TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

ISOPARAMETRIC ELEMENTS

The linear triangular element and the bilinear rectangular element have several important disadvantages.

1. Both elements are unable to accurately represent curved boundaries, and

2. They provide poor approximations for any derived variables.

By utilizing higher-order elements **isoparametric** elements the deficiencies of the linear elements are overcome.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

ISOPARAMETRIC ELEMENTS

The term **isoparametric** means that an equal number of parameters are used to represent the geometry and the dependent variable.

In other words, the same shape functions used to interpolate the dependent variables are used to represent the geometry.

A **subparametric** element is one where the geometry is defined by fewer parameters than used to interpolate the solution.

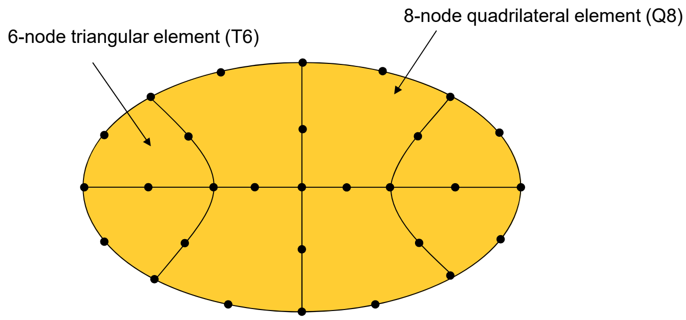
A **superparametric** element is one where the geometry is defined by more parameters than used to interpolate the solution.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

ISOPARAMETRIC ELEMENTS

The term **isoparametric** means that an equal number of parameters are used to represent the geometry and the dependent variable.

$$u_e = \sum u_i N_i \qquad x_e = \sum x_i N_i \qquad y_e = \sum y_i N_i$$



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

The four-node quadrilateral element is very similar to the four-node rectangular element.

Both provide bilinear interpolation over the element for the general solution and a linear approximation of the first derivative.

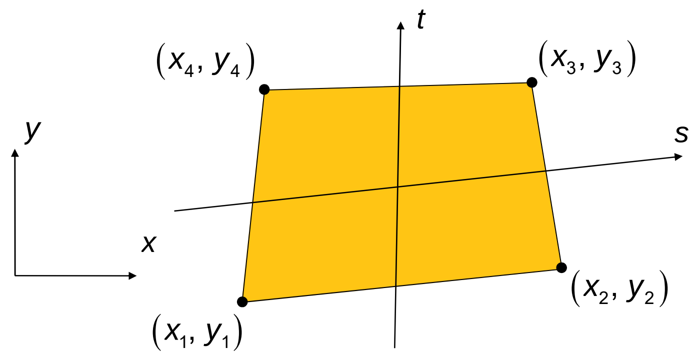
However, the quadrilateral element is not restricted to rectangular shapes; in fact it may represent any number of rectilinear shapes.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

The general approximation over a quadrilateral element is assumed to be:

$$u_e(s,t) = u_1N_1(s,t) + u_2N_2(s,t) + u_3N_3(s,t) + u_4N_4(s,t)$$
$$= \mathbf{u_e}^\mathbf{T}\mathbf{N} = \mathbf{u_e}\mathbf{N}^\mathbf{T}$$



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

The elemental coordinates are *s* and *t*.

The interpolation functions $N_i(s, t)$ are identical to the rectangular element interpolation functions.

$$N_1(x,y) = \frac{(x_2 - x)(y_3 - y)}{(x_2 - x_1)(y_3 - y_1)} \qquad N_2(x,y) = \frac{(x_1 - x)(y_3 - y)}{(x_1 - x_2)(y_3 - y_2)}$$

$$N_3(x,y) = \frac{(x_1 - x)(y_1 - y)}{(x_1 - x_3)(y_1 - y_3)} \qquad N_4(x,y) = \frac{(x - x_2)(y_1 - y)}{(x_4 - x_2)(y_1 - y_4)}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

The elemental coordinates are *s* and *t*.

The interpolation functions $N_i(s, t)$ are identical to the rectangular element interpolation functions.

$$N_1(s,t) = \frac{(1-s)(1-t)}{4} \qquad N_2(s,t) = \frac{(1+s)(1-t)}{4}$$

$$N_3(s,t) = \frac{(1+s)(1+t)}{4} \qquad N_4(s,t) = \frac{(1-s)(1+t)}{4}$$

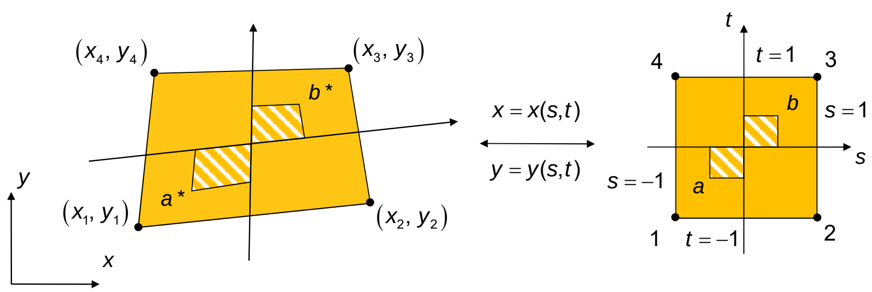
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

The transformation from global coordinates (*x, y*) to elemental or **parental coordinates** (*s, t*) is accomplished by using the following coordinate mapping:

$$x_e(s,t) = x_1N_1(s,t) + x_2N_2(s,t) + x_3N_3(s,t) + x_4N_4(s,t) = \mathbf{x_e}^{\mathbf{T}}\mathbf{N} = \mathbf{x_e}\mathbf{N}^{\mathbf{T}}$$

$$y_e(s,t) = y_1N_1(s,t) + y_2N_2(s,t) + y_3N_3(s,t) + y_4N_4(s,t) = \mathbf{y_e}^{\mathbf{T}}\mathbf{N} = \mathbf{y_e}\mathbf{N}^{\mathbf{T}}$$

