EIGHT-NODE QUADRILATERRAL ELEMENTS (Q8)
The next in our element development is a logical extension of the quadrilateral element to a quadratically interpolated quadrilateral element defined by eight nodes, four at the vertices and four at the middle at each side. The middle node, depending on location, may define a straight line or a quadratic line.

Transformation and Shape Functions - There are two approaches to develop the interpolation or shape functions for elements. The first approach is based on representing the geometry and the dependent variable as a function of the global coordinates $x$ and $y$.

The variation of the dependent variable $u$ over the element may be expressed as:

$$u_i(x, y) = a + bx + cy + dx^2 + exy + fy^2 + gx^2y + hx^2y^2$$

For the first approach, consider a straight-sided rectangular element shown below:

Fitting this expression for $u$ to the definition of the eight-node quadrilateral given above requires:

$$u_i(x, y) = a + bx + cy + dx^2 + exy + fy^2 + gx^2y + hx^2y^2$$

The above equation is written for each nodal value of $x$ and $y$ resulting in eight equations in the eight unknowns $a$, $b$, $c$, $d$, $e$, $f$, $g$, and $h$. 
For the first approach, consider a straight-sided rectangular element shown below:

EIGHT-NODE QUADRILATERAL ELEMENTS (Q8)

Solving this set of equations and collecting terms in $u_i$ results in the interpolation functions.

$$u_e(x, y) = u_i^T N = N^T u_e$$

where $u_e^T = [u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8]$.

The geometry of the quadrilateral element may also be described using the above interpolations as:

$$\sum_{i=1}^{N} x_i N_i = x_e^T N = N^T x_e$$

An isoparametric element may be formed by using a value of $N = 8$ which uses the interpolation functions given above.

However, a subparametric element may also be defined by setting $N = 4$.

In this case, the interpolation functions defined for a four quadrilateral are used.

As described above, the second approach to developing a set of interpolation or shape functions for an eight-node quadrilateral element begins with the parent element in local coordinates.

Consider the following eight-node quadrilateral in local coordinates $s$ and $t$. The eight interpolation or shape functions in global coordinates $x$ and $y$ are mathematically clumsy and rarely used in FEM analysis.
In the parent element the interpolation functions are given as:

\[ \begin{align*}
N_1(s, t) &= \frac{(1-s)(1-t)(1+s-t)}{4} \\
N_2(s, t) &= \frac{(1+s)(1-t)(1+s-t)}{4} \\
N_3(s, t) &= \frac{(1-s)^2(1-t)}{2} \\
N_4(s, t) &= \frac{(1+s)^2(1-t)}{2} \\
N_5(s, t) &= \frac{(1-s)(1+t)(1+s-t)}{4} \\
N_6(s, t) &= \frac{(1+s)(1+t)(1+s-t)}{4} \\
N_7(s, t) &= \frac{(1-s)^2(1+t)}{2} \\
N_8(s, t) &= \frac{(1+s)^2(1+t)}{2} 
\end{align*} \]

The parent element interpolation functions have two basic shapes.

The behavior of the functions \( N_i \), \( N_j \), and \( N_k \) are similar except reference at different nodes. The shape function \( N_1 \) is shown below:

\[
N_1(s, t) = \begin{cases} 
1 & \text{if } (s, t) \in \text{Region 1} \\
0 & \text{otherwise}
\end{cases}
\]

In a manner identical to that used in every element we have developed so far, the nature of the transformation from the parent element to the global element, the chain rule is used to form the differential relationship:

\[
\begin{align*}
\frac{\partial}{\partial s} x &= \frac{\partial x}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial x}{\partial t} \frac{\partial t}{\partial x} \\
\frac{\partial}{\partial t} x &= \frac{\partial x}{\partial s} \frac{\partial s}{\partial t} + \frac{\partial x}{\partial t} \frac{\partial t}{\partial t}
\end{align*}
\]

In matrix notation, these derivatives may be written as:

\[
\begin{bmatrix}
\frac{\partial}{\partial s} \\
\frac{\partial}{\partial t}
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\
\frac{\partial y}{\partial s} & \frac{\partial y}{\partial t}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\
\frac{\partial t}{\partial x} & \frac{\partial t}{\partial y}
\end{bmatrix} = J \begin{bmatrix}
\frac{\partial s}{\partial x} \\
\frac{\partial s}{\partial y}
\end{bmatrix}
\]

When \( |J| \) is positive everywhere in a region, the transformation may be inverted to determine \( s = s(x, y) \) and \( t = t(x, y) \). This means that for a given point \((x, y)\) in the triangle there is a unique corresponding point \((s, t)\) in the parent element.

The \(|J(s, t)|\) is a measure of the expansion or contraction of a differential area:

\[
dx \, dy = |J(s, t)| \, ds \, dt
\]
Consider the eight-node quadrilateral element given below. The coordinate transformation is given as:

\[
\begin{align*}
\hat{x} &= (a - t)(s - t) + 3 - a + 3 \\
\hat{y} &= (a - 1)(s + t) + (1 - a)s + a + 3
\end{align*}
\]

The resulting Jacobian is:

\[
|J(s, t)| = \frac{(a - 1)t^2 - a + 3}{2} \left( \frac{(a - 1)(st - s)}{(a - 1)(st - t)} \right)
\]

Let’s consider several cases where the \(|J|\) may become negative. The mapping at node 1 has the values of \(t = s = -1\). Substituting the corresponding \(s\) and \(t\) coordinates for nodes 1 into the expression for \(|J|\) gives:

\(|J(s, t)| = -4a^2 + 8a - 3\)

