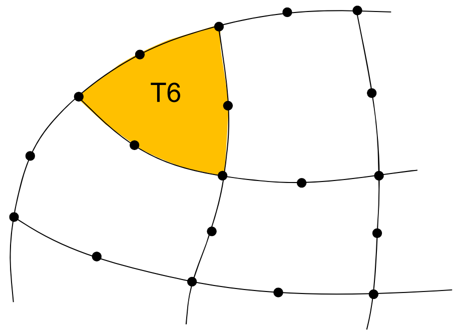
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6)

A quadratically interpolated triangular element is defined by six nodes, three at the vertices and three at the middle at each side.

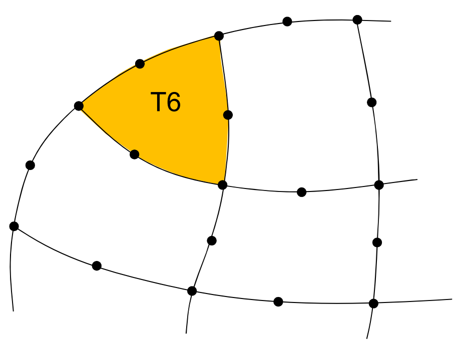
The middle node, depending on location, may define a straight line or a quadratic line.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6)

These elements may be very helpful in modeling some geometries where the use of a quadrilateral element may result in an undesirable deformation of the element and cause problems in the Jacobian mapping.

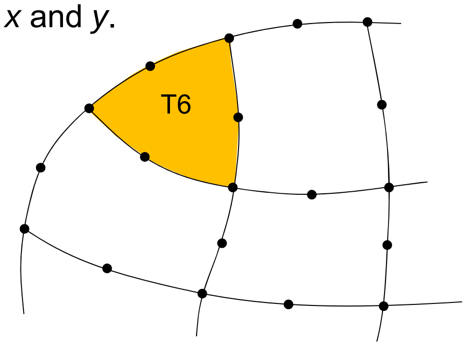


TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6)

Transformation and Shape Functions - There are two approaches to develop the interpolation or shape functions for the quadratic triangular element.

The first approach is based on representing the geometry and the dependent variable as a function of the global coordinates *x* and *y*.

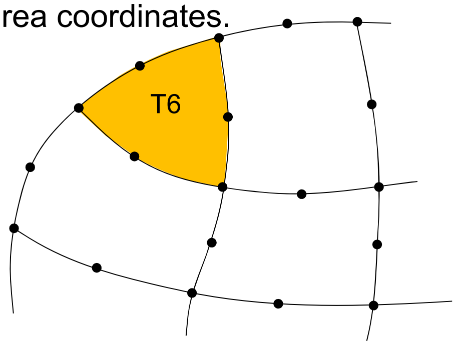


TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6)

Transformation and Shape Functions - There are two approaches to develop the interpolation or shape functions for the quadratic triangular element.

The second approach begins with the parent element with the interpolation and shape functions expressed in terms of the local area coordinates.



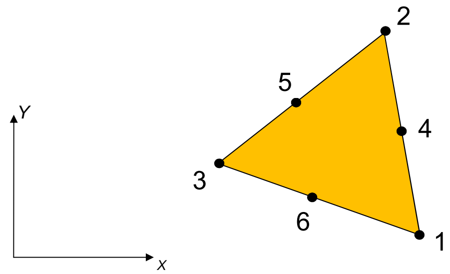
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6)

For the first approach, consider a straight-sided triangular element shown below.

The variation of the dependent variable *u* over the element may be expressed as:

$$u_e(x,y) = a + bx + cy + dx^2 + exy + fy^2$$

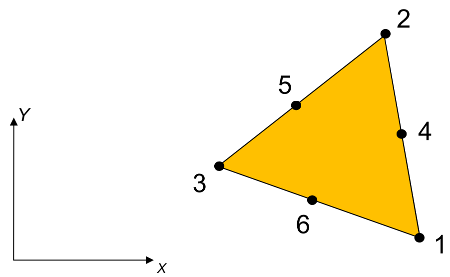


TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6)

Fitting this expression for *u* to the definition of the six-node triangle given above requires:

$$u_e(x_i,y_i) = a + bx_i + cy_i + dx_i^2 + ex_iy_i + fy_i^2 \qquad i = 1, \dots, 6$$



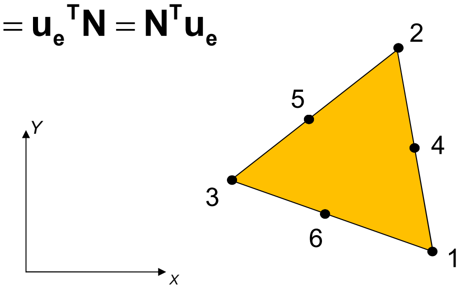
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6)

The above equation is written for each nodal value of x and y resulting in six equations in the six unknowns $a, b, c, d, e,$ and f.

Solving this set of equations gives the following interpolation.

$$u_e(x,y) = \mathbf{u_e}^{\mathbf{T}}\mathbf{N} = \mathbf{N}^{\mathbf{T}}\mathbf{u_e}$$



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6)

Where $\mathbf{u_e}^{\mathbf{T}} = [\ u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ u_6\ ]$ and the interpolation functions $\mathbf{N}$ are:

$$N_1 = \frac{\left[x_{23}(y - y_3) - y_{23}(x - x_3)\right]\left[x_{46}(y - y_6) - y_{46}(x - x_6)\right]}{(x_{23}y_{13} - y_{23}x_{13})(x_{46}y_{16} - y_{46}x_{16})}$$

$$N_2 = \frac{\left[x_{31}(y - y_1) - y_{31}(x - x_1)\right]\left[x_{54}(y - y_4) - y_{54}(x - x_4)\right]}{(x_{31}y_{21} - y_{31}x_{21})(x_{54}y_{24} - y_{54}x_{24})}$$

$$N_3 = \frac{\left[x_{21}(y - y_1) - y_{21}(x - x_1)\right]\left[x_{56}(y - y_6) - y_{56}(x - x_6)\right]}{(x_{21}y_{31} - y_{21}x_{31})(x_{56}y_{36} - y_{56}x_{36})}$$

where $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6)

Where $\mathbf{u_e}^{\mathbf{T}} = [\ u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ u_6\ ]$ and the interpolation functions **N** are:

$$N_4 = \frac{\left[x_{31}(y-y_1)-y_{31}(x-x_1)\right]\left[x_{23}(y-y_3)-y_{23}(x-x_3)\right]}{(x_{31}y_{41}-y_{31}x_{41})(x_{23}y_{43}-y_{23}x_{43})}$$

$$N_5 = \frac{\left[x_{31}(y-y_1)-y_{31}(x-x_1)\right]\left[x_{54}(y-y_4)-y_{54}(x-x_4)\right]}{(x_{31}y_{51}-y_{31}x_{51})(x_{21}y_{51}-y_{21}x_{51})}$$

$$N_6 = \frac{\left[x_{21}(y-y_1)-y_{21}(x-x_1)\right]\left[x_{23}(y-y_3)-y_{23}(x-x_3)\right]}{(x_{21}y_{61}-y_{21}x_{61})(x_{23}y_{63}-y_{23}x_{63})}$$

where $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6)

