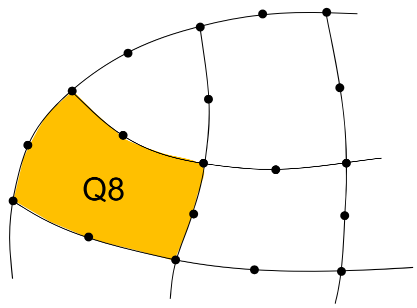
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)

The next in our element development is a logical extension of the quadrilateral element to a quadratically interpolated quadrilateral element is defined by eight nodes, four at the vertices and four at the middle at each side.

The middle node, depending on location, may define a straight line or a quadratic line.

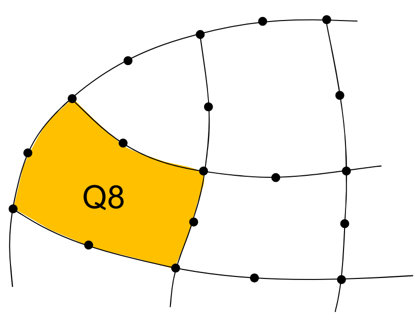


TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)

Transformation and Shape Functions - There are two approaches to develop the interpolation or shape functions for elements.

The first approach is based on representing the geometry and the dependent variable as a function of the global coordinates *x* and *y*.

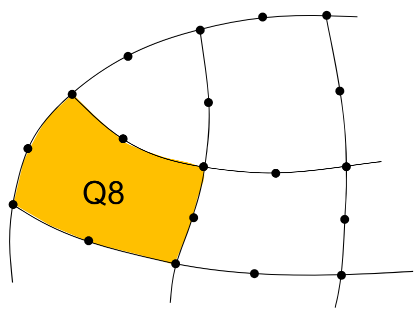


TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)

Transformation and Shape Functions - There are two approaches to develop the interpolation or shape functions for elements.

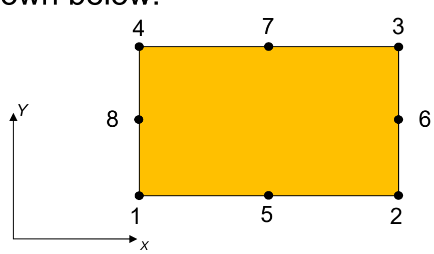
The second approach begins with the parent element with the interpolation and shape functions expressed in terms of the local area coordinates.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)

For the first approach, consider a straight-sided rectangular element shown below:



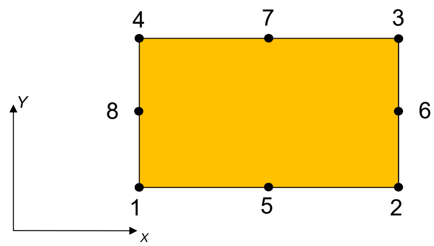
The variation of the dependent variable *u* over the element may be expressed as:

$$u_e(x, y) = a + bx + cy + dx^2 + exy + fy^2 + gx^2y + hxy^2$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)

For the first approach, consider a straight-sided rectangular element shown below:



Fitting this expression for *u* to the definition of the eight-node quadrilateral given above requires:

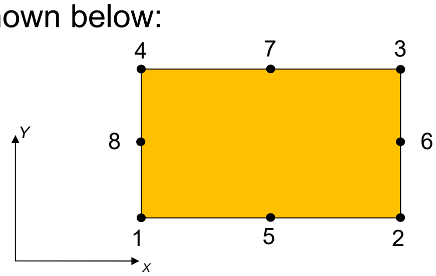
$$u_e(x_i, y_i) = a + bx_i + cy_i + dx_i^2 + ex_iy_i + fy_i^2 + gx_i^2y_i + hx_iy_i^2$$

$$i = 1, 2, \ldots, 8$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)

For the first approach, consider a straight-sided rectangular element shown below:

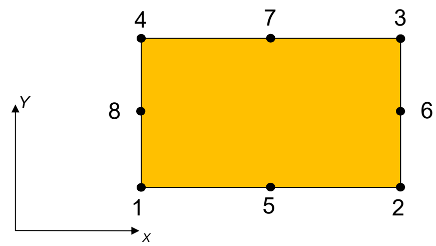


The above equation is written for each nodal value of *x* and *y* resulting in eight equations in the eight unknowns *a, b, c, d, e, f, g,* and *h*.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)

For the first approach, consider a straight-sided rectangular element shown below:



Solving this set of equations and collecting terms in u_i results in the interpolation functions.

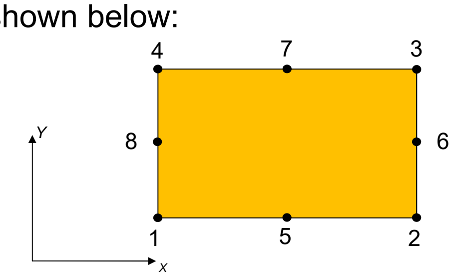
$$u_e(x,y) = \mathbf{u_e}^{\mathbf{T}}\mathbf{N} = \mathbf{N}^{\mathbf{T}}\mathbf{u_e}$$

where $\mathbf{u_e}^{\mathbf{T}}$ = [u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8]

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)

For the first approach, consider a straight-sided rectangular element shown below:



The interpolation functions **N** have the property that $N_{ij}(x_i, y_j) = \delta_{ij}$, in other words, $N_{ij}(x_i, y_j) = 1$ if $i = j$, and zero elsewhere.

$$u_e(x,y) = \mathbf{u_e}^{\mathbf{T}}\mathbf{N} = \mathbf{N}^{\mathbf{T}}\mathbf{u_e}$$

where $\mathbf{u_e}^{\mathbf{T}}$ = [u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8]

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)

The geometry of the quadrilateral element may also be described using the above interpolations as:

$$\sum_{i=1}^{N} x_i N_i = \mathbf{x_e}^{\mathbf{T}}\mathbf{N} = \mathbf{N}^{\mathbf{T}}\mathbf{x_e} \qquad \sum_{i=1}^{N} y_i N_i = \mathbf{y_e}^{\mathbf{T}}\mathbf{N} = \mathbf{N}^{\mathbf{T}}\mathbf{y_e}$$

An ***isoparametric*** element may be formed by using a value of N = 8 which uses the interpolation functions given above.

However, a ***subparametric*** element may also be defined by setting N = 4.

In this case, the interpolation functions defined for a four quadrilateral are used.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)

The geometry of the quadrilateral element may also be described using the above interpolations as:

$$\sum_{i=1}^{N} x_i N_i = \mathbf{x_e}^{\mathbf{T}}\mathbf{N} = \mathbf{N}^{\mathbf{T}}\mathbf{x_e} \qquad \sum_{i=1}^{N} y_i N_i = \mathbf{y_e}^{\mathbf{T}}\mathbf{N} = \mathbf{N}^{\mathbf{T}}\mathbf{y_e}$$

As we found in the development of the quadratic triangular elements, the global coordinates approach is not the most efficient method for describing the interpolation over the element.

