TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

RECTANGULAR ELEMENTS - In some applications, it may be more desirable to use an elemental representation of the domain that has four sides, either rectangular or quadrilateral in shape.

Consider a four-node rectangular element with parallel sides defined as:

\[
\begin{align*}
N_1(x,y) &= \alpha + \beta x + \gamma y + \delta xy \\
N_2(x,y) &= \alpha + \beta x + \gamma y + \delta xy \\
N_3(x,y) &= \alpha + \beta x + \gamma y + \delta xy \\
N_4(x,y) &= \alpha + \beta x + \gamma y + \delta xy
\end{align*}
\]

We will use a bilinear distribution of the \( u \) over the element, therefore:

\[
u(x,y) = \alpha + \beta x + \gamma y + 3 \delta xy
\]

The elemental interpolations for a rectangular element are:

\[
\begin{align*}
N_1(x,y) &= \frac{(x-x_1)(y-y_1)}{(x_2-x_1)(y_2-y_1)} \\
N_2(x,y) &= \frac{(x-x_2)(y-y_2)}{(x_1-x_2)(y_1-y_2)} \\
N_3(x,y) &= \frac{(x-x_3)(y-y_3)}{(x_4-x_3)(y_4-y_3)} \\
N_4(x,y) &= \frac{(x-x_4)(y-y_4)}{(x_1-x_4)(y_1-y_4)}
\end{align*}
\]

The shape functions are visually deceiving. There is no curvature in directions parallel to any side; however, there is a twist due to the \( xy \) term in the element representation.
Two-dimensional Boundary Value Problems

Rectangular Elements - One very important advantage of this type of interpolation is that the derivatives are no longer constant over an element. The derivatives in \( x \) and \( y \) are:

\[
\frac{\partial u}{\partial x} = \left( u_x - u_x \right)(y - y_1) + \left( u_x - u_y \right)(y - y_1)
\]

\[
\frac{\partial u}{\partial y} = \left( u_x - u_y \right)(x - x_1) + \left( u_x - u_y \right)(x - x_1)
\]

Rectangular shape functions also suffer from interelement discontinuities, in particular, the values of normal derivatives at any edges.

Two-dimensional Boundary Value Problems

Rectangular Elements - The shape function may be written in terms of the local coordinates \( \xi \) and \( \eta \):

\[
N_i(\xi, \eta) = \left\{ \frac{(a - \xi)(b - \eta)}{4ab}, \frac{(a + \xi)(b + \eta)}{4ab}, \frac{(a - \xi)(b + \eta)}{4ab}, \frac{(a + \xi)(b - \eta)}{4ab} \right\}
\]

The shape functions may be written in terms of nondimensional local coordinates \( s \) and \( t \), where \( s = \xi/a \), and \( t = \eta/b \):

\[
N_i(s, t) = \left\{ \frac{(1 - s)(1 - t)}{4}, \frac{(1 + s)(1 - t)}{4}, \frac{(1 - s)(1 + t)}{4}, \frac{(1 + s)(1 + t)}{4} \right\}
\]

Two-dimensional Boundary Value Problems

Rectangular Elements - The dimensionless shape functions are defined over a unit square \(-1 \leq s \leq 1, -1 \leq t \leq 1 \).

The variation of the global coordinates \( x \) and \( y \) may be expressed in terms of the shape functions as:

\[
x = \sum x_i N_i, \quad y = \sum y_i N_i
\]

Two-dimensional Boundary Value Problems

Rectangular Elements - This integral may be transformed into the local coordinate space using \( \xi = x - x_0 \) and \( \eta = y - y_0 \):

\[
\int_{A_e} F(x,y) \, dA = \int_{\xi, \eta} \tilde{F}(x,y) \tilde{\xi} \tilde{\eta} \, d\xi \, d\eta
\]

where \( \tilde{F}(x, y, \xi, \eta) \) is the Jacobian of the transformation.

Using the transformation \( \xi = as \) and \( \eta = bt \) the integral may be written as:

\[
\int_{A_e} F(x,y) \, dA = \int_{A_e} F(x_0 + as, y_0 + bt) \frac{A_e}{4} \frac{\partial (x,y)}{\partial (\xi,\eta)} \, ds \, dt
\]

where the Jacobian \( \frac{\partial (x,y)}{\partial (\xi,\eta)} \) has a value of \( A_e/4 \).