The geometry of the triangular element may also be described using the above interpolations as:

$$\sum_{i=1}^{N} x_i N_i = \mathbf{x_e}^{\mathbf{T}}\mathbf{N} = \mathbf{N}^{\mathbf{T}}\mathbf{x_e} \qquad \sum_{i=1}^{N} y_i N_i = \mathbf{y_e}^{\mathbf{T}}\mathbf{N} = \mathbf{N}^{\mathbf{T}}\mathbf{y_e}$$

An ***isoparametric*** element may be formed by using a value of $N = 6$ which uses the interpolation functions given above.

However, a ***subparametric*** element may also be defined by setting $N = 3$.

In this case, the interpolation function defined for a three-node triangular are used.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6)

The geometry of the triangular element may also be described using the above interpolations as:

$$\sum_{i=1}^{N} x_i N_i = \mathbf{x_e}^\mathbf{T}\mathbf{N} = \mathbf{N}^\mathbf{T}\mathbf{x_e} \qquad \sum_{i=1}^{N} y_i N_i = \mathbf{y_e}^\mathbf{T}\mathbf{N} = \mathbf{N}^\mathbf{T}\mathbf{y_e}$$

The six interpolation or shape functions in global coordinates *x* and *y* are mathematically clumsy and rarely used in FEM analysis.

An equivalent form of the shape functions may be derived in terms of the local parental element coordinates.

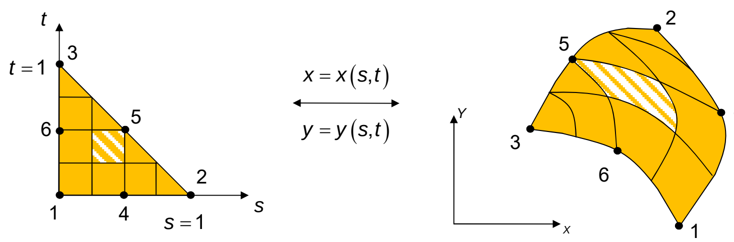
These functions have a relatively simple mathematical form and are more efficient in computing the elemental matrices.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6)

As describe above, the second approach to developing a set of interpolation or shape functions for a six-node quadratic triangular element begins with the parent element in local coordinates.

Consider the following six-node triangle in local coordinates *s* and *t*.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

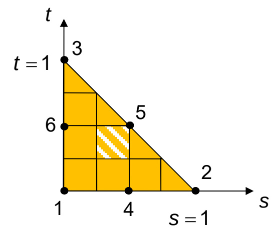
SIX-NODE TRIANGULAR ELEMENTS (T6)

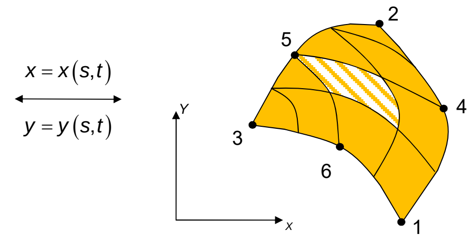
In the parent element the interpolation functions are given as:

$$N_1(s,t) = (1-s-t)(1-2s-2t) \qquad N_2(s,t) = s(2s-1)$$

$$N_3(s,t) = t(2t-1) \qquad N_4(s,t) = 4s(1-s-t)$$

$$N_5(s,t) = 4st \qquad N_6(s,t) = 4t(1-s-t)$$



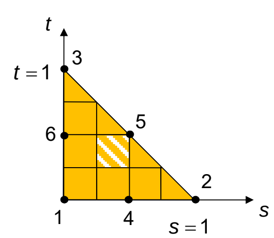


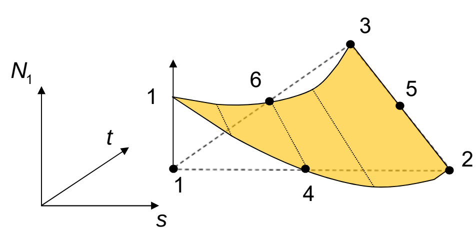
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6)

The parent element interpolation functions $N_i(s, t)$ have two basic shapes.

The behavior of the functions N_1, N_2, and N_3 is similar except reference at different nodes. The shape function N_1 is shown below:





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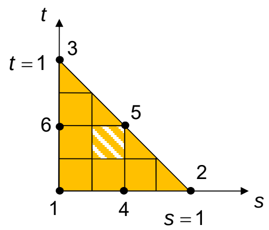
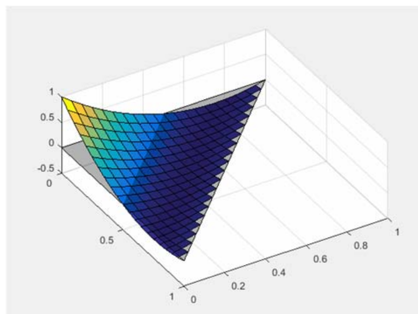


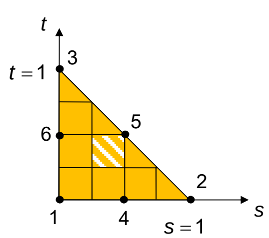


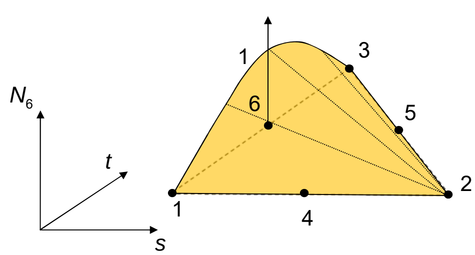
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6)

The parent element interpolation functions $N_i(s, t)$ have two basic shapes.

