two-dimensional Poisson's equations using linear triangular element.

The obvious difference is the *r* coordinate which appears in each integral.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Evaluation of $\mathbf{k_e}$ - Substituting all the pieces of the transformation in the $\mathbf{k_e}$ terms gives:

$$\mathbf{k_e} = \iint_{A_e} \left[\frac{\mathbf{b_e} r \, \mathbf{b_e^T} + \mathbf{c_e} r \, \mathbf{c_e^T}}{4A_e^4} \right] dr\, dz$$

$$= \frac{\mathbf{b_e b_e^T} + \mathbf{c_e c_e^T}}{4A_e^4} \iint_{A_e} r\, dr\, dz \quad = \frac{\mathbf{b_e b_e^T} + \mathbf{c_e c_e^T}}{4A_e^4} R$$

$$R = \frac{r_i + r_j + r_k}{3}$$

The value of R is the r-coordinate of the centroid of the linear triangular element.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Evaluation of $\mathbf{f_e}$ - In general, the integral $\mathbf{f_e}$ is:

$$\mathbf{f_e} = \iint_{A_e} \mathbf{N} r\, f(r,z)\, dA$$

Replacing r by the exact representation $\mathbf{r_e^T N}$, and assuming that the function f may be approximated by the linear interpolation $\mathbf{N^T f}$, the element matrix $\mathbf{f_e}$ becomes:

$$\mathbf{f_e} \approx \iint_{A_e} \mathbf{N r_e^T N N^T f}\, dr\, dz$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Evaluation of $\mathbf{f_e}$ - The formula for integrations of the type is given without proof as:

$$\iint_{A_e} N_I^a N_J^b N_K^c \, dA = a!b!c!\frac{2A_e}{(a+b+c+2)!}$$

Therefore:

$$f_e = \frac{A_e}{60}\begin{bmatrix} 6r_i + 2r_j + 2r_k & 2r_i + 2r_j + r_k & 2r_i + r_j + 2r_k \\ & 2r_i + 6r_j + 2r_k & r_i + 2r_j + 2r_k \\ symmetric & & 2r_i + 2r_j + 6r_k \end{bmatrix}\begin{Bmatrix} f_i \\ f_j \\ f_k \end{Bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Evaluation of $\mathbf{f_e}$ - The formula for integrations of the type is given without proof as:

$$\iint_{A_e} N_I^a N_J^b N_K^c \, dA = a!b!c!\frac{2A_e}{(a+b+c+2)!}$$

If the function *f* is a constant, f_0, the above matrix reduces to:

$$\mathbf{f_e} = \frac{A_e f_0}{12}\begin{bmatrix} 2r_i + r_j + r_k \\ r_i + 2r_j + r_k \\ r_i + r_j + 2r_k \end{bmatrix}$$

The resulting is a 3 x 1 **elemental load vector**

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

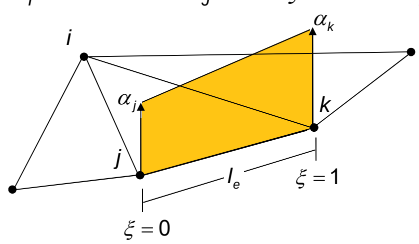
AXISYMMETRIC PROBLEMS

Evaluation of $\mathbf{a_e}$:

$$\mathbf{a_e} = \int_{\gamma_{2e}} \mathbf{N} \alpha r \, \mathbf{N}^{\mathbf{T}} \, ds$$

Since $\mathbf{a_e}$ is evaluated along a segment of the boundary γ_{2e}, the interpolation functions reduce to their one-dimensional counterparts.

$N_I = 0 \qquad N_J = 1-\xi \qquad N_K = \xi$



$$\mathbf{a}_e = \int \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \alpha r \left\langle 1-\xi \quad \xi \right\rangle l_e \, d\xi$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

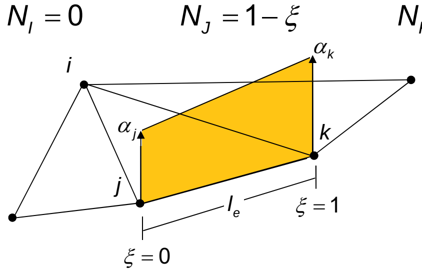
AXISYMMETRIC PROBLEMS

Evaluation of $\mathbf{a_e}$:

$$\mathbf{a_e} = \int_{\gamma_{2e}} \mathbf{N} \alpha r \, \mathbf{N}^{\mathbf{T}} \, ds$$