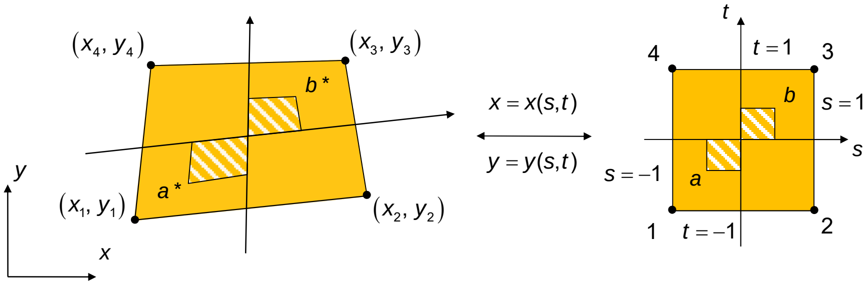


TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

The interpolation functions for a quadrilateral element are referred as **shape functions** since these expressions define the shape of the element.

Let's look at how the lines and points in the parent element (*s, t*) are mapped or transformed to the global system (*x, y*).

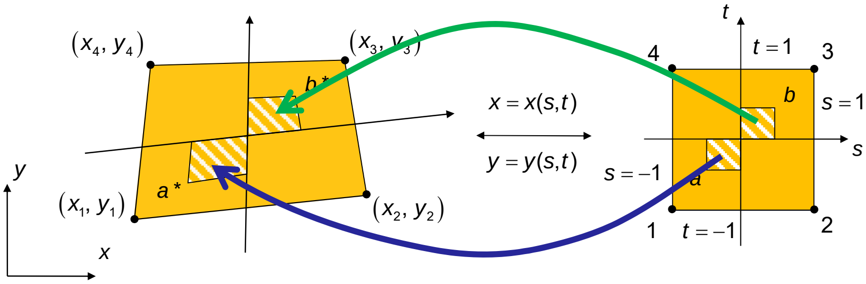


TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

In general straight lines in the *s-t* coordinate system are mapped into straight lines in the quadrilateral element.

For example, points *a* and *b* are mapped to points *a** and *b** in the quadrilateral.

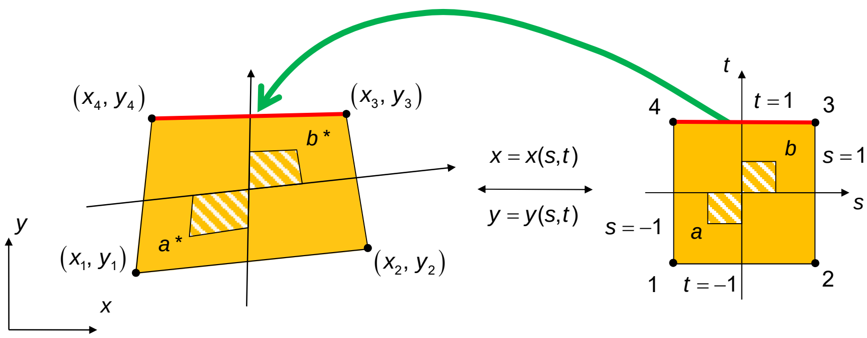


TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

The straight line t = 1 in the parent element, connecting nodes 3 and 4, may be mapped into the quadrilateral as:

$$x = \frac{(x_3 + x_4) + (x_3 - x_4)s}{2} \qquad y = \frac{(y_3 + y_4) + (y_3 - y_4)s}{2}$$



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

In order to understand the nature of the transformation, the chain rule is used to form the differential relationship:

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\frac{\partial y}{\partial t}$$

In matrix notation these derivatives may be written as:

$$\frac{\partial}{\partial \vec{\mathbf{s}}} = \mathbf{J}\frac{\partial}{\partial \vec{\mathbf{x}}} \qquad \frac{\partial}{\partial \vec{\mathbf{s}}} = \left\langle \frac{\partial}{\partial s} \quad \frac{\partial}{\partial t} \right\rangle^T \qquad \frac{\partial}{\partial \vec{\mathbf{x}}} = \left\langle \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \right\rangle^T$$

where

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \qquad |\mathbf{J}| = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

The determinant of the Jacobian matrix, **|J|**, is a test of the invertibility of the transformation $x = x(s, t)$ and $y = y(s, t)$.

When **|J|** is positive everywhere in a region, the transformation may be inverted to determine $s = s(x, y)$ and $t = t(x, y)$.

This means that for a given point (x, y) in the quadrilateral there is a unique corresponding point (s, t) in the parent element.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

The elements of the Jacobian matrix may be computed as:

$$\frac{\partial x}{\partial s} = \frac{(x_2 - x_1)(1-t) + (x_3 - x_4)(1+t)}{4}$$

$$\frac{\partial y}{\partial s} = \frac{(y_2 - y_1)(1-t) + (y_3 - y_4)(1+t)}{4}$$

$$\frac{\partial x}{\partial t} = \frac{(x_3 - x_2)(1+s) + (x_4 - x_1)(1-s)}{4}$$

$$\frac{\partial y}{\partial t} = \frac{(y_3 - y_2)(1+s) + (y_4 - y_1)(1-s)}{4}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Therefore, **|J|** has the form: $|\mathbf{J}| = j_1 + j_2 s + j_3 t + j_4 st$

where the constants j_i are functions of the coordinates of the nodes (functions of the geometry of the element).