The second type of shape function is valid for functions N_4, N_5, and N_6. The function N_6 is shown below:





TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6)

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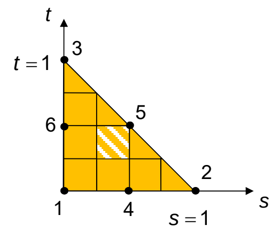
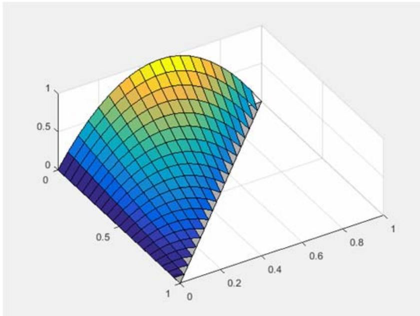




TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6)

In order to understand the nature of the transformation from the parent element to the global element, the chain rule is used to form the differential relationship:

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\frac{\partial y}{\partial t}$$

In matrix notation, these derivatives may be written as:

$$\begin{Bmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \qquad \frac{\partial}{\partial \vec{\mathbf{s}}} = \mathbf{J}\frac{\partial}{\partial \vec{\mathbf{x}}}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6)

The determinate of the Jacobian matrix **|J|:**

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \qquad |\mathbf{J}| = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial y}{\partial s}\frac{\partial x}{\partial t}$$

$$|\mathbf{J}| = \left(\sum_{1}^{6} x_i \frac{\partial N_i}{\partial s}\right)\left(\sum_{1}^{6} y_i \frac{\partial N_i}{\partial t}\right) - \left(\sum_{1}^{6} y_i \frac{\partial N_i}{\partial s}\right)\left(\sum_{1}^{6} x_i \frac{\partial N_i}{\partial t}\right)$$

The determinant of the Jacobian matrix, **|J|**, is a test of the invertibility of the transformation $x = x(s, t)$ and $y = y(s, t)$.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6)

When |**J**| is positive everywhere in the element, the transformation may be inverted to determine $s = s(x, y)$ and $t = t(x, y)$.

This means that for a given point (x, y) in the triangle there is a unique corresponding point (s, t) in the parent element.

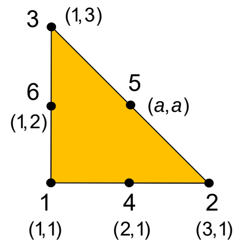
The $|\mathbf{J}(s, t)|$ is a measure of the expansion or contraction of a differential area:

$$dx\,dy = |\,\mathbf{J}(s,t)\,|\,ds\,dt$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6) – Example 1

Consider the quadratic triangular element given below.



$$\mathbf{x_e}^T = \langle 1 \quad 3 \quad 1 \quad 2 \quad a \quad 1 \rangle$$

$$\mathbf{y_e}^T = \langle 1 \quad 1 \quad 3 \quad 1 \quad a \quad 2 \rangle$$

The coordinate transformation is given as:

$$x = (4a - 8)st + 2s + 1 \qquad y = (4a - 8)st + 2t + 1$$

The resulting Jacobian is: $|\,\mathbf{J}(s,t)\,| = (8a - 16)(t + s) + 4$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6) – Example 1

Let's consider several cases where the **|J|** may become negative.

The mapping at nodes 2 and 3 have the value $t + s = 1$ in common.

Substituting the corresponding s and t coordinates for nodes 2 and 3 into the expression for **|J|** gives:

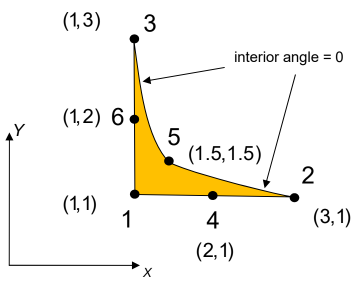
$$|\,\mathbf{J}(s,t)\,| = 8a - 12$$

which is clearly negative at $a \leq 3/2$.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

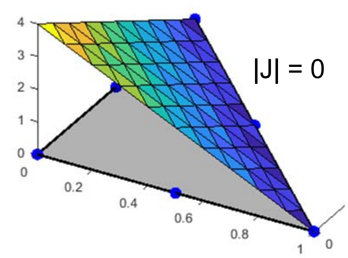
SIX-NODE TRIANGULAR ELEMENTS (T6) – Example 1

For values of x and y less than 3/2, the determinant of the Jacobian is negative.



$$\mathbf{x}_e^T = \langle 1 \quad 3 \quad 1 \quad 2 \quad 1.5 \quad 1 \rangle$$

$$\mathbf{y}_e^T = \langle 1 \quad 1 \quad 3 \quad 1 \quad 1.5 \quad 2 \rangle$$

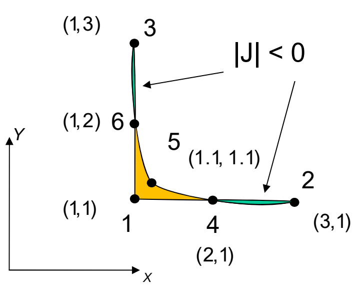


Note that for a = 3/2 the interior angle at nodes 2 and 3 is equal to zero.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

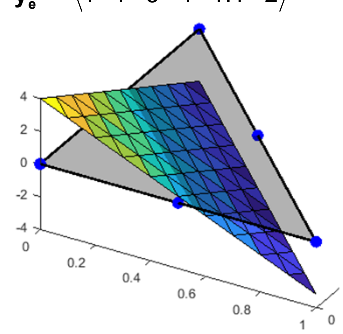
SIX-NODE TRIANGULAR ELEMENTS (T6) – Example 1

For values of x and y less than 3/2, the determinant of the Jacobian is negative.



$$\mathbf{x}_e^T = \langle 1 \quad 3 \quad 1 \quad 2 \quad 1.1 \quad 1 \rangle$$

$$\mathbf{y}_e^T = \langle 1 \quad 1 \quad 3 \quad 1 \quad 1.1 \quad 2 \rangle$$

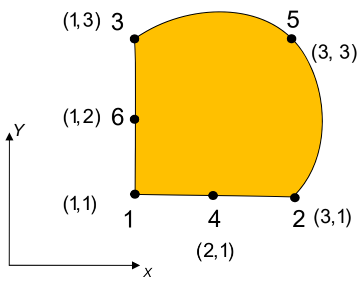


Note that for a = 1.1 the |J| < 0 over about half the element

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

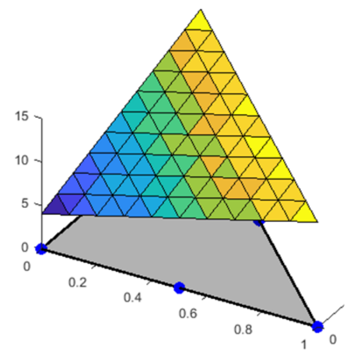
SIX-NODE TRIANGULAR ELEMENTS (T6) – Example 1

For values of *a* > 3/2 the determinant of the Jacobian is always positive.



$$\mathbf{x}_e^T = \langle 1 \quad 3 \quad 1 \quad 2 \quad 3 \quad 1 \rangle$$

$$\mathbf{y}_e^T = \langle 1 \quad 1 \quad 3 \quad 1 \quad 3 \quad 2 \rangle$$

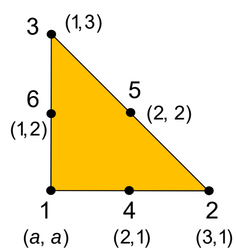


Note that for *a* = 3 the |J| > 0 over the entire element

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6) – Example 1

Consider the quadratic triangular element given below.



$$\mathbf{x}_e^T = \langle a \quad 3 \quad 1 \quad 2 \quad 2 \quad 1 \rangle$$

$$\mathbf{y}_e^T = \langle a \quad 1 \quad 3 \quad 1 \quad 2 \quad 2 \rangle$$

The resulting Jacobian is:

$$|\,\mathbf{J}(s,t)\,| = 16as - 16t - 16t - 12a + 16at + 16$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6) – Example 1

Let's consider several cases where the **|J|** may become negative.