The eight interpolation or shape functions in global coordinates x and y are mathematically clumsy and rarely used in FEM analysis.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)

The geometry of the quadrilateral element may also be described using the above interpolations as:

$$\sum_{i=1}^{N} x_i N_i = \mathbf{x_e}^{\mathbf{T}}\mathbf{N} = \mathbf{N}^{\mathbf{T}}\mathbf{x_e} \qquad \sum_{i=1}^{N} y_i N_i = \mathbf{y_e}^{\mathbf{T}}\mathbf{N} = \mathbf{N}^{\mathbf{T}}\mathbf{y_e}$$

An equivalent form of the shape functions may be derived in terms of the local parental element coordinates.

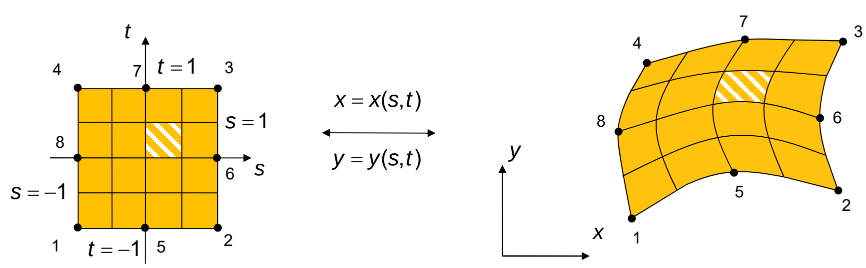
These functions have a relatively simple mathematical form and are more efficient in computing the elemental matrices.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)

As described above, the second approach to developing a set of interpolation or shape functions for an eight-node quadrilateral element begins with the parent element in local coordinates.

Consider the following eight-node quadrilateral in local coordinates *s* and *t*.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)

In the parent element the interpolation functions are given as:

$$N_1(s,t) = \frac{(1-s)(1-t)(-1-s-t)}{4} \qquad N_2(s,t) = \frac{(1+s)(1-t)(-1+s-t)}{4}$$

$$N_3(s,t) = \frac{(1+s)(1+t)(-1+s+t)}{4} \qquad N_4(s,t) = \frac{(1-s)(1+t)(-1-s+t)}{4}$$

$$N_5(s,t) = \frac{(1-s^2)(1-t)}{2} \qquad N_6(s,t) = \frac{(1+s)(1-t^2)}{2}$$

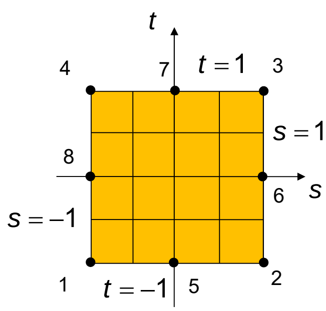
$$N_7(s,t) = \frac{(1-s^2)(1+t)}{2} \qquad N_8(s,t) = \frac{(1-s)(1-t^2)}{2}$$

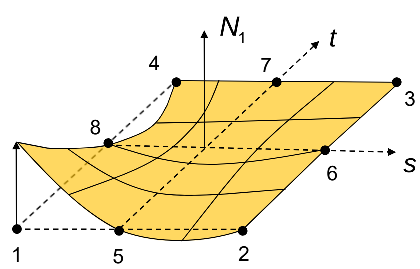
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)

The parent element interpolation functions $N_i(s, t)$ have two basic shapes.

The behavior of the functions N_1, N_2, N_3, and N_4 are similar except reference at different nodes. The shape function N_1 is shown below:





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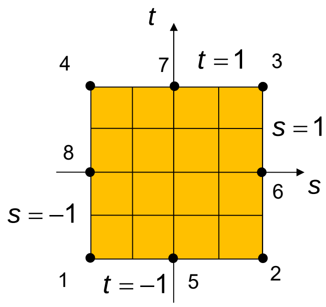
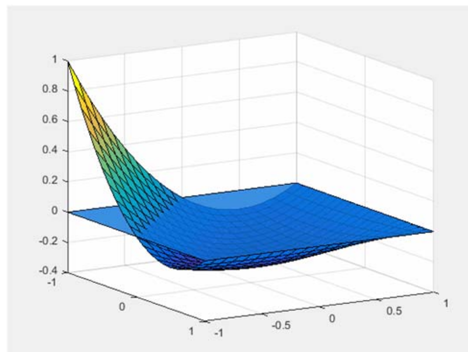


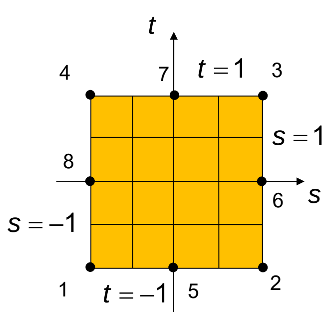


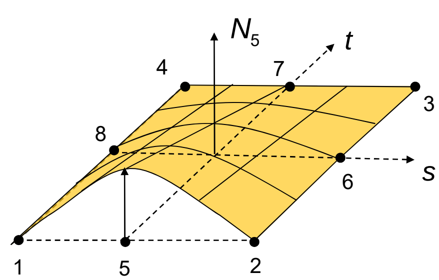
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)

The parent element interpolation functions $N_i(s, t)$ have two basic shapes.

The second type of shape function is valid for functions N_5, N_6, N_7, and N_8. The function N_5 is shown below:





TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)

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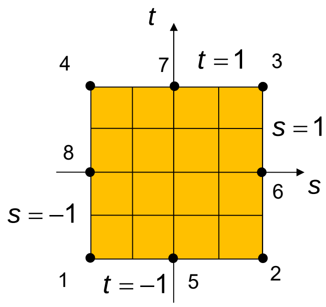
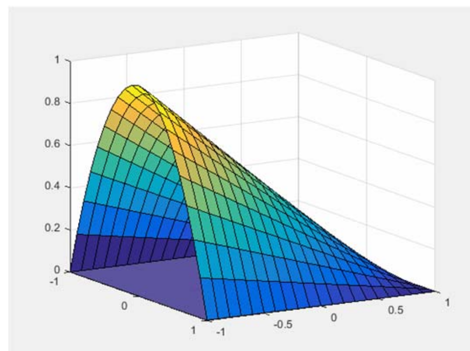




TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)

In a manner identical to that used in every element we have developed so far, the nature of the transformation from the parent element to the global element, the chain rule is used to form the differential relationship:

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\frac{\partial y}{\partial t}$$

In matrix notation, these derivatives may be written as:

$$\begin{Bmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \qquad \frac{\partial}{\partial \vec{\mathbf{s}}} = \mathbf{J}\frac{\partial}{\partial \vec{\mathbf{x}}}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)

The determinate of the Jacobian matrix **|J|:**

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \qquad |\mathbf{J}| = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial y}{\partial s}\frac{\partial x}{\partial t}$$

$$= \left(\sum_{1}^{8} x_i \frac{\partial N_i}{\partial s}\right)\left(\sum_{1}^{8} y_i \frac{\partial N_i}{\partial t}\right) - \left(\sum_{1}^{8} y_i \frac{\partial N_i}{\partial s}\right)\left(\sum_{1}^{8} x_i \frac{\partial N_i}{\partial t}\right)$$

The determinant of the Jacobian matrix, **|J|**, is a test of the invertibility of the transformation $x = x(s, t)$ and $y = y(s, t)$.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)

When |**J**| is positive everywhere in a region, the transformation may be inverted to determine $s = s(x, y)$ and $t = t(x, y)$.