The constant j_4 is actually zero so that **|J|** is a linear function of s and t.

The determinant **|J|** geometrically represents the relationship between the differential area $dA(s, t) = ds\, dt$ and the differential area $dA(x, y)$.

$$dA(x,y) = |\mathbf{J}|\, dA(s,t) = |\mathbf{J}|\, ds\, dt$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

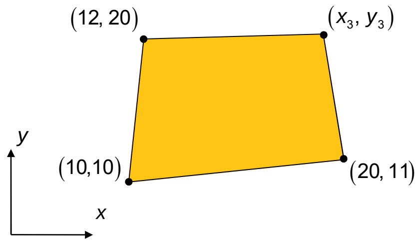
FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

PROBLEM #22 - Show that the constant j_4 defining the determinant of the Jacobian **|J|** is zero.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - Consider the quadrilateral element given below.



The elements of the Jacobian matrix may be calculated as:

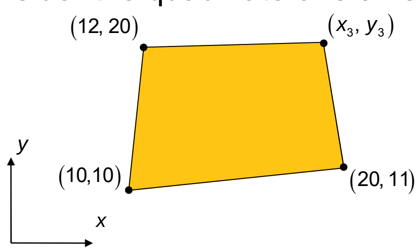
$$\frac{\partial x}{\partial s} = \frac{(x_2 - x_1)(1-t) + (x_3 - x_4)(1+t)}{4} = \frac{(t+1)x_3 - 22t - 2}{4}$$

$$\frac{\partial x}{\partial t} = \frac{(x_3 - x_2)(1+s) + (x_4 - x_1)(1-s)}{4} = \frac{(s+1)x_3 - 22s - 18}{4}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - Consider the quadrilateral element given below.



The elements of the Jacobian matrix may be calculated as:

$$\frac{\partial y}{\partial s} = \frac{(y_2 - y_1)(1-t) + (y_3 - y_4)(1+t)}{4} = \frac{(t+1)y_3 - 21t - 19}{4}$$

$$\frac{\partial y}{\partial t} = \frac{(y_3 - y_2)(1+s) + (y_4 - y_1)(1-s)}{4} = \frac{(s+1)y_3 - 21s - 1}{4}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - The determinant is:

$$|\mathbf{J}| = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}$$

$$= \left[\frac{(t+1)x_3 - 22t - 2}{4}\right]\left[\frac{(s+1)y_3 - 21s - 1}{4}\right] - \left[\frac{(s+1)x_3 - 22s - 18}{4}\right]\left[\frac{(t+1)y_3 - 21t - 19}{4}\right]$$

$$|\mathbf{J}| = \frac{(2t - 10s - 8)y_3 + (s - 10t - 9)x_3 + 178t + 188s + 170}{8}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

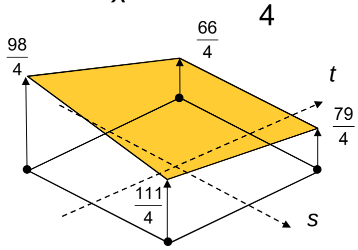
FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - The determinant is:

$$|\mathbf{J}| = \frac{(2t - 10s - 8)y_3 + (s - 10t - 9)x_3 + 178t + 188s + 170}{8}$$

Consider the location of node three as x_3 = 19 and y_3 = 22, then the transformation is given by:

$$x = \frac{61 + 17s + t - 3st}{4} \qquad y = \frac{63 + 3s + 21t + st}{4}$$



$$|\mathbf{J}| = \frac{13s - 32t + 177}{8}$$

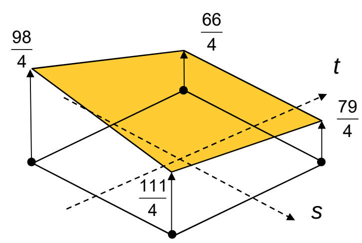
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - Over the interval -1 ≤ *s* ≤ 1 and -1 ≤ *t* ≤ 1 the determinant of the Jacobian, **|J|**, is positive, indicating that there are no problems with the mapping from *s*-*t* to *x*-*y*.

The area of the quadrilateral may be computed as:

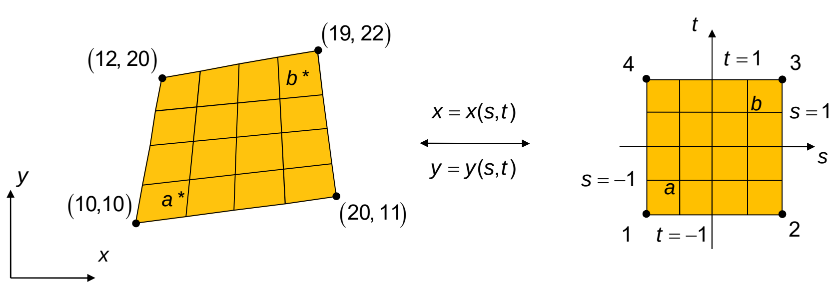
$$A = \iint_{A_e} dx\,dy = \int_{-1}^{1}\int_{-1}^{1} |\mathbf{J}|\,ds\,dt = \int_{-1}^{1}\int_{-1}^{1}\left(\frac{13s - 32t + 177}{8}\right)ds\,dt = \frac{177}{2}$$



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - The parent element in the *s*-*t* system and the shape of the quadrilateral element are shown in the figure below:

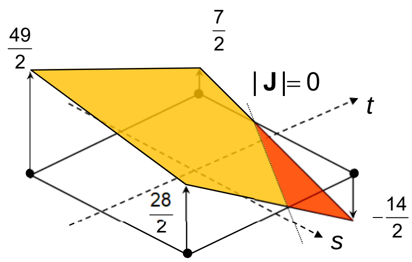


TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

<u>Example</u> - Consider the location of node three as x_3 = 16 and y_3 = 12, then the transformation is given by:

$$x = \frac{29 + 7s - t - 3st}{2} \qquad y = \frac{53 - 7s + 11t - 9st}{4}$$



$$|\mathbf{J}| = -\frac{42s + 21t - 35}{4}$$

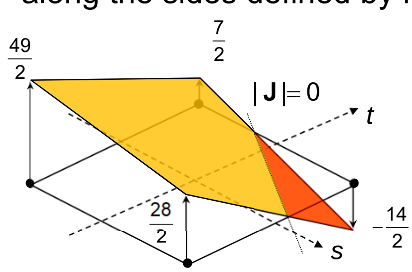
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

<u>Example</u> - Over the interval $-1 \leq s \leq 1$ and $-1 \leq t \leq 1$ the determinant of the Jacobian, **|J|**, is ***<u>not</u>*** always positive.

In fact, **|J|** is negative for $(2s + t) \geq 5/3$. For points (s, t) whose values satisfy $(2s + t) \geq 5/3$, the mapping locates them outside of the quadrilateral.

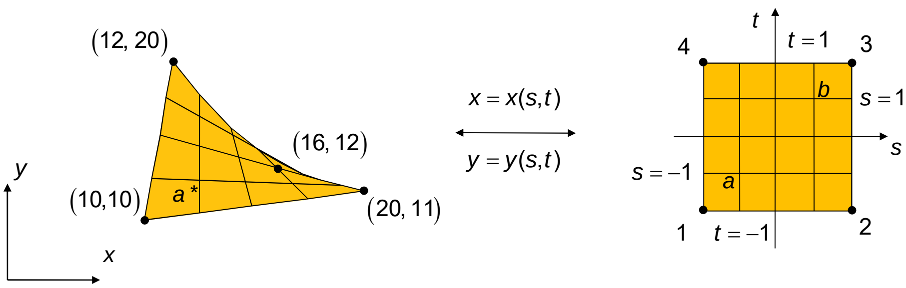
The problem is associated with the fact the region is convex along the sides defined by nodes 2, 3, and 4.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - The parent element in the *s*-*t* system and the shape of the quadrilateral element are shown in the figure below:

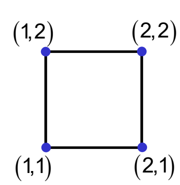


TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - Let's compute the determinant |[J]| for a square global element with the following coordinates:

$$\{X_c\}^T = [1 \quad 2 \quad 2 \quad 1] \qquad \{Y_c\}^T = [1 \quad 1 \quad 2 \quad 2]$$



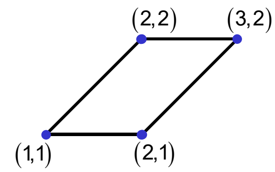
$$\left|[J]\right| = \frac{1}{8}\{X_c\}^T \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \{Y_c\} = \frac{1}{4} = \frac{A}{4}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

<u>Example</u> - Let's compute the determinant |[J]| for a skewed global element with the following coordinates:

$$\{X_c\}^T = [1 \quad 2 \quad 3 \quad 2] \qquad \{Y_c\}^T = [1 \quad 1 \quad 2 \quad 2]$$



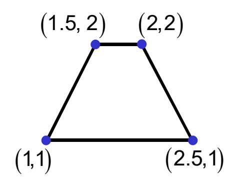
$$\left|[J]\right| = \frac{1}{8}\{X_c\}^T \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \{Y_c\} = \frac{1}{4} = \frac{A}{4}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

<u>Example</u> - Let's compute the determinant |[J]| for a trapezoidal global element with the following coordinates:

$$\{X_c\}^T = [1 \quad 2.5 \quad 2 \quad 1.5] \qquad \{Y_c\}^T = [1 \quad 1 \quad 2 \quad 2]$$



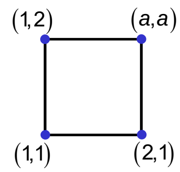
$$\left|[J]\right| = \frac{1}{8}\{X_c\}^T \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \{Y_c\} = \frac{2-t}{8}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - Let's compute the determinant |[J]| for a global element with the following coordinates:

$$\{X_c\}^T = \begin{bmatrix} 1 & 2 & a & 1 \end{bmatrix} \qquad \{Y_c\}^T = \begin{bmatrix} 1 & 1 & a & 2 \end{bmatrix}$$



$$\left|[J]\right| = \frac{1}{8}\{X_c\}^T \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \{Y_c\}$$

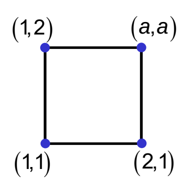
$$= \frac{2a - 2s - 2t + as + at - 2}{8}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - Let's compute the determinant |[J]| for a global element with the following coordinates:

$$\{X_c\}^T = \begin{bmatrix} 1 & 2 & a & 1 \end{bmatrix} \qquad \{Y_c\}^T = \begin{bmatrix} 1 & 1 & a & 2 \end{bmatrix}$$



Let's evaluate the |[J]| at (s, t) = (1, 1) :

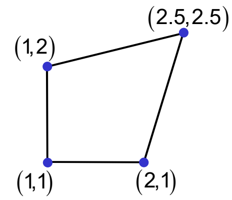
$$\left|[J]\right| = \frac{2a - 2s - 2t + as + at - 2}{8} = \frac{4a - 6}{8}$$

For the |[J]| to be positive, $a > 3/2$. If $a < 3/2$, then the |[J]| is negative.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