The mapping at node 1 has the values *s* = 0 and *t* = 0; substituting these *s* and *t* coordinates for node 1 into the expression for **|J|** gives:

$$|\,\mathbf{J}(s,t)\,| = 16 - 12a$$

which is clearly negative at $a \geq 4/3$.

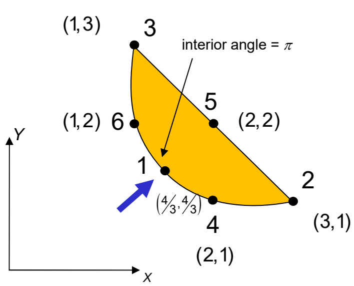
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

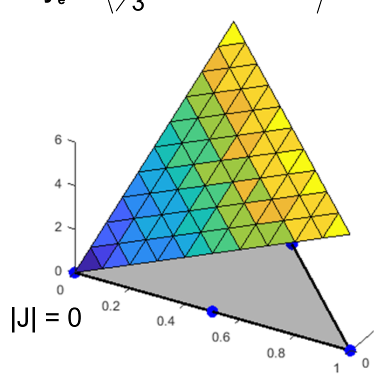
SIX-NODE TRIANGULAR ELEMENTS (T6) – Example 1

For values of *a* greater than 4/3, the determinant of the Jacobian is negative.

$$\mathbf{x}_e^T = \left\langle \tfrac{4}{3} \quad 3 \quad 1 \quad 2 \quad 2 \quad 1 \right\rangle$$

$$\mathbf{y}_e^T = \left\langle \tfrac{4}{3} \quad 1 \quad 3 \quad 1 \quad 2 \quad 2 \right\rangle$$



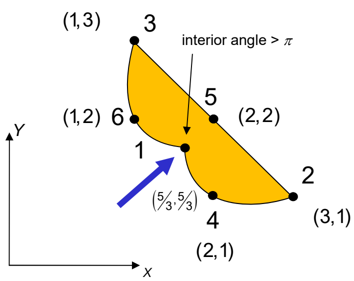


Note that for *a* = 4/3 the interior angle at node 1 is equal to π.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

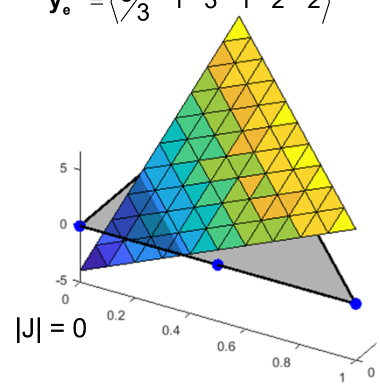
SIX-NODE TRIANGULAR ELEMENTS (T6) – Example 1

When a = 5/3, the determinant of the Jacobian is negative in the region about node 1.



$$\mathbf{x_e}^T = \left\langle \tfrac{5}{3} \quad 3 \quad 1 \quad 2 \quad 2 \quad 1 \right\rangle$$

$$\mathbf{y_e}^T = \left\langle \tfrac{5}{3} \quad 1 \quad 3 \quad 1 \quad 2 \quad 2 \right\rangle$$



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - T6 TRIANGULAR ELEMENTS

The elemental matrices for the Poisson problem are:

$$\mathbf{k_e} = \iint_A \left[\frac{\partial \mathbf{N}}{\partial x}\frac{\partial \mathbf{N}^T}{\partial x} + \frac{\partial \mathbf{N}}{\partial y}\frac{\partial \mathbf{N}^T}{\partial y} \right] dA \qquad \mathbf{f_e} = \iint_{A_e} \mathbf{N} f \, dA$$

$$\mathbf{a_e} = \int_{\gamma_{2e}} \mathbf{N} \alpha \mathbf{N}^T \, ds \qquad \mathbf{h_e} = \int_{\gamma_{2e}} \mathbf{N} h \, ds$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - T6 TRIANGULAR ELEMENTS

The development of both the $\mathbf{k_e}$ and $\mathbf{f_e}$ terms is identical to that presented for other types of elements:

$$\mathbf{k_e} = \int_0^1 \int_0^{1-t} \left[\Delta \mathbf{JJ} \Delta^T \right] ds\, dt$$

$$\mathbf{f_e} = \iint_{A_e} \mathbf{Nf}\, dA \approx \left(\iint_{A_e} \mathbf{NN^T} \left| \mathbf{J} \right| ds\, dt \right) \mathbf{f}$$

where $\mathbf{JJ} = (\mathbf{J_1}^T\mathbf{J_1} + \mathbf{J_2}^T\mathbf{J_2})|\mathbf{J}|$.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - T6 TRIANGULAR ELEMENTS

The terms $\mathbf{J_1}$ and $\mathbf{J_2}$ are the first and second rows of the inverse of the Jacobian matrix.

$$\mathbf{J}^{-1} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \mathbf{y_e}^T \dfrac{\partial \mathbf{N}}{\partial t} & -\mathbf{y_e}^T \dfrac{\partial \mathbf{N}}{\partial s} \\ -\mathbf{x_e}^T \dfrac{\partial \mathbf{N}}{\partial t} & \mathbf{x_e}^T \dfrac{\partial \mathbf{N}}{\partial s} \end{bmatrix}$$

where $|\mathbf{J}|$ is the determinant of $\mathbf{J}$.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - T6 TRIANGULAR ELEMENTS

The values of the matrix $\Delta^{\mathbf{T}}$ may be computed as:

$$\Delta^{\mathbf{T}} = \begin{bmatrix} \frac{\partial \mathbf{N}}{\partial s} \\ \frac{\partial \mathbf{N}}{\partial t} \end{bmatrix} = \left[\begin{array}{c|c|c|c|c|c} 4(s+t)-3 & 4s-1 & 0 & 4(1-t)-8s & 4t & -4t \\ 4(s+t)-3 & 0 & 4t-1 & -4s & 4s & 4(1-s)-8t \end{array}\right]$$

For a general quadratic triangular element, the 6 x 6 matrix **JJ** is a functional of *s* and *t* with the Jacobian **|J|** in the denominator.