Two-dimensional Boundary Value Problems

Rectangular Elements - The elemental matrices for the Poisson problem are:

\[
k_e = \int_A \left[ \frac{\partial N_i/\partial x}{\partial x} \frac{\partial N_j/\partial y}{\partial y} \right] \, dA, \quad f_e = \int_A \tilde{F}(x,y) \, dA
\]

\[
a_e = \int_A N_i N_j \, ds, \quad h_e = \int_A n_i \, ds
\]

Let's first consider how to handle integration over a typical element area \( A_e \). For example:

\[
\int_{A_e} F(x,y) \, dA = \int_{\xi, \eta} F(x,y) \, dA
\]
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS
RECTANGULAR ELEMENTS - Evaluation of \( k_e \)

In matrix form the above expression may be written as:

\[
\begin{bmatrix}
\frac{\partial}{\partial s} \\
\frac{\partial}{\partial t}
\end{bmatrix} = J_s \begin{bmatrix}
\frac{\partial x}{\partial s} \\
\frac{\partial x}{\partial t} \\
\frac{\partial y}{\partial s} \\
\frac{\partial y}{\partial t}
\end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}
\]

The matrix form of the transformation may be inverted.

\[
\begin{bmatrix}
\frac{\partial}{\partial s} \\
\frac{\partial}{\partial t}
\end{bmatrix} \rightarrow J_s^{-1} \begin{bmatrix}
\frac{\partial}{\partial s} \\
\frac{\partial}{\partial t}
\end{bmatrix}
\]

The matrix form of the transformation may be inverted.

\[
\begin{bmatrix}
\frac{\partial}{\partial s} \\
\frac{\partial}{\partial t}
\end{bmatrix} = J_s \begin{bmatrix}
\frac{\partial x}{\partial s} \\
\frac{\partial x}{\partial t} \\
\frac{\partial y}{\partial s} \\
\frac{\partial y}{\partial t}
\end{bmatrix}
\]

where \( J_s \) and \( J_t \) are the first and second rows of \( J^{-1} \).

Substituting all the pieces of the transformation in the \( k_e \) terms gives:

\[
k_e = \int \int \frac{1}{4} \left( J_s^T J_s + J_t^T J_t \right) \left( \frac{\partial \phi}{\partial s} \right) ds dt
\]

Therefore, the partial derivatives of the shape functions may be written as:

\[
\frac{\partial \phi}{\partial s} = J_1 \frac{\partial x}{\partial s} + J_2 \frac{\partial x}{\partial t}
\]

where \( J_1 \) and \( J_2 \) are the first and second rows of \( J^{-1} \).

Transforming the integral into the non-dimensional coordinates \( (s, t) \) yields:

\[
f_s = \frac{1}{4} \int \int \frac{\partial \phi}{\partial s} \frac{\partial \phi}{\partial t} ds dt
\]

The resulting 4 x 4 elemental stiffness matrix \( k_e \) is a 4 x 4 that is added to the global system stiffness matrix \( K_e \) at the corresponding nodal locations.

If the element is square, then \( a = b \), then \( k_e \) becomes:

\[
k_e = \frac{1}{4} \begin{bmatrix}
2 & -2 & -1 & 1 \\
-2 & 2 & 1 & -1 \\
-1 & 1 & 2 & -2 \\
1 & -1 & -2 & 2
\end{bmatrix}
\]

The resulting 4 x 1 elemental load vector contributes to the global system equations at those locations corresponding to the four nodes defining the element.

For a general function \( f(x, y) \) we will assume the function \( f \) varies linearly over the element, \( f(x, y) = N^T f \), where the vector \( f \) contains values of the function \( f \) at the node points.