This means that for a given point (x, y) in the element there is a unique corresponding point (s, t) in the parent element.

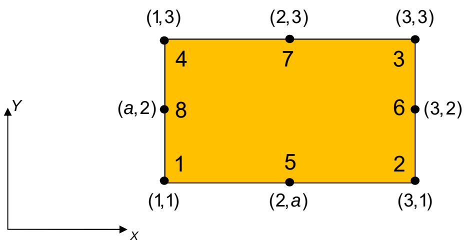
The $|\mathbf{J}(s, t)|$ is a measure of the expansion or contraction of a differential area:

$$dx\,dy = |\,\mathbf{J}(s,t)\,|\,ds\,dt$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS – Example 1

Consider the eight-node quadrilateral element given below.



$$\mathbf{x_e}^T = \langle 1 \quad 3 \quad 3 \quad 1 \quad 2 \quad 3 \quad 2 \quad a \rangle$$

$$\mathbf{y_e}^T = \langle 1 \quad 1 \quad 3 \quad 3 \quad a \quad 2 \quad 3 \quad 2 \rangle$$

The coordinate transformation is given as:

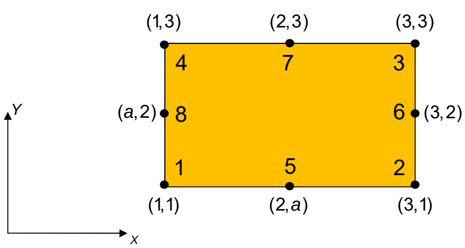
$$x = \frac{\left((a-1)s - a + 1\right)t^2 + (3-a)s + a + 3}{2} \qquad y = \frac{\left((a-1)s^2 - a + 3\right)t + (1-a)s^2 + a + 3}{2}$$

$$\frac{\partial x}{\partial s} = \frac{(a-1)t^2 - a + 3}{2} \qquad \frac{\partial y}{\partial s} = (a-1)(st - s)$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS – Example 1

Consider the eight-node quadrilateral element given below.



$$\mathbf{x_e}^T = \langle 1 \quad 3 \quad 3 \quad 1 \quad 2 \quad 3 \quad 2 \quad a \rangle$$

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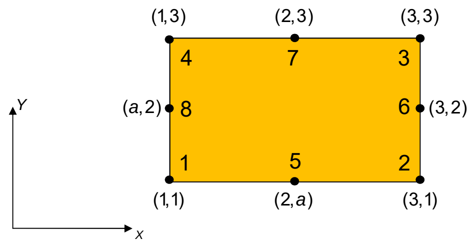
$$x = \frac{\left((a-1)s - a + 1\right)t^2 + (3-a)s + a + 3}{2} \qquad y = \frac{\left((a-1)s^2 - a + 3\right)t + (1-a)s^2 + a + 3}{2}$$

$$\frac{\partial x}{\partial t} = (a-1)(st - 1) \qquad \frac{\partial y}{\partial t} = \frac{(a-1)s^2 - a + 3}{2}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS – Example 1

Consider the eight-node quadrilateral element given below.



$$\mathbf{x}_e^T = \langle 1 \;\; 3 \;\; 3 \;\; 1 \;\; 2 \;\; 3 \;\; 2 \;\; a \rangle$$

$$\mathbf{y}_e^T = \langle 1 \;\; 1 \;\; 3 \;\; 3 \;\; a \;\; 2 \;\; 3 \;\; 2 \rangle$$

The resulting Jacobian is:

$$|\mathbf{J}(s,t)| = \begin{vmatrix} \dfrac{(a-1)t^2 - a + 3}{2} & (a-1)(st-s) \\ (a-1)(st-t) & \dfrac{(a-1)s^2 - a + 3}{2} \end{vmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS – Example 1

Let's consider several cases where the **|J|** may become negative.

The mapping at node 1 has the values of t = *s* = -1.

Substituting the corresponding *s* and *t* coordinates for nodes 1 into the expression for **|J|** gives:

$$|\mathbf{J}(s,t)| = -4a^2 + 8a - 3$$

The resulting Jacobian is:

$$|\mathbf{J}(s,t)| = \begin{vmatrix} \dfrac{(a-1)t^2 - a + 3}{2} & (a-1)(st-s) \\ (a-1)(st-t) & \dfrac{(a-1)s^2 - a + 3}{2} \end{vmatrix}$$

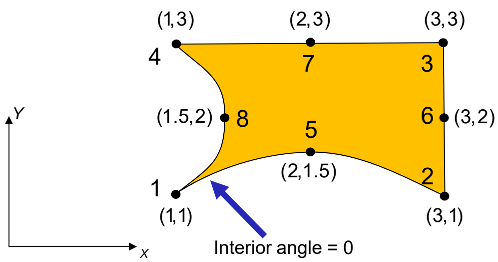
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS – Example 1

The **|J|** is: $|\mathbf{J}(s,t)| = -4a^2 + 8a - 3$

which is zero at a = 3/2 and 1/2.

For values of x and y greater than 3/2, the determinant of the Jacobian is negative.



$\mathbf{x}_e^T = \langle 1 \;\; 3 \;\; 3 \;\; 1 \;\; 2 \;\; 3 \;\; 2 \;\; 1.5 \rangle$

$\mathbf{y}_e^T = \langle 1 \;\; 1 \;\; 3 \;\; 3 \;\; 1.5 \;\; 2 \;\; 3 \;\; 2 \rangle$

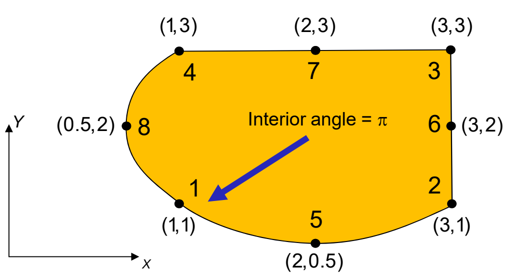
Note that for a = 3/2 the interior angle at node 1 is equal 0.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS – Example 1

The **|J|** is: $|\mathbf{J}(s,t)| = -4a^2 + 8a - 3$

Note that for a = 1/2 the interior angle at node 1 is equal π.