The resulting expression for **JJ** is very difficult to evaluate exactly and the integrations are usually done numerically.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - T6 TRIANGULAR ELEMENTS

Therefore, the terms $\mathbf{k_e}$ and $\mathbf{f_e}$ may be cast in the following form:

$$\mathrm{I} = \int_0^1 \int_0^{1-t} G(s,t)\, ds\, dt$$

where **G**(*s,t*) is a function of the variables *s* and *t*.

In principle, it may be possible to evaluate the $\mathbf{f_e}$ terms, however, numerical integration is typically more practical.

For the $\mathbf{k_e}$ terms, the appearance of the Jacobian **|J|** in the integrand generally indicates the use of numerical quadrature.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - T6 TRIANGULAR ELEMENTS

Therefore, the general expressions for $\mathbf{k_e}$ and $\mathbf{f_e}$ are:

$$\mathbf{k_e} = \int_0^1 \int_0^{1-t} \left[\Delta(s,t)\,\mathbf{JJ}(s,t)\,\Delta^T(s,t) \right] ds\,dt$$

$$\mathbf{f_e} = \int_0^1 \int_0^{1-t} \mathbf{N}(s,t)\,\mathbf{N^T}(s,t)\,|\mathbf{J}|\,ds\,dt$$

The development of Gaussian quadrature for a triangle is identical to that presented in previous sections.

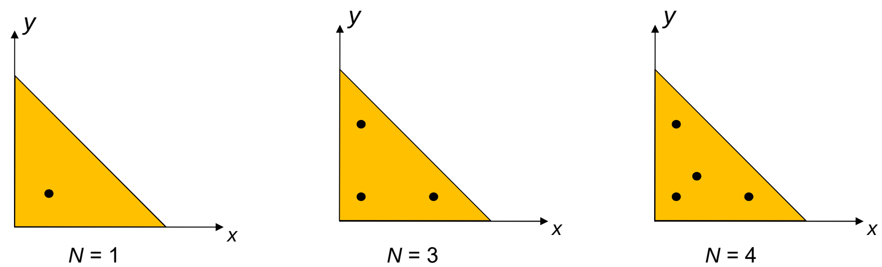
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - T6 TRIANGULAR ELEMENTS

The general form for an *N*-term Gaussian quadrature is:

$$\mathrm{I} = \int_0^1 \int_0^{1-t} G(s,t)\,ds\,dt = \frac{1}{2}\sum_{i=1}^{N} G(s_i,t_i)\,w_i$$

where s_i, t_i are the triangular Gauss points and w_i is the weight.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - T6 TRIANGULAR ELEMENTS

The general form for an N-term Gaussian quadrature is:

$$\mathrm{I} = \int_0^1 \int_0^{1-t} G(s,t)\,ds\,dt = \frac{1}{2}\sum_{i=1}^{N} G(s_i, t_i)\,w_i$$

where s_i, t_i are the triangular Gauss points and w_i is the weight.

Therefore $\mathbf{k_e}$ and $\mathbf{f_e}$ may be evaluated by:

$$\mathbf{k_e} = \int_0^1 \int_0^{1-t} \left[\Delta \mathbf{J} \mathbf{J} \Delta^T\right] ds\,dt = \frac{1}{2}\sum_{i=1}^{N}\left(\Delta_i \mathbf{J} \mathbf{J}_i \Delta_i^T\right) w_i$$

$$\mathbf{f_e} = \left(\int_0^1 \int_0^{1-t} \mathbf{N}\mathbf{N}^{\mathbf{T}} \left|\mathbf{J}\right| ds\,dt\right)\mathbf{f} = \left(\frac{1}{2}\sum_{i=1}^{N}\left(\mathbf{N}_i \mathbf{N}_i^{\mathbf{T}} \left|\mathbf{J}\right|_i\right) w_i\right)\mathbf{f}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - T6 TRIANGULAR ELEMENTS

The following table contains the Gauss points and weights for N = 1, 3, and 4.

N	s_i	t_i	w_i	Accuracy
1	0.33333 33333	0.33333 33333	1.00000 00000	1
3	0.66666 66667	0.16666 66667	0.33333 33333	2
	0.16666 66667	0.66666 66667	0.33333 33333	
	0.16666 66667	0.16666 66667	0.33333 33333	
4	0.33333 33333	0.33333 33333	-0.56250 00000	3
	0.60000 00000	0.20000 00000	0.52083 33333	
	0.20000 00000	0.20000 00000	0.52083 33333	
	0.20000 00000	0.60000 00000	0.52083 33333	

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - T6 TRIANGULAR ELEMENTS

The following table contains the Gauss points and weights for N = 6 and 7.

N	s_i	t_i	w_i	Accuracy
6	0.09157 62135	0.81684 75730	0.10995 17437	4
	0.09157 62135	0.09157 62135	0.10995 17437	
	0.81684 75730	0.09157 62135	0.10995 17437	
	0.44594 84909	0.10810 30182	0.22338 15897	
	0.44594 84909	0.44594 84909	0.22338 15897	
	0.10810 30182	0.44594 84909	0.22338 15897	
7	0.33333 33333	0.33333 33333	0.22500 00000	5
	0.10128 65073	0.79742 69854	0.12593 91805	
	0.10128 65073	0.10128 65073	0.12593 91805	
	0.79742 69854	0.10128 65073	0.12593 91805	
	0.05971 58718	0.47014 20641	0.13239 41528	
	0.47014 20641	0.47014 20641	0.13239 41528	
	0.47014 20641	0.05971 58718	0.13239 41528	

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

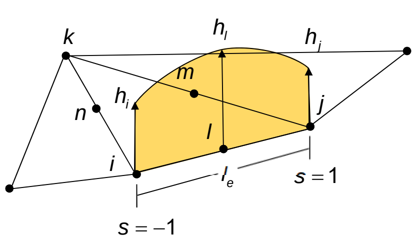
EVALUATION OF MATRICES - T6 TRIANGULAR ELEMENTS

Consider the integrals $\mathbf{a_e}$ and $\mathbf{h_e}$:

$$\mathbf{a_e} = \int_{\gamma_{2e}} \mathbf{N}\alpha\mathbf{N}^{\mathsf{T}}\, ds \qquad \mathbf{h_e} = \int_{\gamma_{2e}} \mathbf{N}h\, ds$$

where the integration is along a boundary segment of the element.