With this assumption the integral becomes:

\[
f_s = \int \int \frac{\partial \phi}{\partial s} \frac{\partial \phi}{\partial t} ds dt = \left( \frac{\partial \phi}{\partial s} \right) f
\]

Two-Dimensional Boundary Value Problems - Rectangular Elements

Rectangular Elements - Evaluation of \( k_e \)

Therefore, the partial derivatives of the shape functions may be written as:

\[
\frac{\partial N^T}{\partial x} = J_1 \frac{\partial x}{\partial s} + J_2 \frac{\partial x}{\partial t}
\]

Substituting all the pieces of the transformation in the \( k_e \) terms gives:

\[
k_e = \int \int \frac{1}{4} \left( J_1^T J_1 + J_2^T J_2 \right) ds dt
\]

where \( JJ = (J_1^T J_1 + J_2^T J_2) \).

The resulting elemental load vector contributes to the global system equations at those locations corresponding to the four nodes defining the element.
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS
RECTANGULAR ELEMENTS - Evaluation of $h_e$

Consider the integral: 

$$h_e = \int_{s=0}^{s=1} Nh ds$$

where the integration is along a boundary segment of the element.

Since the integration is computed along a single side of the rectangular element, the original shape functions reduce to:

$$N_i = 1 - \frac{s}{l_i}, \quad N_j = \frac{s}{l_j}$$

The resulting integral becomes:

$$h_e = \int_{s=0}^{s=1} \left[ \sum_i N_i \left( b_i N_i + h_i N_i \right) \right] ds$$

The resulting 2 x 1 elemental load vector contributes to the global system equations if the element has a side as part of the boundary.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS
RECTANGULAR ELEMENTS - Evaluation of $a_e$

The evaluation of the $a_e$ is very similar to $h_e$ except that there is an extra $N^T$ in the integrand. The variation of the function $\alpha(s)$ will be approximated as $\alpha = \alpha_0 N_0 + \alpha_1 N_1$. Consider the integral $a_e$:

$$a_e = \int_{s=0}^{s=1} N\alpha N^T ds$$

$$a_e = \frac{l}{12} \left[ 3\alpha_0 + \alpha_1 \quad \alpha_0 + \alpha_1 \quad \alpha_0 + 3\alpha_1 \right]$$

The resulting 2 x 2 stiffness matrix contributes to the global system equations when the element has a side as part of the boundary.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS
RECTANGULAR ELEMENTS - Problem #19

Verify the components of the $k_e$, $f_e$, $a_e$, and $h_e$ matrices conform to the results developed in the class notes.

In other words, show all the details of the derivation of the “stiffness” matrix and the loading vectors.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS
RECTANGULAR ELEMENTS - Example

Consider the non-dimensional torsion problem:

$$\nabla^2 \psi (X, Y) + 1 = 0 \quad \text{in} \quad \Omega \quad -1 \leq X \leq 1$$

$$\psi = 0 \quad \text{on} \quad \Gamma \quad -1 \leq Y \leq 1$$

with

$$X = \frac{x}{a} \quad Y = \frac{y}{a} \quad \psi = \frac{\phi}{2G\eta a^2}$$

The stresses and torque for the Prandlt stress function are:

$$\tau_{xx} = 2G\eta a \frac{\partial \psi}{\partial Y} \quad \tau_{yy} = -2G\eta a \frac{\partial \psi}{\partial X}$$

$$T = 4G\eta a \int \psi \ dx \ dy$$
Example - Discretization - The simplest model for torsion of a square bar, utilizing symmetry is a single rectangular element. The general problem domain and the FEM mesh are shown below.

Example - Elemental Formulation - Using a linear rectangular element the elemental stiffness matrix components are:

\[
\begin{bmatrix}
4 & -1 & -1 \\
-1 & 4 & -1 \\
-1 & -1 & 4
\end{bmatrix}
\]

A loading function of \( f = 1 \) gives a load vector of:

\[
\begin{bmatrix}
4f_1 + 2f_2 + 2f_3 + f_4 \\
2f_1 + 2f_2 + 2f_3 + f_4 \\
2f_1 + 2f_2 + 2f_3 + f_4
\end{bmatrix}
\]

Computation of Derived Variables - The partial derivatives with respect to \( x \) and \( y \) that define the stress components are:

\[
\begin{aligned}
\sigma_{xx} &= \frac{\partial^2 \phi}{\partial x^2} \\
\sigma_{yy} &= \frac{\partial^2 \phi}{\partial y^2} \\
\tau_{xy} &= \frac{\partial^2 \phi}{\partial x \partial y}
\end{aligned}
\]