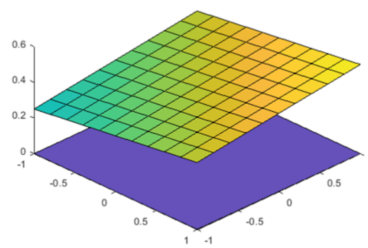
FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

<u>Example</u> - Let's compute the determinant |[J]| for a global element with the following coordinates:



Let's assume *a* > 3/2, say *a* = 2.5, then the |[J]| is:

$$\left\|[J]\right\| = \frac{1}{16}(6 + s + t)$$

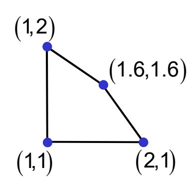


|[J]| is positive over the entire element.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

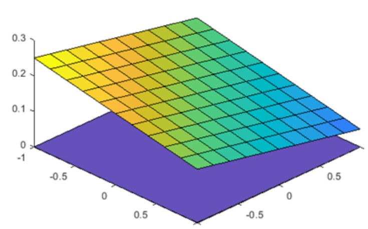
FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

<u>Example</u> - Let's compute the determinant |[J]| for a global element with the following coordinates:



Let's assume *a* > 3/2, say *a* = 1.6, then the |[J]| is:

$$\left\|[J]\right\| = \frac{1}{20}(3 - t - s)$$

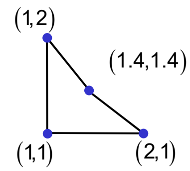


|[J]| is positive over the entire element.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

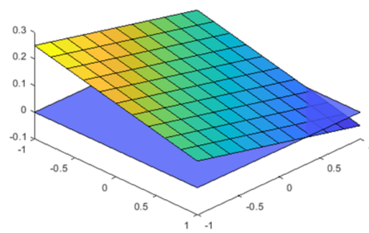
Example - Let's compute the determinant |[J]| for a global element with the following coordinates:



Let's assume *a* < 3/2, say *a* = 1.4, then the |[J]| is:

$$\|[J]\| = \frac{1}{40}(4 - 3t - 3s)$$

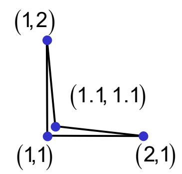
|[J]| is negative at (*s*, *t*) = (1, 1).



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

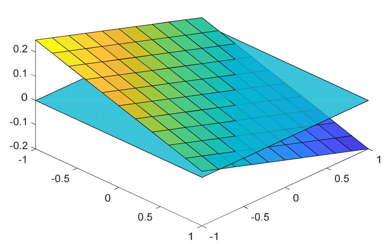
Example - Let's compute the determinant |[J]| for a global element with the following coordinates:



Let's assume *a* < 3/2, say *a* = 1.1, then the |[J]| is:

$$\|[J]\| = \frac{1}{80}(2 - 9t - 9s)$$

|[J]| is negative at (*s*, *t*) = (1, 1).

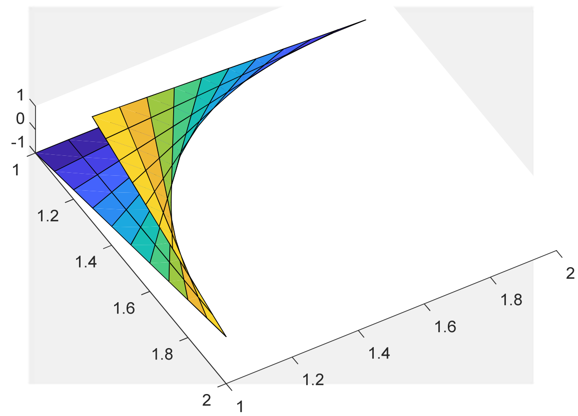


TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

When the |[J]| is negative the mapping between local coordinates to global coordinates is not 1-to-1.

Here is a plot of the mapping as the vales of *a* range from 2 to 1.1:



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

The elemental matrices for the Poisson problem are:

$$\mathbf{k_e} = \iint_A \left[\frac{\partial \mathbf{N}}{\partial x}\frac{\partial \mathbf{N}^T}{\partial x} + \frac{\partial \mathbf{N}}{\partial y}\frac{\partial \mathbf{N}^T}{\partial y} \right] dA \qquad \mathbf{f_e} = \iint_{A_e} \mathbf{N} f \, dA$$

$$\mathbf{a_e} = \int_{\gamma_{2e}} \mathbf{N} \alpha \mathbf{N}^T \, ds \qquad \mathbf{h_e} = \int_{\gamma_{2e}} \mathbf{N} h \, ds$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

The development of both the $\mathbf{k_e}$ and $\mathbf{f_e}$ terms is identical to that presented for rectangular elements:

$$\mathbf{k_e} = \int_{-1}^{1}\int_{-1}^{1}\left[\Delta \mathbf{JJ} \Delta^{T}\right] ds\, dt \qquad \mathbf{f_e} = \int_{-1}^{1}\int_{-1}^{1} \mathbf{N} f \left|\mathbf{J}\right| ds\, dt$$

where $\mathbf{JJ} = (\mathbf{J_1}^T\mathbf{J_1} + \mathbf{J_2}^T\mathbf{J_2})|\mathbf{J}|$. The values of the matrix $\Delta^{\mathbf{T}}$ may be computed as:

$$\Delta^{\mathbf{T}} = \begin{bmatrix} \dfrac{\partial \mathbf{N}}{\partial s} \\ \dfrac{\partial \mathbf{N}}{\partial t} \end{bmatrix} = \begin{bmatrix} -\dfrac{1-t}{4} & \dfrac{1-t}{4} & \dfrac{1+t}{4} & -\dfrac{1+t}{4} \\ -\dfrac{1-s}{4} & -\dfrac{1+s}{4} & \dfrac{1+s}{4} & \dfrac{1-s}{4} \end{bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Recall, $\mathbf{J_1}$ and $\mathbf{J_2}$ are the first and second rows of the inverse of the Jacobian matrix.