Since, the integration is computed along a single side of the quadratic element, the interpolation functions are quadratic.



$$h(s) = h_i N_i + h_j N_j + h_l N_l$$

$$\alpha(s) = \alpha_i N_i + \alpha_j N_j + \alpha_l N_l$$

$$N_i = \frac{s(s-1)}{2} \qquad N_l = 1 - s^2 \qquad N_j = \frac{s(s+1)}{2}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - T6 TRIANGULAR ELEMENTS

The variation of *x* and *y* as function of *s* along the boundary is given as:

$$x(s) = x_i N_I + x_j N_J + x_l N_l \qquad y(s) = y_i N_I + y_j N_J + y_l N_l$$

The differential arc length dl_e is:

$$dl_e^{\,2} = dx^2 + dy^2 \qquad \rightarrow \qquad dl_e = \sqrt{\left[x'(s)\right]^2 + \left[y'(s)\right]^2}\; ds$$

$$l_e = \int_{-1}^{1} dl_e = \int_{-1}^{1} \sqrt{\left[x'(s)\right]^2 + \left[y'(s)\right]^2}\; ds$$

$$x'(s) = \frac{x_j - x_i}{2} + s\left(x_i - 2x_l + x_j\right) \qquad y'(s) = \frac{y_j - y_i}{2} + s\left(y_i - 2y_l + y_j\right)$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - T6 TRIANGULAR ELEMENTS

The integrals defining $\mathbf{a_e}$ and $\mathbf{h_e}$ are:

$$\mathbf{a_e} = \int_{-1}^{1} \mathbf{N}\left(\alpha_i N_I + \alpha_j N_J + \alpha_l N_l\right)\mathbf{N^T}\, l_e\, ds$$

The resulting 3 x 3 elemental stiffness matrix contributes to the global system equations if the element has a side as part of the boundary.

$$\mathbf{h_e} = \int_{\gamma_{2e}} \mathbf{N} h\, ds \approx \left(\int_{-1}^{1} \mathbf{N}\mathbf{N}^T\, l_e ds\right)\mathbf{h}$$

The resulting 3 x 1 elemental load vector contributes to the global system equations if the element has a side as part of the boundary.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

EVALUATION OF MATRICES - T6 TRIANGULAR ELEMENTS

The global system equations are composed from the following summations:

$$\mathbf{K_G} = \sum_e \mathbf{k_G} + \sum_e{}' \mathbf{a_G} \qquad \mathbf{F_G} = \sum_e \mathbf{f_G} + \sum_e{}' \mathbf{h_G}$$

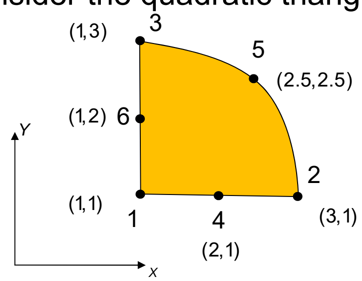
The resulting system equations are, in matrix form, given as:

$$\boxed{\mathbf{K_G u_G} = \mathbf{F_G}}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6) – Example 2

Consider the quadratic triangular element given below.



$$\mathbf{x_e}^{\mathbf{T}} = \langle 1 \quad 3 \quad 1 \quad 2 \quad 2.5 \quad 1 \rangle$$

$$\mathbf{y_e}^{\mathbf{T}} = \langle 1 \quad 1 \quad 3 \quad 1 \quad 2.5 \quad 2 \rangle$$

Calculate the values of the stiffness matrix $\mathbf{k_e}$ for the above element when node 5 is <u>$x = 2.5$</u> and <u>$y = 2.5$</u>.

$$\mathbf{k_e} = \int_0^1 \int_0^{1-t} \left[\Delta \mathbf{JJ} \Delta^T \right] ds\, dt \qquad \mathbf{JJ} = (\mathbf{J_1}^T\mathbf{J_1} + \mathbf{J_2}^T\mathbf{J_2})|\mathbf{J}|.$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6) – Example 2

$$\mathbf{k_e} = \int_0^1 \int_0^{1-t} \left[\Delta \mathbf{J} \mathbf{J} \Delta^T \right] ds\, dt$$

The values of the matrix $\Delta^{\mathbf{T}}$ may be computed as:

$$\Delta^{\mathbf{T}} = \begin{bmatrix} \frac{\partial \mathbf{N}}{\partial s} \\ \frac{\partial \mathbf{N}}{\partial t} \end{bmatrix} = \left[\begin{array}{c|c|c|c|c|c} 4(s+t)-3 & 4s-1 & 0 & 4(1-t)-8s & 4t & -4t \\ 4(s+t)-3 & 0 & 4t-1 & -4s & 4s & 4(1-s)-8t \end{array}\right]$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6) – Example 2

$$\mathbf{k_e} = \int_0^1 \int_0^{1-t} \left[\Delta \mathbf{J} \mathbf{J} \Delta^T \right] ds\, dt$$

$\mathbf{J_1}$ and $\mathbf{J_2}$ are the first and second rows of the inverse of the Jacobian matrix.

$$\mathbf{J}^{-1} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \frac{\partial y}{\partial t} & -\frac{\partial y}{\partial s} \\ -\frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \end{bmatrix} \qquad |\mathbf{J}| = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}$$

$$\mathbf{J_1} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \frac{\partial y}{\partial t} & -\frac{\partial y}{\partial s} \end{bmatrix} \qquad \mathbf{J}_2 = \frac{1}{|\mathbf{J}|} \begin{bmatrix} -\frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \end{bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6) – Example 2

Therefore, the general expressions for $\mathbf{k_e}$ is:

$$\mathbf{k_e} = \int_0^1 \int_0^{1-t} \left[\Delta \mathbf{J} \mathbf{J} \Delta^T \right] ds\, dt = \frac{1}{2} \sum_{i=1}^{N} \left(\Delta_i \mathbf{J} \mathbf{J}_i \Delta_i^T \right) w_i$$

$N = 1$

$$\mathbf{k_e} = \begin{bmatrix} 0.066667 & -0.033333 & -0.033333 & 0.133333 & -0.266667 & -0.133333 \\ & 0.062963 & -0.029630 & 0.118519 & 0.133333 & -0.251852 \\ & & 0.062963 & -0.251852 & 0.133333 & 0.118519 \\ & & & 1.007407 & -0.533333 & -0.474074 \\ & & & & 1.066667 & -0.533333 \\ \textit{symmetric} & & & & & 1.007407 \end{bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6) – Example 2

Therefore, the general expressions for $\mathbf{k_e}$ is:

$$\mathbf{k_e} = \int_0^1 \int_0^{1-t} \left[\Delta \mathbf{J} \mathbf{J} \Delta^T \right] ds\, dt = \frac{1}{2} \sum_{i=1}^{N} \left(\Delta_i \mathbf{J} \mathbf{J}_i \Delta_i^T \right) w_i$$

$N = 3$

$$\mathbf{k_e} = \begin{bmatrix} 0.744949 & -0.175505 & 0.175505 & -0.479798 & -0.136364 & -0.479798 \\ & 0.847012 & 0.101221 & -0.697811 & -0.219697 & -0.206229 \\ & & 0.847012 & -0.206229 & -0.219697 & -0.697811 \\ & & & 2.077441 & -0.454545 & -0.239057 \\ & & & & 1.484848 & -0.454545 \\ \textit{symmetric} & & & & & 2.077441 \end{bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6) – Example 2