Example - Solution - In this case, the solution is quite simple:

\[
\psi_1 = \frac{3}{8} \quad \Rightarrow \quad \phi_1 = \frac{3G\alpha^2}{4}
\]

Example - Solution - Repeat the previous non-dimensional torsion problem using four rectangular elements:

\[
\begin{aligned}
\psi_1 &= \frac{3}{8} \\
\phi_1 &= \frac{3G\alpha^2}{4}
\end{aligned}
\]
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

RECTANGULAR ELEMENTS

Example – Discretization - Discretization - The simplest model for torsion of a square bar, utilizing symmetry and using four rectangular elements is shown below.

Example - Elemental Formulation - Using a linear rectangular element the elemental stiffness matrix components are:

\[
\begin{bmatrix}
4 & -1 & -2 & -1 \\
-1 & 4 & -1 & -2 \\
-2 & -1 & 4 & -1 \\
2 & -1 & 2 & 1
\end{bmatrix}
\]

For element 1 & 2:

\[
\begin{bmatrix}
4 & -1 & -2 & -1 \\
-1 & 4 & -1 & -2 \\
-2 & -1 & 4 & -1 \\
2 & -1 & 2 & 1
\end{bmatrix}
\]

For element 3 & 4:

\[
\begin{bmatrix}
4 & -1 & -2 & -1 \\
-1 & 4 & -1 & -2 \\
-2 & -1 & 4 & -1 \\
2 & -1 & 2 & 1
\end{bmatrix}
\]

Example - Constraints - For this model, \( \psi = 0 \) on the boundary, therefore, \( \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8, \) and \( \psi_9 = 0. \)

Example - Assembly - Compiling each elemental matrix into the system equations yields:

\[
\begin{bmatrix}
4 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 8 & -2 & 0 & 0 & 0 & 0 & 0 & 2 \\
-1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & -1 & 0 & 4 & 0 & 0 & 0 & 0 & 2 \\
-1 & -2 & 0 & 2 & 0 & -2 & 0 & -1 & 2 \\
2 & -1 & 4 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 4 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & -1 & 2 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
\end{bmatrix}
\]
Example - Solution - In this case, the solution is quite simple:

\[ \begin{align*}
\Psi_1 &= \frac{174}{560} \\
\Psi_2 &= \frac{135}{560} \\
\Psi_3 &= \frac{135}{560} \\
\Psi_5 &= \frac{108}{560}
\end{align*} \]

\[ X = \frac{x}{a} \quad \Psi = \frac{\phi}{2G\ell a^2} \]

\[ \begin{align*}
\phi_1 &= \frac{348G\ell a^3}{560} \\
\phi_2 &= \frac{270G\ell a^3}{560} \\
\phi_3 &= \frac{270G\ell a^3}{560} \\
\phi_4 &= \frac{216G\ell a^3}{560}
\end{align*} \]

Example - Computation of Derived Variables - The partial derivatives with respect to \( x \) and \( y \) that define the stress components are:

\[ T = 4G\ell a^4 \int \int \Psi dXdY = 4 \left[ \frac{2A_0}{4} \sum \phi_i \right] \]

\[ T = \frac{2.292G\ell a^4}{1.120} = 2.0464G\ell a^4 \]

\[ T_{\text{exact}} = \frac{16G\ell a^4}{6} = 2.6667G\ell a^4 \]

Example - Computation of Derived Variables - The partial derivatives with respect to \( x \) and \( y \) that define the stress components are:

\[ \begin{align*}
\Pi(X,Y) &= 1 \quad \text{in } \Omega \\
\Psi &= 0 \quad \text{on } \Gamma \\
X &= \frac{x}{a} \\
Y &= \frac{y}{a}
\end{align*} \]

PROBLEM #20 - Consider the non-dimensional torsion problem:

\[ \begin{align*}
\nabla \cdot \Psi(X, Y) &= 0 & \text{in } \Omega \\
\Psi &= 0 & \text{on } \Gamma \\
X &= \frac{x}{a} & \quad Y = \frac{y}{a} & \quad \Psi = \frac{\phi}{2G\ell a^2}
\end{align*} \]

End of Rectangular Elements