$\mathbf{x}_e^T = \langle 1 \;\; 3 \;\; 3 \;\; 1 \;\; 2 \;\; 3 \;\; 2 \;\; 0.5 \rangle$

$\mathbf{y}_e^T = \langle 1 \;\; 1 \;\; 3 \;\; 3 \;\; 0.5 \;\; 2 \;\; 3 \;\; 2 \rangle$

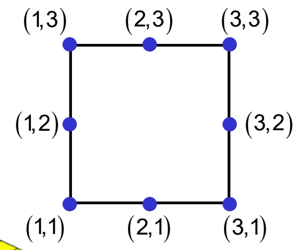
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS – Example 1

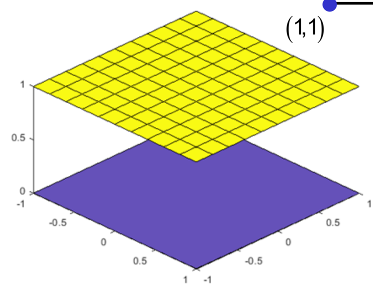
Let's compute the determinant |[J]| for a element as shown:

$$\mathbf{x}_e^T = \langle 1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2\ \ 1 \rangle$$

$$\mathbf{y}_e^T = \langle 1\ \ 1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2 \rangle$$



The |[J]| is:



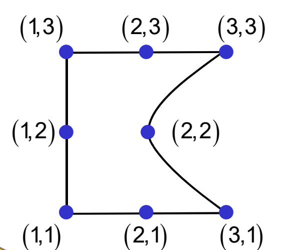
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS – Example 1

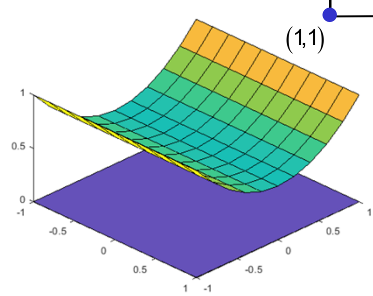
Let's compute the determinant |[J]| for a element with $(x_6, y_6) = (2, 2)$:

$$\mathbf{x}_e^T = \langle 1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 2\ \ 2\ \ 1 \rangle$$

$$\mathbf{y}_e^T = \langle 1\ \ 1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2 \rangle$$



The |[J]| is:



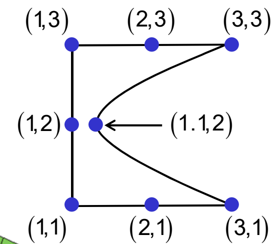
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS – Example 1

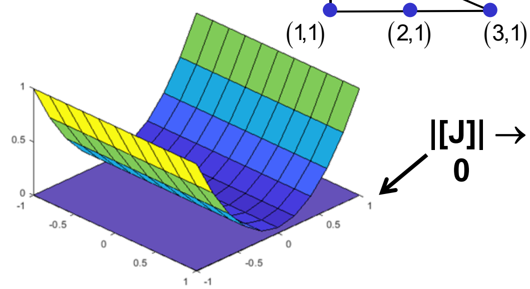
Let's compute the determinant |[J]| for a element with $(x_6, y_6) = (1.1, 2)$:

$$\mathbf{x}_e^T = \langle 1 \;\; 3 \;\; 3 \;\; 1 \;\; 2 \;\; 1.1 \;\; 2 \;\; 1 \rangle$$

$$\mathbf{y}_e^T = \langle 1 \;\; 1 \;\; 3 \;\; 3 \;\; 1 \;\; 2 \;\; 3 \;\; 2 \rangle$$



The |[J]| is:



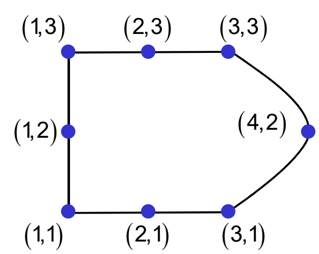
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS – Example 1

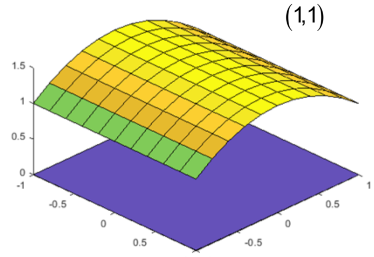
Let's compute the determinant |[J]| for a element with $(x_6, y_6) = (4, 2)$:

$$\mathbf{x}_e^T = \langle 1 \;\; 3 \;\; 3 \;\; 1 \;\; 2 \;\; 4 \;\; 2 \;\; 1 \rangle$$

$$\mathbf{y}_e^T = \langle 1 \;\; 1 \;\; 3 \;\; 3 \;\; 1 \;\; 2 \;\; 3 \;\; 2 \rangle$$



The |[J]| is:



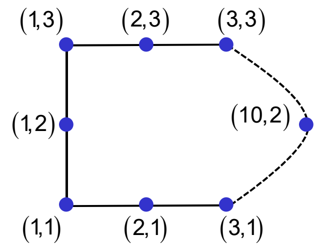
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS – Example 1

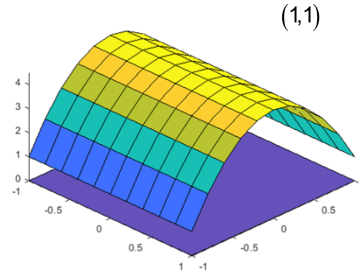
Let's compute the determinant |[J]| for a element with (x_6, y_6) = (10, 2):

$$\mathbf{x_e}^T = \langle 1 \;\; 3 \;\; 3 \;\; 1 \;\; 2 \;\; 10 \;\; 2 \;\; 1 \rangle$$

$$\mathbf{y_e}^T = \langle 1 \;\; 1 \;\; 3 \;\; 3 \;\; 1 \;\; 2 \;\; 3 \;\; 2 \rangle$$



The |[J]| is:



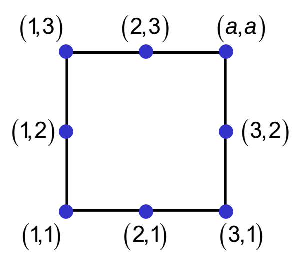
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS – Example 1

Let's compute the determinant |[J]| for a element with the following coordinates:

$$\mathbf{x_e}^T = \langle 1 \;\; 3 \;\; a \;\; 1 \;\; 2 \;\; 3 \;\; 2 \;\; 1 \rangle$$

$$\mathbf{y_e}^T = \langle 1 \;\; 1 \;\; a \;\; 3 \;\; 1 \;\; 2 \;\; 3 \;\; 2 \rangle$$



Let's evaluate the |[J]| at (s, t) = (1, 1) :

$$\left|[J]\right| = 3a - 8$$

For the |[J]| to be positive, a > 8/3. If a < 8/3, then the |[J]| is negative.

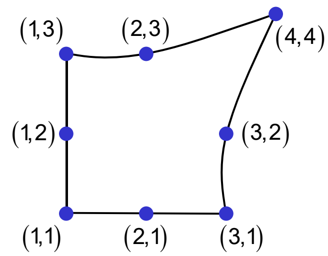
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS – Example 1

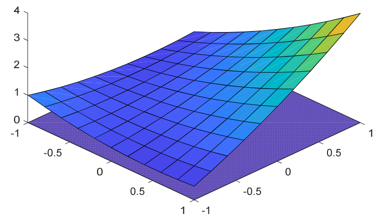
Let's compute the determinant |[J]| for a element with (x_3, y_3) = (4, 4):

$$\mathbf{x_e}^T = \langle 1 \ \ 3 \ \ 4 \ \ 1 \ \ 2 \ \ 3 \ \ 2 \ \ 1 \rangle$$

$$\mathbf{y_e}^T = \langle 1 \ \ 1 \ \ 4 \ \ 3 \ \ 1 \ \ 2 \ \ 3 \ \ 2 \rangle$$



At (s, t) = (1, 1) |[J]| > 0.