Therefore, the general expressions for $\mathbf{k_e}$ is:

$$\mathbf{k_e} = \int_0^1 \int_0^{1-t} \left[\Delta \mathbf{J} \mathbf{J} \Delta^T \right] ds\, dt = \frac{1}{2} \sum_{i=1}^{N} \left(\Delta_i \mathbf{J} \mathbf{J}_i \Delta_i^T \right) w_i$$

$N = 4$

$$\mathbf{k_e} = \begin{bmatrix} 0.718519 & 0.144444 & 0.144444 & -0.444444 & -0.118519 & -0.444444 \\ & 0.812169 & 0.097354 & -0.633862 & -0.215873 & -0.204233 \\ & & 0.812169 & -0.204233 & -0.215873 & -0.633862 \\ & & & 1.972487 & -0.469841 & -0.220106 \\ & & & & 1.489947 & -0.469841 \\ \textit{symmetric} & & & & & 1.972487 \end{bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6) – Example 2

Therefore, the general expressions for $\mathbf{k_e}$ is:

$$\mathbf{k_e} = \int_0^1 \int_0^{1-t} \left[\Delta \mathbf{J} \mathbf{J} \Delta^T \right] ds\, dt = \frac{1}{2} \sum_{i=1}^{N} \left(\Delta_i \mathbf{J} \mathbf{J}_i \Delta_i^T \right) w_i$$

$N = 6$

$$\mathbf{k_e} = \begin{bmatrix} 0.783183 & 0.158301 & 0.158301 & -0.499871 & -0.100043 & -0.499871 \\ & 0.836785 & 0.078678 & -0.689032 & -0.211914 & -0.172818 \\ & & 0.836785 & -0.172818 & -0.211914 & -0.689032 \\ & & & 2.106579 & -0.485677 & -0.259181 \\ & & & & 1.495226 & -0.485677 \\ \textit{symmetric} & & & & & 2.106579 \end{bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6) – Example 2

Therefore, the general expressions for $\mathbf{k_e}$ is:

$$\mathbf{k_e} = \int_0^1 \int_0^{1-t} \left[\Delta \mathbf{J} \mathbf{J} \Delta^T \right] ds\, dt = \frac{1}{2} \sum_{i=1}^{N} \left(\Delta_i \mathbf{J} \mathbf{J}_i \Delta_i^T \right) w_i$$

$N = 7$

$$\mathbf{k_e} = \begin{bmatrix} 0.783169 & 0.158298 & 0.158298 & -0.499871 & -0.100047 & -0.499859 \\ & 0.837239 & 0.078222 & -0.689939 & -0.211915 & -0.171906 \\ & & 0.837239 & -0.171906 & -0.211915 & -0.689939 \\ & & & 2.108389 & -0.485674 & -0.261010 \\ & & & & 1.495225 & -0.485674 \\ symmetric & & & & & 2.108389 \end{bmatrix}$$

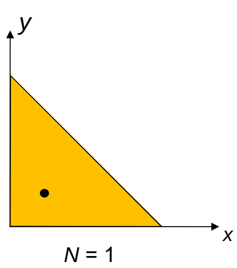
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

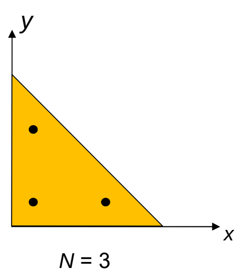
SIX-NODE TRIANGULAR ELEMENTS (T6) – Example 2

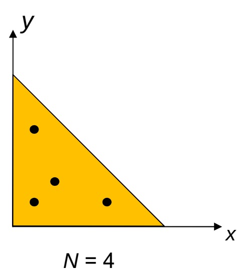
The results for N = 1 are not very accurate.

Since only one Gauss point is used in the quadrature, this result is not unexpected.

The values for $\mathbf{k_e}$ using Gaussian quadrature with N = 3 and 4 indicate that the accuracy of the evaluations increases as the number of Gauss points increases.







TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

SIX-NODE TRIANGULAR ELEMENTS (T6)

PROBLEM #23 - Write a compute subroutine called **T6QUAD** that calculates the components of the $\mathbf{k_e}$ matrix for a general quadratic triangular element using Gaussian quadrature.

While you will not write a complete finite element program, this subroutine is a critical component of the computational procedure.

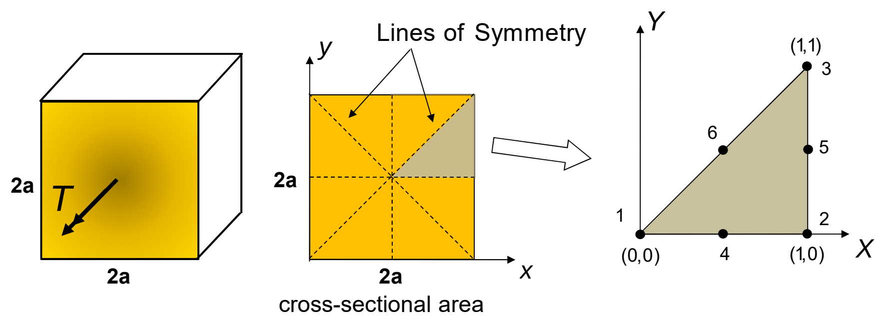
A simple driver program will be required to debug and test your subroutine and pass geometric information.

Your program should allow the user to specify the number of Gaussian quadrature points and the coordinate information.

Check your work with the problem in your textbook on page 359 and the example presented in the notes.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

<u>Example</u> - Consider the same problem of torsion of a homogeneous isotropic prismatic bar we solved before, except using a single T6 element.



$$\mathbf{x_e}^T = \langle 0 \quad 1 \quad 1 \quad 0.5 \quad 1 \quad 0.5 \rangle$$

$$\mathbf{y_e}^T = \langle 0 \quad 0 \quad 1 \quad 0 \quad 0.5 \quad 0.5 \rangle$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - Recall, the non-dimensional Poisson equation governing this problem.

$$\nabla^2\Psi\left(X,Y\right)+1=0 \qquad \text{in } \Omega \qquad -1\le X\le 1$$
$$\Psi=0 \qquad \text{on } \Gamma \qquad -1\le Y\le 1$$

with

$$X=\frac{x}{a} \qquad Y=\frac{y}{a} \qquad \Psi=\frac{\phi}{2G\theta a^2}$$

The stresses and torque for the Prandlt stress function are:

$$\tau_{xz}=2G\theta a\frac{\partial\Psi}{\partial Y} \qquad \tau_{yz}=-2G\theta a\frac{\partial\Psi}{\partial X}$$

$$T=4G\theta a^4\iint\Psi\ dXdY$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - **Elemental Formulation** - Using a linear triangular element the elemental stiffness matrix components are:

$$\mathbf{k_e}=\int_0^1\int_0^{1-t}\left[\Delta\mathbf{JJ}\Delta^T\right]ds\,dt=\frac{1}{2}\sum_{i=1}^{N}\left(\Delta_i\mathbf{JJ}_i\Delta_i^{\ T}\right)w_i$$