$$\mathbf{J}^{-1} = \frac{1}{|\mathbf{J}|}\begin{bmatrix} \dfrac{\partial y}{\partial t} & -\dfrac{\partial y}{\partial s} \\ -\dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial s} \end{bmatrix} \qquad \mathbf{J_1} = \frac{1}{|\mathbf{J}|}\left\langle \frac{\partial y}{\partial t} \quad -\frac{\partial y}{\partial s} \right\rangle \qquad \mathbf{J_2} = \frac{1}{|\mathbf{J}|}\left\langle -\frac{\partial x}{\partial t} \quad \frac{\partial x}{\partial s} \right\rangle$$

The elements of the inverse of the Jacobian are:

$$\frac{\partial x}{\partial s} = \frac{(x_2 - x_1)(1-t) + (x_3 - x_4)(1+t)}{4} \qquad \frac{\partial y}{\partial s} = \frac{(y_2 - y_1)(1-t) + (y_3 - y_4)(1+t)}{4}$$

$$\frac{\partial x}{\partial t} = \frac{(x_3 - x_2)(1+s) + (x_4 - x_1)(1-s)}{4} \qquad \frac{\partial y}{\partial t} = \frac{(y_3 - y_2)(1+s) + (y_4 - y_1)(1-s)}{4}$$

$$|\mathbf{J}| = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

For a general quadrilateral element, the 2 x 2 matrix **JJ** is a function of *s* and *t* with the Jacobian **|J|** in the denominator.

The resulting expression for **JJ** is very difficult to evaluate exactly and the integrations are usually done numerically.

Therefore, the terms $\mathbf{k_e}$ and $\mathbf{f_e}$ may be cast in the following form:

$$I = \int_{-1}^{1}\int_{-1}^{1} G(s,t)\,ds\,dt$$

where **G**(*s,t*) is a complicated function of the variables *s* and *t*.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

In principle, it may be possible to evaluate the $\mathbf{f_e}$ terms, however, numerical integration is typically more practical.

For the $\mathbf{k_e}$ terms, the appearance of the Jacobian **|J|** in the integrand generally indicates the use of numerical quadrature.

It is fortunate that the limits of the integral conform to the format used in Gauss-Legendre quadrature.

Since the integrations are in both the *s* and *t*-directions, one approach is to apply Gaussian quadrature separately in each direction.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Therefore, the general expression for $\mathbf{k_e}$ and $\mathbf{f_e}$ an *N*-term Gaussian quadrature is:

$$\mathbf{k_e} = \int_{-1}^{1}\int_{-1}^{1}\left[\Delta \mathbf{JJ} \Delta^T\right] ds\, dt \approx \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \Delta_{ij} \mathbf{JJ}_{ij} \Delta_{ij}^T$$

$$\mathbf{f_e} = \int_{-1}^{1}\int_{-1}^{1} \mathbf{N} f \left|\mathbf{J}\right| ds\, dt \approx \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j N\left(s_i, t_j\right) f\left[x\left(s_i, t_j\right), y\left(s_i, t_j\right)\right] \left|\mathbf{J}\right|_{ij}$$

where the subscripts *ij* refer to the evaluation of the quantity at s_i and t_j.

$$\Delta_{ij} = \Delta\left(s_i, t_j\right) \qquad \mathbf{JJ}_{ij} = \mathbf{JJ}\left(s_i, t_j\right)$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Consider the integrals $\mathbf{a_e}$ and $\mathbf{h_e}$:

$$\mathbf{a_e} = \int_{\gamma_{2e}} \mathbf{N} \alpha \mathbf{N^T}\, ds \qquad \mathbf{h_e} = \int_{\gamma_{2e}} \mathbf{N} h\, ds$$

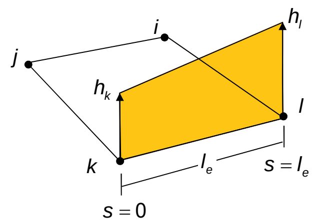
where the integration is along a boundary segment of the element.

Since, the integration is computed along a single side of the quadrilateral element, the quadrilateral shape functions reduce to linear functions.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

The results for this integration are identical to expressions obtained for linear triangular and rectangular elements.



$$N_k = 1 - \frac{s}{l_e}$$

$$N_l = \frac{s}{l_e}$$

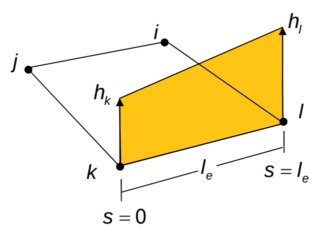
$$\mathbf{a_e} \approx \int_{\gamma_{2e}} \begin{Bmatrix} N_k \\ N_l \end{Bmatrix} \left(\alpha_k N_k + \alpha_l N_l\right) \langle N_k \quad N_l \rangle \, ds$$

$$\mathbf{h_e} \approx \int_{\gamma_{2e}} \begin{Bmatrix} N_k \\ N_l \end{Bmatrix} \left(h_k N_k + h_l N_l\right) ds$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Evaluating the integrals, the $\mathbf{a_e}$ and $\mathbf{h_e}$ terms reduce to:



$$N_k = 1 - \frac{s}{l_e}$$

$$N_l = \frac{s}{l_e}$$

$$\mathbf{a_e} \approx \frac{l_e}{12} \begin{bmatrix} 3\alpha_k + \alpha_l & \alpha_k + \alpha_l \\ \alpha_k + \alpha_l & \alpha_k + 3\alpha_l \end{bmatrix} \qquad \mathbf{h_e} \approx \frac{l_e}{6} \begin{Bmatrix} 2h_k + h_l \\ h_k + 2h_l \end{Bmatrix}$$