The resulting Jacobian is:

\[
\begin{align*}
\frac{(a - 1)t^2 - a + 3}{2} \left( \frac{(a - 1)(st - s)}{(a - 1)(st - t)} \right)
\end{align*}
\]

Derived Variables - The derived variables for the Poisson or Laplace’s equation are the partial derivatives \(\frac{\partial u}{\partial x}\) and \(\frac{\partial u}{\partial y}\). These terms are computed as functions of position in an element using the shape functions and the nodal values. The partial derivatives may be combined to give the normal or directional derivative:

\[
\hat{n} \hat{u} = (n_x J_1 + n_y J_2) \hat{\Delta}{\mathbf{u}}
\]
EVALUATION OF MATRICES - Q8 ELEMENTS

The elemental matrices for the Poisson problem are:

\[
\begin{bmatrix}
\frac{\partial N}{\partial x} \\
\frac{\partial N}{\partial y}
\end{bmatrix}
\begin{bmatrix}
\int_{A} \frac{\partial N}{\partial x} \frac{\partial N}{\partial x} \, dA \\
\int_{A} \frac{\partial N}{\partial x} \frac{\partial N}{\partial y} \, dA \\
\int_{A} \frac{\partial N}{\partial y} \frac{\partial N}{\partial x} \, dA \\
\int_{A} \frac{\partial N}{\partial y} \frac{\partial N}{\partial y} \, dA
\end{bmatrix}
\begin{bmatrix}
\frac{\partial N}{\partial x} \\
\frac{\partial N}{\partial y}
\end{bmatrix}
\]

The terms \(J_1\) and \(J_2\) are the first and second rows of the inverse of the Jacobian matrix.

\[
J^{-1} = \frac{1}{|J|} \begin{bmatrix}
\frac{\partial y}{\partial t} & -\frac{\partial y}{\partial s} \\
-\frac{\partial x}{\partial t} & \frac{\partial x}{\partial s}
\end{bmatrix}
\]

where \(|J|\) is the determinant of \(J\).

The values of the matrix \(\Delta\) may be computed as:

\[
\Delta = \begin{bmatrix}
\frac{\partial N}{\partial x} & \frac{\partial N}{\partial y} \\
\frac{\partial X}{\partial t} & \frac{\partial X}{\partial s}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial N}{\partial x} & \frac{\partial N}{\partial y} \\
\frac{\partial X}{\partial t} & \frac{\partial X}{\partial s}
\end{bmatrix}^T
\]

Therefore, the terms \(k_e\) and \(f_e\) may be cast in the following form:

\[
k_e = \int_{-1}^{1} \int_{-1}^{1} [\Delta JJ^T] \, ds \, dt
\]

For a general quadratic quadrilateral element, the 8x8 matrix \(JJ\) is a functional of \(s\) and \(t\) with the Jacobian \(|J|\) in the denominator.

The resulting expression for \(JJ\) is very difficult to evaluate exactly and the integrations are usually done numerically.
Therefore, the general expressions for $k_e$ and $f_e$ are:

$$
k_e = \int \int \Delta(s,t) J J^T(s,t) ds dt,
$$

$$
f_e = \int \int N(s,t) N^T(s,t) ds dt.
$$

We can use the Gaussian quadrature developed for one-dimensional integrals and apply that approximation for both $s$ and $t$.

The general form for an $N$-term Gaussian quadrature is:

$$
\sum_{i=1}^{N} w_i s_i \Delta(s_i) \Delta^T(s_i) = \Delta J J^T
$$

$$
\sum_{i=1}^{N} w_i t_j N(t_j) N^T(t_j) = N^T J f
$$

Therefore $k_e$ and $f_e$ may be evaluated by:

$$
k_e = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \Delta(s_i,s_j) \Delta^T(s_i,s_j)
$$

$$
f_e = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j N(s_i,t_j) N^T(s_i,t_j)
$$

Consider the integrals $a_e$ and $h_e$:

$$
a_e = \int N dN^T ds
$$

$$
h_e = \int N h ds
$$

where the integration is along a boundary segment of the element. Since, the integration is computed along a single side of the quadratic element, the interpolation functions are quadratic.

The variation of $x$ and $y$ as function of $s$ along the boundary is given as:

$$
x(s) = x_i N_i + x_j N_j + x_k N_k
$$

$$
y(s) = y_i N_i + y_j N_j + y_k N_k
$$

The differential arc length $dl_s$ is:

$$
dl_s = ds^2 + dy^2 \rightarrow dl_s = \sqrt{x'(s)^2 + y'(s)^2} \ ds
$$

$$
I_s = \int_0^1 ds = \int \sqrt{x'(s)^2 + y'(s)^2} \ ds
$$

$$
x'(s) = \frac{x_i - x_j}{2} + s(x_k - x_i)
$$

$$
y'(s) = \frac{y_i - y_j}{2} + s(y_k - y_i)
$$

The resulting 3 x 1 elemental load vector contributes to the global system equations if the element has a side as part of the boundary.
The global system equations are composed from the following summations:

\[ K_0 = \sum_k K_k + \sum_a a_k \quad \text{and} \quad F_0 = \sum f_0 + \sum h_0 \]

The resulting system equations are, in matrix form, given as:

\[ K_0 u_0 = F_0 \]

**Problem #23** - Write a computer program subroutine called `Q8QUAD` that calculates the components of the \( k_0 \) matrix for a general quadratic quadrilateral element using Gaussian quadrature.

This assignment is similar to Problem #22 except an eight-node quadrilateral is used.

Check your work with the problem in your textbook on pages 336 and 338.

---

**End of 8-Node Quadrilateral Elements (Q8)**