For element 1:

$N=7$

$$\mathbf{k_e}=\begin{bmatrix} 0.500000 & 0.166667 & 0.000000 & -0.666667 & 0.000000 & 0.000000 \\ & 1.000000 & 0.166667 & -0.666667 & -0.666667 & 0.000000 \\ & & 0.500000 & 0.000000 & -0.666667 & 0.000000 \\ & & & 2.666667 & 0.000000 & -1.333333 \\ & & & & 2.666667 & -1.333333 \\ \textit{symmetric} & & & & & 2.666667 \end{bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - **Elemental Formulation** - Using a linear triangular element the elemental stiffness matrix components are:

$$\mathbf{k_e} = \int_0^1 \int_0^{1-t} \left[\Delta \mathbf{J} \mathbf{J} \Delta^T \right] ds\, dt = \frac{1}{2} \sum_{i=1}^{N} \left(\Delta_i \mathbf{J} \mathbf{J}_i \Delta_i^T \right) w_i$$

For element 1:

$$\mathbf{k_e} = 6 \begin{bmatrix} 3 & 1 & 0 & -4 & 0 & 0 \\ 1 & 6 & 1 & -4 & -4 & 0 \\ 0 & 1 & 3 & 0 & -4 & 0 \\ -4 & -4 & 0 & 16 & 0 & -8 \\ 0 & -4 & -4 & 0 & 16 & -8 \\ 0 & 0 & 0 & -8 & -8 & 16 \end{bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - **Elemental Formulation** - The loading function of f = 1 gives a series of elemental load vectors of:

$$\mathbf{f_e} = \left(\frac{1}{2} \sum_{i=1}^{N} \left(\mathbf{N}_i \mathbf{N}_i^{\mathbf{T}} \left| \mathbf{J} \right|_i \right) w_i \right) \mathbf{f}$$

For element 1:

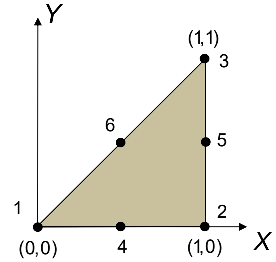
$$N = 7$$

$$\mathbf{f_e} = \frac{1}{6} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Assembly - Since there is one element assembly is not difficult:

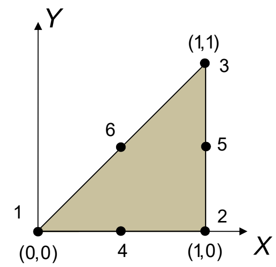
$$6\begin{bmatrix} 3 & 1 & 0 & -4 & 0 & 0 \\ 1 & 6 & 1 & -4 & -4 & 0 \\ 0 & 1 & 3 & 0 & -4 & 0 \\ -4 & -4 & 0 & 16 & 0 & -8 \\ 0 & -4 & -4 & 0 & 16 & -8 \\ 0 & 0 & 0 & -8 & -8 & 16 \end{bmatrix}\begin{Bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \Psi_6 \end{Bmatrix} = \frac{1}{6}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - **Constraints** - For this model, $\Psi = 0$ on the boundary, therefore, Ψ_2, Ψ_5, and $\Psi_3 = 0$.

$$6\begin{bmatrix} 3 & 0 & 0 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 0 & 16 & 0 & -8 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -8 & 0 & 16 \end{bmatrix}\begin{Bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \Psi_6 \end{Bmatrix} = \frac{1}{6}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{Bmatrix}$$



Solution - Solving the above equations gives:

$$\Psi_1 = \frac{12}{40} \qquad \Psi_4 = \frac{9}{40} \qquad \Psi_6 = \frac{7}{40} \qquad \Psi = \frac{\phi}{2G\theta a^2}$$

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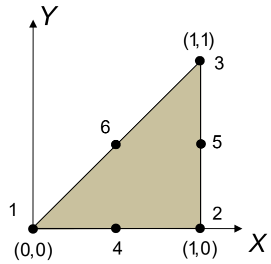

Solution - Solving the above equations gives:

$$\Psi_1 = \frac{12}{40} \qquad \Psi_4 = \frac{9}{40} \qquad \Psi_6 = \frac{7}{40} \qquad \text{T6 element}$$

$$\Psi_1 = \frac{14}{48} \qquad \Psi_4 = \frac{10}{48} \qquad \Psi_6 = \frac{9}{48} \qquad \text{4 T3 elements}$$

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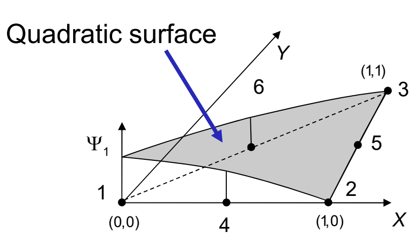
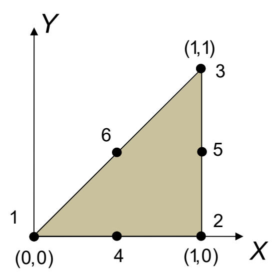
Solution – Converting into the Prandtl stress function gives:

$$\Psi_1 = \frac{12}{40} \qquad \Psi_4 = \frac{9}{40} \qquad \Psi_6 = \frac{7}{40} \qquad \Psi = \frac{\phi}{2G\theta a^2}$$

$$\phi_1 = \frac{24G\theta a^2}{40} \qquad \phi_4 = \frac{18G\theta a^2}{40} \qquad \phi_4 = \frac{14G\theta a^2}{44}$$





TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

<u>Example</u> - Computation of Derived Variables - The total torque may be calculated as:

$$T = 4G\theta a^4 \iint \Psi \, dXdY = 8\left(4G\theta a^4 \iint_{A_e} \Psi_e^T N \, dX \, dY\right)$$

$$= 8\left(4G\theta a^4 \frac{1}{15}\right) = \frac{32G\theta a^4}{15}$$

$$T = 2.133G\theta a^4 \qquad \rightarrow \qquad T_{exact} = 2.2496G\theta a^4$$

$T = 1.9444G\theta a^4$ Four T3 elements

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

<u>Example</u> – Results from various FE meshes:

Number of T6 elements	Torque ($G\theta a^4$)	% Error
1	2.1333	5.170
2	2.2222	1.218
4	2.2364	0.587
8	2.2468	0.124
16	2.2483	0.058
32	2.2491	0.022
144	2.2493	0.013
1,024	2.2493	0.013

$$T_{exact} = 2.2496G\theta a^4$$

End of
6-Node Triangular
Elements (T6)