The resulting 2 x 2 $\mathbf{a_e}$ elemental stiffness matrix and the 2 x 1 $\mathbf{h_e}$ elemental load vector contribute to the global system equations if the element has a side as part of the boundary.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

The global system equations are composed from the following summations:

$$\mathbf{K_G} = \sum_e \mathbf{k_G} + \sum_e {}'\mathbf{a_G} \qquad \mathbf{F_G} = \sum_e \mathbf{f_G} + \sum_e {}'\mathbf{h_G}$$

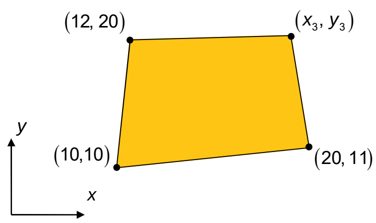
The resulting system equations are, in matrix form, given as:

$$\boxed{\mathbf{K_G u_G} = \mathbf{F_G}}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - Consider the quadrilateral element given below.



Calculate the values of the stiffness matrix $\mathbf{k_e}$ for the above element when x_3 = 19 and y_3 = 21.

$$\mathbf{k_e} = \int_{-1}^{1}\int_{-1}^{1}\left[\Delta \mathbf{J} \mathbf{J} \Delta^T\right] ds\,dt \qquad \mathbf{f_e} = \int_{-1}^{1}\int_{-1}^{1} \mathbf{N} f \left|\mathbf{J}\right| ds\,dt$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - Recall $\mathbf{JJ} = (\mathbf{J}_1^T\mathbf{J}_1 + \mathbf{J}_2^T\mathbf{J}_2)|\mathbf{J}|$. The values of the matrix Δ^T may be computed as:

$$\Delta^{\mathbf{T}} = \begin{bmatrix} \dfrac{\partial \mathbf{N}}{\partial s} \\ \dfrac{\partial \mathbf{N}}{\partial t} \end{bmatrix} = \begin{bmatrix} -\dfrac{1-t}{4} & \dfrac{1-t}{4} & \dfrac{1+t}{4} & -\dfrac{1+t}{4} \\ -\dfrac{1-s}{4} & -\dfrac{1+s}{4} & \dfrac{1+s}{4} & \dfrac{1-s}{4} \end{bmatrix}$$

where $\mathbf{J}_1$ and $\mathbf{J}_2$ are the first and second rows of the inverse of the Jacobian matrix.

$$\mathbf{J}^{-1} = \frac{1}{|\mathbf{J}|}\begin{bmatrix} \dfrac{\partial y}{\partial t} & -\dfrac{\partial y}{\partial s} \\ -\dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial s} \end{bmatrix} \qquad \mathbf{J}_1 = \frac{1}{|\mathbf{J}|}\left\langle \frac{\partial y}{\partial t} \quad -\frac{\partial y}{\partial s} \right\rangle \qquad \mathbf{J}_2 = \frac{1}{|\mathbf{J}|}\left\langle -\frac{\partial x}{\partial t} \quad \frac{\partial x}{\partial s} \right\rangle$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - The elements of the inverse of the Jacobian are:

$$\frac{\partial x}{\partial s} = \frac{(x_2 - x_1)(1-t) + (x_3 - x_4)(1+t)}{4}$$

$$\frac{\partial y}{\partial s} = \frac{(y_2 - y_1)(1-t) + (y_3 - y_4)(1+t)}{4}$$

$$\frac{\partial x}{\partial t} = \frac{(x_3 - x_2)(1+s) + (x_4 - x_1)(1-s)}{4}$$

$$\frac{\partial y}{\partial t} = \frac{(y_3 - y_2)(1+s) + (y_4 - y_1)(1-s)}{4}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - The calculations are performed by computing the value of each component of Δ, Δ^T, $\mathbf{J_1}$, $\mathbf{J_2}$, and $|\mathbf{J}|$ matrices at each Gauss point, performing the matrix multiplications, and finally summing the results for all the Gauss points.

The use of numerical quadrature at this point in the formulations may generate some error.

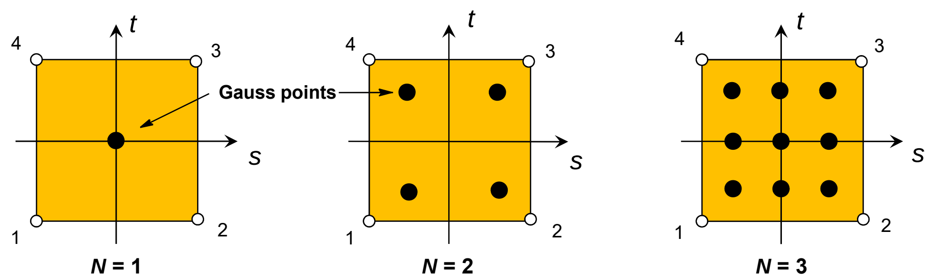
However, if enough Gauss points are used the numerical results are quite accurate when compared to the exact value for the integrations.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - A two-dimensional quadrature using N = 2 is actually 4 Gauss points and using N = 4 is 16 quadrature points.

$$\mathbf{k_e} = \int_{-1}^{1}\int_{-1}^{1}\left[\Delta \mathbf{J}\mathbf{J}\Delta^T\right] ds\, dt \approx \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \Delta_{ij}\mathbf{J}\mathbf{J}_{ij}\Delta_{ij}^T$$



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - This fact is evident in the following calculations using Gaussian quadrature with *N* = 1, 2, 3, and 4.