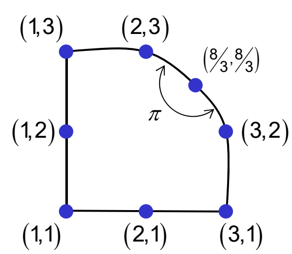
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS – Example 1

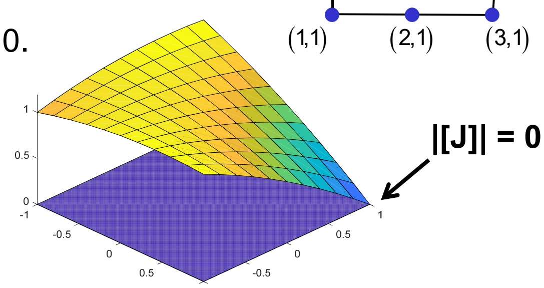
Let's compute the determinant |[J]| for a element with (x_3, y_3) = (8/3, 8/3):

$$\mathbf{x_e}^T = \langle 1 \ \ 3 \ \ \tfrac{8}{3} \ \ 1 \ \ 2 \ \ 3 \ \ 2 \ \ 1 \rangle$$

$$\mathbf{y_e}^T = \langle 1 \ \ 1 \ \ \tfrac{8}{3} \ \ 3 \ \ 1 \ \ 2 \ \ 3 \ \ 2 \rangle$$



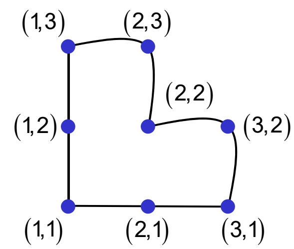
At (s, t) = (1, 1) |[J]| = 0.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS – Example 1

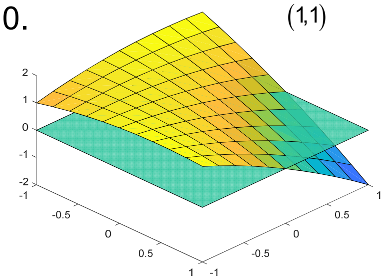
Let's compute the determinant |[J]| for a element with (x_3, y_3) = (2, 2):



$$\mathbf{x}_e^T = \langle 1\ \ 3\ \ 2\ \ 1\ \ 2\ \ 3\ \ 2\ \ 1 \rangle$$

$$\mathbf{y}_e^T = \langle 1\ \ 1\ \ 2\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2 \rangle$$

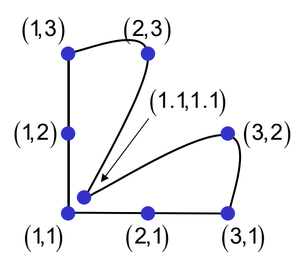
At (s, t) = (1, 1) |[J]| < 0.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS – Example 1

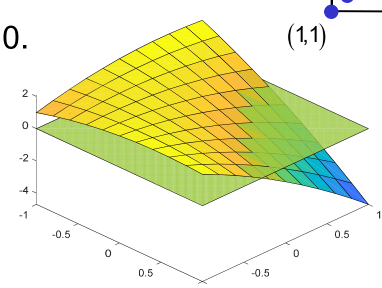
Let's compute the determinant |[J]| for a element with (x_3, y_3) = (1.1, 1.1):



$$\mathbf{x}_e^T = \langle 1\ \ 3\ \ 1.1\ \ 1\ \ 2\ \ 3\ \ 2\ \ 1 \rangle$$

$$\mathbf{y}_e^T = \langle 1\ \ 1\ \ 1.1\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2 \rangle$$

At (s, t) = (1, 1) |[J]| < 0.

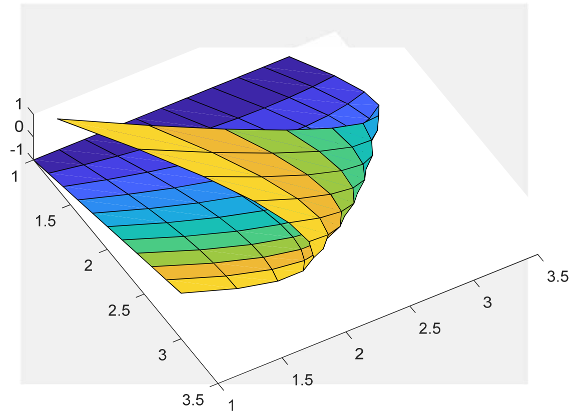


TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS – Example 1

When the |[J]| is negative the mapping between local coordinates to global coordinates is not 1-to-1.

Here is a plot of the mapping as the vales of *a* range from 3 to 1.1:



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EIGHT-NODE QUADRILATERRAL ELEMENTS

Derived Variables - The derived variables for the Poisson or Laplace's equation are the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$.

These terms are computed as functions of position in an element using the shape functions and the nodal values.

The partial derivatives may be combined to give the normal or directional derivative:

$$\frac{\partial u}{\partial n} = \left(n_x \mathbf{J_1} + n_y \mathbf{J_2}\right) \Delta^{\mathbf{T}} \mathbf{u_e}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - Q8 ELEMENTS

The elemental matrices for the Poisson problem are:

$$\mathbf{k_e} = \iint_A \left[\frac{\partial \mathbf{N}}{\partial x}\frac{\partial \mathbf{N^T}}{\partial x} + \frac{\partial \mathbf{N}}{\partial y}\frac{\partial \mathbf{N^T}}{\partial y} \right] dA \qquad \mathbf{f_e} = \iint_{A_e} \mathbf{N} f \, dA$$

$$\mathbf{a_e} = \int_{\gamma_{2e}} \mathbf{N} \alpha \mathbf{N^T} \, ds \qquad \mathbf{h_e} = \int_{\gamma_{2e}} \mathbf{N} h \, ds$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - Q8 ELEMENTS

The development of both the $\mathbf{k_e}$ and $\mathbf{f_e}$ terms is identical to that presented for other types of elements:

$$\mathbf{k_e} = \int_{-1}^{1} \int_{-1}^{1} \left[\Delta \mathbf{JJ} \Delta^T \right] ds \, dt$$

$$\mathbf{f_e} = \iint_{A_e} \mathbf{Nf} \, dA \approx \left(\iint_{A_e} \mathbf{NN^T} \left| \mathbf{J} \right| ds \, dt \right) \mathbf{f}$$

where $\mathbf{JJ} = (\mathbf{J_1^T J_1} + \mathbf{J_2^T J_2})|\mathbf{J}|$.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - Q8 ELEMENTS

The terms $\mathbf{J}_1$ and $\mathbf{J}_2$ are the first and second rows of the inverse of the Jacobian matrix.

$$\mathbf{J}^{-1} = \frac{1}{|\mathbf{J}|}\begin{bmatrix} \mathbf{y_e}^{\mathbf{T}} \dfrac{\partial \mathbf{N}}{\partial t} & -\mathbf{y_e}^{\mathbf{T}} \dfrac{\partial \mathbf{N}}{\partial s} \\ -\mathbf{x_e}^{\mathbf{T}} \dfrac{\partial \mathbf{N}}{\partial t} & \mathbf{x_e}^{\mathbf{T}} \dfrac{\partial \mathbf{N}}{\partial s} \end{bmatrix}$$

where **|J|** is the determinant of **J**.