As before, *r* may be replace by the exact expression $\mathbf{r_e^T N}$ and assuming that α may be approximated by a linear interpolation as $\mathbf{N^T}\boldsymbol{\alpha}$, therefore, $\mathbf{a_e}$ becomes:

$N_I = 0 \qquad N_J = 1-\xi \qquad N_K = \xi$



$$\mathbf{a_e} = \int_0^1 \mathbf{N r_e^T N N^T} \boldsymbol{\alpha} \, \mathbf{N^T} \, l_e \, d\xi$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Evaluation of $\mathbf{a_e}$: $$\mathbf{a_e} = \int_{\gamma_{2e}} \mathbf{N} \alpha r \, \mathbf{N^T} \, ds$$

$$\mathbf{a_e} = \int_0^1 \mathbf{N} \mathbf{r_e^T} \mathbf{N} \mathbf{N^T} \alpha \, \mathbf{N^T} \, l_e \, d\xi$$

$$= \int_0^1 \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix}_{2x1} \langle r_j \quad r_k \rangle_{1x2} \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix}_{2x1} \langle 1-\xi \quad \xi \rangle_{1x2} \begin{Bmatrix} \alpha_j \\ \alpha_k \end{Bmatrix}_{2x1} \langle 1-\xi \quad \xi \rangle_{1x2} \, l_e \, d\xi$$

$$= \int_0^1 \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix}_{2x1} \left(\alpha_j \left[r_j (\xi - 1) - r_k \xi \right] - \alpha_k \xi \left[r_j (\xi - 1) - r_k \xi \right] \right)_{1x1} \langle 1-\xi \quad \xi \rangle_{1x2} \, l_e \, d\xi$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Evaluation of $\mathbf{a_e}$ - The integration formula for the type of integrals is:

$$\int_{\gamma_{2e}} N_I^a N_J^b \, ds = a! \, b! \frac{l_e}{(a+b+1)!}$$

$$[\mathbf{a_e}]_{11} = \frac{l_e}{60} \left(12 \alpha_j r_j + 3 \left(\alpha_j r_k + \alpha_k r_j \right) + 2 \alpha_k r_k \right)$$

$$[\mathbf{a_e}]_{12} = \frac{l_e}{60} \left(3 \alpha_j r_j + 2 \left(\alpha_j r_k + \alpha_k r_j \right) + 3 \alpha_k r_k \right)$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Evaluation of $\mathbf{a_e}$ - The integration formula for the type of integrals is:

$$\int_{\gamma_{2e}} N_I^a N_J^b \, ds = a!\,b!\,\frac{l_e}{(a+b+1)!}$$

$$[\mathbf{a_e}]_{21} = \frac{l_e}{60}\left(3\alpha_j r_j + 2\left(\alpha_j r_k + \alpha_k r_j\right) + 3\alpha_k r_k\right)$$

$$[\mathbf{a_e}]_{22} = \frac{l_e}{60}\left(2\alpha_j r_j + 3\left(\alpha_j r_k + \alpha_k r_j\right) + 12\alpha_k r_k\right)$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

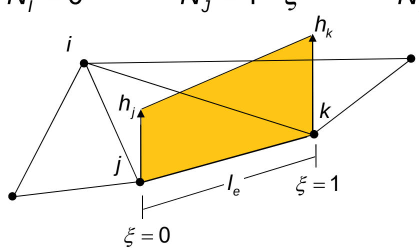
AXISYMMETRIC PROBLEMS

Evaluation of $\mathbf{h_e}$ - Consider the integral: $\mathbf{h_e} = \int_{\gamma_{2e}} \mathbf{N} r\, h\, ds$

where the integration is along a boundary segment of the element.