$N = 1$

$$\mathbf{k_e} = \begin{bmatrix} 0.409605 & -0.101695 & -0.409605 & 0.101695 \\ & 0.635593 & 0.101695 & -0.635593 \\ & & 0.409605 & -0.101695 \\ \textit{symmetric} & & & 0.635593 \end{bmatrix}$$

Exact

$$\mathbf{k_e} = \begin{bmatrix} 0.551339 & -0.220106 & -0.233781 & -0.097452 \\ & 0.734519 & 0.045195 & -0.469217 \\ & & 0.627714 & -0.348738 \\ \textit{symmetric} & & & 0.915407 \end{bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - This fact is evident in the following calculations using Gaussian quadrature with *N* = 1, 2, 3, and 4.

$N = 2$

$$\mathbf{k_e} = \begin{bmatrix} 0.550481 & -0.219389 & -0.234846 & -0.096246 \\ & 0.733920 & 0.044306 & -0.470225 \\ & & 0.626339 & -0.347241 \\ \textit{symmetric} & & & 0.913712 \end{bmatrix}$$

Exact

$$\mathbf{k_e} = \begin{bmatrix} 0.551339 & -0.220106 & -0.233781 & -0.097452 \\ & 0.734519 & 0.045195 & -0.469217 \\ & & 0.627714 & -0.348738 \\ \textit{symmetric} & & & 0.915407 \end{bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

<u>Example</u> - This fact is evident in the following calculations using Gaussian quadrature with *N* = 1, 2, 3, and 4.

$N = 3$

$$\mathbf{k_e} = \begin{bmatrix} 0.551333 & -0.220101 & -0.233790 & -0.097442 \\ & 0.734515 & 0.045188 & -0.469225 \\ & & 0.627704 & -0.348726 \\ \textit{symmetric} & & & 0.915394 \end{bmatrix}$$

Exact

$$\mathbf{k_e} = \begin{bmatrix} 0.551339 & -0.220106 & -0.233781 & -0.097452 \\ & 0.734519 & 0.045195 & -0.469217 \\ & & 0.627714 & -0.348738 \\ \textit{symmetric} & & & 0.915407 \end{bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

<u>Example</u> - This fact is evident in the following calculations using Gaussian quadrature with *N* = 1, 2, 3, and 4.

$N = 4$

$$\mathbf{k_e} = \begin{bmatrix} 0.551339 & -0.220106 & -0.233781 & -0.097452 \\ & 0.734519 & 0.045195 & -0.469217 \\ & & 0.627714 & -0.348738 \\ \textit{symmetric} & & & 0.915407 \end{bmatrix}$$

Exact

$$\mathbf{k_e} = \begin{bmatrix} 0.551339 & -0.220106 & -0.233781 & -0.097452 \\ & 0.734519 & 0.045195 & -0.469217 \\ & & 0.627714 & -0.348738 \\ \textit{symmetric} & & & 0.915407 \end{bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - The results for N = 1 are not very accurate. Since only one Gauss point is used in the quadrature, this result is not unexpected.

The values for $\mathbf{k_e}$ using Gaussian quadrature with N = 2 and 3 indicate that the accuracy of the evaluations increases as the number of Gauss points increases.

The results using N = 4 are exact to five or six decimal places.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

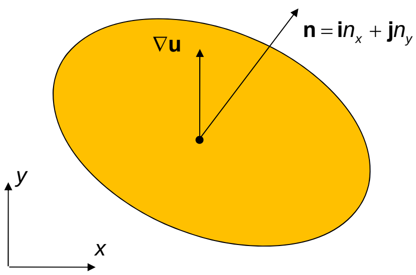
Example - **Derived Variables** - The derived variables for the Poisson or Laplace's equation are the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$.

These terms are computed as functions of position in an element using the shape functions and the nodal values.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - **Derived Variables** - The partial derivatives may be combined to give the normal or directional derivative:



$$\frac{\partial u}{\partial n} = \vec{\mathbf{n}} \cdot \nabla \vec{\mathbf{u}} = n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y}$$

$$\frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{s}}} = \mathbf{J} \frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{x}}} \qquad \frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{x}}} = \mathbf{J}^{-1} \frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{s}}}$$

$$\frac{\partial u}{\partial x} = \mathbf{J_1} \frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{s}}} \qquad \frac{\partial u}{\partial y} = \mathbf{J_2} \frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{s}}}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

FOUR-NODE QUADRILATERAL ELEMENTS (Q4)

Example - **Derived Variables** - Recall $\mathbf{J_1}$ and $\mathbf{J_2}$ are the first and second rows of $\mathbf{J}^{-1}$, respectively.

Recall that $\partial \vec{\mathbf{u}} / \partial \vec{\mathbf{s}}$ is:

$$\frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{s}}} = \begin{Bmatrix} \dfrac{\partial u}{\partial s} \\ \dfrac{\partial u}{\partial t} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial [\mathbf{N^T u_e}]}{\partial s} \\ \dfrac{\partial [\mathbf{N^T u_e}]}{\partial t} \end{Bmatrix} = \Delta^{\mathbf{T}} \mathbf{u_e}$$

Therefore:

$$\frac{\partial u}{\partial x} = \mathbf{J_1} \Delta^{\mathbf{T}} \mathbf{u_e} \qquad \frac{\partial u}{\partial y} = \mathbf{J_2} \Delta^{\mathbf{T}} \mathbf{u_e}$$

The directional or normal derivative is:

$$\frac{\partial u}{\partial n} = \left(n_x \mathbf{J_1} + n_y \mathbf{J_2} \right) \Delta^{\mathbf{T}} \mathbf{u_e}$$

End of Quadrilateral elements (Q4)