$$\mathbf{J}_1 = \frac{1}{|\mathbf{J}|}\begin{bmatrix} \dfrac{\partial y}{\partial t} & -\dfrac{\partial y}{\partial s} \end{bmatrix} \qquad \mathbf{J}_2 = \frac{1}{|\mathbf{J}|}\begin{bmatrix} -\dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial s} \end{bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - Q8 ELEMENTS

The values of the matrix Δ may be computed as:

$$\Delta = \begin{bmatrix} \dfrac{\partial \mathbf{N}}{\partial s} & \dfrac{\partial \mathbf{N}}{\partial t} \end{bmatrix} = \begin{bmatrix} \dfrac{(1-t)(2s+t)}{4} & \dfrac{(1-s)(s+2t)}{4} \\ \dfrac{(1-t)(2s-t)}{4} & \dfrac{(1+s)(2t-s)}{4} \\ \dfrac{(1+t)(2s+t)}{4} & \dfrac{(1+s)(s+2t)}{4} \\ \dfrac{(1+t)(2s-t)}{4} & \dfrac{(1-s)(2t-s)}{4} \\ -s(1-t) & -\dfrac{1-s^2}{2} \\ \dfrac{1-t^2}{2} & -t(1+s) \\ -s(1+t) & \dfrac{1-s^2}{2} \\ -\dfrac{1-t^2}{2} & -t(1-s) \end{bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - Q8 ELEMENTS

The $\mathbf{k_e}$ terms: $$\mathbf{k_e} = \int_{-1}^{1}\int_{-1}^{1}\left[\Delta \mathbf{JJ} \Delta^T\right] ds\, dt$$

For a general quadratic quadrilateral element, the 8x8 matrix **JJ** is a functional of *s* and *t* with the Jacobian **|J|** in the denominator.

The resulting expression for **JJ** is very difficult to evaluate exacting and the integrations are usually done numerically.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - Q8 ELEMENTS

Therefore, the terms $\mathbf{k_e}$ and $\mathbf{f_e}$ may be cast in the following form:

$$\mathrm{I} = \int_{-1}^{1}\int_{-1}^{1} G(s,t)\, ds\, dt$$

where **G**(*s,t*) is a complicated function of the variables *s* and *t*.

In principle, it may be possible to evaluate the $\mathbf{f_e}$ terms, however, numerical integration is typically more practical.

For the $\mathbf{k_e}$ terms, the appearance of the Jacobian **|J|** in the integrand generally indicates the use of numerical quadrature.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - Q8 ELEMENTS

Therefore, the general expressions for $\mathbf{k_e}$ and $\mathbf{f_e}$ are:

$$\mathbf{k_e} = \int_{-1}^{1}\int_{-1}^{1}\left[\Delta(s,t)\,\mathbf{JJ}(s,t)\,\Delta^T(s,t)\right]ds\,dt$$

$$\mathbf{f_e} = \int_{-1}^{1}\int_{-1}^{1}\mathbf{N}(s,t)\,\mathbf{N^T}(s,t)\left|\mathbf{J}\right|ds\,dt$$

We can use the Gaussian quadrature developed for one dimensional integrals and apply that approximation for both *s* and *t*.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - Q8 ELEMENTS

The general form for an *N*-term Gaussian quadrature is:

$$\mathrm{I} = \int_{-1}^{1}\int_{-1}^{1}G(s,t)\,ds\,dt = \sum_{i=1}^{N}\sum_{j=1}^{N}G(s_i,t_j)\,w_i\,w_j$$

where s_i, t_j are the Gauss points and w_i and w_j are the corresponding weights.

Therefore $\mathbf{k_e}$ and $\mathbf{f_e}$ may be evaluated by:

$$\mathbf{k_e} = \int_{-1}^{1}\int_{-1}^{1}\left[\Delta\mathbf{JJ}\Delta^T\right]ds\,dt = \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\Delta_{ij}\mathbf{JJ}_{ij}\Delta_{ij}{}^T\right)w_i\,w_j$$

$$\mathbf{f_e} = \left(\int_{-1}^{1}\int_{-1}^{1}\mathbf{NN^T}\left|\mathbf{J}\right|ds\,dt\right)\mathbf{f} = \left(\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\mathbf{N}_{ij}\mathbf{N}_{ij}{}^{\mathbf{T}}\left|\mathbf{J}\right|_{ij}\right)w_i\,w_j\right)\mathbf{f}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

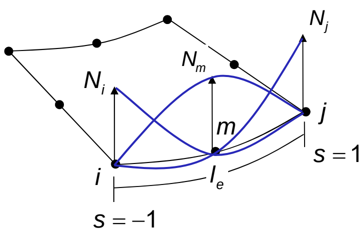
EVALUATION OF MATRICES - Q8 ELEMENTS

Consider the integrals $\mathbf{a_e}$ and $\mathbf{h_e}$:

$$\mathbf{a_e} = \int_{\gamma_{2e}} \mathbf{N}\alpha\mathbf{N}^{\mathbf{T}}\, ds \qquad \mathbf{h_e} = \int_{\gamma_{2e}} \mathbf{N} h\, ds$$

where the integration is along a boundary segment of the element.

Since, the integration is computed along a single side of the quadratic element, the interpolation functions are quadratic.



$$N_i = \frac{s(s-1)}{2} \qquad N_m = 1 - s^2 \qquad N_j = \frac{s(s+1)}{2}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

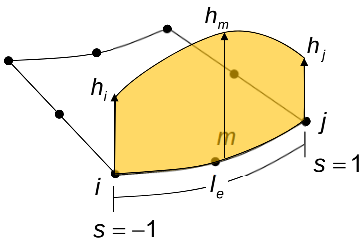
EVALUATION OF MATRICES - Q8 ELEMENTS

Consider the integrals $\mathbf{a_e}$ and $\mathbf{h_e}$:

$$\mathbf{a_e} = \int_{\gamma_{2e}} \mathbf{N}\alpha\mathbf{N}^{\mathbf{T}}\, ds \qquad \mathbf{h_e} = \int_{\gamma_{2e}} \mathbf{N} h\, ds$$

where the integration is along a boundary segment of the element.