Since, the integration is computed along a single side of the triangular element, the original shape functions reduce to:

$N_I = 0 \qquad N_J = 1-\xi \qquad N_K = \xi$



$$\mathbf{h_e} = \int \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} r\, h(\xi)\, l_e \, d\xi$$

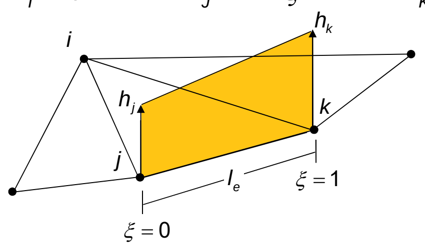
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Evaluation of $\mathbf{h_e}$ - Replacing *r* by the exact representation $\mathbf{r_e^T N}$, and assuming that the function *h* may be approximated by the linear interpolation $\mathbf{N^T h}$, the element matrix $\mathbf{h_e}$ becomes:

$$\mathbf{h_e} = \int_{\gamma_{2e}} \mathbf{N r_e^T N N^T h}\, ds = \int_{\gamma_{2e}} \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} \langle r_j \quad r_k \rangle \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} \langle 1-\xi \quad \xi \rangle \begin{Bmatrix} h_j \\ h_k \end{Bmatrix} l_e\, d\xi$$

$N_I = 0 \qquad N_J = 1-\xi \qquad N_K = \xi$



$$\mathbf{h_e} = \int \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} r\, h(\xi)\, l_e\, d\xi$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Evaluation of $\mathbf{h_e}$ - With this assumption the integral becomes:

$$\mathbf{h_e} = \frac{l_e}{12} \begin{bmatrix} 3r_j + r_k & r_j + r_k \\ r_j + r_k & r_j + 3r_k \end{bmatrix} \begin{bmatrix} h_j \\ h_k \end{bmatrix}$$

Recall that $\mathbf{h_e}$ in *x* and *y* is:

$$\mathbf{h_e} = \frac{l_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} h_j \\ h_k \end{Bmatrix} \qquad r_j = r_k = 1$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Evaluation of $\mathbf{h_e}$ - If the function h is a constant, h_0, the above matrix reduces to:

$$\mathbf{h_e} = \frac{l_e h_0}{6}\begin{bmatrix} 2r_j + r_k \\ r_j + 2r_k \end{bmatrix}$$

Recall that $\mathbf{h_e}$ in x and y is:

$$\mathbf{h_e} = \frac{l_e h_0}{6}\begin{bmatrix} 3 \\ 3 \end{bmatrix} \qquad r_j = r_k = 1$$

The resulting is a 2 x 1 **elemental load vector**

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

AXISYMMETRIC PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Recall the elemental matrices have the following form:

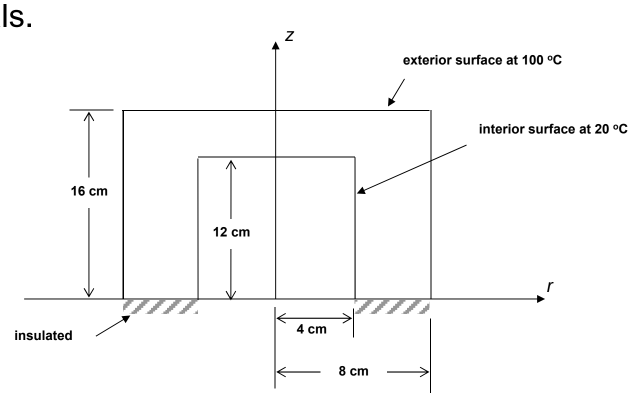
$$\mathbf{k_e} = \iint_A \left[\frac{\mathbf{b_e} r\, \mathbf{b_e}^{\mathbf{T}} + \mathbf{c_e} r\, \mathbf{c_e}^{\mathbf{T}}}{4A_e}\right] dr\, dz \qquad \mathbf{f_e} = \iint_{A_e} \mathbf{N} r\, f\, dr\, dz$$

$$\mathbf{a_e} = \int_{\gamma_{2e}} \mathbf{N} \alpha r\, \mathbf{N}^{\mathbf{T}}\, ds \qquad \mathbf{h_e} = \int_{\gamma_{2e}} \mathbf{N} r\, h\, ds$$

It should be clear that any of the elements we have discussed, (quadratic triangles, quadrilaterals, etc.) may be used in connection with the axisymmetric functional to develop a finite element model.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

PROBLEM #25 - Use the axisymmetric form of **POIS36** to find the temperature distribution in the problem shown below. Justify your discrimination of the problem and present your solution as a plot of isothermal lines at 10 °C intervals.



End of Axisymmetric Problems