Since, the integration is computed along a single side of the quadratic element, the interpolation functions are quadratic.



$$\alpha(s) = \alpha_i N_i + \alpha_j N_j + \alpha_m N_m$$

$$h(s) = h_i N_i + h_j N_j + h_m N_m$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - Q8 ELEMENTS

The variation of *x* and *y* as function of *s* along the boundary is given as:

$$x(s) = x_i N_I + x_j N_J + x_m N_m \qquad y(s) = y_i N_I + y_j N_J + y_m N_m$$

The differential arc length dl_e is:

$$dl_e^2 = dx^2 + dy^2 \qquad \rightarrow \qquad dl_e = \sqrt{\left[x'(s)\right]^2 + \left[y'(s)\right]^2}\ ds$$

$$l_e = \int_{-1}^{1} dl_e = \int_{-1}^{1} \sqrt{\left[x'(s)\right]^2 + \left[y'(s)\right]^2}\ ds$$

$$x'(s) = \frac{x_j - x_i}{2} + s\left(x_i - 2x_m + x_j\right) \qquad y'(s) = \frac{y_j - y_i}{2} + s\left(y_i - 2y_m + y_j\right)$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - Q8 ELEMENTS

The integrals defining $\mathbf{a_e}$ and $\mathbf{h_e}$ are:

$$\mathbf{a_e} = \int_{-1}^{1} \mathbf{N}\left(\alpha_i N_I + \alpha_j N_J + \alpha_m N_m\right)\mathbf{N^T}\, l_e\, ds$$

$$\mathbf{h_e} = \int_{\gamma_{2e}} \mathbf{N} h\, ds \approx \left(\int_{-1}^{1} \mathbf{N}\mathbf{N}^T\, l_e ds\right)\mathbf{h}$$

The resulting 3 x 1 elemental load vector contributes to the global system equations if the element has a side as part of the boundary.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - Q8 ELEMENTS

The global system equations are composed from the following summations:

$$\mathbf{K_G} = \sum_e \mathbf{k_G} + \sum_e \mathbf{'a_G} \qquad \mathbf{F_G} = \sum_e \mathbf{f_G} + \sum_e \mathbf{'h_G}$$

The resulting system equations are, in matrix form, given as:

$$\boxed{\mathbf{K_G u_G} = \mathbf{F_G}}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - Q8 ELEMENTS

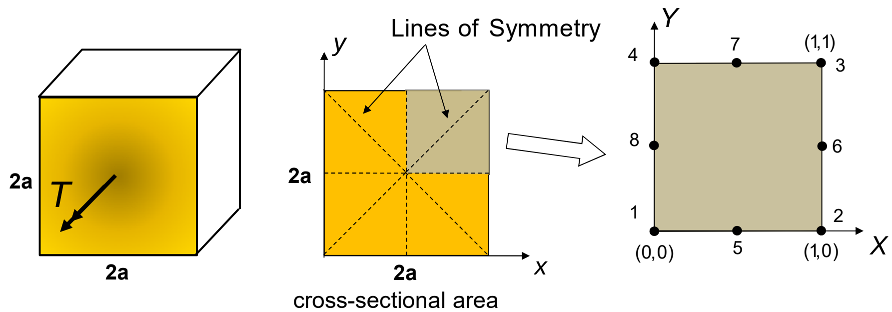
PROBLEM #24 - Write a computer program subroutine called **Q8QUAD** that calculates the components of the $\mathbf{k_e}$ matrix for a general quadratic quadrilateral element using Gaussian quadrature.

This assignment is similar to **Problem #23** except an eight-node quadrilateral is used.

Check your work with the problem in the your textbook on pages 336 and 338.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - Consider the same problem of torsion of a homogeneous isotropic prismatic bar we solved before, except using a single Q8 element.



$$\mathbf{x}_e^T = \langle 0 \quad 1 \quad 1 \quad 0 \quad 0.5 \quad 1 \quad 0.5 \quad 0 \rangle$$

$$\mathbf{y}_e^T = \langle 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0.5 \quad 1 \quad 0.5 \rangle$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - Recall, the non-dimensional Poisson equation governing this problem.

$$\nabla^2 \Psi(X, Y) + 1 = 0 \quad \text{in } \Omega \qquad -1 \le X \le 1$$

$$\Psi = 0 \quad \text{on } \Gamma \qquad -1 \le Y \le 1$$

with

$$X = \frac{x}{a} \qquad Y = \frac{y}{a} \qquad \Psi = \frac{\phi}{2G\theta a^2}$$

The stresses and torque for the Prandtl stress function are:

$$\tau_{xz} = 2G\theta a \frac{\partial \Psi}{\partial Y} \qquad \tau_{yz} = -2G\theta a \frac{\partial \Psi}{\partial X}$$

$$T = 4G\theta a^4 \iint \Psi \, dXdY$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - **Elemental Formulation** - Using a linear triangular element the elemental stiffness matrix components are:

$$\mathbf{k_e} = \int_{-1}^{1}\int_{-1}^{1}\left[\Delta \mathbf{J}\mathbf{J}\Delta^T\right] ds\, dt = \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\Delta_{ij}\mathbf{J}\mathbf{J}_{ij}\Delta_{ij}^{\ T}\right) w_i\, w_j$$

For element 1:

$N = 7$

$$\mathbf{k_e} = \begin{bmatrix} 1.1556 & 0.5000 & 0.5111 & 0.5000 & -0.8222 & -0.5111 & -0.5111 & -0.8222 \\ & 1.1556 & 0.5000 & 0.5111 & -0.8222 & -0.8222 & -0.5111 & -0.5111 \\ & & 1.1556 & 0.5000 & -0.5111 & -0.8222 & -0.8222 & -0.5111 \\ & & & 1.1556 & -0.5111 & -0.5111 & -0.8222 & -0.8222 \\ & & & & 2.3111 & 0.0000 & 0.3556 & 0.0000 \\ & & & & & 2.3111 & 0.0000 & 0.3556 \\ & & & & & & 2.3111 & 0.0000 \\ \textit{symetric} & & & & & & & 2.3111 \end{bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - **Elemental Formulation** - The loading function of $f(X, Y) = 1$ gives a series of elemental load vectors of:

$$\mathbf{f_e} = \left(\int_{-1}^{1}\int_{-1}^{1}\mathbf{N}\mathbf{N}^\mathbf{T}\left|\mathbf{J}\right| ds\, dt\right)\mathbf{f} = \left(\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\mathbf{N}_{ij}\mathbf{N}_{ij}^{\ \mathbf{T}}\left|\mathbf{J}\right|_{ij}\right) w_i\, w_j\right)\mathbf{f}$$

For element 1:

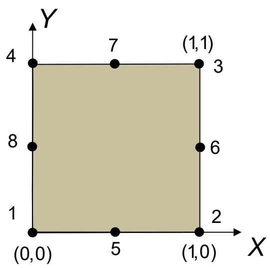
$N = 7$

$$\mathbf{f_e} = \begin{Bmatrix} -0.0833 \\ -0.0833 \\ -0.0833 \\ -0.0833 \\ 0.3333 \\ 0.3333 \\ 0.3333 \\ 0.3333 \end{Bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Assembly - Since there is one element assembly is not difficult:

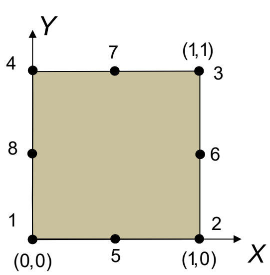
$$\begin{bmatrix} 1.1556 & 0.5000 & 0.5111 & 0.5000 & -0.8222 & -0.5111 & -0.5111 & -0.8222 \\ 0.5000 & 1.1556 & 0.5000 & 0.5111 & -0.8222 & -0.8222 & -0.5111 & -0.5111 \\ 0.5111 & 0.5000 & 1.1556 & 0.5000 & -0.5111 & -0.8222 & -0.8222 & -0.5111 \\ 0.5000 & 0.5111 & 0.5000 & 1.1556 & -0.5111 & -0.5111 & -0.8222 & -0.8222 \\ -0.8222 & -0.8222 & -0.5111 & -0.5111 & 2.3111 & 0.0000 & 0.3556 & 0.0000 \\ -0.5111 & -0.8222 & -0.8222 & -0.5111 & 0.0000 & 2.3111 & 0.0000 & 0.3556 \\ -0.5111 & -0.5111 & -0.8222 & -0.8222 & 0.3556 & 0.0000 & 2.3111 & 0.0000 \\ -0.8222 & -0.5111 & -0.5111 & -0.8222 & 0.0000 & 0.3556 & 0.0000 & 2.3111 \end{bmatrix} \begin{Bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \Psi_6 \\ \Psi_7 \\ \Psi_8 \end{Bmatrix} = \begin{Bmatrix} -0.0833 \\ -0.0833 \\ -0.0833 \\ -0.0833 \\ 0.3333 \\ 0.3333 \\ 0.3333 \\ 0.3333 \end{Bmatrix}$$



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - **Constraints** - For this model, $\Psi = 0$ on the boundary, therefore, Ψ_2, Ψ_3, Ψ_4, Ψ_6, and $\Psi_7 = 0$.

$$\begin{bmatrix} 1.1556 & 0.5000 & 0.5111 & 0.5000 & -0.8222 & -0.5111 & -0.5111 & -0.8222 \\ 0.5000 & 1.1556 & 0.5000 & 0.5111 & -0.8222 & -0.8222 & -0.5111 & -0.5111 \\ 0.5111 & 0.5000 & 1.1556 & 0.5000 & -0.5111 & -0.8222 & -0.8222 & -0.5111 \\ 0.5000 & 0.5111 & 0.5000 & 1.1556 & -0.5111 & -0.5111 & -0.8222 & -0.8222 \\ -0.8222 & -0.8222 & -0.5111 & -0.5111 & 2.3111 & 0.0000 & 0.3556 & 0.0000 \\ -0.5111 & -0.8222 & -0.8222 & -0.5111 & 0.0000 & 2.3111 & 0.0000 & 0.3556 \\ -0.5111 & -0.5111 & -0.8222 & -0.8222 & 0.3556 & 0.0000 & 2.3111 & 0.0000 \\ -0.8222 & -0.5111 & -0.5111 & -0.8222 & 0.0000 & 0.3556 & 0.0000 & 2.3111 \end{bmatrix} \begin{Bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \Psi_6 \\ \Psi_7 \\ \Psi_8 \end{Bmatrix} = \begin{Bmatrix} -0.0833 \\ -0.0833 \\ -0.0833 \\ -0.0833 \\ 0.3333 \\ 0.3333 \\ 0.3333 \\ 0.3333 \end{Bmatrix}$$



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

<u>Example</u> - **Constraints** - For this model, $\Psi = 0$ on the boundary, therefore, Ψ_2, Ψ_3, Ψ_4, Ψ_6, and $\Psi_7 = 0$.

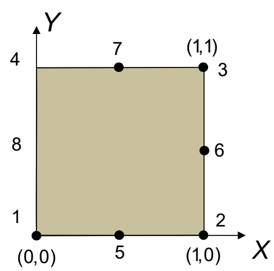
$$\begin{bmatrix} 1.1556 & 0.5000 & 0.5111 & 0.5000 & -0.8222 & -0.5111 & -0.5111 & -0.8222 \\ 0.5000 & 1.1556 & 0.5000 & 0.5111 & -0.8222 & -0.8222 & -0.5111 & -0.5111 \\ 0.5111 & 0.5000 & 1.1556 & 0.5000 & -0.5111 & -0.8222 & -0.8222 & -0.5111 \\ 0.5000 & 0.5111 & 0.5000 & 1.1556 & -0.5111 & -0.5111 & -0.8222 & -0.8222 \\ -0.8222 & -0.8222 & -0.5111 & -0.5111 & 2.3111 & 0.0000 & 0.3556 & 0.0000 \\ -0.5111 & -0.8222 & -0.8222 & -0.5111 & 0.0000 & 2.3111 & 0.0000 & 0.3556 \\ -0.5111 & -0.5111 & -0.8222 & -0.8222 & 0.3556 & 0.0000 & 2.3111 & 0.0000 \\ -0.8222 & -0.5111 & -0.5111 & -0.8222 & 0.0000 & 0.3556 & 0.0000 & 2.3111 \end{bmatrix} \begin{Bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \Psi_6 \\ \Psi_7 \\ \Psi_8 \end{Bmatrix} = \begin{Bmatrix} -0.0833 \\ -0.0833 \\ -0.0833 \\ -0.0833 \\ 0.3333 \\ 0.3333 \\ 0.3333 \\ 0.3333 \end{Bmatrix}$$

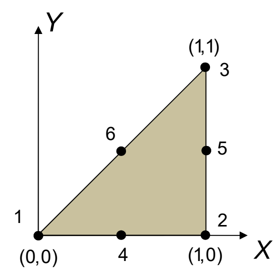
<u>Solution</u> - Solving the above equations gives:

$\Psi_1 = 0.2696$ $\Psi_5 = 0.2401$ $\Psi_8 = 0.2401$ $\Psi = \dfrac{\phi}{2G\theta a^2}$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

<u>Example</u> - **Constraints** - For this model, $\Psi = 0$ on the boundary, therefore, Ψ_2, Ψ_3, Ψ_4, Ψ_6, and $\Psi_7 = 0$.





<u>Solution</u> - Solving the above equations gives:

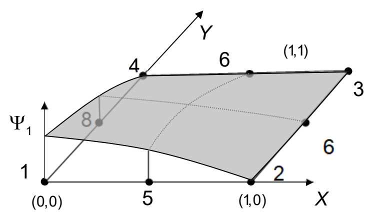
$\Psi_1 = 0.2696$ $\Psi_5 = 0.2401$ $\Psi_8 = 0.2401$ Q8 element

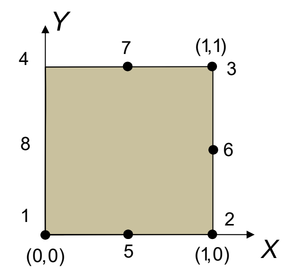
$\Psi_1 = 0.3000$ $\Psi_4 = 0.2250$ T6 element

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Solution - Solving the above equations gives:

$$\Psi_1 = 0.2696 \qquad \Psi_5 = 0.2401 \qquad \Psi_8 = 0.2401$$





TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - **Computation of Derived Variables** - The total torque may be calculated as:

$$T = 4G\theta a^4 \iint \Psi \, dXdY = 4\left(4G\theta a^4 \iint_{A_e} \Psi_e^T N \, dX \, dY\right)$$

$$= 4\left(4G\theta a^4 \frac{86}{625}\right)$$

$$\boxed{T = 2.2016G\theta a^4} \qquad \rightarrow \qquad T_{exact} = 2.2496G\theta a^4$$

$T = 2.133G\theta a^4$ T6 element

End of
8-Node Quadrilateral
Elements (Q8)